CRYSTAL BASES OF MODIFIED QUANTIZED ENVELOPING ALGEBRA

MASAKI KASHIWARA

0. Introduction.

0.1. G. Lusztig gives the crystal base on the modified quantized enveloping algebra $\tilde{U}_q(g)$ in [L2]. The algebra $\tilde{U}_q(g)$ is obtained from the quantized universal enveloping algebra $U_q(g)$ by modifying the torus part $\bigoplus_{h \in P_+} \mathbb{Q}(q)q^h$ to $\bigoplus_i \mathbb{Q}(q)a_i$, where $a_i$ is the projector to the weight space of weight $\lambda$ (see §1.2). He gives also several conjectures on its properties in [L3]. The purpose of this paper is to study the structure of crystal bases $B(\tilde{U}_q(g))$ of $\tilde{U}_q(g)$ and to give an affirmative answer to some of his conjectures.

0.2. Let us explain the results obtained here more precisely. We establish that the crystal structure of $\tilde{U}_q(g)$ is described by those of $U_q^-(g)$ and $U_q^+(g)$. Namely, let $B(\infty)$ be the crystal base of $U_q^-(g)$ and $B(-\infty)$ the one of $U_q^+(g)$. Let $T_\lambda$ be the crystal consisting of a single element of weight $\lambda$. Then the crystal base of $\tilde{U}_q(g)$ is isomorphic to the direct sum of $B(\infty) \otimes T_\lambda \otimes B(-\infty)$ (Theorem 3.1.1). This fact is a reflection of $\tilde{U}_q(g) = \bigoplus_\lambda U_q^-(g) \otimes U_q^+(g) \otimes \mathbb{Q}(q)a_\lambda$. The algebra $\tilde{U}_q(g)$ has the antiautomorphism $^*$ that sends $e_i$, $f_i$ to themselves and $a_\lambda$ to $a_{-\lambda}$. We prove that the crystal base is stable by $^*$ (Theorem 4.3.2). This is one of the conjectures of Lusztig [L3]. This automorphism sends $b_1 \otimes t_\lambda \otimes b_2 \in B(\infty) \otimes T_\lambda \otimes B(-\infty) \subset B(\tilde{U}_q(g))$ to $b_1^* \otimes t_{-\lambda-w_{b_1}-w_{b_2}} \otimes b_2^*$. By this automorphism, $B(\tilde{U}_q(g))$ has another crystal structure. These two structures are compatible (see §5), and $B(\tilde{U}_q(g))$ may be regarded as a crystal over $g \oplus g$. This is a reflection of the $U_q(g)$-bimodule structure of $\tilde{U}_q(g)$.

0.3. In [K2], the author introduces "the dual algebra" $A_q(g)$ of $U_q(g)$ and its crystal base $B(A_q(g))$. This algebra has the Peter-Weyl–type decomposition

$$A_q(g) = \bigoplus_{\lambda \in P_+} V'(\lambda) \otimes V(\lambda).$$

Here $P_+$ is the set of dominant integral weights and $V(\lambda)$ and $V'(\lambda)$ are the left and right highest-weight module with highest weight $\lambda$. Accordingly $B(A_q(g))$ has the crystal structure

$$B(A_q(g)) = \bigoplus_{\lambda \in P_+} B(\lambda) \otimes B(\lambda).$$

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383
When \( g \) is finite-dimensional, \( B(\bar{U}_q(g)) \) is isomorphic to \( B(A_q(g)) \). When \( g \) is affine, \( \bar{U}_q(g) \) is the direct sum of three algebras \( \bar{U}_q(g)_+, \bar{U}_q(g)_0 \), and \( \bar{U}_q(g)_- \). They consist of elements of positive level, level 0, and negative level, respectively. Accordingly \( B(\bar{U}_q(g)) \) is the direct sum of \( B(\bar{U}_q(g)_+, B(\bar{U}_q(g)_0), \) and \( B(\bar{U}_q(g)_-) \). The crystal structure of \( B(\bar{U}_q(g)_+) \) is rather simple. Namely, \( B(\bar{U}_q(g)_+) \oplus B(0) \) is isomorphic to \( B(A_q(g)) \). Here \( B(0) \) is the crystal base of the trivial representation. However, the author does not know much about the structure of \( B(\bar{U}_q(g)_0) \). The result of [IIJMNT] suggests that \( B(\bar{U}_q(g)_0) \) is a direct sum of the affinization of the crystals of finite-dimensional \( \bar{U}_q(g) \)-modules. We show here one property: for any connected component \( B' \) of \( B(\bar{U}_q(g)) \), \( \{(\text{wt}(b), \text{wt}(b)); b \in B' \} \) is bounded from above (see §9.3).

0.4. We show also that the Weyl group operates on \( B(\bar{U}_q(g)) \) or more generally on the crystal base of integrable \( U_q(g) \)-modules. The action \( S_i \) of simple reflection is given by

\[
S_i b = \begin{cases} f_i^{\langle h_i, \text{wt}b \rangle} b & \text{if } \langle h_i, \text{wt}b \rangle > 0, \\ e_i^{\langle h_i, \text{wt}b \rangle} b & \text{if } \langle h_i, \text{wt}b \rangle \leq 0. \end{cases}
\]

We show (Theorem 7.2.2) that \( \{S_i\} \) satisfies the braid relation. We call a crystal base \( b \) an extremal vector if, for any \( w \in W \) and \( i \in I \), \( S_w b \) is killed by either \( e_i \) or \( f_i \). We show another remarkable property of \( B(\bar{U}_q(g)) \). Let \( B' \) be a connected component of \( B(\bar{U}_q(g)) \). Then \( B' \) may not contain either a highest-weight vector or a lowest-weight vector, but it always contains an extremal vector.

1. Notations.

1.1. Definition of quantized enveloping algebra. Let us recall the definition of \( \bar{U}_q(g) \) (cf. [K]). We prepare the following data:

(1.1.1) a free \( \mathbb{Z} \)-module \( P \) (weight lattice),

(1.1.2) an index set \( I \) and \( \alpha_i \in P \) and \( h_i \in P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \) for \( i \in I \),

(1.1.3) a \( \mathbb{Q} \)-valued symmetric bilinear form \( (\ , \ ) \) on \( P \).

We assume for the sake of simplicity that there exist \( \Lambda_i \in P \) such that \( \langle h_j, \Lambda_i \rangle = \delta_{ij} \) for any \( j \in I \). We call \( \Lambda_i \) the fundamental weight. We assume further that \( \{\alpha_i\}_{i \in I} \) is linearly independent. Many of the results in this paper still hold without these assumptions. Assume that they satisfy the conditions

(1.1.4) \( (\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} \) for \( i \in I \),

(1.1.5) \( \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \) for \( i \in I \) and \( \lambda \in P \),

(1.1.6) \( (\alpha_i, \alpha_j) \leq 0 \) for \( i \neq j \in I \).
The quantized enveloping algebra $U_q(g)$ is the $Q(q)$-algebra generated by $e_i, f_i (i \in I)$, and $q^h (h \in P^*)$ satisfying the defining relations

\begin{align}
(1.1.7) \quad q^h &= 1 \quad \text{for } h = 0; \\
(1.1.8) \quad q^{h_1} q^{h_2} &= q^{h_1 + h_2} \quad \text{for } h_1, h_2 \in P^*; \\
(1.1.9) \quad q^h e_i q^{-h} &= q^{\langle h, \alpha_i \rangle} e_i \quad \text{and} \\
q^h f_i q^{-h} &= q^{-\langle h, \alpha_i \rangle} f_i; \\
(1.1.10) \quad [e_i, f_j] &= \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}
\end{align}

where $q_i = q^{(\alpha_i, \alpha_i)/2}$ and $t_i = q^{(\alpha_i, \alpha_i) h_i/2}$;

\begin{equation}
(1.1.11) \quad \text{for } i \neq j \in I, \quad \text{setting } c = 1 - \langle h_i, \alpha_j \rangle,
\end{equation}

\[ \sum_{n=0}^{\xi} (-)^n e_i^{(n)} e_j e_i^{(c-n)} = \sum (-)^n f_i^{(n)} f_j f_i^{(c-n)} = 0. \]

Here $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^{n} [k]_i$, $e_i^{(n)} = e_i^n/[n]_i!$, and $f_i^{(n)} = f_i^n/[n]_i!$.

