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CRYSTAL BASES OF MODIFIED QUANTIZED ENVELOPING ALGEBRA

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0. Introduction.

0.1. G. Lusztig gives the crystal base on the modified quantized enveloping algebra $\tilde{U}_q(\mathfrak{g})$ in [L2]. The algebra $\tilde{U}_q(\mathfrak{g})$ is obtained from the quantized universal enveloping algebra $U_q(\mathfrak{g})$ by modifying the torus part $\bigoplus_{h\in P^*}\mathbf{Q}(q)q^h$ to $\bigoplus_\lambda \mathbf{Q}(q)a_\lambda$, where a_λ is the projector to the weight space of weight λ (see §1.2). He gives also several conjectures on its properties in [L3]. The purpose of this paper is to study the structure of crystal bases $B(\tilde{U}_q(\mathfrak{g}))$ of $\tilde{U}_q(\mathfrak{g})$ and to give an affirmative answer to some of his conjectures.

0.2. Let us explain the results obtained here more precisely. We establish that the crystal structure of $\tilde{U}_q(\mathfrak{g})$ is described by those of $U_q^-(\mathfrak{g})$ and $U_q^+(\mathfrak{g})$. Namely, let $B(\infty)$ be the crystal base of $U_q^-(\mathfrak{g})$ and $B(-\infty)$ the one of $U_q^+(\mathfrak{g})$. Let T_λ be the crystal consisting of a single element of weight λ. Then the crystal base of $\tilde{U}_q(\mathfrak{g})$ is isomorphic to the direct sum of $B(\infty)\otimes T_\lambda\otimes B(-\infty)$ (Theorem 3.1.1). This fact is a reflection of $\tilde{U}_q(\mathfrak{g}) = \bigoplus_\lambda U_q^-(\mathfrak{g})\otimes U_q^+(\mathfrak{g})\otimes \mathbf{Q}(q)a_\lambda$. The algebra $\tilde{U}_q(\mathfrak{g})$ has the antiautomorphism $*$ that sends e_i, f_i to themselves and a_λ to $a_{-\lambda}$. We prove that the crystal base is stable by $*$ (Theorem 4.3.2). This is one of the conjectures of Lusztig [L3]. This automorphism sends $b_1\otimes t_\lambda\otimes b_2\in B(\infty)\otimes T_\lambda\otimes B(-\infty)\subset B(\tilde{U}_q(\mathfrak{g}))$ to $b_1^*\otimes t_{-\lambda-wtb_1-wtb_2}\otimes b_2^*$. By this automorphism, $B(\tilde{U}_q(\mathfrak{g}))$ has another crystal structure. These two structures are compatible (see §5), and $B(\tilde{U}_q(\mathfrak{g}))$ may be regarded as a crystal over $\mathfrak{g}\oplus\mathfrak{g}$. This is a reflection of the $U_q(\mathfrak{g})$-bimodule structure of $\tilde{U}_q(\mathfrak{g})$.

0.3. In [K2], the author introduces "the dual algebra" $A_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$ and its crystal base $B(A_q(\mathfrak{g}))$. This algebra has the Peter-Weyl–type decomposition

$$A_q(\mathfrak{g}) = \bigoplus_{\lambda\in P_+} V^r(\lambda)\otimes V(\lambda).$$

Here P_+ is the set of dominant integral weights and $V(\lambda)$ and $V^r(\lambda)$ are the left and right highest-weight module with highest weight λ. Accordingly $B(A_q(\mathfrak{g}))$ has the crystal structure

$$B(A_q(\mathfrak{g})) = \bigoplus_{\lambda\in P_+} B(\lambda)\otimes B(\lambda).$$

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When $\mathfrak{g}$ is finite-dimensional, $B(\tilde{U}_q(\mathfrak{g}))$ is isomorphic to $B(A_q(\mathfrak{g}))$. When $\mathfrak{g}$ is affine, $\tilde{U}_q(\mathfrak{g})$ is the direct sum of three algebras $\tilde{U}_q(\mathfrak{g})_+$, $\tilde{U}_q(\mathfrak{g})_0$, and $\tilde{U}_q(\mathfrak{g})_-$. They consist of elements of positive level, level 0, and negative level, respectively. Accordingly $B(\tilde{U}_q(\mathfrak{g}))$ is the direct sum of $B(\tilde{U}_q(\mathfrak{g})_+)$, $B(\tilde{U}_q(\mathfrak{g})_0)$, and $B(\tilde{U}_q(\mathfrak{g})_-)$. The crystal structure of $B(\tilde{U}_q(\mathfrak{g})_\pm)$ is rather simple. Namely, $B(\tilde{U}_q(\mathfrak{g})_+) \oplus B(0)$ is isomorphic to $B(A_q(\mathfrak{g}))$. Here $B(0)$ is the crystal base of the trivial representation. However, the author does not know much about the structure of $B(\tilde{U}_q(\mathfrak{g})_0)$. The result of [IIJMNT] suggests that $B(\tilde{U}_q(\mathfrak{g})_0)$ is a direct sum of the affinization of the crystals of finite-dimensional $\tilde{U}_q(\mathfrak{g})$-modules. We show here one property: for any connected component B' of $B(\tilde{U}_q(\mathfrak{g}))$, $\{(wt(b), wt(b)); b \in B'\}$ is bounded from above (see §9.3).

0.4. We show also that the Weyl group operates on $B(\tilde{U}_q(\mathfrak{g}))$ or more generally on the crystal base of integrable $U_q(\mathfrak{g})$-modules. The action S_i of simple reflection is given by

$$S_i b = \begin{cases} \tilde{f}_i^{\langle h_i, wtb\rangle} b & \text{if } \langle h_i, wtb\rangle \geqslant 0, \\ \tilde{e}_i^{-\langle h_i, wtb\rangle} b & \text{if } \langle h_i, wtb\rangle \leqslant 0. \end{cases}$$

We show (Theorem 7.2.2) that $\{S_i\}$ satisfies the braid relation. We call a crystal base b an extremal vector if, for any $w \in W$ and $i \in I$, $S_w b$ is killed by either $\tilde{e}_i$ or $\tilde{f}_i$. We show another remarkable property of $B(\tilde{U}_q(\mathfrak{g}))$. Let B' be a connected component of $B(\tilde{U}_q(\mathfrak{g}))$. Then B' may not contain either a highest-weight vector or a lowest-weight vector, but it always contains an extremal vector.

1. Notations.

1.1. Definition of quantized enveloping algebra. Let us recall the definition of $\tilde{U}_q(\mathfrak{g})$ (cf. [K]). We prepare the following data:

(1.1.1) a free $\mathbf{Z}$-module P (weight lattice),

(1.1.2) an index set I and $\alpha_i \in P$ and $h_i \in P^* = \mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{Z})$ for $i \in I$,

(1.1.3) a $\mathbf{Q}$-valued symmetric bilinear form $(\ ,\ )$ on P.

We assume for the sake of simplicity that there exist $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for any $j \in I$. We call Λ_i the fundamental weight. We assume further that $\{\alpha_i\}_{i\in I}$ is linearly independent. Many of the results in this paper still hold without these assumptions. Assume that they satisfy the conditions

(1.1.4) $$(\alpha_i, \alpha_i) \in 2\mathbf{Z}_{>0} \quad \text{for } i \in I,$$

(1.1.5) $$\langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for } i \in I \text{ and } \lambda \in P,$$

(1.1.6) $$(\alpha_i, \alpha_j) \leqslant 0 \quad \text{for } i \neq j \in I.$$

The quantized enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbf{Q}(q)$-algebra generated by e_i, f_i ($i \in I$), and q^h ($h \in P^*$) satisfying the defining relations

(1.1.7) $$q^h = 1 \quad \text{for } h = 0;$$

(1.1.8) $$q^{h_1}q^{h_2} = q^{h_1+h_2} \quad \text{for } h_1, h_2 \in P^*;$$

(1.1.9) $$q^h e_i q^{-h} = q^{\langle h, \alpha_i\rangle} e_i \quad \text{and}$$

$$q^h f_i q^{-h} = q^{-\langle h, \alpha_i\rangle} f_i;$$

(1.1.10) $$[e_i, f_j] = \delta_{ij}\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$$

$$\text{where } q_i = q^{(\alpha_i,\alpha_i)/2} \quad \text{and} \quad t_i = q^{(\alpha_i,\alpha_i)h_i/2};$$

(1.1.11) $$\text{for } i \neq j \in I, \quad \text{setting } c = 1 - \langle h_i, \alpha_j\rangle,$$

$$\sum_{n=0}^{c} (-)^n e_i^{(n)} e_j e_i^{(c-n)} = \sum (-)^n f_i^{(n)} f_j f_i^{(c-n)} = 0.$$

Here $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^{n} [k]_i$, $e_i^{(n)} = e_i^n/[n]_i!$, and $f_i^{(n)} = f_i^n/[n]_i!$. We set $\{x\} = (x - x^{-1})/(q - q^{-1})$ and $\left\{\begin{matrix} x \\ n \end{matrix}\right\} = \prod_{k=1}^{n} \frac{\{q^{1-k}x\}}{[k]}$. We denote by $U_q^{\mathbf{Z}}(\mathfrak{g})$ the $\mathbf{Z}[q, q^{-1}]$-algebra generated by $e_i^{(n)}$, $f_i^{(n)}$, q^h, $\left\{\begin{matrix} q^h \\ n \end{matrix}\right\}$ ($i \in I$, $h \in P^*$, $n \in \mathbf{Z}_{\geqslant 0}$).

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the $\mathbf{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by the e_i's (resp. the f_i's) and $U_q^+(\mathfrak{g})_{\mathbf{Z}}$ (resp. $U_q^-(\mathfrak{g})_{\mathbf{Z}}$) the $\mathbf{Z}[q, q^{-1}]$-subalgebra generated by the $e_i^{(n)}$'s (resp. the $f_i^{(n)}$'s). We set $U_q^{\mathbf{Q}}(\mathfrak{g}) = \mathbf{Q}[q, q^{-1}] \otimes_{\mathbf{Z}[q]} U_q^{\mathbf{Z}}(\mathfrak{g})$, etc. For $\xi \in Q = \oplus \mathbf{Z}\alpha_i$, we set

$$U_q(\mathfrak{g})_\xi = \{P \in U_q(\mathfrak{g});\, q^h P q^{-h} = q^{\langle h, \xi\rangle} \text{ for any } h \in P^*\},$$

$$U_q^{\pm}(\mathfrak{g})_\xi = U_q^{\pm}(\mathfrak{g}) \cap U_q(\mathfrak{g})_\xi.$$

We set $|\xi| = \sum |n_i|$ for $\xi = \sum n_i\alpha_i \in Q$. We define the filtration F of $U_q^{\pm}(\mathfrak{g})$ by $F_n(U_q^{\pm}(\mathfrak{g})) = \bigoplus_{|\xi| \leqslant n} U_q^{\pm}(\mathfrak{g})_\xi$.

We say $\lambda \in P$ is *dominant* (resp. *antidominant*) if $\langle h_i, \lambda\rangle \geqslant 0$ (resp. $\langle h_i, \lambda\rangle \leqslant 0$) for any i. We write $P_{\pm}$ for the set of dominant (resp. antidominant) integral weights.

1.2. Definition of modified quantized enveloping algebra. Let $\mathrm{Mod}(\mathfrak{g}, P)$ denote the category of left $U_q(\mathfrak{g})$-modules M with the weight decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda$$

where

$$M_\lambda = \{u \in M;\ q^h u = q^{\langle h, \lambda \rangle} u \text{ for any } h \in P^*\}.$$

Let (forget) be the functor from Mod(g, P) to the category of vector spaces over $\mathbf{Q}(q)$, forgetting the $U_q(\mathfrak{g})$-module structure. Let R denote the endomorphism ring of (forget). Hence to give an element of R is to associate an endomorphism $\varphi(M)$ of M with each M in Mod(g, P) such that, for any morphism $f\colon M \to M'$ in Mod(g, P), $\varphi(M') \circ f = f \circ \varphi(M)$ holds. Note that R contains $U_q(\mathfrak{g})$. For $\lambda \in P$ let $a_\lambda \in R$ denote the projector $M \to M_\lambda$ to the weight space. Then the defining relation of a_λ (as a left $U_q(\mathfrak{g})$-module) is

(1.2.1) $$q^h a_\lambda = q^{\langle h, \lambda \rangle} a_\lambda .$$

We have

(1.2.2) $$a_\lambda P = P a_{\lambda - \xi} \qquad \text{for } \xi \in Q \text{ and } P \in U_q(\mathfrak{g})_\xi ,$$

(1.2.3) $$a_\lambda a_\mu = \delta_{\lambda, \mu} a_\lambda .$$

Then R is isomorphic to the direct product $\prod_{\lambda \in P} U_q(\mathfrak{g}) a_\lambda$. We set

$$\tilde{U}_q(\mathfrak{g}) = \bigoplus_{\lambda \in P} U_q(\mathfrak{g}) a_\lambda .$$

Then $\tilde{U}_q(\mathfrak{g})$ is a subring of R by (1.2.2) and (1.2.3). Hence any object M in Mod(g, P) may be regarded as a left $\tilde{U}_q(\mathfrak{g})$-module. We set

(1.2.4) $$\tilde{U}_q^{\mathbf{Z}}(\mathfrak{g}) = \bigoplus_\lambda U_q^{\mathbf{Z}}(\mathfrak{g}) a_\lambda ,$$

(1.2.5) $$\tilde{U}_q^{\pm}(\mathfrak{g}) = \bigoplus_\lambda U_q^{\pm}(\mathfrak{g}) a_\lambda = \bigoplus_\lambda a_\lambda U_q^{\pm}(\mathfrak{g}).$$

They are subrings of $\tilde{U}_q(\mathfrak{g})$.

1.3. The automorphisms of $U_q(\mathfrak{g})$. Let $*$ denote the antiautomorphism of $U_q(\mathfrak{g})$ given by

(1.3.1) $$q^* = q, \qquad (q^h)^* = q^{-h}, \qquad e_i^* = e_i, \qquad f_i^* = f_i .$$

Let φ be the antiautomorphism of $U_q(\mathfrak{g})$ given by

(1.3.2) $$\varphi(q) = q, \qquad \varphi(q^h) = q^h, \qquad \varphi(f_i) = f_i, \qquad \varphi(e_i) = e_i .$$

Let us denote by $\vee$ the automorphism $\varphi \circ * = * \circ \varphi$ of $U_q(\mathfrak{g})$. Hence we have

$$(1.3.3) \qquad q^\vee = q, \qquad (q^h)^\vee = q^{-h}, \qquad e_i^\vee = f_i, \qquad f_i^\vee = e_i.$$

Let $-$ be the automorphism of $U_q(\mathfrak{g})$ given by

$$(1.3.4) \qquad \bar{q} = q^{-1}, \qquad (q^h)^- = q^{-h}, \qquad \bar{e}_i = e_i, \qquad \bar{f}_i = f_i.$$

They commute with each other.