We set \( \{ x \} = (x - x^{-1})/(q - q^{-1}) \) and \( \{ x \} = \prod_{k=1}^{n} \{ q_1^{k-1} x \} \). We denote by $U_q^Z(g)$ the $Z[q, q^{-1}]$-algebra generated by $e_i^{(n)}, f_i^{(n)}, q^h, \{ q^h \}(i \in I, h \in P^*, n \in Z_{\geq 0})$.

Let $U_q^+(g)$ (resp. $U_q^-(g)$) be the $Q(q)$-subalgebra of $U_q(g)$ generated by the $e_i$'s (resp. the $f_i$'s) and $U_q^+(g)_Z$ (resp. $U_q^-(g)_Z$) the $Z[q, q^{-1}]$-subalgebra generated by the $e_i^{(n)}$'s (resp. the $f_i^{(n)}$'s). We set $U_q^B(g) = Q[q, q^{-1}] \otimes Z_{\alpha \in I} U_q^B(g)$, etc. For $\zeta \in Q = \bigoplus Z_{\alpha \in I}$, we set

\[ U_q(g)_{\zeta} = \{ P \in U_q(g); q^h P q^{-h} = q^{\langle h, \zeta \rangle} \text{ for any } h \in P^* \}, \]

\[ U_q^\pm(g)_{\zeta} = U_q^\pm(g) \cap U_q(g)_{\zeta}. \]

We set $|\zeta| = \sum |n_i|$, $\zeta = \sum n_i \alpha_i \in Q$. We define the filtration $F$ of $U_q^\pm(g)$ by

$F_n(U_q^\pm(g)) = \bigoplus_{|\zeta| \leq n} U_q^\pm(g)_\zeta$.

We say $\lambda \in P$ is dominant (resp. antidominant) if $\langle h_i, \lambda \rangle \geq 0$ (resp. $\langle h_i, \lambda \rangle \leq 0$) for any $i$. We write $P_\pm$ for the set of dominant (resp. antidominant) integral weights.

\section{Definition of modified quantized enveloping algebra}

Let $\text{Mod}(g, P)$ denote the category of left $U_q(g)$-modules $M$ with the weight decomposition

\[ M = \bigoplus_{\lambda \in P} M_\lambda. \]
where

\[ M_\lambda = \{ u \in M ; q^h u = q^{(h,\lambda)} u \text{ for any } h \in P^* \}. \]

Let (forget) be the functor from \text{Mod}(g, P) to the category of vector spaces over \( \mathbb{Q}(g) \), forgetting the \( U_q(g) \)-module structure. Let \( R \) denote the endomorphism ring of (forget). Hence to give an element of \( R \) is to associate an endomorphism \( \varphi(M) \) of \( M \) with each \( M \) in \text{Mod}(g, P) \) such that, for any morphism \( f : M \to M' \) in \text{Mod}(g, P), \( \varphi(M') \circ f = f \circ \varphi(M) \) holds. Note that \( R \) contains \( U_q(g) \). For \( \lambda \in P \) let \( a_\lambda \in R \) denote the projector \( M \to M_\lambda \) to the weight space. Then the defining relation of \( a_\lambda \) (as a left \( U_q(g) \)-module) is

\[ (1.2.1) \quad q^h a_\lambda = q^{(h,\lambda)} a_\lambda. \]

We have

\[ (1.2.2) \quad a_\lambda P = P a_{\lambda - \xi} \quad \text{for } \xi \in Q \text{ and } P \in U_q(g)_{\xi}, \]

\[ (1.2.3) \quad a_\lambda a_\mu = \delta_{\lambda,\mu} a_\lambda. \]

Then \( R \) is isomorphic to the direct product \( \prod_{\lambda \in P} U_q(g)_{a_\lambda} \). We set

\[ \tilde{U}_q(g) = \bigoplus_{\lambda \in P} U_q(g)_{a_\lambda}. \]

Then \( \tilde{U}_q(g) \) is a subring of \( R \) by (1.2.2) and (1.2.3). Hence any object \( M \) in \text{Mod}(g, P) may be regarded as a left \( \tilde{U}_q(g) \)-module. We set

\[ (1.2.4) \quad \tilde{U}^Z_q(g) = \bigoplus_{\lambda} U_q^Z(g)_{a_\lambda}, \]

\[ (1.2.5) \quad \tilde{U}^\pm_q(g) = \bigoplus_{\lambda} U_q^\pm(g)_{a_\lambda} = \bigoplus_{\lambda} a_\lambda U_q^\pm(g). \]

They are subrings of \( \tilde{U}_q(g) \).

1.3. The automorphisms of \( U_q(g) \). Let * denote the antiautomorphism of \( U_q(g) \) given by

\[ (1.3.1) \quad q^* = q, \quad (q^h)^* = q^{-h}, \quad e_i^* = e_i, \quad f_i^* = f_i. \]

Let \( \varphi \) be the antiautomorphism of \( U_q(g) \) given by

\[ (1.3.2) \quad \varphi(q) = q, \quad \varphi(q^h) = q^h, \quad \varphi(f_i) = f_i, \quad \varphi(e_i) = e_i. \]
Let us denote by $\vee$ the automorphism $\varphi \circ \ast \circ \ast \circ \varphi$ of $U_q(g)$. Hence we have

$$q^{\vee} = q, \quad (q^h)^{\vee} = q^{-h}, \quad e_i^{\vee} = f_i, \quad f_i^{\vee} = e_i.$$  

Let $-\ast$ be the automorphism of $U_q(g)$ given by

$$\bar{q} = q^{-1}, \quad (q^h)^{-} = q^{-h}, \quad \bar{e}_i = e_i, \quad \bar{f}_i = f_i.$$  

They commute with each other.

These automorphisms and anti-automorphisms are extended to those of $\bar{U}_q(g)$. We shall denote them by the same letter. We have

$$a^\pm = a_{\pm}, \quad \varphi(a_{\pm}) = a_{\mp}, \quad a^{\vee}_{\pm} = a_{\mp}, \quad \bar{a}_{\pm} = a_{\pm}.$$  

1.4. Tensor product. In this article, we take the comultiplication $\Delta$ of $U_q(g)$ given by

$$\Delta q^h = q^h \otimes q^h,$$

$$\Delta e_i = e_i \otimes t_i^{-1} + 1 \otimes e_i,$$

$$\Delta f_i = f_i \otimes 1 + t_i \otimes f_i.$$  

By this comultiplication, the tensor product of $U_q(g)$-modules has a structure of $U_q(g)$-module.

For an automorphism $g$ of $U_q(g)$ and a $U_q(g)$-module $M$, let us denote by $M^g$ the $U_q(g)$-module $\{u^g; u \in M\}$ with $Pu^g = (g(P)u)^g$ for $u \in M$ and $P \in U_q(g)$. With this notation we have

$$\left( M \otimes N \right)^g \cong N^g \otimes M^g$$  

by $(u \otimes v)^g \leftrightarrow v^g \otimes u^g$ for $u \in M$, $v \in N$.

1.5. Crystals. Let us recall the definition of crystals (cf. [K3]).

Definition 1.5.1. A crystal $B$ is a set with the following data:

$$a \text{ map } wt: B \to P,$$

$$e_i: B \to \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i: B \to \mathbb{Z} \sqcup \{-\infty\} \quad \text{for } i \in I,$$

$$\bar{e}_i: B \to B \sqcup \{0\} \quad \text{and} \quad \bar{f}_i: B \to B \sqcup \{0\} \quad \text{for } i \in I.$$  

They satisfy the following axioms.
(1.5.4) For $b \in B$, 
\[ \varphi_i(b) = \epsilon_i(b) + \langle h_i, \text{wt}(b) \rangle. \]

(1.5.5) For $b \in B$ with $\tilde{\epsilon}_i b \in B$,
\[ \text{wt}(\tilde{\epsilon}_i b) = \text{wt}(b) + \alpha_i. \]

(1.5.6) For $b \in B$ with $\tilde{f}_i b \in B$,
\[ \text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i. \]

(1.5.7) For $b_1, b_2 \in B$, \quad $\tilde{f}_i b_2 = b_1$ if and only if $\tilde{\epsilon}_i b_1 = b_2$.

(1.5.8) If $\epsilon_i(b) = -\infty$ then $\tilde{\epsilon}_i b = \tilde{f}_i b = 0$.