These automorphisms and antiautomorphisms are extended to those of $\tilde{U}_q(\mathfrak{g})$. We shall denote them by the same letter. We have

$$(1.3.5) \qquad a_\lambda^* = a_{-\lambda}, \qquad \varphi(a_\lambda) = a_\lambda, \qquad a_\lambda^\vee = a_{-\lambda}, \qquad \bar{a}_\lambda = a_\lambda.$$

1.4. Tensor product. In this article, we take the comultiplication Δ of $U_q(\mathfrak{g})$ given by

$$(1.4.1) \qquad \begin{aligned} \Delta q^h &= q^h \otimes q^h, \\ \Delta e_i &= e_i \otimes t_i^{-1} + 1 \otimes e_i, \\ \Delta f_i &= f_i \otimes 1 + t_i \otimes f_i. \end{aligned}$$

By this comultiplication, the tensor product of $U_q(\mathfrak{g})$-modules has a structure of $U_q(\mathfrak{g})$-module.

For an automorphism g of $U_q(\mathfrak{g})$ and a $U_q(\mathfrak{g})$-module M, let us denote by M^g the $U_q(\mathfrak{g})$-module $\{u^g; u \in M\}$ with $Pu^g = (g(P)u)^g$ for $u \in M$ and $P \in U_q(\mathfrak{g})$. With this notation we have

$$(1.4.2) \qquad (M \otimes N)^\wedge \cong N^\wedge \otimes M^\wedge$$

by $(u \otimes v)^\wedge \leftrightarrow v^\wedge \otimes u^\wedge$ for $u \in M$, $v \in N$.

1.5. Crystals. Let us recall the definition of crystals (cf. [K3]).

Definition 1.5.1. A crystal B is a set with the following data:

$$(1.5.1) \qquad \text{a map } wt\colon B \to P,$$

$$(1.5.2) \qquad \varepsilon_i\colon B \to \mathbf{Z} \sqcup \{-\infty\}, \qquad \varphi_i\colon B \to \mathbf{Z} \sqcup \{-\infty\} \qquad \text{for } i \in I,$$

$$(1.5.3) \qquad \tilde{e}_i\colon B \to B \sqcup \{0\} \quad \text{and} \quad \tilde{f}_i\colon B \to B \sqcup \{0\} \qquad \text{for } i \in I.$$

They satisfy the following axioms.

(1.5.4) For $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle$.

(1.5.5) For $b \in B$ with $\tilde{e}_i b \in B$,

$$wt(\tilde{e}_i b) = wt(b) + \alpha_i .$$

(1.5.6) For $b \in B$ with $\tilde{f}_i b \in B$,

$$wt(\tilde{f}_i b) = wt(b) - \alpha_i .$$

(1.5.7) For $b_1, b_2 \in B$, $\tilde{f}_i b_2 = b_1$ if and only if $\tilde{e}_i b_1 = b_2$.

(1.5.8) If $\varepsilon_i(b) = -\infty$ then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Definition 1.5.2. A morphism $\psi: B_1 \to B_2$ from a crystal B_1 to a crystal B_2 is a map $\psi: B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ satisfying the following axioms:

(1.5.9) $\psi(0) = 0$;

(1.5.10) if $b \in B_1$ and $\psi(b) \in B_2$, then

$$wt(\psi(b)) = wt(b), \qquad \varepsilon_i(\psi(b)) = \varepsilon_i(b) \qquad \text{and} \qquad \varphi_i(\psi(b)) = \varphi_i(b);$$

(1.5.11) if $b \in B_1$ satisfies $\psi(\tilde{e}_i b) \neq 0$ and $\psi(b) \neq 0$, then $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$;

(1.5.12) if $b \in B_1$ satisfies $\psi(\tilde{f}_i b) \neq 0$ and $\psi(b) \neq 0$, then $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

The definition of morphisms is slightly different from [K3].

Let $\mathscr{C}(I, P)$ denote the category of crystals.

A morphism $\psi: B_1 \to B_2$ of crystals is called *strict* if the associated map from $B_1 \sqcup \{0\}$ to $B_2 \sqcup \{0\}$ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$. If the associated map is injective, then ψ is called *embedding*.

A crystal B is called *seminormal* if, for any $b \in B$ and $i \in I$, $\varepsilon_i(b)$ and $\varphi_i(b)$ are nonnegative integers and

$$\varepsilon_i(b) = \max \{n \geqslant 0; \tilde{e}_i^n b \in B\},$$

$$\varphi_i(b) = \max \{n \geqslant 0; \tilde{f}_i^n b \in B\}.$$

In such a case, we set

$$\tilde{e}_i^{\max} b = \tilde{e}_i^{\varepsilon_i(b)} b \qquad \text{and}$$

$$\tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b .$$

A crystal B is called *normal* if, for any subset J of I such that $\{(\alpha_i, \alpha_j)\}_{i,j\in J}$ is a poisitive-definite symmetric matrix, B is isomorphic (in $\mathscr{C}(J, P)$) to a crystal base of an integrable $U_q(\mathfrak{g}_J)$-module. Here $U_q(\mathfrak{g}_J)$ is the quantized universal enveloping algebra generated by e_j, f_j ($j \in J$), and q^h ($h \in P^*$).

For crystals B_1 and B_2, let us define their tensor product $B_1 \otimes B_2$ by

(1.5.13) $$B_1 \otimes B_2 = \{b_1 \otimes b_2; b_1 \in B_1, b_2 \in B_2\},$$

(1.5.14) $$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2),$$

(1.5.15) $$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle),$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2)\rangle),$$

(1.5.16) $$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_i \otimes b_2 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_i \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0. Then $\otimes$ is a functor from $\mathscr{C}(I, P) \times \mathscr{C}(I, P)$ to $\mathscr{C}(I, P)$ and satisfies the associative law: $(B_1 \otimes B_2) \otimes B_3 \cong B_1 \otimes (B_2 \otimes B_3)$ by $(b_1 \otimes b_2) \otimes b_3 \leftrightarrow b_1 \otimes (b_2 \otimes b_3)$.

For a crystal B, let us denote by $B^\vee$ the crystal defined by

(1.5.17) $$B^\vee = \{b^\vee; b \in B\},$$

(1.5.18) $$wt(b^\vee) = -wt(b), \qquad \varepsilon_i(b^\vee) = \varphi_i(b), \qquad \varphi_i(b^\vee) = \varepsilon_i(b),$$

$$\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee \quad \text{and} \quad \tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee.$$

Then we have (cf. (1.4.2))

(1.5.19) $$(B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee \quad \text{by} \quad (b_1 \otimes b_2)^\vee \leftrightarrow b_2^\vee \otimes b_1^\vee.$$

Example 1.5.3.

1. $C = \{c\}$ with

$$wt(c) = 0, \qquad \varepsilon_i(c) = \varphi_i(c) = 0,$$

$$\tilde{e}_i c = \tilde{f}_i c = 0.$$

For any seminormal crystal B, $B \otimes C$ and $C \otimes B$ are isomorphic to B.

2. For $\lambda \in P$, $T_\lambda = \{t_\lambda\}$ with

$$wt(t_\lambda) = \lambda, \qquad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty,$$

$$\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0.$$

We have $T_\lambda \otimes T_\mu \cong T_{\lambda+\mu}$ and $B \otimes T_0 \cong T_0 \otimes B \cong B$ for any crystal B.

3. For $i \in I$, $B_i = \{b_i(n); n \in \mathbf{Z}\}$ with $wt(b_i(n)) = n\alpha_i$,

$$\varepsilon_i(b_i(n)) = -n, \qquad \varphi_i(b_i(n)) = n,$$

$$\varepsilon_j(b_i(n)) = \varphi_j(b_i(n)) = -\infty \qquad \text{for } j \neq i,$$

$$\text{and} \qquad \tilde{e}_i b_i(n) = b_i(n+1), \qquad \tilde{f}_i b_i(n) = b_i(n-1),$$

$$\tilde{e}_j b_i(n) = \tilde{f}_j b_i(n) = 0 \qquad \text{for } j \neq i.$$

We write b_i for $b_i(0)$.

4. $B(\infty)$ denotes the crystal associated with $U_q^-(\mathfrak{g})$. We denote by u_∞ the vector of weight 0.

5. $B(-\infty) = B(\infty)^\vee$. This is regarded as the crystal associated with $U_q^+(\mathfrak{g})$. We set $u_{-\infty} = u_\infty^\vee$.

6. For $\lambda \in P_+$, let $B(\lambda)$ denote the crystal associated with the irreducible module $V(\lambda)$ of highest weight λ. Set $B(-\lambda) = B(\lambda)^\vee$. Then $B(-\lambda)$ is isomorphic to the crystal associated with the irreducible module $V(-\lambda)$ of lowest weight $-\lambda$. Then C is isomorphic to $B(0)$.

Let us recall that the automorphism $*$ of $U_q^-(\mathfrak{g})$ induces the automorphism of $B(\infty)$ (cf. [K3]). We shall also denote it by $*$. We set $\varepsilon_i^*(b) = \varepsilon_i(b^*)$, $\varphi_i^*(b) = \varphi_i(b^*)$, $\tilde{e}_i^* b = (\tilde{e}_i b^*)^*$, and $\tilde{f}_i^* b = (\tilde{f}_i b^*)^*$. For $\lambda \in P_+$, there exists a unique embedding $B(\lambda)$ into $B(\infty) \otimes T_\lambda$, whose image is $\{b \otimes t_\lambda \in B(\infty) \otimes T_\lambda; \varepsilon_i^*(b) \leqslant \langle h_i, \lambda\rangle\}$. Similarly, we define $*$, ε_i^*, etc., for $B(-\infty)$. Then, for $\lambda \in P_-$, $B(\lambda)$ is isomorphic to the subcrystal $\{t_\lambda \otimes b \in T_\lambda \otimes B(-\infty); \varphi_i^*(b) \leqslant -\langle h_i, \lambda\rangle\}$ of $T_\lambda \otimes B(-\infty)$.

Let us also recall that, for any $i \in I$, there is a unique strict embedding

$$\Phi_i \colon B(\infty) \to B(\infty) \otimes B_i$$

such that $\Phi_i(u_\infty) = u_\infty \otimes b_i$. We have

$$\Phi_i(b) = \tilde{e}_i^{*\varepsilon_i^*(b)} b \otimes \tilde{f}_i^{\varepsilon_i^*(b)} b_i.$$

Also we have

$$B(\infty) \otimes B_i \cong \bigoplus_{n \geqslant 0} B(\infty) \otimes T_{n\alpha_i}. \tag{1.5.20}$$

Taking $\vee$ we obtain

$$\Phi_i^\vee : B(-\infty) \to B_i \otimes B(-\infty) \qquad \text{and}$$

$$B_i \otimes B(-\infty) \cong \bigoplus_{n \geqslant 0} T_{-n\alpha_i} \otimes B(-\infty).$$

1.6. Balanced triples. Let us recall the definition of balanced triple.

Let V be a vector space over $\mathbf{Q}(q)$. For a subring B of $\mathbf{Q}(q)$, a B-lattice of V is a B-submodule M of V such that $V \cong \mathbf{Q}(q) \otimes_B M$.

Let A (resp. $\bar{A}$) be the subring of $\mathbf{Q}(q)$ consisting of functions regular at $q = 0$ (resp. $q = \infty$). Let $V_{\mathbf{Z}}$ be a $\mathbf{Z}[q, q^{-1}]$-lattice of V, L an A-lattice of V, and $\bar{L}$ an $\bar{A}$-lattice of V. Then we have the following lemma.

LEMMA 1.6.1 [K1]. *Set $E = V_{\mathbf{Z}} \cap L \cap \bar{L}$. Then the following conditions are equivalent.*

(i) *$E \to V_{\mathbf{Z}} \cap L/V_{\mathbf{Z}} \cap qL$ is an isomorphism.*

(ii) *$E \to V_{\mathbf{Z}} \cap \bar{L}/V_{\mathbf{Z}} \cap q^{-1}\bar{L}$ is an isomorphism.*

(iii) *$(V_{\mathbf{Z}} \cap qL) \oplus (V_{\mathbf{Z}} \cap \bar{L}) \to V_{\mathbf{Z}}$ is an isomorphism.*

(iv) *$A \otimes_{\mathbf{Z}} E \to L$, $\bar{A} \otimes_{\mathbf{Z}} E \to L$, $\mathbf{Z}[q, q^{-1}] \otimes_{\mathbf{Z}} E \to V_{\mathbf{Z}}$, $\mathbf{Q}(q) \otimes_{\mathbf{Z}} E \to V$ are isomorphisms.*

We call $(L, \bar{L}, V_{\mathbf{Z}})$ *balanced* if these equivalent conditions are satisfied. Let us denote by G the inverse of the isomorphism $E \to V_{\mathbf{Z}} \cap L/V_{\mathbf{Z}} \cap qL$. If B is a base of $V_{\mathbf{Z}} \cap L/V_{\mathbf{Z}} \cap qL$, then $\{G(b); b \in B\}$ is a base of V. The following proposition is easily proven (e.g., by (iii)).

PROPOSITION 1.6.2 (Triangular property). *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence of vector spaces over $\mathbf{Q}(q)$. Let $V_{i\mathbf{Z}}$ (resp. L_i, $\bar{L}_i$) be a $\mathbf{Z}[q, q^{-1}]$-lattice (resp. A-lattice, $\bar{A}$-lattice) of V_i $(i = 1, 2, 3)$. Assume that*

$$0 \to V_{1\mathbf{Z}} \to V_{2\mathbf{Z}} \to V_{3\mathbf{Z}} \to 0,$$

$$0 \to V_{1\mathbf{Z}} \cap L_1 \to V_{2\mathbf{Z}} \cap L_2 \to V_{3\mathbf{Z}} \cap L_3 \to 0 \qquad \textit{and}$$

$$0 \to V_{1\mathbf{Z}} \cap \bar{L}_1 \to V_{2\mathbf{Z}} \cap \bar{L}_2 \to V_{3\mathbf{Z}} \cap \bar{L}_3 \to 0$$

are exact. If two of $(V_{i\mathbf{Z}}, L_i, \bar{L}_i)$ are balanced, then so is the other.

2. Result of Lusztig [L2].

2.1. Global base of $\tilde{U}_q(\mathfrak{g})$. Let us recall the result of Lusztig on the crystal base of $\tilde{U}_q(\mathfrak{g})$. For a dominant integral weight $\lambda \in P_+$, let us denote by $V(\lambda)$ (resp. $V(-\lambda)$) the irreducible module with highest- (resp. lowest-) weight λ (resp. $-\lambda$). Let u_λ (resp. $u_{-\lambda}$) be the highest- (resp. lowest-) weight vector. For $\lambda \in P_+, \mu \in P_-$, we set $V(\lambda, \mu) = V(\lambda) \otimes V(\mu)$. Then $V(\lambda, \mu)$ is generated by $u_\lambda \otimes u_\mu$ as a $U_q(\mathfrak{g})$-module, and the defining relation of $u_\lambda \otimes u_\mu$ is

(2.1.1) $$q^h(u_\lambda \otimes u_\mu) = q^{\langle h, \lambda+\mu\rangle}(u_\lambda \otimes u_\mu),$$

(2.1.2) $$e_i^{1-\langle h_i, \mu\rangle}(u_\lambda \otimes u_\mu) = 0,$$

$$f_i^{1+\langle h_i, \lambda\rangle}(u_\lambda \otimes u_\mu) = 0.$$

Let us define the automorphism $-$ of $V(\lambda, \mu)$ by

(2.1.3) $$(P(u_\lambda \otimes u_\mu))^- = \bar{P}(u_\lambda \otimes u_\mu) \qquad \text{for } P \in U_q(\mathfrak{g}).$$

We set $L(\lambda, \mu) = L(\lambda) \otimes_A L(\mu)$ and $B(\lambda, \mu) = B(\lambda) \otimes B(\mu) \subset L(\lambda, \mu)/qL(\lambda, \mu)$. Then $(L(\lambda, \mu), B(\lambda, \mu))$ is a crystal base of $V(\lambda, \mu)$.