Definition 1.5.2. A morphism $\psi: B_1 \to B_2$ from a crystal $B_1$ to a crystal $B_2$ is a map $\psi: B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ satisfying the following axioms:

(1.5.9) $\psi(0) = 0$;

(1.5.10) if $b \in B_1$ and $\psi(b) \in B_2$, then
\[ \text{wt}(\psi(b)) = \text{wt}(b), \quad \epsilon_i(\psi(b)) = \epsilon_i(b) \quad \text{and} \quad \varphi_i(\psi(b)) = \varphi_i(b); \]

(1.5.11) if $b \in B_1$ satisfies $\psi(\tilde{\epsilon}_i b) \neq 0$ and $\psi(b) \neq 0$, then $\psi(\tilde{\epsilon}_i b) = \tilde{\epsilon}_i \psi(b)$;

(1.5.12) if $b \in B_1$ satisfies $\psi(\tilde{f}_i b) \neq 0$ and $\psi(b) \neq 0$, then $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

The definition of morphisms is slightly different from [K3].

Let $\mathcal{C}(I, P)$ denote the category of crystals.

A morphism $\psi: B_1 \to B_2$ of crystals is called strict if the associated map from $B_1 \sqcup \{0\}$ to $B_2 \sqcup \{0\}$ commutes with all $\tilde{\epsilon}_i$ and $\tilde{f}_i$. If the associated map is injective, then $\psi$ is called embedding.

A crystal $B$ is called seminormal if, for any $b \in B$ and $i \in I$, $\epsilon_i(b)$ and $\varphi_i(b)$ are nonnegative integers and
\[
\epsilon_i(b) = \max \{ n \geq 0; \tilde{\epsilon}_i^n b \in B \},
\]
\[
\varphi_i(b) = \max \{ n \geq 0; \tilde{f}_i^n b \in B \}.
\]

In such a case, we set
\[
\tilde{\epsilon}_i^{\max} b = \tilde{\epsilon}_i^{\varphi_i(b)} b \quad \text{and} \quad \tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b.
\]
A crystal $B$ is called normal if, for any subset $J$ of $I$ such that $\{ (\alpha_i, \alpha_j) \}_{i,j \in J}$ is a positive-definite symmetric matrix, $B$ is isomorphic (in $\mathcal{O}(J, P)$) to a crystal base of an integrable $U_q(g_J)$-module. Here $U_q(g_J)$ is the quantized universal enveloping algebra generated by $e_j, f_j$ ($j \in J$), and $q^h$ ($h \in P^*$).

For crystals $B_1$ and $B_2$, let us define their tensor product $B_1 \otimes B_2$ by

\begin{align}
B_1 \otimes B_2 &= \{ b_1 \otimes b_2; b_1 \in B_1, b_2 \in B_2 \}, \\
wt(b_1 \otimes b_2) &= wt(b_1) + wt(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle), \\
\varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2) \rangle), \\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\
b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), 
\end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\
b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). 
\end{cases}
\end{align}

Here $0 \otimes b$ and $b \otimes 0$ are understood to be $0$. Then $\otimes$ is a functor from $\mathcal{O}(I, P) \times \mathcal{O}(I, P)$ to $\mathcal{O}(I, P)$ and satisfies the associative law: $(B_1 \otimes B_2) \otimes B_3 \cong B_1 \otimes (B_2 \otimes B_3)$ by $(b_1 \otimes b_2) \otimes b_3 \leftrightarrow b_1 \otimes (b_2 \otimes b_3)$.

For a crystal $B$, let us denote by $B^\vee$ the crystal defined by

\begin{align}
B^\vee &= \{ b^\vee; b \in B \}, \\
wt(b^\vee) &= -wt(b), \\
\varepsilon_i(b^\vee) &= \varphi_i(b), \\
\varphi_i(b^\vee) &= \varepsilon_i(b), \\
\tilde{e}_i(b^\vee) &= (\tilde{f}_i b)^\vee \quad \text{and} \quad \tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee.
\end{align}

Then we have (cf. (1.4.2))

\begin{align}
(B_1 \otimes B_2)^\vee &\cong B_2^\vee \otimes B_1^\vee \\
(b_1 \otimes b_2)^\vee &\leftrightarrow b_2^\vee \otimes b_1^\vee.
\end{align}

**Example 1.5.3.**

1. $C = \{ c \}$ with

\begin{align}
wt(c) &= 0, \\
\varepsilon_i(c) &= \varphi_i(c) = 0, \\
\tilde{e}_i c &= \tilde{f}_i c = 0.
\end{align}

For any seminormal crystal $B$, $B \otimes C$ and $C \otimes B$ are isomorphic to $B$. 
2. For \( \lambda \in P \), \( T_{\lambda} = \{ t_{\lambda} \} \) with
\[
 \text{wt}(t_{\lambda}) = \lambda, \quad \varepsilon_i(t_{\lambda}) = \varphi_i(t_{\lambda}) = -\infty,
\]
\[
 \bar{e}_i t_{\lambda} = \bar{f}_i t_{\lambda} = 0.
\]
We have \( T_i \otimes T_{\lambda} \simeq T_{\lambda + \mu} \) and \( B \otimes T_0 \simeq T_0 \otimes B \simeq B \) for any crystal \( B \).

3. For \( i \in I \), \( B_i = \{ b_i(n); n \in \mathbb{Z} \} \) with \( \text{wt}(b_i(n)) = n \alpha_i \),
\[
 \varepsilon_i(b_i(n)) = -n, \quad \varphi_i(b_i(n)) = n,
\]
\[
 \varepsilon_j(b_i(n)) = \varphi_j(b_i(n)) = -\infty \quad \text{for } j \neq i,
\]
and \( \bar{e}_i b_i(n) = b_i(n + 1) \), \( \bar{f}_i b_i(n) = b_i(n - 1) \), \( \bar{e}_j b_i(n) = \bar{f}_j b_i(n) = 0 \) for \( j \neq i \).

We write \( b_i \) for \( b_i(0) \).

4. \( B(\infty) \) denotes the crystal associated with \( U_q^{-}(g) \). We denote by \( u_{\infty} \) the vector of weight 0.

5. \( B(-\infty) = B(\infty)^\vee \). This is regarded as the crystal associated with \( U_q^{+}(g) \). We set \( u_{-\infty} = u_{\infty}^\vee \).

6. For \( \lambda \in P_+ \), let \( B(\lambda) \) denote the crystal associated with the irreducible module \( V(\lambda) \) of highest weight \( \lambda \). Set \( B(-\lambda) = B(\lambda)^\vee \). Then \( B(-\lambda) \) is isomorphic to the crystal associated with the irreducible module \( V(-\lambda) \) of lowest weight \( -\lambda \). Then \( C \) is isomorphic to \( B(0) \).

Let us recall that the automorphism \( \ast \) of \( U_q^{-}(g) \) induces the automorphism of \( B(\infty) \) (cf. [K3]). We shall also denote it by \( \ast \). We set \( \varepsilon_i^\ast(b) = \varepsilon_i(b^\ast) \), \( \varphi_i^\ast(b) = \varphi_i(b^\ast) \),
\[
 \bar{e}_i^\ast b = (\bar{e}_i b)^\ast, \quad \text{and } \bar{f}_i^\ast b = (\bar{f}_i b)^\ast.
\]
For \( \lambda \in P_+ \), there exists a unique embedding \( B(\lambda) \)
into \( B(\infty) \otimes T_\lambda \), whose image is \( \{ b \otimes t_{\lambda} \in B(\infty) \otimes T_\lambda; \varepsilon_i^\ast(b) \leq \langle h_i, \lambda \rangle \} \). Similarly, we define \( \ast \), \( \varepsilon_i^\ast \), etc., for \( B(-\infty) \). Then, for \( \lambda \in P_- \), \( B(\lambda) \) is isomorphic to the subcrystal \( \{ t_{\lambda} \otimes b \in T_\lambda \otimes B(-\infty); \varepsilon_i^\ast(b) \leq -\langle h_i, \lambda \rangle \} \) of \( T_\lambda \otimes B(-\infty) \).