Set $V_{\mathbf{Z}}(\lambda, \mu) = U_q^{\mathbf{Z}}(\mathfrak{g})(u_\lambda \otimes u_\mu) = U_q^{\mathbf{Z}}(\mathfrak{g})u_\lambda \otimes U_q^{\mathbf{Z}}(\mathfrak{g})u_\mu$ and $V_{\mathbf{Q}}(\lambda, \mu) = U_q^{\mathbf{Q}}(\mathfrak{g})(u_\lambda \otimes u_\mu)$. The following results are due to Lusztig.

PROPOSITION 2.1.1 [L2]. *$(L(\lambda, \mu), L(\lambda, \mu)^-, V_{\mathbf{Z}}(\lambda, \mu))$ is balanced.*

Let G be the inverse of the isomorphism $L(\lambda, \mu) \cap L(\lambda, \mu)^- \cap V_{\mathbf{Q}}(\lambda, \mu) \to L(\lambda, \mu)/qL(\lambda, \mu)$. Then $V_{\mathbf{Z}}(\lambda, \mu) = \bigoplus_{b \in B(\lambda, \mu)} \mathbf{Z}[q, q^{-1}]G(b)$.

THEOREM 2.1.2 [L2]. *There exist a unique A-lattice $L(\tilde{U}_q(\mathfrak{g}))$ of $\tilde{U}_q(\mathfrak{g})$ and a unique base $B(\tilde{U}_q(\mathfrak{g}))$ of $L(\tilde{U}_q(\mathfrak{g}))/qL(\tilde{U}_q(\mathfrak{g}))$ satisfying the following properties.*

(i) *$(L(\tilde{U}_q(\mathfrak{g})), L(\tilde{U}_q(\mathfrak{g}))^-, \tilde{U}_q^{\mathbf{Z}}(\mathfrak{g}))$ is balanced.*

(ii) *Let G denote the inverse of $L(\tilde{U}_q(\mathfrak{g})) \cap L(\tilde{U}_q(\mathfrak{g}))^- \cap \tilde{U}_q^{\mathbf{Q}}(\mathfrak{g}) \xrightarrow{\sim} L(\tilde{U}_q(\mathfrak{g}))/qL(\tilde{U}_q(\mathfrak{g}))$. $L(\tilde{U}_q(\mathfrak{g})) = \bigoplus_{\lambda \in P} L(\tilde{U}_q(\mathfrak{g})a_\lambda)$ and $B(\tilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda \in P} B(\tilde{U}_q(\mathfrak{g})a_\lambda)$ where $L(\tilde{U}_q(\mathfrak{g}))a_\lambda = L(\tilde{U}_q(\mathfrak{g})) \cap U_q(\mathfrak{g})a_\lambda$ and $B(U_q(\mathfrak{g})a_\lambda) = B(\tilde{U}_q(\mathfrak{g})) \cap (L(U_q(\mathfrak{g})a_\lambda)/qL(U_q(\mathfrak{g})a_\lambda))$.*

(iii) *For any $\xi \in P_+$ and $\eta \in P_-$, let $\Phi(\xi, \eta)$ denote the $U_q(\mathfrak{g})$-linear map $U_q(\mathfrak{g})a_{\xi+\eta} \to V(\xi) \otimes V(\eta)$ sending $a_{\xi+\eta}$ to $u_\xi \otimes u_\eta$. Then $\Phi(\xi, \eta)L(U_q(\mathfrak{g})a_{\xi+\eta}) = L(\xi) \otimes L(\eta)$.*

(iv) *Let $\bar{\Phi}(\xi, \eta)$ be the induced homomorphism $L(U_q(\mathfrak{g})a_{\xi+\eta})/qL(U_q(\mathfrak{g})a_{\xi+\eta})$ to $L(\xi) \otimes L(\eta)/qL(\xi) \otimes L(\eta)$. Then $\{b \in B(U_q(\mathfrak{g})a_{\xi+\eta});\ \bar{\Phi}(\xi, \eta)b \neq 0\} \xrightarrow{\sim} B(\xi) \otimes B(\eta)$ and $\Phi(\xi, \eta)G(b) = G(\bar{\Phi}(\xi, \eta)b)$ for any $b \in B(U_q(\mathfrak{g})a_{\xi+\eta})$.*

(v) *$B(\tilde{U}_q(\mathfrak{g}))$ has a structure of crystal such that*

$$B(\xi) \otimes B(\eta) \to B(U_q(\mathfrak{g})a_{\xi+\eta}) \subset B(\tilde{U}_q(\mathfrak{g}))$$

is a strict embedding.

The crystal $B(\tilde{U}_q(\mathfrak{g}))$ is therefore a normal crystal.

2.2. Arguments of Lusztig. Since we use the arguments of Lusztig later, we shall review his argument briefly. The following lemma is easily checked.

LEMMA 2.2.1. *Let $l_+, l_- \in \mathbf{Z}_{\geqslant 0}$ and $\lambda \in P_+$, $\mu \in P_-$.*

(i) *$F_{l_-}(U_q^-(\mathfrak{g}))F_{l_+}(U_q^+(\mathfrak{g}))(u_\lambda \otimes u_\mu) = F_{l_+}(U_q^+(\mathfrak{g}))F_{l_-}(U_q^-(\mathfrak{g}))(u_\lambda \otimes u_\mu)$ $= (F_{l_-}(U_q^-(\mathfrak{g}))u_\lambda) \otimes (F_{l_+}(U_q^+(\mathfrak{g}))u_\mu)$.*

(ii) *For* $P_{\pm} \in F_{l_{\pm}}(U_q^{\pm}(\mathfrak{g}))$, *we have*

$$(P_- u_\lambda) \otimes (P_+ u_\mu) \equiv P_- P_+ (u_\lambda \otimes u_\mu) \equiv P_+ P_- (u_\lambda \otimes u_\mu)$$
$$\mod F_{l_- -1}(U_q^-(\mathfrak{g})) u_\lambda \otimes F_{l_+ -1}(U_q^+(\mathfrak{g})) u_\mu .$$

(iii)
$$L(\lambda, \mu)^- \cap F_{l_-}(U_q^-(\mathfrak{g})) u_\lambda \otimes F_{l_+}(U_q^+(\mathfrak{g})) u_\mu$$
$$\equiv (\bar{L}(\lambda) \cap F_{l_-}(U_q^-(\mathfrak{g})) u_\lambda) \otimes (\bar{L}(\mu) \cap F_{l_+}(U_q^+(\mathfrak{g})) u_\mu)$$
$$\mod F_{l_- -1}(\mathfrak{g}) u_\lambda \otimes F_{l_+ -1}(\mathfrak{g}) u_\mu .$$

(iv) $\overline{u \otimes v} - \bar{u} \otimes \bar{v} \in (U_q^+(\mathfrak{g})_{>0}\, \bar{u}) \otimes (U_q^-(\mathfrak{g})_{>0}\, \bar{v})$. *Here we set* $U_q^+(\mathfrak{g})_{>0} = \sum_i U_q^+(\mathfrak{g}) e_i$ *and* $U_q^-(\mathfrak{g})_{>0} = \sum_i U_q^-(\mathfrak{g}) f_i$.

The property (iv) follows from the existence of the universal R-matrix (cf. [L2]).

PROPOSITION 2.2.2. *Let* $\lambda \in P_+$ *and* $\mu \in P_-$, *and let* N *be a* $U_q^+(\mathfrak{g})$*-submodule of* $V(\lambda)$ *and* N' *a* $U_q^-(\mathfrak{g})$*-submodule of* $V(\mu)$. *Assume that* N *and* N' *are generated by global bases, i.e.,* $N = \bigoplus_{b \in B_N} \mathbf{Q}(q) G(b)$ *and* $N' = \bigoplus_{b \in B_{N'}} \mathbf{Q}(q) G(b')$ *for some* $B_N \subset B(\lambda)$ *and* $B_{N'} \subset B(\mu)$. *Then* $(L(\lambda, \mu) \cap N \otimes N', \bar{L}(\lambda, \mu) \cap N \otimes N', N_{\mathbf{Z}} \otimes N'_{\mathbf{Z}})$ *is balanced. Here* $N_{\mathbf{Z}} = \bigoplus_{b \in B_N} \mathbf{Z}[q, q^{-1}] G(b)$, *etc.*

Proof. Set $F_l(N) = N \cap F_l(U_q^-(\mathfrak{g})) u_\lambda$ and $F_l(N') = N' \cap F_l(U_q^+(\mathfrak{g})) u_\mu$. Then $F_l(N)$ (resp. $F_l(N')$) is a $U_q^+(\mathfrak{g})$- (resp. $U_q^-(\mathfrak{g})$-) submodule and is generated by global bases. Hence it is enough to show

$$(2.2.1)_l \quad (L(\lambda, \mu) \cap F_l(N) \otimes F_l(N'), L(\lambda, \mu)^- \cap F_l(N) \otimes F_l(N'), F_l(N_{\mathbf{Z}}) \otimes F_l(N'_{\mathbf{Z}}))$$

is balanced. We prove this by the induction on l. Assume that $(2.2.1)_{l-1}$ is satisfied. Let $Gr_l N$, etc., be the gradation with respect to the filtration F. Then Lemma 2.2.1 implies

$$Gr_l(L(\lambda, \mu) \cap N \otimes N') = Gr_l(L(\lambda) \cap N) \otimes Gr_l(L(\mu) \cap N')$$

$$Gr_l(\bar{L}(\lambda, \mu) \cap N \otimes N') = Gr_l(\bar{L}(\lambda) \cap N) \otimes Gr_l(\bar{L}(\mu) \cap N').$$

Hence $(Gr_l(L(\lambda, \mu) \cap N \otimes N'), Gr_l(\bar{L}(\lambda, \mu) \cap N \otimes N'), Gr_l(N_{\mathbf{Z}} \otimes N'_{\mathbf{Z}}))$ is balanced. Then we obtain $(2.2.1)_l$ by the triangular property of balancedness (Proposition 1.6.2). Q.E.D.

Applying this lemma to $N = V(\lambda)$ and $N' = V(\mu)$ we obtain Proposition 2.1.1. We also obtain the following lemma.

LEMMA 2.2.3. *Let* $\lambda \in P_+$ *and* $\mu \in P_-$. *Then for* $b \in B(\lambda)$, $b' \in B(\mu)$

$$G(b \otimes b') \equiv G(b) \otimes G(b') \mod \bigoplus_{\xi \in Q_+ \setminus \{0\}} V(\lambda)_{wt(b)+\xi} \otimes V(\mu)_{wt(b')-\xi} .$$

Here Q_+ is the set of linear combinations of simple roots with nonnegative integer coefficients.

3. Description of $B(U_q(\mathfrak{g})a_\lambda)$.

3.1. Relation with $B(\infty)$ and $B(-\infty)$. Let λ be an integral weight. For $\xi \in P_+$, $\eta \in P_-$ such that $\lambda = \xi + \eta$, $B(\xi) \otimes B(\eta)$ is embedded into $B(U_q(\mathfrak{g})a_\lambda)$. Since we have $B(\xi) \subset B(\infty) \otimes T_\xi$ and $B(\eta) \subset T_\eta \otimes B(-\infty)$, $B(\xi) \otimes B(\eta)$ is embedded into the crystal $B(\infty) \otimes T_\lambda \otimes B(-\infty)$ through $T_\lambda \cong T_\xi \otimes T_\eta$. Now take $\zeta \in P_+$. Then it is easy to see that

$$\begin{array}{ccc} B(\xi+\zeta)\otimes B(\eta-\zeta) & \hookrightarrow & B(\infty)\otimes T_\lambda \otimes B(-\infty) \\ \uparrow\!\cup & \nearrow & \\ B(\xi)\otimes B(\eta) & & \end{array}$$

commutes. Thus we obtain the following theorem.

THEOREM 3.1.1. $B(U_q(\mathfrak{g})a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty)$.

Note that $B(\xi) \otimes B(\eta)$ is a strict subcrystal of $B(\infty) \otimes T_\lambda \otimes B(-\infty)$.

By Lemma 2.2.1 (ii) and Lemma 2.2.3, we have, for $b_1 \in B(\infty)$ and $b_2 \in B(-\infty)$,

(3.1.1) $$G(b_1 \otimes t_\lambda \otimes b_2) \equiv G(b_1)G(b_2)a_\lambda \mod F_{|wt b_1|-1}(U_q^-(\mathfrak{g}))F_{|wt b_2|-1}(U_q^+(\mathfrak{g}))a_\lambda .$$

3.2. Filtration by Bruhat order. Let W be the Weyl group. For $w \in W$, let us take a reduced expression $w = s_{i_1} \cdots s_{i_l}$. We define a subset $B_w(\infty)$ of $B(\infty)$ by

(3.2.1) $$B_w(\infty) = \{\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_l}^{a_l} u_\infty;\, a_1, \ldots, a_l \geqslant 0\}.$$

Then $B_w(\infty)$ does not depend on the choice of reduced expression (see [K3]), and

(3.2.2) $$\bigoplus_{b \in B_w(\infty)} \mathbf{Z}[q, q^{-1}]G(b) = \sum_{a_1,\ldots,a_l \in \mathbf{Z}_{\geqslant 0}} \mathbf{Z}[q, q^{-1}]f_{i_1}^{(a_1)} \cdots f_{i_l}^{(a_l)}.$$

Set $B_w(-\infty) = B_w(\infty)^\vee \subset B(-\infty)$. Then we have

(3.2.3) $$\bigoplus_{b \in B_w(-\infty)} \mathbf{Z}[q, q^{-1}]G(b) = \sum \mathbf{Z}[q, q^{-1}]e_{i_1}^{(a_1)} \cdots e_{i_l}^{(a_l)}.$$

We set for any $i \in I$

(3.2.4) $$\tilde{U}_q^+(\mathfrak{g})_i^{\mathbf{Z}} = \bigoplus \mathbf{Z}[q, q^{-1}]e_i^{(n)}a_\lambda \qquad \text{and}$$

$$\tilde{U}_q^-(\mathfrak{g})_i^{\mathbf{Z}} = \bigoplus \mathbf{Z}[q, q^{-1}]f_i^{(n)}a_\lambda .$$

THEOREM 3.2.1. *Let $w, w' \in W$, and let $w = s_{i_1} \cdots s_{i_l}$, $w' = s_{j_1} \cdots s_{j_{l'}}$ be their reduced expressions. Set $B_{w,w'}(\tilde{U}_q(\mathfrak{g})) = \bigsqcup_\lambda B_w(\infty) \otimes T_\lambda \otimes B_{w'}(-\infty) \subset B(\tilde{U}_q(\mathfrak{g}))$. Then*

$$\bigoplus_{b \in B_{w,w'}(\tilde{U}_q(\mathfrak{g}))} \mathbf{Z}[q, q^{-1}]G(b)$$

is equal to the $\mathbf{Z}[q, q^{-1}]$-module $\tilde{U}_q^-(\mathfrak{g})_{i_1}^{\mathbf{Z}} \cdots \tilde{U}_q^-(\mathfrak{g})_{i_l}^{\mathbf{Z}} \tilde{U}_q^+(\mathfrak{g})_{j_1}^{\mathbf{Z}} \cdots \tilde{U}_q^+(\mathfrak{g})_{j_{l'}}^{\mathbf{Z}}$.