Let us also recall that, for any \( i \in I \), there is a unique strict embedding
\[
 \Phi_i; B(\infty) \to B(\infty) \otimes B_i
\]
such that \( \Phi_i(u_{\infty}) = u_{\infty} \otimes b_i \). We have
\[
 \Phi_i(b) = \bar{e}_i^\ast \varepsilon_i(b) b \otimes \bar{f}_i^\ast \varphi_i(b) b_i.
\]
Also we have
\[
 (1.5.20) \quad B(\infty) \otimes B_i \cong \bigoplus_{n \geq 0} B(\infty) \otimes T_{n \alpha_i}.
\]
Taking ∨ we obtain

\[ \Phi_\gamma^* : B(-\infty) \to B_l \otimes B(-\infty) \quad \text{and} \]

\[ B_l \otimes B(-\infty) \cong \bigoplus_{n \geq 0} T_{-n \delta_1} \otimes B(-\infty). \]

1.6. Balanced triples. Let us recall the definition of balanced triple.

Let \( V \) be a vector space over \( \mathbb{Q}(q) \). For a subring \( B \) of \( \mathbb{Q}(q) \), a \( B \)-lattice of \( V \) is a \( B \)-submodule \( M \) of \( V \) such that \( V \cong \mathbb{Q}(q) \otimes_B M \).

Let \( A \) (resp. \( \bar{A} \)) be the subring of \( \mathbb{Q}(q) \) consisting of functions regular at \( q = 0 \) (resp. \( q = \infty \)). Let \( V_Z \) be a \( \mathbb{Z}[q, q^{-1}] \)-lattice of \( V \) an \( A \)-lattice of \( V \), and \( \bar{L} \) an \( \bar{A} \)-lattice of \( V \). Then we have the following lemma.

**Lemma 1.6.1 [K1].** Set \( E = V_{Z} \cap L \cap \bar{L} \). Then the following conditions are equivalent.

(i) \( E \to V_{Z} \cap L/V_{Z} \cap qL \) is an isomorphism.

(ii) \( E \to V_{Z} \cap \bar{L}/V_{Z} \cap q^{-1}\bar{L} \) is an isomorphism.

(iii) \( (V_{Z} \cap qL) \oplus (V_{Z} \cap \bar{L}) \to V_{Z} \) is an isomorphism.

(iv) \( A \otimes_{\mathbb{Z}} E \to L, \quad A \otimes_{\mathbb{Z}} E \to L, \quad \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} E \to V_{Z}, \quad \mathbb{Q}(q) \otimes_{\mathbb{Z}} E \to V \) are isomorphisms.

We call \((L, \bar{L}, V_{Z})\) balanced if these equivalent conditions are satisfied. Let us denote by \( G \) the inverse of the isomorphism \( E \to V_{Z} \cap L/V_{Z} \cap qL \). If \( B \) is a base of \( V_{Z} \cap L/V_{Z} \cap qL \), then \( \{ G(b); b \in B \} \) is a base of \( V \). The following proposition is easily proven (e.g., by (iii)).

**Proposition 1.6.2 (Triangular property).** Let \( 0 \to V_{i} \to V_{j} \to V_{k} \to 0 \) be an exact sequence of vector spaces over \( \mathbb{Q}(q) \). Let \( V_{iZ} \) (resp. \( L_{i}, \bar{L}_{i} \)) be a \( \mathbb{Z}[q, q^{-1}] \)-lattice (resp. \( A \)-lattice, \( \bar{A} \)-lattice) of \( V_{i} \) (\( i = 1, 2, 3 \)). Assume that

\[ 0 \to V_{iZ} \to V_{jZ} \to V_{kZ} \to 0, \]

\[ 0 \to V_{iZ} \cap L_{1} \to V_{jZ} \cap L_{2} \to V_{kZ} \cap L_{3} \to 0 \quad \text{and} \]

\[ 0 \to V_{iZ} \cap \bar{L}_{1} \to V_{jZ} \cap \bar{L}_{2} \to V_{kZ} \cap \bar{L}_{3} \to 0 \]

are exact. If two of \((V_{iZ}, L_{i}, \bar{L}_{i})\) are balanced, then so is the other.

2. Result of Lusztig [L2].

2.1. Global base of \( \tilde{U}_{\mathfrak{g}}(q) \). Let us recall the result of Lusztig on the crystal base of \( \tilde{U}_{\mathfrak{g}}(q) \). For a dominant integral weight \( \lambda \in P_{+} \), let us denote by \( V(\lambda) \) (resp. \( V(-\lambda) \)) the irreducible module with highest (resp. lowest) weight \( \lambda \) (resp. \( -\lambda \)). Let \( u_{\lambda} \) (resp. \( u_{-\lambda} \)) be the highest (resp. lowest) weight vector. For \( \lambda \in P_{+}, \mu \in P_{-} \), we set \( V(\lambda, \mu) = V(\lambda) \otimes V(\mu) \). Then \( V(\lambda, \mu) \) is generated by \( u_{\lambda} \otimes u_{\mu} \) as a \( U_{q}(\mathfrak{g}) \)-module, and the
defining relation of \( u_\lambda \otimes u_\mu \) is

\[
\begin{align*}
(2.1.1) & \quad q^h(u_\lambda \otimes u_\mu) = q^{\langle h, \lambda + \mu \rangle}(u_\lambda \otimes u_\mu), \\
(2.1.2) & \quad e_i^{1-\langle h, \mu \rangle}(u_\lambda \otimes u_\mu) = 0, \\
& \quad f_i^{1+\langle h, \lambda \rangle}(u_\lambda \otimes u_\mu) = 0.
\end{align*}
\]

Let us define the automorphism – of \( V(\lambda, \mu) \) by

\[
(2.1.3) \quad (P(u_\lambda \otimes u_\mu))^{-} = \overline{P}(u_\lambda \otimes u_\mu) \quad \text{for } P \in U_q(g).
\]

We set \( L(\lambda, \mu) = L(\lambda) \otimes A \otimes L(\mu) \) and \( B(\lambda, \mu) = B(\lambda) \otimes B(\mu) \subset L(\lambda, \mu)/qL(\lambda, \mu) \). Then \( (L(\lambda, \mu), B(\lambda, \mu)) \) is a crystal base of \( V(\lambda, \mu) \).

Set \( V_Z(\lambda, \mu) = U_q^Z(g)(u_\lambda \otimes u_\mu) = U_q^Z(g)u_\lambda \otimes U_q^Z(g)u_\mu \) and \( V_0(\lambda, \mu) = U_q^0(g)(u_\lambda \otimes u_\mu) \). The following results are due to Lusztig.

**Proposition 2.1.1 [L2].** \( (L(\lambda, \mu), L(\lambda, \mu)^-, V_Z(\lambda, \mu)) \) is balanced.

Let \( G \) be the inverse of the isomorphism \( L(\lambda, \mu) \otimes L(\lambda, \mu)^- \otimes V_0(\lambda, \mu) \to L(\lambda, \mu)/qL(\lambda, \mu) \). Then \( V_Z(\lambda, \mu) = \bigoplus_{b \in B(\lambda, \mu)} Z[q, q^{-1}]G(b) \).

**Theorem 2.1.2 [L2].** There exist a unique \( A \)-lattice \( L(U_q(g)) \) of \( \tilde{U}_q(g) \) and a unique base \( B(U_q(g)) \) of \( L(U_q(g))/qL(U_q(g)) \) satisfying the following properties.

(i) \( (L(U_q(g)), L(U_q(g))^-, \tilde{U}_q^0(g)) \) is balanced.

(ii) \( L \) denote the inverse of \( L(U_q(g)) \cap L(U_q(g))^- \otimes \tilde{U}_q^0(g) \otimes L(U_q(g))/qL(U_q(g)) \).

(iii) For any \( \xi, \eta \in P_+ \), \( \Phi(\xi, \eta) \) denote the \( U_q^0(g) \)-linear map \( U_q^0(g)a_{\xi+\eta} \to V(\xi) \otimes V(\eta) \) sending \( a_{\xi+\eta} \) to \( u_\xi \otimes u_\eta \). Then \( \Phi(\xi, \eta)L(U_q(g)a_{\xi+\eta}) = L(\xi) \otimes L(\eta) \).