This follows immediately from Proposition 2.2.2. by taking

$$\lambda = \xi + \eta \text{ with } \xi,\ -\eta \in P_+ \text{ and}$$

$$N = \bigoplus_{b \in B_w(\infty)} \mathbf{Q}(q)G(b)u_\xi \qquad \text{and } N' = \bigoplus_{b \in B_{w'}(-\infty)} \mathbf{Q}(q)G(b)u_\eta .$$

This theorem shows an affirmative answer to a conjecture of Lusztig [L3].

4. A metric of $\tilde{U}_q(\mathfrak{g})$.

4.1. General facts. Let us define a metric on $\tilde{U}_q(\mathfrak{g})$ that behaves well with crystal bases. Let ψ be the antiautomorphism of the $\mathbf{Q}(q)$-algebra $U_q(\mathfrak{g})$ defined by

(4.1.1)
$$\psi(e_i) = q_i^{-1}t_i^{-1}f_i,$$
$$\psi(f_i) = q_i^{-1}t_ie_i,$$
$$\psi(q^h) = q^h.$$

Let M_1 and M_2 be $U_q(\mathfrak{g})$-modules, and let $(\ ,\ )_{M_1}$ and $(\ ,\ )_{M_2}$ be a symmetric form on M_1 and M_2 satisfying

$$(Pu, v)_{M_i} = (u, \psi(P)v)_{M_i} \qquad \text{for } u, v \in M_i, \qquad P \in U_q(\mathfrak{g}).$$

We define the symmetric form $(\ ,\ )$ on $M = M_1 \otimes M_2$ by

$$(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_{M_1}(u_2, v_2)_{M_2} \qquad \text{for } u_i, v_i \in M_i .$$

Then it is known (cf. [K1]) that $(\ ,\ )$ satisfies

(4.1.1)
$$(Pu, v) = (u, \psi(P)v) \qquad \text{for } u, v \in M .$$

Let us call this metric the tensor product of $(\ ,\ )_{M_1}$ and $(\ ,\ )_{M_2}$.

4.2. Definition of a metric on $\tilde{U}_q(\mathfrak{g})$. For $\lambda \in P_+ \cup (-P_+)$, there exists a unique nondegenerate symmetric bilinear form $(\ ,\ )$ on $V(\lambda)$ such that

(4.2.1) $(u_\lambda, u_\lambda) = 1$

$$(Pu, v) = (P, \psi(P)v) \qquad \text{for any } u, v \in V(\lambda) \qquad \text{and} \qquad P \in U_q(\mathfrak{g}).$$

Therefore, for λ, $\mu \in P_+$, the tensor product of those metrics gives a metric on $V(\lambda) \otimes V(-\mu)$. Let us take an arbitrary $\lambda \in P$. For $\xi \in P_+ \cap (\lambda + P_+)$, $V(\xi) \otimes V(\lambda - \xi)$ has a metric.

LEMMA 4.2.2. *For any P, $Q \in U_q(\mathfrak{g})$, there exists a unique polynomial $f(x)$ in $x = (x_i)_{i \in I}$ such that, for any $\xi \in P_+ \cap (\lambda + P_+)$,*

$$(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = f(x) \qquad \text{with } x_i = q_i^{2\langle h_i, \xi\rangle}.$$

Proof. Let us take $P_\pm^{(\nu)} \in U_q^\pm(\mathfrak{g}) \otimes \mathbf{Q}(q)[q^h; h \in P^*]$ such that $\psi(Q)P = \sum P_+^{(\nu)} P_-^{(\nu)}$. Then

$$(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = \sum (P_-^{(\nu)}(u_\xi \otimes u_{\lambda-\xi}), \psi(P_+^{(\nu)})(u_\xi \otimes u_{\lambda-\xi})).$$

Hence we may assume that P and Q belong to $U_q^-(\mathfrak{g}) \otimes \mathbf{Q}(q)[q^h; h \in P^*]$. Since $q^h(u_\xi \otimes u_{\lambda-\xi}) = q^{\langle \lambda, h\rangle}(u_\xi \otimes u_{\lambda-\xi})$, we may assume $P, Q \in U_q^-(\mathfrak{g})$. In this case $P(u_\xi \otimes u_{\lambda-\xi}) = Pu_\xi \otimes u_{\lambda-\xi}$ and hence $(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = (Pu_\xi, Qu_\xi)$. Then the result follows from [K1, Lemma 4.7.1]. Q.E.D.

We define a metric on $U_q(\mathfrak{g})a_\lambda$ by

$$(Pa_\lambda, Qa_\lambda) = f(0)$$

where f is the polynomial given in Lemma 4.2.2. Hence (Pa_λ, Qa_λ) is the limit of $(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi}))$ when all $\langle h_i, \xi\rangle$ tend to infinity. Here we regard $|q| < 1$. We extend this metric to the metric of $\tilde{U}_q(\mathfrak{g})$ such that $U_q(\mathfrak{g})a_\lambda$ and $U_q(\mathfrak{g})a_\mu$ are orthogonal for different λ, μ.

In [K1], we define a metric on $U_q^-(\mathfrak{g})$. The relation is given by the following formula (4.2.3).

$$(4.2.3) \quad (Pa_\lambda, Qa_\lambda) = \prod_i (1 - q_i^2)^{n_i}(P, Q) \qquad \text{for } P, Q \in U_q^-(\mathfrak{g})_\xi \qquad \text{with } \xi = \sum n_i \alpha_i.$$

Hence, we can apply the result of [K1]. The relation $(P, Q) = (P^*, Q^*)$ [K1, Proposition 5.2.1] implies

$$(4.2.4) \qquad (u, v) = (u^*, v^*) \qquad \text{for } u, v \in \tilde{U}_q^-(\mathfrak{g}) = \bigoplus_\lambda U_q^-(\mathfrak{g})a_\lambda.$$

LEMMA 4.2.3. *For $u \in \tilde{U}_q^-(\mathfrak{g})$ and $v \in \tilde{U}_q^-(\mathfrak{g})$, $(u, vf_i) = (uq_i t_i^{-1} e_i, v)$.*

Proof. Set $u = Pa_{\lambda+\alpha_i}$, $v = Qa_\lambda$ for $\lambda \in P$, $P, Q \in U_q^-(\mathfrak{g})$. Take $\xi \in P_+$, $\eta \in P_-$ such that $\lambda = \xi + \eta$. Then $Pe_i(u_\xi \otimes u_\eta) = P(u_\xi \otimes e_i u_\eta)$. On the other hand, we have

$$\Delta P = P \otimes 1 + t_i e_i'' P \otimes f_i \bmod \bigoplus_{\zeta \neq 0, -\alpha_i} U_q^-(\mathfrak{g}) \otimes U_q^-(\mathfrak{g})_\zeta .$$

For the definition of e_i'', see [K1]. Hence we obtain with $\xi_i = \langle h_i \xi \rangle$, etc.

$$Pe_i(u_\xi \otimes u_\eta) = Pu_\xi \otimes u_\eta + t_i e_i'' Pu_\xi \otimes f_i e_i u_\eta$$
$$= Pu_\xi \otimes u_\eta + q_i^{\xi_i}[-\eta_i]_i (Adt_i e_i'' P) u_\xi \otimes u_\eta .$$

Thus

$$(Pe_i(u_\xi \otimes u_\eta), Q(u_\xi \otimes u_\eta)) = \frac{q_i^{\xi_i+\eta_i+1} - q_i^{\xi_i-\eta_i+1}}{1-q_i^2}((Adt_i e_i'' P)u_\xi, Qu_\xi)$$

Letting ξ go to infinity, we obtain

$$(Pe_i a_\lambda, Qa_\lambda) = \frac{q_i^{\lambda_i+1}}{1-q_i^2} a(q)((Adt_i)e_i''P, Q).$$

Here $a(q) = \prod (1-q_i^2)^{n_i}$ and $wtQ = \sum n_i \alpha_i$. By [K1, Lemma 5.2.2], we have

$$((Adt_i)e_i''P, Q) = (P, Qf_i). \tag{4.2.5}$$

Finally we obtain

$$(Pa_{\lambda+\alpha_i} q_i t_i^{-1} e_i, Qa_\lambda) = q_i^{-1-\lambda_i}(Pe_i a_\lambda, Qa_\lambda)$$
$$= (Pa_{\lambda+\alpha_i}, Qf_i a_{\lambda+\alpha_i}).$$
Q.E.D.

LEMMA 4.2.4. *$(u, vf_i) = (uq_i^{-1} t_i e_i, v)$ for any $u, v \in \tilde{U}_q(\mathfrak{g})$.*

Proof. Assume first $u \in \tilde{U}_q^-(\mathfrak{g})$. Let us write $v = Pw$ for $P \in U_q^+(\mathfrak{g})$, $w \in \tilde{U}_q^-(\mathfrak{g})$. Then $(ue_i, v) = (\psi(P)ue_i, w) = (\psi(P)u, wq_i t_i f_i) = (u, Pwq_i t_i f_i) = (u, vq_i t_i f_i)$ by the last lemma. Hence the lemma is true if $u \in \tilde{U}_q^-(\mathfrak{g})$. In general case, writing $u = Pw$ with $P \in U_q^+(\mathfrak{g})$ and $w \in \tilde{U}_q^-(\mathfrak{g})$, we can argue similarly. Q.E.D.

Set $\psi^*(P) = (\psi(P^*))^*$. Then $\psi^*(f_i) = q_i t_i^{-1} e_i$. Hence Lemma 4.2.3 implies

$$(uf_i, v) = (u, v\psi^*(f_i)).$$

This implies easily the following lemma.

LEMMA 4.2.6. *$(uP, u) = (u, v\psi^*(P))$ for any $u, v \in \hat{U}_q(\mathfrak{g})$ and $P \in U_q(\mathfrak{g})$.*

PROPOSITION 4.2.5. *$(u, v) = (u^*, v^*)$ for any $u, v \in \tilde{U}_q(\mathfrak{g})$.*

Proof. This is already shown for $u, v \in \tilde{U}_q^-(\mathfrak{g})$. If $u \in U_q^-(\mathfrak{g})$, then, writing $v = Pw$ with $P \in U_q^+(\mathfrak{g})$ and $w \in \tilde{U}_q^-(\mathfrak{g})$, we have

$$(u, v) = (\psi(P)u, w) = (u^*\psi(P)^*, w^*) = (u^*, w^*P^*) = (u^*, v^*).$$

The general case can be argued similarly. Q.E.D.

4.3. Metric and crystal base. The relation of the metric of $V(\lambda)$ and its crystal base implies the properties

$$(L(\tilde{U}_q(\mathfrak{g})), L(\tilde{U}_q(\mathfrak{g}))) \subset A, \tag{4.3.1}$$

$$(G(b), G(b')) \equiv \delta_{b,b'} \bmod qA. \tag{4.3.2}$$

Thus (,) is a nondegenerate metric on $\tilde{U}_q(\mathfrak{g})$. Let $A_{\mathbf{Z}}$ be the subring of $\mathbf{Q}(q)$ generated by q, $(1-q^n)^{-1}$ $(n>0)$. Then one can see easily

$$(\tilde{U}_q^{\mathbf{Z}}(\mathfrak{g}), \tilde{U}_q^{\mathbf{Z}}(\mathfrak{g})) \subset A_{\mathbf{Z}}[q^{-1}]. \tag{4.3.3}$$

Hence we can apply the similar arguments as in [K2, §6.1]. Thus we obtain the following.

PROPOSITON 4.3.1. (i) $L(\tilde{U}_q(\mathfrak{g})) = \{u \in \tilde{U}_q(\mathfrak{g}); (u, u) \in A\}$.
(ii) *If* $u \in \tilde{U}_q^{\mathbf{Z}}(\mathfrak{g})$ *and* $(u, u) \in 1 + qA$, *then* $u \equiv G(b) \bmod qL(\tilde{U}_q(\mathfrak{g}))$ *for some* $b \in B((\tilde{U}_q(\mathfrak{g})) \sqcup -B(\tilde{U}_q(\mathfrak{g}))$.

We have Theorem 4.3.2 below as a corollary.

THEOREM 4.3.2. (i) $L(\tilde{U}_q(\mathfrak{g}))$ *is invariant by* $*$.
(ii) $B(\tilde{U}_q(\mathfrak{g}))^* = B(\tilde{U}_q(\mathfrak{g}))$.
(iii) $G(b^*) = G(b)^*$ *for* $b \in B(\tilde{U}_q(\mathfrak{g}))$.

The proofs of (i) and (iii) are similar to [K1]; we will prove only (ii). For $b \in B(\tilde{U}_q(\mathfrak{g}))$, $b^* \in B(\tilde{U}_q(\mathfrak{g})) \sqcup (-B(\tilde{U}_q(\mathfrak{g}))$. Write $b = b_1 \otimes t_\lambda \otimes b_2$, $l_\nu = |wtb_\nu|$. Then

$$G(b) \equiv G(b_1)G(b_2)a_\lambda \bmod F_{l_1-1}(U_q^-(\mathfrak{g}))F_{l_2-1}(U_q^+(\mathfrak{g}))a_\lambda.$$

Hence we obtain

$$\begin{aligned} G(b)^* &\equiv a_{-\lambda}G(b_2)^*G(b_1)^* = a_{-\lambda}G(b_2^*)G(b_1^*) = G(b_2^*)G(b_1^*)a_\mu \\ &\equiv G(b_1^*)G(b_2^*)a_\mu \bmod F_{l_1-1}(U_q^-(\mathfrak{g}))F_{l_2-1}(U_q^+(\mathfrak{g}))a_\mu \end{aligned}$$

with

$$\mu = -\lambda - wtb_1 - wtb_2.$$

Since $G(b)^* = G(b')$ for some $b' \in B(\tilde{U}_q(\mathfrak{g})) \sqcup -B(\tilde{U}_q(\mathfrak{g}))$, b' must be $b_1^* \otimes t_\mu \otimes b_2^*$. (See (3.1.1).)

As seen in the course of the proof above, we have the following.

COROLLARY 4.3.3. *For* $b_1 \in B(\infty)$, $b_2 \in B(-\infty)$, *we have*

$$(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{-\lambda - wtb_1 - wtb_2} \otimes b_2^*.$$

Theorem 4.3.2 is conjectured by Lusztig [L3].

5. Right structure.

5.1. Two crystal structures on $B(\tilde{U}_q(\mathfrak{g}))$. We define for $b \in B(\tilde{U}_q(\mathfrak{g}))$

(5.1.1)
$$\varepsilon_i^*(b) = \varepsilon_i(b^*),$$
$$\varphi_i^*(b) = \varphi_i(b^*),$$
$$\tilde{e}_i^* b = (\tilde{e}_i b^*)^* \quad \text{and}$$
$$\tilde{f}_i^* b = (\tilde{f}_i b^*)^*.$$

Then this defines another crystal structure on $B(\tilde{U}_q(\mathfrak{g}))$. By Corollary 4.3.3 we obtain easily the following formula for $b_1 \in B(\infty)$, $b_2 \in B(-\infty)$, $\lambda \in P$:

(5.1.2)
$$\varepsilon_i^*(b_1 \otimes t_\lambda \otimes b_2) = \max(\varepsilon_i^*(b_1), \varphi_i^*(b_2) + \langle h_i, \lambda\rangle),$$
$$\varphi_i^*(b_1 \otimes t_\lambda \otimes b_2) = \max(\varepsilon_i^*(b_1) - \langle h_i, \lambda\rangle, \varphi_i^*(b_2)),$$
$$\varphi_i^*(b_1 \otimes t_\lambda \otimes b_2) - \varepsilon_i^*(b_1 \otimes t_\lambda \otimes b_2) = -\langle h_i, \lambda\rangle,$$

(5.1.3)
$$\tilde{e}_i^*(b_1 \otimes t_\lambda \otimes b_2) = \begin{cases} \tilde{e}_i^* b_1 \otimes t_{\lambda-\alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) \geqslant \varphi_i^*(b_2) + \langle h_i, \lambda\rangle \\ b_1 \otimes t_{\lambda-\alpha_i} \otimes \tilde{e}_i^* b_2 & \text{if } \varepsilon_i^*(b_1) < \varphi_i^*(b_2) + \langle h_i, \lambda\rangle, \end{cases}$$

(5.1.4)
$$\tilde{f}_i^*(b_1 \otimes t_\lambda \otimes b_2) = \begin{cases} \tilde{f}_i^* b_1 \otimes t_{\lambda+\alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) > \varphi_i^*(b_2) + \langle h_i, \lambda\rangle \\ b_1 \otimes t_{\lambda+\alpha_i} \otimes \tilde{f}_i^* b_2 & \text{if } \varepsilon_i^*(b_1) \leqslant \varphi_i^*(b_2) + \langle h_i, \lambda\rangle. \end{cases}$$

We prove that these two structures of crystal may be regarded as a crystal structure over $\mathfrak{g} \oplus \mathfrak{g}$. This is compared with the fact that the bimodule structure of $\tilde{U}_q(\mathfrak{g})$ may be regarded as a left $U_q(\mathfrak{g} \oplus \mathfrak{g})$-module structure.

In order to see this, it is enough to show the following theorem.

THEOREM 5.1.1. *$\tilde{e}_i^*$ and $\tilde{f}_i^*$ are strict morphisms of crystals (with respect to $\tilde{e}_i$ and $\tilde{f}_i$).*

In order to prove this theorem, we use the following lemma. The proof being straightforward, we omit it.

LEMMA 5.1.2. *For any i and $\lambda \in P$, let us define the map $E_i\colon B_i \otimes T_\lambda \otimes B_i \to B_i \otimes T_{\lambda-\alpha_i} \otimes B_i$ by*

$$E_i(b_i(n) \otimes t_\lambda \otimes b_i(m)) = \begin{cases} b_i(n+1) \otimes t_{\lambda-\alpha_i} \otimes b_i(m) & \text{if } n + m + \langle h_i, \lambda\rangle \leqslant 0, \\ b_i(n) \otimes t_{\lambda-\alpha_i} \otimes b_i(m+1) & \text{if } n + m + \langle h_i, \lambda\rangle > 0. \end{cases}$$

Then E_i is a strict morphism of crystal.

Now, there are strict embeddings from $B(\infty)$ into $B(\infty)\otimes B_i$ and from $B(-\infty)$ into $B_i\otimes B(-\infty)$. There are also strict morphisms from $B(\infty)\otimes B_i$ to $B(\infty)$ and from $B_i\otimes B(-\infty)$ to $B(-\infty)$ (cf. (1.5.20)). Thus we obtain a chain of strict morphisms of crystals

(5.1.5)
$$\begin{aligned}B(\infty)\otimes T_\lambda\otimes B(-\infty)&\to B(\infty)\otimes B_i\otimes T_\lambda\otimes B_i\otimes B(-\infty)\\&\xrightarrow{E_i} B(\infty)\otimes B_i\otimes T_{\lambda-\alpha_i}\otimes B_i\otimes B(-\infty)\\&\to B(\infty)\otimes T_{\lambda-\alpha_i}\otimes B(-\infty).\end{aligned}$$

LEMMA 5.1.3. *The composition of the morphisms* (5.1.5) *coincides with* $\tilde{e}_i^*$.

This follows immediately from the formulas (5.1.2) and (5.1.3). Thus $\tilde{e}_i^*$ is a strict morphism of crystal. Similarly $\tilde{f}_i^*$ is a strict morphism of crystals.

6. Properties of global bases.

6.1. Preliminary. We study in [K2] the properties of global bases of integrable modules. They give the properties of the global bases of $\tilde{U}_q(\mathfrak{g})$ reducing to those of $V(\xi)\otimes V(\eta)$. For $N\geqslant 0$, we set

$$I_N=\sum_i \tilde{U}_q(\mathfrak{g})e_i^N+\sum_i \tilde{U}_q(\mathfrak{g})f_i^N.$$

PROPOSITION 6.1.1. *Let n be a nonnegative integer and $i\in I$. Then $u\in\tilde{U}_q(\mathfrak{g})$ satisfies $u\in f_i^n\tilde{U}_q(\mathfrak{g})+I_N$ (resp. $u\in e_i^n\tilde{U}_q(\mathfrak{g})+I_N$) for any N if and only if u is a linear combination of $G(b)$ with $\varepsilon_i(b)\geqslant n$.*

Proof. Assuming $u\in U_q(\mathfrak{g})a_\lambda$, let us take $\xi\in P_+$, $\eta\in P_-$ such that $\lambda=\xi+\eta$. Then $\Phi_{\xi,\eta}(u)\in f_i^n(V(\xi)\otimes V(\eta))$ if and only if $u\in f_i^n\tilde{U}_q(\mathfrak{g})+\sum_j\tilde{U}_q(\mathfrak{g})e_j^{1-\langle h_j,\eta\rangle}+\sum_j\tilde{U}_q(\mathfrak{g})f_j^{1+\langle h_j,\xi\rangle}$. Hence taking ξ such that $\langle h_j,\xi\rangle\gg 0$ for all j, we obtain the lemma by the corresponding result of global bases of $V(\xi)\otimes V(\eta)$. Q.E.D.

6.2. Definition of $V_i(\lambda)$. To get more precise results than Proposition 6.1.1, let us generalize the results of Lusztig little bit.

For $i\in I$ and $\lambda\in P_+$, let $V_i(\lambda)$ be the $U_q(\mathfrak{g})$-module generated by $u_{i,\lambda}$ with the defining relation $q^h u_{i,\lambda}=q^{\langle h,\lambda\rangle}u_{i,\lambda}$, $e_j u_{i,\lambda}=0$ for any $j\in I$ and $f_i^{1+\langle h_i,\lambda\rangle}u_{i,\lambda}=0$. Similarly for $\mu\in P_-$, let $V_i(\mu)$ be the $U_q(\mathfrak{g})$-module generated by $u_{i,\mu}$ with the defining relation $q^h u_{i,\mu}=q^{\langle h,\mu\rangle}u_{i,\mu}$, $f_j u_{i,\mu}=0$ for any $j\in I$ and $e_i^{1-\langle h_i,\mu\rangle}u_{i,\mu}=0$. Then

$$V_i(\lambda)\cong U_q^-(\mathfrak{g})/U_q^-(\mathfrak{g})f_i^{1+\langle h_i,\lambda\rangle},\tag{6.2.1}$$

$$V_i(\mu)\cong U_q^+(\mathfrak{g})/U_q^+(\mathfrak{g})e_i^{1-\langle h_i,\mu\rangle}.\tag{6.2.2}$$

Now consider the $U_q(\mathfrak{g})$-module $V_i(\lambda)\otimes V_i(\mu)$. One can see easily that this is a $U_q(\mathfrak{g})$-

module generated by $u_{i,\lambda} \otimes u_{i,\mu}$ with the defining relation

$$q^h(u_{i,\lambda} \otimes u_{i,\mu}) = q^{\langle h, \lambda+\mu \rangle}(u_{i,\lambda} \otimes u_{i,\mu}) \qquad \text{for any } h \in P^*,$$

$$f_i^{1+\langle h_i, \lambda \rangle}(u_{i,\lambda} \otimes u_{i,\mu}) = 0 \qquad \text{and} \qquad e_i^{1-\langle h_i, \mu \rangle}(u_{i,\lambda} \otimes u_{i,\mu}) = 0 .$$

Hence there is a chain of surjective homomorphisms

$$U_q(\mathfrak{g})a_{\lambda+\mu} \xrightarrow{\Phi^i_{\lambda,\mu}} V_i(\lambda) \otimes V_i(\mu) \to V(\lambda) \otimes V(\mu).$$

6.3. *Refinement.* For $\xi \in Q_\pm$, let us denote

$$F_\xi U_q^\pm(\mathfrak{g}) = \bigoplus_{\eta \in \xi \mp Q_+} U_q^\pm(\mathfrak{g})_\eta .$$

For $\xi_\pm \in P_\pm$, we set

$$F_{\xi_+, \xi_-} U_q(\mathfrak{g}) = F_{\xi_+}(U_q^+(\mathfrak{g})) F_{\xi_-}(U_q^-(\mathfrak{g}))\mathscr{T}$$

$$= F_{\xi_-}(U_q^-(\mathfrak{g})) F_{\xi_+}(U_q^+(\mathfrak{g}))\mathscr{T} ,$$

where $\mathscr{T} = \bigoplus_{h \in P^*} \mathbf{Q}(q)q^h$. For $\xi \in Q_+$, we set $F_\xi U_q(\mathfrak{g}) = F_{\xi, -\xi} U_q(\mathfrak{g})$. Then, by Proposition 2.2.2, $F_{\xi_+, \xi_-} U_q(\mathfrak{g})a_{\lambda+\mu}$ is generated by global bases. The purpose of this section is to prove the following proposition.

PROPOSITON 6.3.1. *Set $L(V_i(\lambda) \otimes V_i(\mu)) = \Phi^i_{\lambda,\mu}(L(U_q(\mathfrak{g})a_{\lambda+\mu}))$. Let $\overline{\Phi}^i_{\lambda,\mu}$ be the induced morphism $L(U_q(\mathfrak{g})a_{\lambda+\mu})/qL(U_q(\mathfrak{g})a_{\lambda+\mu}) \to L(V_i(\lambda) \otimes V_i(\mu))/qL(V_i(\lambda) \otimes V_i(\mu))$. Set $B(V_i(\lambda) \otimes V_i(\mu)) = \{\overline{\Phi}^i_{\lambda,\mu}(b);\, b \in B(U_q(\mathfrak{g})a_{\lambda+\mu})\} \setminus \{0\}$. Then:*

(i) *$(L(V_i(\lambda) \otimes V_i(\mu)), B(V_i(\lambda) \otimes V_i(\mu)))$ is a crystal base of the integrable $U_q(\mathfrak{g}_i)$-module $V_i(\lambda) \otimes V_i(\mu)$. Here $U_q(\mathfrak{g}_i)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, and q^h ($h \in P^*$).*

(ii) *For $b \in B(U_q(\mathfrak{g})a_{\lambda+\mu})$, $\overline{\Phi}^i_{\lambda,\mu}(b) \neq 0$ if and only if $\varepsilon_i^*(b) \leqslant \langle h_i, \lambda \rangle$.*

(iii) *$\{b \in B(U_q(\mathfrak{g})a_{\lambda+\mu});\, \overline{\Phi}^i_{\lambda,\mu}(b) \neq 0\}$ is isomorphic to $B(V_i(\lambda) \otimes V_i(\mu))$.*

(iv) *$\{G(b)(u_{i\lambda} \otimes u_{i\mu})\}$ forms a base of $V_i(\lambda) \otimes V_i(\mu)$ where b ranges over $\{b \in B(U_q(\mathfrak{g})a_{\lambda+\mu});\, \overline{\Phi}^i_{\lambda,\mu}(b) \neq 0\}$.*

(v) *$\overline{\Phi}^i_{\lambda,\mu}$ is a morphism (in $\mathscr{C}(\{i\}, P)$) from $B(U_q(\mathfrak{g})a_{\lambda+\mu})$ to $B(V_i(\lambda) \otimes V_i(\mu))$.*

Proof. Let us take $\xi \in Q_+$. Since $V_i(\lambda) \otimes V_i(\mu)$ is integrable as $U_q(\mathfrak{g}_i)$-module, $F_{\xi+n\alpha_i} U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu})$ is stationary when n increases. Take N such that $F_{\xi+N\alpha_i} U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) = F_{\xi+n\alpha_i} U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu})$ for $n \geqslant N$. Then $F_{\xi+N\alpha_i} U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu})$ is a finite-dimensional $U_q(\mathfrak{g}_i)$-module. Taking $\eta \in P_+$ such that $\langle h_j, \eta \rangle \gg 0$ for $j \neq i$ and $\langle h_i, \eta \rangle = 0$,

$$F_{\xi+N\alpha_i} U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) \to V(\lambda + \eta) \otimes V(\mu - \eta)$$

is injective. Now consider the chain of homomorphisms

$$U_q(\mathfrak{g})a_{\lambda+\mu} \xrightarrow{\Phi^i_{\lambda,\mu}} V_i(\lambda) \otimes V_i(\mu) \xrightarrow{p} V(\lambda+\eta) \otimes V(\mu-\eta).$$

Then $p\Phi^i_{\lambda,\mu}(F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu})) = F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{\lambda+\eta} \otimes u_{\mu-\eta}) \cap L(\lambda+\eta) \otimes L(\mu-\eta)$. Hence $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) \cap L(V_i(\lambda) \otimes V_i(\mu))$ is contained in $p^{-1}p\Phi^i_{\lambda,\mu}(F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu})) \subset p^{-1}(0) + \Phi^i_{\lambda,\mu}(F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu}))$. Since $p^{-1}(0) \cap F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) = 0$, we obtain

$$F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) \cap L(V_i(\lambda) \otimes V_i(\mu)) = \Phi^i_{\lambda,\mu}(F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu})).$$

Thus, we obtain

$$\begin{aligned}
&(F_{\xi+N\alpha_i}U_q(\mathfrak{g}))a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu}) \\
&\twoheadrightarrow (F_{\xi+N\alpha_i}U_q(\mathfrak{g}))(u_{i\lambda} \otimes u_{i\mu}) \cap L(V_i(\lambda) \otimes V_i(\mu)) \\
&\xrightarrow{\sim} (F_{\xi+N\alpha_i}U_q(\mathfrak{g}))(u_{\lambda+\eta} \otimes u_{\mu-\eta}) \cap L(\lambda+\eta) \otimes L(\mu-\eta).
\end{aligned}$$

Now, let us prove Proposition 6.2.1. For example, let us show that $L(V_i(\lambda) \otimes V_i(\mu))$ is a crystal lattice. Since $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{\lambda+\eta} \otimes u_{\mu-\eta}) \cap L(V(\lambda+\eta) \otimes V(\mu-\eta))$ is a crystal lattice of the integrable $U_q(\mathfrak{g}_i)$-module $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{\lambda+\eta} \otimes u_{\mu-\eta})$, $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) \cap L(V_i(\lambda) \otimes V_i(\mu))$ is a crystal lattice of $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu})$. Since $V_i(\lambda) \otimes V_i(\mu)$ is a union of $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu})$, $L(V_i(\lambda) \otimes V_i(\mu))$ is a crystal lattice. The other statements can be proven similarly. Q.E.D.