(iv) \( \Phi(\xi, \eta) \) be the induced homomorphism \( L(U_q(g)a_{\xi+\eta})/qL(U_q(g)a_{\xi+\eta}) \) to \( L(\xi) \otimes L(\eta)/qL(\xi) \otimes L(\eta) \). Then \( \{ b \in B(U_q(g)a_{\xi+\eta}) \mid \Phi(\xi, \eta)b \neq 0 \} \cong B(\xi) \otimes B(\eta) \) and \( \Phi(\xi, \eta)G(b) = G(\Phi(\xi, \eta)b) \) for any \( b \in B(U_q(g)a_{\xi+\eta}) \).

(v) \( B(U_q(g)) \) has a structure of crystal such that

\[
B(\xi) \otimes B(\eta) \to B(U_q(g)a_{\xi+\eta}) \subset B(U_q(g))
\]

is a strict embedding.

The crystal \( B(U_q(g)) \) is therefore a normal crystal.

2.2. **Arguments of Lusztig.** Since we use the arguments of Lusztig later, we shall review his argument briefly. The following lemma is easily checked.

**Lemma 2.2.1.** Let \( l_+, l_- \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in P_+, \mu \in P_- \).

(i) \( F_i(U_q^-(g))F_i(U_q^+(g))(u_\lambda \otimes u_\mu) = F_i(U_q^+(g))F_i(U_q^-(g))(u_\lambda \otimes u_\mu) \)

\[
= (F_i(U_q^-(g))u_\lambda) \otimes (F_i(U_q^+(g))u_\mu).
\]
For $P_{\pm} \in F_{-}(U_{q}^{\pm}(g))$, we have

$$\left( P_{-}u_{\lambda} \right) \otimes \left( P_{+}u_{\mu} \right) \equiv P_{-}P_{+}(u_{\lambda} \otimes u_{\mu}) \equiv P_{-}(u_{\lambda} \otimes u_{\mu}) \mod F_{-1}(U_{q}^{-(g)})u_{\lambda} \otimes F_{-1}(U_{q}^{+(g)})u_{\mu}.$$ 

Moreover, \( F_{-1}(U_{q}^{-(g)})u_{\lambda} \otimes F_{-1}(U_{q}^{+(g)})u_{\mu} \equiv (\overline{L}(\lambda) \cap F_{-}(U_{q}^{-(g)})u_{\lambda}) \otimes (\overline{L}(\mu) \cap F_{+}(U_{q}^{+(g)})u_{\mu}) \mod F_{-1}(g)u_{\lambda} \otimes F_{-1}(g)u_{\mu}. \)

(iv) \( \overline{u}_{\lambda} \otimes \overline{v} - \overline{u} \otimes \overline{v} \in (U_{q}^{+(g)}) \otimes (U_{q}^{-}(g)) \). Here we set \( U_{q}^{+(g)}_{>0} = \sum_{i} U_{q}^{+(g)}e_{i} \) and \( U_{q}^{-}(g)_{>0} = \sum_{i} U_{q}^{-}(g)^{i} \).

The property (iv) follows from the existence of the universal \( R \)-matrix (cf. [L2]).

**Proposition 2.2.2.** Let \( \lambda \in P_{+} \) and \( \mu \in P_{-} \), and let \( N \) be a \( U_{q}^{+(g)} \)-submodule of \( V(\lambda) \) and \( N' \) a \( U_{q}^{-}(g) \)-submodule of \( V(\mu) \). Assume that \( N \) and \( N' \) are generated by global bases, i.e., \( N = \bigoplus_{b \in B_{N}} Q(q)G(b) \) and \( N' = \bigoplus_{b' \in B_{N'}} Q(q)G(b') \) for some \( B_{N} \supset B(\lambda) \) and \( B_{N'} \subset B(\mu) \). Then \( (L(\lambda, \mu) \cap N \otimes N', \overline{L}(\lambda, \mu) \cap N \otimes N', N_{Z} \otimes N_{Z}) \) is balanced. Here \( N_{Z} = \bigoplus_{b \in B_{N}} Z[q, q^{-1}]G(b) \), etc.

**Proof.** Set \( F_{i}(N) = N \cap F_{i}(U_{q}^{-(g)})u_{\lambda} \) and \( F_{i}(N') = N' \cap F_{i}(U_{q}^{+(g)})u_{\mu} \). Then \( F_{i}(N) \) (resp. \( F_{i}(N') \)) is a \( U_{q}^{+(g)} \)- (resp. \( U_{q}^{-}(g) \))-submodule and is generated by global bases. Hence it is enough to show

\[(2.2.1)_{l} \quad (L(\lambda, \mu) \cap F_{i}(N) \otimes F_{i}(N'), L(\lambda, \mu)^{-} \cap F_{i}(N) \otimes F_{i}(N'), F_{i}(N_{Z} \otimes F_{i}(N_{Z}'))) \]

is balanced. We prove this by the induction on \( l \). Assume that \((2.2.1)_{l-1}\) is satisfied. Let \( Gr_{l}N \), etc., be the gradation with respect to the filtration \( F \). Then Lemma 2.2.1 implies

\[
Gr_{l}(L(\lambda, \mu) \cap N \otimes N') = Gr_{l}(L(\lambda) \cap N) \otimes Gr_{l}(L(\mu) \cap N')
\]

and

\[
Gr_{l}(\overline{L}(\lambda, \mu) \cap N \otimes N') = Gr_{l}(\overline{L}(\lambda) \cap N) \otimes Gr_{l}(\overline{L}(\mu) \cap N').
\]

Hence \((Gr_{l}(L(\lambda, \mu) \cap N \otimes N'), Gr_{l}(\overline{L}(\lambda, \mu) \cap N \otimes N'), Gr_{l}(N_{Z} \otimes N_{Z}') \) is balanced. Then we obtain \((2.2.1)_{l}\) by the triangular property of balancedness (Proposition 1.6.2).

Q.E.D.

Applying this lemma to \( N = V(\lambda) \) and \( N' = V(\mu) \) we obtain Proposition 2.1.1. We also obtain the following lemma.

**Lemma 2.2.3.** Let \( \lambda \in P_{+} \) and \( \mu \in P_{-} \). Then for \( b \in B(\lambda), b' \in B(\mu) \)

\[
G(b \otimes b') \equiv G(b) \otimes G(b') \mod \bigoplus_{\xi \in Q_{-} \setminus \{0\}} V(\lambda)_{\text{w}(b)+\xi} \otimes V(\mu)_{\text{w}(b')-\xi}.
\]
Here $Q_+$ is the set of linear combinations of simple roots with nonnegative integer coefficients.

3. Description of $B(U_q(g) a_\lambda)$.

3.1. Relation with $B(\infty)$ and $B(-\infty)$. Let $\lambda$ be an integral weight. For $\xi \in P_+$, $\eta \in P_-$ such that $\lambda = \xi + \eta$, $B(\xi) \otimes B(\eta)$ is embedded into $B(U_q(g) a_\lambda)$. Since we have $B(\xi) \subset B(\infty) \otimes T_\xi$ and $B(\eta) \cong T_\eta \otimes B(-\infty)$, $B(\xi) \otimes B(\eta)$ is embedded into the crystal $B(\infty) \otimes T_\xi \otimes B(-\infty)$ through $T_\xi \cong T_\xi \otimes T_\eta$. Now take $\xi \in P_+$. Then it is easy to see that

$$B(\xi + \zeta) \otimes B(\eta - \zeta) \hookrightarrow B(\infty) \otimes T_\xi \otimes B(-\infty)$$

$$\cup$$

$$B(\xi) \otimes B(\eta)$$

commutes. Thus we obtain the following theorem.

**Theorem 3.1.1.** $B(U_q(g) a_\lambda) \cong B(\infty) \otimes T_\xi \otimes B(-\infty)$.

Note that $B(\xi) \otimes B(\eta)$ is a strict subcrystal of $B(\infty) \otimes T_\xi \otimes B(-\infty)$.

By Lemma 2.2.1 (ii) and Lemma 2.2.3, we have, for $b_1 \in B(\infty)$ and $b_2 \in B(-\infty)$,

(3.1.1) $G(b_1 \otimes t_\lambda \otimes b_2) \equiv G(b_1) G(b_2) a_\lambda \mod F_{\text{arb}, \lambda-1}(U_q^-(g)) F_{\text{arb}, -\lambda-1}(U_q^+(g)) a_\lambda$.