6.4. *Interpretation of ε and φ.* By using Proposition 6.3.1, we can sharpen Proposition 6.1.1. Set

$$I^i_N = \tilde{U}_q(\mathfrak{g})e_i^N + \tilde{U}_q(\mathfrak{g})f_i^N.$$

Then the proof of the following proposition is similar to the one of Proposition 6.1.1.

PROPOSITION 6.4.1. *Let n be a nonnegative integer and $i \in I$. Then $u \in \tilde{U}_q(\mathfrak{g})$ satisfies $u \in \tilde{f}_i^n\tilde{U}_q(\mathfrak{g}) + I^i_N$ (resp. $u \in e_i^n\tilde{U}_q(\mathfrak{g}) + I^i_N$) for any N if and only if u is a linear combination of $G(b)$ with $\varepsilon_i(b) \geqslant n$.*

PROPOSITION 6.4.2. *For $a, c \in \mathbb{Z}_{\geqslant 0}$ and $i \in I$,*

$$e_i^a\tilde{U}_q(\mathfrak{g}) + f_i^c\tilde{U}_q(\mathfrak{g}) = \bigoplus_b \mathbb{Q}(q)G(b). \tag{6.4.1}$$

Here b ranges over $\{b \in B(\tilde{U}_q(\mathfrak{g})); \varphi_i(b) \geqslant a \text{ or } \varepsilon_i(b) \geqslant c\}$.

Proof. Taking $*$, it is enough to show

$$U_q(\mathfrak{g})e_i^a a_\lambda + U_q(\mathfrak{g})f_i^c a_\lambda = \bigoplus_b \mathbf{Q}(q)G(b). \tag{6.4.2}$$

Here b ranges over $\{b \in B(\tilde{U}_q(\mathfrak{g})a_\lambda),\ \varphi_i^*(b) \geqslant a \text{ or } \varepsilon_i^*(b) \geqslant c\}$.

(Step I) The case where $\langle h_i, \lambda\rangle = c - a$. Let us take $\xi \in P_+$ and $\eta \in P_-$ such that

$$\lambda = \xi + \eta, \qquad \langle h_i, \xi\rangle = 1 + c, \qquad \langle h_i, \eta\rangle = -1 - a.$$

Then the left-hand side of (6.4.2) coincides with the kernel of $\Phi^i(\xi, \eta)$. Since the kernel of $\Phi^i(\xi, \eta)$ is generated by $G(b)$ with $b \in B(\tilde{U}_q(\mathfrak{g})a_\lambda)\backslash B(\xi) \otimes B(\eta) = \{b_1 \otimes t_\lambda \otimes b_2;\ \varepsilon_i^*(b_1) \leqslant c,\ \varphi_i^*(b_2) \leqslant a\}$, we obtain the desired result.

(Step II) General case. We define $\tilde{a}$, $\tilde{c}$ by

$$\tilde{c} = c, \qquad \tilde{a} = c - \langle h_i, \lambda\rangle \qquad \text{if } \langle h_i, \lambda\rangle \leqslant c - a,$$

$$\tilde{a} = a, \qquad \tilde{c} = \langle h_i, \lambda\rangle + a \qquad \text{if } \langle h_i, \lambda\rangle \geqslant c - a.$$

Then $\tilde{c} \geqslant c$ and $\tilde{a} \geqslant a$. On the other hand, $\varepsilon_i^*(b) - \varphi_i^*(b) = \langle h_i, \lambda\rangle$ implies that the condition $\varphi_i^*(b) \geqslant a$ or $\varepsilon_i^*(b) \geqslant c$ is equivalent to $\varphi_i^*(b) \geqslant \tilde{a}$ or $\varepsilon_i^*(b) \geqslant \tilde{c}$. Hence the right-hand side of (6.4.2) is equal to $\tilde{U}_q(\mathfrak{g})e_i^{\tilde{a}}a_\lambda + U_q(\mathfrak{g})f_i^{\tilde{c}}a_\lambda$. Then it is enough to apply

$$U_q(\mathfrak{g})e_i^{\tilde{a}}a_\lambda + U_q(\mathfrak{g})f_i^{\tilde{c}}a_\lambda = U_q(\mathfrak{g})e_i^a a_\lambda + U_q(\mathfrak{g})f_i^c a_\lambda. \qquad \text{Q.E.D.}$$

Let us remark that $\varepsilon_i(b) \geqslant n$ does not imply $G(b) \in f_c^n\tilde{U}_q(\mathfrak{g})$ in general. This is only true modulo I_N^i or modulo $e_i^N\tilde{U}_q(\mathfrak{g})$.

Proposition 6.4.3. *For $b \in B(\tilde{U}_q(\mathfrak{g}))$,*

$$f_i^{(n)}G(b) = \begin{bmatrix}\varepsilon_i(b) + n\\ n\end{bmatrix}_i G(\tilde{f}_i^n b) + \sum_{b'} F_{b,b'}^{i,n}(q)G(b'). \tag{6.4.3}$$

Here b' ranges over the set of $B(\tilde{U}_q(\mathfrak{g}))$ such that $\varepsilon_i(b') > \varepsilon_i(b) + n$ and $F_{b,b'}^{i,n}(q) \in \mathbf{Z}[q, q^{-1}]$. In addition,

$$e_i^{(n)}G(B) = \begin{bmatrix}\varphi_i(b) + n\\ n\end{bmatrix}_i G(\tilde{e}_i^n b) + \sum_{b'} E_{b,b'}^{i,n}(q)G(b'). \tag{6.4.4}$$

Here b' ranges over the set of $B(\tilde{U}_q(\mathfrak{g}))$ such that $\varphi_i(b') > \varphi_i(b) + n$.

Formulas (6.4.3) and (6.4.4) follow immediately from the corresponding results on global bases of integrable modules (see [K2]).

6.5. Further property. The following results are used later.

LEMMA 6.5.1. *If* $\langle h_i, \lambda\rangle \geqslant k$, *then*

$$L(U_q(\mathfrak{g})a_{\lambda-k\alpha_i})f_i^{(k)} \subset q_i^{k(k-\langle h_i,\lambda\rangle)}L(\tilde{U}_q(\mathfrak{g})) + \tilde{U}_q(\mathfrak{g})e_i + \tilde{U}_q(\mathfrak{g})f_i^{1+\langle h_i,\lambda\rangle}.$$

Proof. Taking $*$ and changing λ with $-\lambda$, it is enough to show

$$a_\lambda f_i^{(k)}L(\tilde{U}_q(\mathfrak{g})) \subset q_i^{k(k+\langle h_i,\lambda\rangle)}L(\tilde{U}(\mathfrak{g})) + e_i\tilde{U}_q(\mathfrak{g}) + f_i^{1-\langle h_i,\lambda\rangle}\tilde{U}_q(\mathfrak{g})$$

$$\text{when} \qquad \langle h_i, \lambda\rangle + k \leqslant 0,$$

or for $b \in B(\tilde{U}_q(\mathfrak{g}))$ of weight $\lambda + k\alpha_i$, writing $f_i^{(k)}G(b) = \sum c_{b'}G(b')$,

(6.5.1) $$\varphi_i(b') = 0 \qquad \text{implies } c_{b'} \in q_i^{k(k+\langle h_i,\lambda\rangle)}.$$

This reduces to the same statement for $b \in B(V(\xi) \otimes V(\eta))$ for $\xi \in P_+$ and $\eta \in P_-$. Write $G(b) = \sum f_i^{(n)}u_n$ with $e_iu_n = 0$ and $wt(u_n) = \lambda + (n+k)\alpha_i$. Then $u_n \in L(\xi) \otimes L(\eta)$ and $f_i^{(k)}G(b) = \sum \begin{bmatrix} n+k \\ k \end{bmatrix}_i f_i^{(k+n)}u_n$. If $k + n \neq \langle h_i, \lambda + (n+k)\alpha_i\rangle$, then $f_i^{(k+n)}u_n$ belongs to $e_i(V(\xi) \otimes V(\eta))$. If $k + n = \langle h_i, \lambda + (n+k)\alpha_i\rangle$, then $\begin{bmatrix} n+k \\ k \end{bmatrix}_i f_i^{(k+n)}u_n$ belongs to $q_i^{-nk}L(\xi) \otimes L(\eta) = q_i^{k(\langle h_i,\lambda\rangle+k)}L(\xi) \otimes L(\eta)$. Hence $f_i^{(k)}G(b) \in q_i^{k(\langle h_i,\lambda\rangle+k)}L(\xi) \otimes L(\eta) + e_i(V(\xi) \otimes V(\eta))$. This implies (6.5.1). Q.E.D.

7. The Weyl group action on crystal bases.

7.1. Action of simple reflections. Let B be a normal crystal. We define the action of the Weyl group W on the underlying set B. For $i \in I$ and $b \in B$, we set

(7.1.1) $$S_ib = \begin{cases} \tilde{f}_i^{\langle h_i, wt(b)\rangle}b & \text{if } \langle h_i, wt(b)\rangle \geqslant 0 \\ \tilde{e}_i^{-\langle h_i, wt(b)\rangle}b & \text{if } \langle h_i, wt(b)\rangle \leqslant 0 \cdot \end{cases}$$

Then we have the obvious relation

(7.1.2) $$wt(S_ib) = s_i(wt(b))$$

where $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle\, \alpha_i$ is the simple reflection,

(7.1.3) $$S_i^2 = id,$$

(7.1.4) $$S_i\tilde{e}_iS_i^{-1} = \tilde{f}_i.$$

We show that this extends to the action of the Weyl group. In order to see this, it is enough to check the braid relation for a finite-dimensional $\mathfrak{g}$ of rank 2.

7.2. Braid relation. Set $I = \{1, 2\}$ and assume $\mathfrak{g}$ is finite-dimensional. Let $w_0 = s_{i_1} \cdots s_{i_l}$ be a reduced expression of the longest element of W. There are two choices. We show that

(7.2.1) $\quad S_{i_1} \cdots S_{i_l} b$ does not depend on the choice of reduced expression.

In order to see this we may assume that $wt(b)$ is dominant. If $wt(b)$ is not regular, (7.2.1) is trivial. Hence we may assume $wt(b)$ is regular and dominant.

For any normal crystal B, set $\tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b$. Then we have that

(7.2.2) $\tilde{f}_{i_1}^{\max} \cdots \tilde{f}_{i_l}^{\max} b$ does not depend on the choice of reduced expression.

In fact this vector is the unique lowest-weight vector in the connected component containing b (cf. [K3]).

Now we remark

$$(7.2.3) \qquad \tilde{f}_i^{\max}(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i^{\varphi_i(b_1) - \varepsilon_i(b_2)} b_1 \otimes \tilde{f}_i^{\max} b_2 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i^{\max} b_2 & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases}$$

LEMMA 7.2.1. *Let b be a vector with $\langle h_i, wt(b) \rangle > 0$. Then, for any b' in any crystal B', there exists an integer m and $b'' \in B^{\otimes m} \otimes B'$ such, that for any $n \geqslant m$,*

$$\tilde{f}_i^{\max}(b^{\otimes n} \otimes b') = (S_i b)^{\otimes (n-m)} \otimes b''.$$

Proof. We have for $1 \leqslant \nu$

$$\varepsilon_i(b^{\otimes \nu} \otimes b') = \max\{\varepsilon_i(b) - k\langle h_i, wt(b) \rangle (0 \leqslant k < \nu), \varepsilon_i(b') - \nu\langle h_i, wt(b) \rangle\}$$

$$= \max(\varepsilon_i(b), \varepsilon_i(b') - \nu\langle h_i, wt(b) \rangle).$$

Hence if $\varepsilon_i(b') - \nu\langle h_i, wt(b) \rangle \leqslant \varepsilon_i(b)$, (7.2.3) and $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, wt(b) \rangle$ imply $f_i^{\max}(b^{\otimes(\nu+1)} \otimes b') = S_i b \otimes f_i^{\max}(b^{\otimes \nu} \otimes b')$. Thus we obtain the desired result. Q.E.D.

This lemma implies that $\tilde{f}_{i_1}^{\max} \cdots \tilde{f}_{i_l}^{\max}(b^{\otimes n}) = S_{i_1} \cdots S_{i_l} b \otimes b'$ for some $b' \in B^{\otimes(n-1)}$ if $n \gg 0$. Thus (7.2.2) implies the following result.

THEOREM 7.2.2. *$\{S_i\}$ satisfies the braid relation.*

7.3. Application. For $\lambda \in P$ and $w \in W$, take a reduced expression $w = s_{i_l} \cdots s_{i_1}$ of w. Then the condition $\langle h_{i_l}, s_{i_{l-1}} \cdots s_{i_1} \lambda \rangle \geqslant 0, \ldots, \langle h_{i_1}, \lambda \rangle \geqslant 0$ does not depend on the choice of reduced expression. If this condition is satisfied, we say that λ is *w-dominant*. If λ is w-dominant, we set $\tilde{f}_{w,\lambda} = \tilde{f}_{i_l}^{\langle h_{i_l}, s_{i_{l-1}} \cdots s_{i_1} \lambda \rangle} \cdots \tilde{f}_{i_1}^{\langle h_{i_1}, \lambda \rangle}$.

PROPOSITION 7.4.1. *If $\lambda \in P$ is w-dominant, the definition of $\tilde{f}_{w,\lambda}$ does not depend on the choice of reduced expression of w. Here we regard $\tilde{f}_{w,\lambda}$ as an operator on normal crystals, $B(\infty)$ or $B(-\infty)$.*

Proof. It is enough to show this on $B(\infty)$. Then it follows from

(7.4.2) $$S_{i_1} \cdots S_{i_l}(b \otimes t_{\lambda - wt(b)} \otimes u_{-\infty}) = \tilde{f}_{w,\lambda} b \otimes t_{\lambda - wt(b)} \otimes u_{-\infty}.$$ Q.E.D

Remark 7.4.2. $w\alpha_i = \alpha_j$ does not imply $S_w \tilde{e}_i S_w^{-1} = \tilde{e}_j$ even in the A_2-case.