3.2. Filtration by Bruhat order. Let $W$ be the Weyl group. For $w \in W$, let us take a reduced expression $w = s_{i_1} \cdots s_{i_l}$. We define a subset $B_w(\infty)$ of $B(\infty)$ by

(3.2.1) $B_w(\infty) = \{ f_{i_1}^{a_1} \cdots f_{i_l}^{a_l} u_{\infty}; a_1, \ldots, a_l \geq 0 \}$.

Then $B_w(\infty)$ does not depend on the choice of reduced expression (see [K3]), and

(3.2.2) $\bigoplus_{b \in B_w(\infty)} \mathbb{Z}[q, q^{-1}] G(b) = \sum_{a_1, \ldots, a_l \geq 0} \mathbb{Z}[q, q^{-1}] f_{i_1}^{a_1} \cdots f_{i_l}^{a_l}$.

Set $B_w(-\infty) = B_w(\infty)^{\vee} \subset B(-\infty)$. Then we have

(3.2.3) $\bigoplus_{b \in B_w(-\infty)} \mathbb{Z}[q, q^{-1}] G(b) = \sum \mathbb{Z}[q, q^{-1}] e_{i_1}^{(a_1)} \cdots e_{i_l}^{(a_l)}$.

We set for any $i \in I$

(3.2.4) $\tilde{U}_q^+(g_i) \mathbb{Z} = \bigoplus \mathbb{Z}[q, q^{-1}] e_{i_1}^{(a_1)} a_\lambda$ and

$\tilde{U}_q^-(g_i) \mathbb{Z} = \bigoplus \mathbb{Z}[q, q^{-1}] f_{i_1}^{(a_1)} a_\lambda$. 

**Theorem 3.2.1.** Let \( w, w' \in W \), and let \( w = s_{i_1} \cdots s_{i_l} \), \( w' = s_{j_1} \cdots s_{j_l} \), be their reduced expressions. Set \( B_{w, w}(\tilde{U}_q(\mathfrak{g})) = \bigoplus_{b \in B_{w, w}(\tilde{U}_q(\mathfrak{g}))} \mathbb{Z}[q, q^{-1}] G(b) \) is equal to the \( \mathbb{Z}[q, q^{-1}] \)-module \( \tilde{U}_q^- \mathfrak{g} \mathfrak{g}^- \cdots \tilde{U}_q^- \mathfrak{g} \mathfrak{g}^- \cdots \tilde{U}_q^+ \mathfrak{g} \mathfrak{g}^+ \cdots \tilde{U}_q^+ \mathfrak{g} \mathfrak{g}^+ \).

This follows immediately from Proposition 2.2.2. by taking

\[ \lambda = \xi + \eta \text{ with } \xi, -\eta \in P_+ \text{ and } \]

\[ N = \bigoplus_{b \in B_{w}(\infty)} \mathbb{Q}(q)G(b)u_{\lambda} \quad \text{and} \quad N' = \bigoplus_{b \in B_{w}(\infty)} \mathbb{Q}(q)G(b)u_{\eta}. \]

This theorem shows an affirmative answer to a conjecture of Lusztig [L3].

**4. A metric of \( \tilde{U}_q(\mathfrak{g}) \).**

4.1. General facts. Let us define a metric on \( \tilde{U}_q(\mathfrak{g}) \) that behaves well with crystal bases. Let \( \psi \) be the antiautomorphism of the \( \mathbb{Q}(q) \)-algebra \( U_q(\mathfrak{g}) \) defined by

\[
(4.1.1) \quad \psi(e_i) = q^{-1}t_i^{-1}f_i, \quad \psi(f_i) = q^{-1}t_i e_i, \quad \psi(q^h) = q^h.
\]

Let \( M_1 \) and \( M_2 \) be \( U_q(\mathfrak{g}) \)-modules, and let \( ( , )_{M_1} \) and \( ( , )_{M_2} \) be a symmetric form on \( M_1 \) and \( M_2 \) satisfying

\[
(Pu, v)_{M_1} = (u, \psi(P)v)_{M_1} \quad \text{for } u, v \in M_1, \quad P \in U_q(\mathfrak{g}).
\]

We define the symmetric form \( ( , ) \) on \( M = M_1 \otimes M_2 \) by

\[
(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_{M_1}(u_2, v_2)_{M_2} \quad \text{for } u_i, v_i \in M_i.
\]

Then it is known (cf. [K1]) that \( ( , ) \) satisfies

\[
(4.1.1) \quad (Pu, v) = (u, \psi(P)v) \quad \text{for } u, v \in M.
\]

Let us call this metric the tensor product of \( ( , )_{M_1} \) and \( ( , )_{M_2} \).

4.2. Definition of a metric on \( \tilde{U}_q(\mathfrak{g}) \). For \( \lambda \in P_+ \cup (-P_+) \), there exists a unique nondegenerate symmetric bilinear form \( ( , ) \) on \( V(\lambda) \) such that

\[
(4.2.1) \quad (u_{\lambda}, u_{\lambda}) = 1
\]

\[
(Pu, v) = (P, \psi(P)v) \quad \text{for any } u, v \in V(\lambda) \quad \text{and} \quad P \in U_q(\mathfrak{g}).
\]
Therefore, for $\lambda, \mu \in P_+$, the tensor product of those metrics gives a metric
on $V(\lambda) \otimes V(-\mu)$. Let us take an arbitrary $\lambda \in P$. For $\xi \in P_+ \cap (\lambda + P_+)$,
$V(\xi) \otimes V(\lambda - \xi)$ has a metric.

**Lemma 4.2.2.** For any $P, Q \in U_q(g)$, there exists a unique polynomial $f(x)$ in $x = (x_i)_{i \in I}$ such that, for any $\xi \in P_+ \cap (\lambda + P_+)$,

\[
(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = f(x) \quad \text{with} \quad x_i = q_i^{2\langle h_i, \xi \rangle}.
\]

**Proof.** Let us take $P^{(\nu)} \in U_q^+(g) \otimes Q(q)[q^h; h \in P^*]$ such that $\psi(Q)P = \sum P^{(\nu)}P^{(\nu)}$. Then

\[
(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = \sum (P^{(\nu)}(u_\xi \otimes u_{\lambda-\xi}), \psi(P^{(\nu)})(u_\xi \otimes u_{\lambda-\xi})).
\]

Hence we may assume that $P$ and $Q$ belong to $U_q^+(g) \otimes Q(q)[q^h; h \in P^*]$. Since $q^h(u_\xi \otimes u_{\lambda-\xi}) = q^{(\lambda, h)}(u_\xi \otimes u_{\lambda-\xi})$, we may assume $P, Q \in U_q^+(g)$. In this case $P(u_\xi \otimes u_{\lambda-\xi}) = Pu_\xi \otimes u_{\lambda-\xi}$ and hence $(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = (Pu_\xi, Qu_\xi)$. Then the result follows from [K1, Lemma 4.7.1]. Q.E.D.

We define a metric on $U_q(g)\alpha_\lambda$ by

\[
(P_{\alpha, \lambda}, Q_{\alpha, \lambda}) = f(0)
\]

where $f$ is the polynomial given in Lemma 4.2.2. Hence $(P_{\alpha, \lambda}, Q_{\alpha, \lambda})$ is the limit of $(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi}))$ when all $\langle h_i, \xi \rangle$ tend to infinity. Here we regard $|q| < 1$.

We extend this metric to the metric of $U_q^-(g)$ such that $U_q(g)\alpha_\lambda$ and $U_q(g)\alpha_\mu$ are orthogonal for different $\lambda, \mu$.

In [K1], we define a metric on $U_q^-(g)$. The relation is given by the following formula (4.2.3).

\[
(4.2.3) \quad (P_{\alpha, \lambda}, Q_{\alpha, \lambda}) = \prod_i (1 - q_i^2)^{n_i}(P, Q) \quad \text{for} \ P, Q \in U_q^-(g)_{\xi} \quad \text{with} \ \xi = \sum n_i \alpha_i.
\]

Hence, we can apply the result of [K1]. The relation $(P, Q) = (P, Q)[K1, Proposition 5.2.1]$ implies

\[
(4.2.4) \quad (u, v) = (u^*, v^*) \quad \text{for} \ u, v \in U_q^-(g) = \bigoplus \alpha U_q^-(g)\alpha_\lambda.
\]

**Lemma 4.2.3.** For $u \in U_q^-(g)$ and $v \in U_q^-(g)$, $(u, \psi v) = (u \psi, v)$.

**Proof.** Set $u = P_{a_{\lambda + \eta}, \alpha_{\lambda - \eta}}, v = Q_{a_{\lambda}, \alpha_{\lambda}}$ for $\lambda \in P, P, Q \in U_q^-(g)$. Take $\xi \in P_+, \eta \in P_-$ such that $\lambda = \xi + \eta$. Then $P e_i(u_\xi \otimes u_\eta) = P(u_\xi \otimes e_i u_\eta)$. On the other hand, we have

\[
\Delta P = P \otimes 1 + t_i e_i^* P \otimes f_i \mod \bigoplus_{\zeta \neq 0, -a_i} U_q^-(g) \otimes U_q^-(g)_{\zeta}.
\]
For the definition of \( e'_t \), see [K1]. Hence we obtain with \( \xi = \langle h, \xi \rangle \), etc.

\[
Pe_i(u_\xi \otimes u_\eta) = Pu_\xi \otimes u_\eta + t_\xi e'_t Pu_\xi \otimes f_i e_\xi u_\eta \\
= Pu_\xi \otimes u_\eta + q^{\xi}_t \{ -n_i \}(Ad t_i e'_t P)u_\xi \otimes u_\eta.
\]

Thus

\[
(Pe_i(u_\xi \otimes u_\eta), Q(u_\xi \otimes u_\eta)) = \frac{q^{\xi + n_i + 1}_t - q^{\xi - n_i + 1}_t}{1 - q^2_t} ((Ad t_i e'_t P)u_\xi, Qu_\xi).
\]

Letting \( \xi \) go to infinity, we obtain

\[
(Pe_i a_\lambda, Qa_\lambda) = \frac{q^{\lambda + 1}_i}{1 - q^2_i} a(q)((Ad t_i e'_t P, Q).
\]

Here \( a(q) = \prod (1 - q^2_i)^{n_i} \) and \( \text{wt} Q = \sum n_i x_i \). By [K1, Lemma 5.2.2], we have

\[
(Ad t_i e'_t P, Q) (P, Qf_i).
\]

Finally we obtain

\[
(Pa_{\lambda + x_i} q_\eta t_i^{-1} e_i, Qa_\lambda) = q^{-1 - \lambda}_i (Pe_i a_\lambda, Qa_\lambda) = (Pa_{\lambda + x_i}, Qf_i a_{\lambda + x_i}).
\]

**Q.E.D.**

**Lemma 4.2.4.** \( (u, v f_i) = (u q_\eta^{-1} t_i e_i, v) \) for any \( u, v \in \tilde{U}^-_q(g) \).

**Proof.** Assume first \( u \in \tilde{U}^-_q(g) \). Let us write \( v = Pw \) for \( P \in U^+_q(g) \), \( w \in \tilde{U}^-_q(g) \). Then \( (ue_i, v) = (\psi(P)u e_i, w) = (\psi(P)u q_i t_i f_i) = (u, P w q_i t_i f_i) = (u, v q_i t_i f_i) \). By the last lemma. Hence the lemma is true if \( u \in \tilde{U}^-_q(g) \). In general case, writing \( u = Pw \) with \( P \in U^+_q(g) \) and \( w \in \tilde{U}^-_q(g) \), we can argue similarly.

**Q.E.D.**

Set \( \psi^*(P) = (\psi(P^*))^* \). Then \( \psi^*(f_i) = q_t t_i^{-1} e_i \). Hence Lemma 4.2.3 implies

\[
(uf_i, v) = (u, v \psi^*(f_i)).
\]

This implies easily the following lemma.

**Lemma 4.2.6.** \( (uP, u) = (u, v \psi^*(P)) \) for any \( u, v \in \tilde{U}^-_q(g) \) and \( P \in U_q(g) \).

**Proposition 4.2.5.** \( (u, v) = (u^*, v^*) \) for any \( u, v \in \tilde{U}^-_q(g) \).

**Proof.** This is already shown for \( u, v \in \tilde{U}^-_q(g) \). If \( u \in U^+_q(g) \), then, writing \( v = Pw \) with \( P \in U^+_q(g) \) and \( w \in \tilde{U}^-_q(g) \), we have

\[
(u, v) = (\psi(P)u, w) = (u^* \psi^*(P^*), w^*) = (u^*, w^* P^*) = (u^*, v^*).
\]

The general case can be argued similarly. **Q.E.D.**
4.3. Metric and crystal base. The relation of the metric of \( V(\lambda) \) and its crystal base implies the properties

\[
\text{(4.3.1) } (L(\tilde{U}_q(g)), L(\tilde{U}_q(g))) \subset A,
\]
\[
\text{(4.3.2) } (G(b), G(b')) \equiv \delta_{b, b'} \mod qA.
\]

Thus \(( , , )\) is a nondegenerate metric on \( \tilde{U}_q(g) \). Let \( A_Z \) be the subring of \( Q(g) \) generated by \( q, (1 - q^n)^{-1} \ (n > 0) \). Then one can see easily

\[
\text{(4.3.3) } (\tilde{U}_q^Z(g), \tilde{U}_q^Z(g)) \subset A_Z[q^{-1}].
\]

Hence we can apply the similar arguments as in [K2, §6.1]. Thus we obtain the following.

**Proposition 4.3.1.** (i) \( L(\tilde{U}_q(g)) = \{ u \in \tilde{U}_q(g); (u, u) \in A \} \).

(ii) If \( u \in \tilde{U}_q^Z(g) \) and \( (u, u) = 1 + qA \), then \( u \equiv G(b) \mod qL(\tilde{U}_q(g)) \) for some \( b \in B(\tilde{U}_q(g)) \triangleleft B(\tilde{U}_q(g)) \).

We have Theorem 4.3.2 below as a corollary.

**Theorem 4.3.2.** (i) \( L(\tilde{U}_q(g)) \) is invariant by \(*\).

(ii) \( B(\tilde{U}_q(g))^* = B(\tilde{U}_q(g)) \).

(iii) \( G(b^*) = G(b)^* \) for \( b \in B(\tilde{U}_q(g)) \).

The proofs of (i) and (iii) are similar to [K1]; we will prove only (ii). For \( b \in B(\tilde{U}_q(g)), b^* \in B(\tilde{U}_q(g)) \triangleleft (B(\tilde{U}_q(g)) \).

Write \( b = b_1 \otimes t_\lambda \otimes b_2, t_\nu = |wb_v| \). Then

\[
G(b) \equiv G(b_1)G(b_2)a_\lambda \mod F_{1, -1}(U_q^-(g))F_{1, -1}(U_q^+(g))a_\lambda.
\]

Hence we obtain

\[
G(b)^* \equiv a_{-\lambda}G(b_2)^*G(b_1)^* = a_{-\lambda}G(b_2^\#)G(b_1^\#) = G(b_2^\#)G(b_1^\#)a_\mu
\]

\[
\equiv G(b_1^\#)G(b_2^\#)a_\mu \mod F_{1, -1}(U_q^-(g))F_{1, -1}(U_q^+(g))a_\mu
\]

with

\[
\mu = -\lambda - wt_b - wt_{b^2}.
\]

Since \( G(b)^* = G(b') \) for some \( b' \in B(\tilde{U}_q(g)) \triangleleft B(\tilde{U}_q(g)) \), \( b' \) must be \( b_1^\# \otimes t_\mu \otimes b_2^\# \).

(See (3.1.1).)

As seen in the course of the proof above, we have the following.

**Corollary 4.3.3.** For \( b_1 \in B(\infty), b_2 \in B(-\infty), \) we have

\[
(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^\# \otimes t_{-\lambda - wt_{b_1} - wt_{b_2}} \otimes b_2^\#.
\]

Theorem 4.3.2 is conjectured by Lusztig [L3].
5. Right structure.