8. Extremal vectors.

8.1. Definition of extremal vectors. Let M be an integral $U_q(\mathfrak{g})$-module. A weight vector u of weight $\lambda \in P$ of M is called i-extremal if $e_i u = 0$ or $f_i u = 0$. In this case, we set $S_i u = f_i^{(\langle h_i, \lambda \rangle)} u$ or $e_i^{(-\langle h_i, \lambda \rangle)} u$, respectively.

Definition 8.1.1. A weight vector u is called extremal if, for any $l \geqslant 0$, $S_{i_1} \cdots S_{i_l} u$ is i-extremal for any $i, i_1, \ldots, i_l \in I$.

This notion generalizes that of highest-weight vector. A similar definition is possible for an element of a normal crystal. An element b of a normal crystal B is called i-extremal if $\tilde{e}_i b = 0$ or $\tilde{f}_i b = 0$.

Definition 8.1.2. An element b of B is called extremal if, for any $l \geqslant 0$, $S_{i_1} \cdots S_{i_l} b$ is i-extremal for any $i, i_1, \ldots, i_l \in I$.

8.2. Modules generated by extremal vectors. Let $\lambda \in P$ be an integral weight. Set $\lambda_+ = \sum_{\langle h_i, \lambda \rangle \geqslant 0} \langle h_i, \lambda \rangle \Lambda_i$ and $\lambda_- = \lambda_+ - \lambda \in P_+$. Then $V'(\lambda) = V(\lambda_+) \otimes V(-\lambda_-)$ is isomorphic to

$$U_q(\mathfrak{g}) a_\lambda \Big/ \sum_{\langle h_i, \lambda \rangle \geqslant 0} U_q(\mathfrak{g}) e_i^{1 - \langle h_i, \lambda \rangle} + \sum_{\langle h_i, \lambda \rangle \leqslant 0} U_q(\mathfrak{g}) f_i^{1 + \langle h_i, \lambda \rangle},$$

and $V'(\lambda)$ has a global base: $V'(\lambda) = \bigoplus_{\varepsilon_i^*(b) = \langle h_i, \lambda \rangle_+} \mathbf{Q}(q) G(b)$. Let $p_\lambda \colon U_q(\mathfrak{g}) a_\lambda \to V'(\lambda)$ be the projection.

LEMMA 8.2.1. *For $i \in I$ and $\lambda \in P$, consider the commutative diagram*

$$\begin{array}{ccc} U_q(\mathfrak{g}) a_\lambda & \xrightarrow{\psi_1} & U_q(\mathfrak{g}) a_{s_i \lambda} \\ & \searrow^{\psi} & \downarrow p_{s_i \lambda} \\ & & V'(s_i \lambda). \end{array}$$

Here ψ_1 is given by

$$\psi_1(a_\lambda) = \begin{cases} e_i^{(\langle h_i, \lambda \rangle)} a_{s_i \lambda} & \text{if } \langle h_i, \lambda \rangle \geqslant 0, \\ f_i^{(-\langle h_i, \lambda \rangle)} a_{s_i \lambda} & \text{if } \langle h_i, \lambda \rangle \leqslant 0. \end{cases}$$

Then for $b \in B(U_q(\mathfrak{g}) a_\lambda)$ we have $\psi(G(b)) = p_{s_i \lambda} G(S_i^* b)$. Here $S_i^* b = (S_i(b^*))^*$.

Proof. The other case being similarly proved, let us assume $\langle h_i, \lambda \rangle \geqslant 0$. Then $\psi(G(b)) = p_{s_i \lambda} G(b) e_i^{(\langle h_i, \lambda \rangle)} a_{s_i \lambda}$. If $\varepsilon_i^*(b) > 0$, then $G(b) \in \tilde{U}_q(\mathfrak{g}) e_i + I_N$ for any $N \geqslant 0$

and hence $p_{s_i\lambda}(e_i^{\langle h_i\lambda\rangle+1}a_{s_i\lambda})=0$ implies $\psi(G(b))=0$. If $\varepsilon_i^*(b)=0$ then

$$G(b)e_i^{(\langle h_i,\lambda\rangle)} \equiv G(\tilde{e}_i^{*\langle h_i,\lambda\rangle}b) \bmod \tilde{U}_q(\mathfrak{g})e_i^{\langle h_i,\lambda\rangle+1}+I_N.$$

Therefore we have

$$\psi(G(b)) = p_{s_i\lambda}(G(\tilde{e}_i^{*\langle h_i\lambda\rangle}b)a_{s_i\lambda}). \qquad \text{Q.E.D.}$$

Thus, repeating this procedure, we obtain the following.

PROPOSITION 8.2.2. *For $\lambda\in P$, set $B^{\max}(\lambda)=\{b\in B(U_q(\mathfrak{g})a_\lambda);\ b^*$ is extremal$\}$ and*

$$I_\lambda = \bigoplus_{b\in B(U_q(\mathfrak{g})a_\lambda)\setminus B^{\max}(\lambda)} \mathbf{Q}(q)G(b).$$

Then we have:
(i) *I_λ is a left $U_q(\mathfrak{g})$-submodule of $U_q(\mathfrak{g})a_\lambda$.*
(ii) *$V^{\max}(\lambda)=U_q(\mathfrak{g})a_\lambda/I_\lambda$ is an integrable $U_q(\mathfrak{g})$-module.*
(iii) *Let $p_\lambda\colon U_q(\mathfrak{g})a_\lambda\to V^{\max}(\lambda)$ be the projection and $u_\lambda=p_\lambda(a_\lambda)$. Then u_λ is an extremal vector of weight λ.*
(iv) *For any $i\in I$, we have an isomorphism*

$$V^{\max}(\lambda) \xrightarrow{\sim} V^{\max}(s_i\lambda)$$

$$by \qquad u_\lambda \mapsto S_i u_{s_i\lambda}.$$

Moreover this isomorphism sends the global base to the global base.
(v) *For any i, S_i^* gives an isomorphism $B^{\max}(\lambda)\xrightarrow{\sim}B^{\max}(s_i\lambda)$.*

We have

$$V^{\max}(\lambda) \cong \bigoplus_{b\in B^{\max}(\lambda)} \mathbf{Q}(q)G(b). \tag{8.2.2}$$

Thus $B^{\max}(\lambda)$ is a crystal base of $V^{\max}(\lambda)$.

If λ is dominant (resp. antidominant), then $V^{\max}(\lambda)$ is the irreducible $U_q(\mathfrak{g})$-module with highest- (resp. lowest-) weight λ.

8.3. A proof of the Parthasarathy-Varadarajan-Rao conjecture. For $\lambda\in P$, let us denote by $B(\lambda)$ the connected component of $B(U_q(\mathfrak{g})a_\lambda)$ containing a_λ. Let us write $u_\lambda\in B(\lambda)$ for a_λ. Then $S_w^*=*S_w*$ gives an isomorphism

$$B(\lambda)\cong B(w\lambda) \qquad \text{for any } w\in W. \tag{8.3.1}$$

If λ is dominant (resp. antidominant), $B(\lambda)$ and $B^{\max}(\lambda)$ coincide with the sixth example in Example 1.5.3. (cf. the proof of Lemma 10.2.1.).

LEMMA 8.3.1. *For $\lambda \in P_+$ and $\mu \in P$, $B(\lambda+\mu)$ is isomorphic to the connected component of $B(\lambda)\otimes B(\mu)$ containing $u_\lambda \otimes u_\mu$.*

Proof. This follows immediately from the following chain of morphisms $B(\lambda+\mu) \hookrightarrow B(\lambda+\mu_+)\otimes B(-\mu_-) \hookrightarrow B(\lambda)\otimes B(\mu_+)\otimes B(-\mu_-) \hookleftarrow B(\lambda)\otimes B(\mu)$.
Q.E.D.

As an application, we can obtain a new proof of the Parthasarathy-Varadarajan-Rao conjecture:

(8.3.2) For $\lambda, \mu \in P_+$, $w, w' \in W$, if $w'(\lambda + w\mu) \in P_+$, then $V(w'(\lambda+w\mu))$ appears in $V(\lambda)\otimes V(\mu)$.

In fact, it is enough to show that $B(w'(\lambda+w\mu))$ appears in $B(\lambda)\otimes B(\mu)$. However this follows from $B(w'(\lambda+w\mu)) \cong B(\lambda+w\mu) \subset B(\lambda)\otimes B(w\mu) \cong B(\lambda)\otimes B(\mu)$.

9. A property of $L(\tilde{U}_q(\mathfrak{g}))$.

9.1. Property of $L(\tilde{U}_q(\mathfrak{g}))u$. Let us investigate properties of $L(\tilde{U}_q(\mathfrak{g}))u$ for an element u of an integrable $U_q(\mathfrak{g})$-module.

PROPOSITION 9.1.1. *Let M be an integrable $U_q(\mathfrak{g})$-module, $\lambda \in P$, and u an element of M_λ. Then $L(\tilde{U}_q(\mathfrak{g})a_\lambda)u$ is invariant by $\tilde{e}_i$ and $\tilde{f}_i$.*

Proof. Take N such that $e_i^N u = f_i^N u = 0$ for any i. Then take $\xi \in P_+, \eta \in P_-$ such that $\lambda = \xi + \eta$ and $\langle h_i, \xi\rangle > N$, $-\langle h_i, \eta\rangle > N$. Then the morphism $\tilde{U}_q(\mathfrak{g})a_\lambda \to M$ splits through $V(\xi)\otimes V(\eta)$. The result follows from the fact that any $U_q(\mathfrak{g})$-linear homomorphism commutes with $\tilde{e}_i$ and $\tilde{f}_i$.
Q.E.D.

PROPOSITION 9.1.2. *Let L be a crystal base of an integrable $U_q(\mathfrak{g})$-module M. For $\lambda \in P$, set*

$$N = \{u \in M;\ L(\tilde{U}_q(\mathfrak{g}))u \subset q^c L \text{ for some } c\}.$$

Then N is a $U_q(\mathfrak{g})$-module.

Proof. It is trivial that N is a $\mathbf{Q}(q)$-vector space. Hence it is enough to show that N is invariant by e_i and f_i. Thus the proposition is reduced to the following statement.

(9.1.1) If $u \in M_\lambda$ satisfies $L(U_q(\mathfrak{g})a_\lambda)u \subset L$, then $L(U_q(\mathfrak{g})a_{\lambda-\alpha_i})f_i u \subset q^c L$ for some c.

Let us take N such that $f_i^{N+1}u = 0$. We shall show

$$L(\infty)u \subset q_i^{-N}L. \tag{9.1.2}$$

For any $P \in L(\infty)$, there exists $Q_n \in L(\infty) \cap \operatorname{Ker} e_i''$ such that $P = \sum Q_n f_i^{(n)}$ (cf. [K1]).

Then

$$Pu = \sum Q_n f_i^{(n)} f_i u$$
$$= \sum [n+1]_i Q_n f_i^{(n+1)} u.$$

Since $f_i^{N+1}u = 0$, we may assume that n ranges over the integers $n \leqslant N$. In this case $[n+1]_i Q_n f_i^{(n+1)} u \subset q_i^{-n} L(U_q(\mathfrak{g})a_\lambda)u \subset q_i^{-N}L$. Thus we obtain (9.1.2). To complete the proof, it is enough to apply the following proposition. Q.E.D.

PROPOSITION 9.1.3. *Let M be an integrable $U_q(\mathfrak{g})$-module and L an A submodule of M invariant by $\tilde{e}_i$ and $\tilde{f}_i$. Let $\psi\colon U_q(\mathfrak{g})a_\lambda \to M$ be a $U_q(\mathfrak{g})$-linear homomorphism such that $\psi(L(\infty)a_\lambda) \subset L$. Then $\psi(\tilde{U}_q(\mathfrak{g})a_\lambda) \subset L$.*

Proof. Since ψ splits $U_q(\mathfrak{g})a_\lambda \to V(\xi) \otimes V(\eta)$ for some $\xi, -\eta \in P_+$ with $\lambda = \xi + \eta$, the result follows from Lemma 9.1.4. below.

LEMMA 9.1.4. *Let $\xi \in P_+$ and $\eta \in P_-$. Then $L(\xi) \otimes L(\eta)$ is the smallest A-module of $V(\xi) \otimes V(\eta)$ that is invariant by $\tilde{e}_i$ and $\tilde{f}_i$ and that contains $L(\lambda) \otimes u_\eta$.*

Proof. Let L be an A-submodule of $V(\xi) \otimes V(\eta)$ invariant by $\tilde{e}_i$ and $\tilde{f}_i$. For $w \in W$, with reduced expression $w = s_{i_1} \cdots s_{i_l}$, set $V_w(\lambda) = \sum \mathbf{Q}(q) f_{i_1}^{(a_1)} \cdots f_{i_l}^{a_l} u_\lambda$ and $V_w(\eta) = \sum \mathbf{Q}(q) l_{i_1}^{(a_1)} \cdots e_{i_l}^{(a_l)} u_\eta$. Then $V_w(\lambda)$ and $V_w(\eta)$ are finite-dimensional submodules. Set $L_w(\lambda) = L(\lambda) \cap V_w(\lambda)$ and $L_w(\eta) = L(\mu) \cap V_w(\lambda)$.

In order to prove the lemma it is enough to show that,

(9.1.3) if $s_i w > w, s_i w' < w'$ and if $L \supset L_{w'}(\xi) \otimes (L(\eta) \cap V_w(\eta))$, then $L \supset L_{w'}(\xi) \otimes (L(\eta) \cap V_{s_i w}(\eta))$.

Let us take $b_1 \in B_{w'}(\xi)$ and $b_2 \in B_{s_i w}(\eta)$. Then $\tilde{f}_i^{\max} b_2 \in B_w(\eta)$, and there is c such that

$$\tilde{f}_i^{\max}(b_1 \otimes b_2) = \tilde{f}_i^c b_1 \otimes \tilde{f}_i^{\max} b_2 .$$

Hence $\tilde{f}_i^c b_1 \otimes \tilde{f}_i^{\max} b_2$ belongs to $L_{w'}(\xi) \otimes L_w(\eta)$ mod $qL_{w'}(\xi) \otimes L_{s_i w}(\eta)$. Hence $b_1 \otimes b_2$ belongs to L mod $qL_{w'}(\xi) \otimes L_{s_i w}(\eta)$. This shows that

$$L_{w'}(\xi) \otimes L_{s_i w}(\eta) \subset L + qL_{w'}(\xi) \otimes L_{s_i w}(\eta).$$

Nakayama's lemma implies the desired result: $L_{w'}(\xi) \otimes L_{s_i w}(\eta) \subset L$. Q.E.D.

9.2. Crystal lattice and $L(\tilde{U}(\mathfrak{g}))$.

THEOREM 9.2.1. *Let M be an integrable $U_q(\mathfrak{g})$-module and L a lower crystal lattice of M. Then*

$$L' = \bigoplus_\lambda \{u \in M_\lambda;\ L(U_q(\mathfrak{g})a_\lambda)u \subset L\}$$

is invariant by $\tilde{e}_i^{\mathrm{up}}$ and $\tilde{f}_i^{\mathrm{up}}$. (For $\tilde{e}_i^{\mathrm{up}}$ and $\tilde{f}_i^{\mathrm{up}}$ see [K2]*).*

We remark that, by the relation of upper and lower crystal bases, the statement above is equivalent to the statement that

$$\bigoplus_{\lambda} \{u \in M_\lambda;\ L(U_q(\mathfrak{g})a_\lambda)u \subset q^{1/2(\|\lambda\|^2-c)}L\}$$

is invariant by $\tilde{e}_i$ and $\tilde{f}_i$. Here c is a number such that $\|\lambda\|^2 - c \in 2\mathbf{Z}$ for any weight λ of M. Here $\|\lambda\|^2 = (\lambda, \lambda)$. Note that, if $\lambda, \mu \in P$ satisfy $\lambda - \mu \in \sum \mathbf{Z}\alpha_i$, then $\|\lambda\|^2 - \|\mu\|^2 \in 2\mathbf{Z}$.

We shall prove the theorem under the last form. Set $L' = \bigoplus_\lambda \{u \in M_\lambda;\ L(U_q(\mathfrak{g})a_\lambda)u \in q^{1/2(\|\lambda\|^2-c)}L\}$

LEMMA 9.2.2. *If $e_i u = 0$ and if $u \in L'_\lambda$ then $f_i^{(k)}u \in L'$.*

Proof. We may assume $k \leqslant \langle h_i, \lambda\rangle$. Since $\|\lambda - k\alpha_i\|^2 = \|\lambda\|^2 + k(\alpha_i, \alpha_i)\cdot(k - \langle h_i, \lambda\rangle)$, it is enough to show

$$L(U_q(\mathfrak{g})a_{\lambda-k\alpha_i})f_i^{(k)}u \subset q_i^{k(k-\langle h_i,\lambda\rangle)}L(U_q(\mathfrak{g})a_\lambda)u\,.$$

This is reduced to the statement

$$\text{(9.2.1)}\quad L(U_q(\mathfrak{g})a_{\lambda-k\alpha_i})f_i^{(k)} \subset q_i^{k(k-\langle h_i,\lambda\rangle)}L(U_q(\mathfrak{g})a_\lambda) + \tilde{U}_q(\mathfrak{g})e_i + \tilde{U}_q(\mathfrak{g})f_i^{1+\langle h_i,\lambda\rangle}$$

This follows from Lemma 6.5.1. Q.E.D.

Proof of Theorem 9.2.1. Let us take an element u in L' of weight λ. We write $u = \sum_{n\leqslant N} f_i^{(n)}u_n$ with $e_i u_n = 0$ and $wt(u_n) = \lambda + n\alpha_i$. By the preceding lemma, it is enough to show $u_n \in L'$. We begin with induction on N. We have

$$e_i^{(N)}u = e_i^{(N)}f_i^{(N)}u_N = \begin{bmatrix}\langle h_i, \lambda + N\alpha_i\rangle \\ N\end{bmatrix}_i u_N$$

$$\text{and}\qquad e_i^{N+1}u = 0\,.$$

The crystal lattice of $\tilde{U}(\mathfrak{g})$ has the property

$$\text{(9.2.2)}\quad L(U_q(\mathfrak{g})a_{\lambda+N\alpha_i})e_i^{(N)} \subset L(U_q(\mathfrak{g})a_\lambda) + \tilde{U}_q(\mathfrak{g})e_i^{N+1} + I_m \qquad \text{for any } m\,.$$

Admitting this, let us finish the proof of Theorem 9.2.1. We have

$$\begin{aligned} L(U_q(\mathfrak{g})a_{\lambda+N\alpha_i})u_N &= q_i^{N(\langle h_i,\lambda\rangle+N)}L(U_q(\mathfrak{g})a_{\lambda+N\alpha_i})e_i^{(N)}u \\ &\subset q_i^{N(\langle h_i,\lambda\rangle+N)}L(U_q(\mathfrak{g})a_\lambda)u \subset q_i^{N(\langle h_i,\lambda\rangle+N)}q^{1/2(\|\lambda\|^2-c)}L \\ &= q^{1/2(\|\lambda+N\alpha_i\|^2-c)}L\,. \end{aligned}$$

Thus u_N belongs to L'. This implies $f_i^{(N)}u_N \in L'$ and hence $\sum_{n<N} f_i^{(n)}u_n$ also belongs to L'. Thus the induction proceeds.

Now it remains to prove (9.2.2.). For $b \in B(\tilde{U}_q(\mathfrak{g}))$, if $\psi_i^*(b) > 0$, then $G(b)e_i^{(N)} \subset \tilde{U}_q(\mathfrak{g})e_i^{N+1} + I_m$ for any m by Proposition 6.4.2. If $\psi_i^*(b) = 0$, then Proposition 6.4.4 implies $G(b)e_i^{(N)} \equiv G(\tilde{e}_i^{*N}b) \bmod \tilde{U}_q(\mathfrak{g})e_i^{N+1} + I_m$. This completes the proof of Theorem 9.2.1.

9.3. Applications. Let us give applications of Theorem 9.2.1.

PROPOSITION 9.3.1. *For $\lambda \in P_+$, let $L^{up}(\lambda)$ be the upper crystal lattice of $V(\lambda)$. Then $L(\tilde{U}_q(\mathfrak{g}))L^{up}(\lambda) \subset L(\lambda)$.*

Proof. Set $L' = \{u \in V(\lambda); L(\tilde{U}_q(\mathfrak{g}))u \subset L(\lambda)\}$. Then $L'_\lambda = Au_\lambda$, and L' is invariant by $\tilde{e}_i^{up}$ and $\tilde{f}_i^{up}$. Hence $L' = L^{up}(\lambda)$. Q.E.D.

PROPOSITION 9.3.2. *For any connected component B' of $B(\tilde{U}_q(\mathfrak{g}))$, $\{\|wt(b)\|^2; b \in B'\}$ is bounded from above.*

Proof. Let us take $\xi \in P_+$ and $\eta \in P_-$ such that $B' \subset B(\xi) \otimes B(\eta)$. Let us take $b_0 \in B'$ and let λ_0 be the weight of b_0. Then by Lemma 9.1.2 there exists $c \in \|\lambda_0\|^2 + 2\mathbf{Z}$ such that

$$L(\tilde{U}_q(\mathfrak{g}))G(b_0) \subset q^{1/2(\|\lambda_0\|^2 - c)}L(\xi) \otimes L(\eta).$$

Set

$$L' = \bigoplus_\lambda \{u \in (L(\xi) \otimes L(\eta))_\lambda; L(U_q(\mathfrak{g})a_\lambda)u \subset q^{1/2(\|\lambda\|^2 - c)}L(\xi) \otimes L(\eta)\}.$$

Then L' is invariant by $\tilde{e}_i$ and $\tilde{f}_i$ by Theorem 9.2.1. Let ψ be the map $L' \to L(\lambda) \otimes L(\eta)/qL(\xi) \otimes L(\eta)$. Then ψ is invariant by $\tilde{e}_i$ and $\tilde{f}_i$, and hence the image of ψ contains B'. For any $b \in B'$, let us take $v \in L'$ such that $\psi(v) = b$. Then $v \notin qL(\xi) \otimes L(\eta)$ and $v \in L(\tilde{U}_q(\mathfrak{g}))v \subset q^{1/2(\|wt(b)\|^2 - c)}L(\xi) \otimes L(\eta)$. They imply $\|wt(b)\|^2 \leqslant c$. Q.E.D.

For a connected component B' of $B(\tilde{U}_q(\mathfrak{g}))$, an element $b \in B'$ is an extremal vector if $\|wt(b)\|^2$ is maximal. Hence we obtain the following corollary.

COROLLARY 9.3.3. *Any connected component of $B(\tilde{U}_q(\mathfrak{g}))$ contains an extremal vector.*

We obtain Corollary 9.3.4 by applying this to b^*.

COROLLARY 9.3.4. *Any connected component of $B(\tilde{U}_q(\mathfrak{g}))$ can be embedded into some $B^{\max}(\lambda)$.*

In the course of the proof of Proposition 9.3.2, if B' is the connected component $B(\lambda)$ of $B(\tilde{U}_q(\mathfrak{g}))$ containing a_λ, we can take $b_0 = a_\lambda$ and $c = 0$. Thus we obtain the following.

PROPOSITION 9.3.5. *For any $b \in B(\lambda)$, $\|wt(b)\|^2 \leqslant \|\lambda\|^2$.*

Remark. The result of Proposition 9.3.2 gives a strong constraint on the crystal structure of $B(\tilde{U}_q(\mathfrak{g}))$. For example, for $\lambda \in P_-$ and $\mu \in P_+$, the connected component B of $B(\lambda) \otimes B(\mu)$ containing $u_\lambda \otimes u_\mu$ does not satisfy the bounded condition in Proposition 9.3.2. In fact, taking $\tilde{e}_i^{\max}$ successively, B contains $u_{w\lambda} \otimes u_\mu$ for any $w \in W$. However $\{\|w\lambda + \mu\|^2; w \in W\}$ is not bounded from above even in the affine case (if λ, μ are regular).

10. Comparison with the result of [K2].

10.1. Relation of $A_q(\mathfrak{g})$ and $\tilde{U}_q(\mathfrak{g})$. In [K2], we define the crystal base of $A_q(\mathfrak{g})$. Let us recall that

$$A_q(\mathfrak{g}) = \bigoplus_{\lambda \in P} \{u \in (U_q(\mathfrak{g})^*)_\lambda; \text{there exists } l \geqslant 0 \text{ such that } e_{i_1} \cdots e_{i_l} u = u f_{i_1} \cdots f_{i_l} = 0$$
$$\text{for any } i_1, \ldots, i_l \in I\}.$$

$A_q(\mathfrak{g}) \cong \bigoplus_{\lambda \in P_+} V^r(\lambda) \otimes V(\lambda)$, and $A_q(\mathfrak{g})$ has an upper global base. Here $V^r(\lambda)$ is the irreducible right highest-weight $U_q(\mathfrak{g})$-module generated by the highest-weight vector v_λ. We have $B(A_q(\mathfrak{g})) = \bigoplus_\lambda B^r(\lambda) \otimes B(\lambda)$. There exists a canonical coupling

$$\langle\ ,\ \rangle\colon A_q(\mathfrak{g}) \otimes \tilde{U}_q(\mathfrak{g}) \to \mathbf{Q}(q). \tag{10.1.1}$$

Set $A_q^{\mathbf{Z}}(\mathfrak{g}) = \{u \in A_q(\mathfrak{g}); \langle u, U_q^{\mathbf{Z}}(\mathfrak{g})\rangle \subset \mathbf{Z}[q, q^{-1}]\}$. Then we can see easily

$$\langle A_q^{\mathbf{Z}}(\mathfrak{g}), \tilde{U}_q(\mathfrak{g})_{\mathbf{Z}}\rangle \subset \mathbf{Q}[q, q^{-1}].$$

THEOREM 10.1.1. *There exists a unique embedding $\psi\colon B(A_q(\mathfrak{g})) \subset B(\tilde{U}_q(\mathfrak{g}))$ such that $(G(b), G(b')) = \delta_{\psi(b), b'}$ for any $b \in B(A_q(\mathfrak{g}))$ and $b' \in B(\tilde{U}_q(\mathfrak{g}))$.*

Proof. There exists a unique embedding of crystals over $\mathfrak{g} \oplus \mathfrak{g}$

$$B^r(\lambda) \otimes B(\lambda) \to B(\tilde{U}_q(\mathfrak{g}))$$

that sends $v_\lambda \otimes u_\lambda$ to a_λ for $\lambda \in P_+$. This gives an embedding $B(A_q(\mathfrak{g})) \subset B(\tilde{U}_q(\mathfrak{g}))$. In order to see that this satisfies the required property, let us remark the following lemma.

LEMMA 10.1.2. $\langle L(A_q(\mathfrak{g})), L(\tilde{U}_q(\mathfrak{g}))\rangle \subset A$.

Proof. By the definition, it is enough to show $\langle L^{up}(V^r(\lambda)), L(\tilde{U}_q(\mathfrak{g}))L^{up}(\lambda)\rangle \subset A$. Since $\{u \in V(\lambda); \langle L^{up}(\lambda), u\rangle \subset A\} = L^{low}(\lambda)$, this follows from Proposition 9.3.1.

Q.E.D.

Thus we obtain $\langle G(b'), G(b)\rangle \subset A \cap \bar{A} \cap \mathbf{Q}[q, q^{-1}] = \mathbf{Q}$ for $b \in B(A_q(\mathfrak{g}))$, $b' \in B(\tilde{U}_q(\mathfrak{g}))$. Since $\langle\ ,\ \rangle$ is invariant by $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{e}_i^*$ and $\tilde{f}_i^*$, we obtain Theorem 10.1.1, because $\langle G(v_\lambda \otimes u_\lambda), a_\lambda\rangle = \langle v_\lambda, u_\lambda\rangle = 1$.

10.2. Finite-dimensional and affine case. Let us regard $B(A_q(\mathfrak{g}))$ as a subset of $B(\tilde{U}_q(\mathfrak{g}))$. Then $B(A_q(\mathfrak{g}))$ is the smallest subcrystal (with respect $\mathfrak{g} \oplus \mathfrak{g}$) of $B(\tilde{U}_q(\mathfrak{g}))$ that contains all $a_\lambda (\lambda \in P)$.

Let us denote by T the Tits cone; i.e., $T = \bigcup_{w \in W} wP_+$.

LEMMA 10.2.1. *If $\lambda \in T$ then $B^{\max}(\lambda) \subset B(A_q(\mathfrak{g}))$.*

Proof. For $w \in W$, S_w^* sends $B^{\max}(\lambda)$ onto $B^{\max}(w\lambda)$. Hence we may assume $\lambda \in P_+$ from the beginning. If $b = b_1 \otimes t_\lambda \otimes b_2 \in B^{\max}(\lambda)$, then $\varphi_i^*(b) = \max(\varphi_i^*(b_2), \varepsilon_i^*(b_1) - \langle h_i, \lambda \rangle) = 0$. Hence $\varphi_i^*(b_2) = 0$ for all i and hence $b_2 = u_{-\infty}$ and $\varepsilon_i^*(b_1) \leqslant \langle h_i, \lambda \rangle$. This shows $B^{\max}(\lambda) = B(\lambda) \otimes t_\lambda \otimes u_{-\infty}$. Then the desired result follows from the connectedness of $B(\lambda)$. Combining this with Corollary 9.3.3 and 9.3.4 we obtain the following result.

PROPOSITION 10.2.2. (i) *If $\mathfrak{g}$ is finite dimensional, $B(A_q(\mathfrak{g})) = B(\tilde{U}_q(\mathfrak{g}))$.*

(ii) *If $\mathfrak{g}$ is affine, let $B(\tilde{U}_q(\mathfrak{g})_+)$ be the subcrystal of $B(\tilde{U}_q(\mathfrak{g}))$ consisting of vectors with positive level. Then $B(A_q(\mathfrak{g})) = B(\tilde{U}_q(\mathfrak{g}))_+ \bigsqcup \{a_0\}$.*

Recall that when $\mathfrak{g}$ is affine we take $c \in \sum \mathbf{Z}_{>0} h_i$ with $\langle c, \alpha_i \rangle = 0$ for any i. Then $\langle c, \lambda \rangle$ is called the level of $\lambda \in P$.

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