5.1. Two crystal structures on $B(\tilde{U}_q(g))$. We define for $b \in B(\tilde{U}_q(g))$

\begin{equation}
\varepsilon_i^*(b) = \varepsilon_i(b^*),
\end{equation}
\begin{equation}
\varphi_i^*(b) = \varphi_i(b^*),
\end{equation}
\begin{equation}
\tilde{e}_i^* b = (\tilde{e}_i b^*)^* \quad \text{and}
\end{equation}
\begin{equation}
\tilde{f}_i^* b = (\tilde{f}_i b^*)^*.
\end{equation}

Then this defines another crystal structure on $B(\tilde{U}_q(g))$. By Corollary 4.3.3 we obtain easily the following formula for $b_1 \in B(\infty)$, $b_2 \in B(-\infty)$, $\lambda \in P$:

\begin{equation}
\varepsilon_i^*(b_1 \otimes t_{\lambda} \otimes b_2) = \max(\varepsilon_i^*(b_1), \varphi_i^*(b_2) + \langle h_i, \lambda \rangle),
\end{equation}
\begin{equation}
\varphi_i^*(b_1 \otimes t_{\lambda} \otimes b_2) = \max(\varepsilon_i^*(b_1) - \langle h_i, \lambda \rangle, \varphi_i^*(b_2)),
\end{equation}
\begin{equation}
\varepsilon_i^*(b_1 \otimes t_{\lambda} \otimes b_2) - \varepsilon_i^*(b_1 \otimes t_{\lambda} \otimes b_2) = -\langle h_i, \lambda \rangle,
\end{equation}
\begin{equation}
\varepsilon_i^*(b_1 \otimes t_{\lambda} \otimes b_2) = \begin{cases}
\tilde{e}_i^* b_1 \otimes t_{\lambda - \alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) \geq \varphi_i^*(b_2) + \langle h_i, \lambda \rangle, \\
\varepsilon_i^*(b_1 \otimes t_{\lambda - \alpha_i} \otimes \tilde{e}_i^* b_2) & \text{if } \varepsilon_i^*(b_1) < \varphi_i^*(b_2) + \langle h_i, \lambda \rangle,
\end{cases}
\end{equation}
\begin{equation}
\tilde{f}_i^*(b_1 \otimes t_{\lambda} \otimes b_2) = \begin{cases}
\tilde{f}_i^* b_1 \otimes t_{\lambda + \alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) > \varphi_i^*(b_2) + \langle h_i, \lambda \rangle, \\
\varepsilon_i^*(b_1 \otimes t_{\lambda + \alpha_i} \otimes \tilde{f}_i^* b_2) & \text{if } \varepsilon_i^*(b_1) \leq \varphi_i^*(b_2) + \langle h_i, \lambda \rangle.
\end{cases}
\end{equation}

We prove that these two structures of crystal may be regarded as a crystal structure over $g \oplus g$. This is compared with the fact that the bimodule structure of $\tilde{U}_q(g)$ may be regarded as a left $U_q(g \oplus g)$-module structure.

In order to see this, it is enough to show the following theorem.

**Theorem 5.1.1.** $\tilde{e}_i^*$ and $\tilde{f}_i^*$ are strict morphisms of crystals (with respect to $\tilde{e}_i$ and $\tilde{f}_i$).

In order to prove this theorem, we use the following lemma. The proof being straightforward, we omit it.

**Lemma 5.1.2.** For any $i$ and $\lambda \in P$, let us define the map $E_i : B_i \otimes T_{\lambda} \otimes B_i \to B_i \otimes T_{\lambda - \alpha_i} \otimes B_i$ by

\[
E_i(b_i(n) \otimes t_{\lambda} \otimes b_i(m)) = \begin{cases}
b_i(n + 1) \otimes t_{\lambda - \alpha_i} \otimes b_i(m) & \text{if } n + m + \langle h_i, \lambda \rangle \leq 0, \\
b_i(n) \otimes t_{\lambda - \alpha_i} \otimes b_i(m + 1) & \text{if } n + m + \langle h_i, \lambda \rangle > 0.
\end{cases}
\]

Then $E_i$ is a strict morphism of crystal.
Now, there are strict embeddings from $B(\infty)$ into $B(\infty) \otimes B_1$ and from $B(-\infty)$ into $B_i \otimes B(-\infty)$. There are also strict morphisms from $B_i \otimes B_i$ to $B(\infty)$ and from $B_i \otimes B(-\infty)$ to $B(-\infty)$ (cf. (1.5.20)). Thus we obtain a chain of strict morphisms of crystals

\[(5.1.5) \quad B(\infty) \otimes T_\lambda \otimes B(-\infty) \to B(\infty) \otimes B_i \otimes T_\lambda \otimes B_i \otimes B(-\infty) \]

\[\to B(\infty) \otimes B_i \otimes B(-\infty).\]

**Lemma 5.1.3.** The composition of the morphisms (5.1.5) coincides with $\tilde{e}_i^*$. This follows immediately from the formulas (5.1.2) and (5.1.3). Thus $\tilde{e}_i^*$ is a strict morphism of crystal. Similarly $\tilde{f}_i^*$ is a strict morphism of crystals.


6.1. Preliminary. We study in [K2] the properties of global bases of integrable modules. They give the properties of the global bases of $\tilde{U}_q(g)$ reducing to those of $V(\xi) \otimes V(\eta)$. For $N \geq 0$, we set

\[I_N = \sum \tilde{U}_q(g)e_i^N + \sum \tilde{U}_q(g)f_i^N.\]

**Proposition 6.1.1.** Let $n$ be a nonnegative integer and $i \in I$. Then $u \in \tilde{U}_q(g)$ satisfies $u \in f_i^n\tilde{U}_q(g) + I_N$ (resp. $u \in e_i^n\tilde{U}_q(g) + I_N$) for any $N$ if and only if $u$ is a linear combination of $G(b)$ with $e_i(b) \geq n$.

**Proof.** Assuming $u \in U_q(g)_{\alpha_k}$, let us take $\xi \in P_+$, $\eta \in P_-$ such that $\lambda = \xi + \eta$. Then $\Phi_{\xi,\eta}(u) = f_i^n(V(\xi) \otimes V(\eta))$ if and only if $u \in f_i^n \tilde{U}_q(g) + \sum_j \tilde{U}_q(g)e_j^{1+<h_j,\xi>} + \sum_j \tilde{U}_q(g)f_j^{1+<h_j,\xi>}$. Hence taking $\xi$ such that $<h_j,\xi> \gg 0$ for all $j$, we obtain the lemma by the corresponding result of global bases of $V(\xi) \otimes V(\eta)$. Q.E.D.

6.2. Definition of $V_i(\lambda)$. To get more precise results than Proposition 6.1.1, let us generalize the results of Lusztig little bit.

For $i \in I$ and $\lambda \in P_+$, let $V_i(\lambda)$ be the $U_q(g)$-module generated by $u_{i,\lambda}$ with the defining relation $q^h u_{i,\lambda} = q^{<h_i,\lambda>} u_{i,\lambda}, e_j u_{i,\lambda} = 0$ for any $j \in I$ and $f_i^{1+<h_i,\lambda>} u_{i,\lambda} = 0$. Similarly for $\mu \in P_-$, let $V_i(\mu)$ be the $U_q(g)$-module generated by $u_{i,\mu}$ with the defining relation $q^h u_{i,\mu} = q^{<h_i,\mu>} u_{i,\mu}, f_j u_{i,\mu} = 0$ for any $j \in I$ and $e_i^{1+<h_i,\mu>} u_{i,\mu} = 0$. Then

\[(6.2.1) \quad V_i(\lambda) \cong U_q^-(g)/U_q^-(g)f_i^{1+<h_i,\lambda>},\]

\[(6.2.2) \quad V_i(\mu) \cong U_q^+(g)/U_q^+(g)e_i^{1+<h_i,\mu>}.\]

Now consider the $U_q(g)$-module $V_i(\lambda) \otimes V_i(\mu)$. One can see easily that this is a $U_q(g)$-
module generated by \( u_{i,\lambda} \otimes u_{i,\mu} \) with the defining relation
\[
q^h(u_{i,\lambda} \otimes u_{i,\mu}) = q^{\langle h_i, \lambda + \mu \rangle}(u_{i,\lambda} \otimes u_{i,\mu}) \quad \text{for any } h \in P^*,
\]
\[
f_i^{1+\langle h_i, \lambda \rangle}(u_{i,\lambda} \otimes u_{i,\mu}) = 0 \quad \text{and} \quad e_i^{1-\langle h_i, \mu \rangle}(u_{i,\lambda} \otimes u_{i,\mu}) = 0.
\]

Hence there is a chain of surjective homomorphisms
\[
U_q(g) a_{\lambda + \mu} \xrightarrow{\Phi_{\lambda, \mu}} V(\lambda) \otimes V(\mu) \rightarrow V(\lambda) \otimes V(\mu).
\]

6.3. Refinement. For \( \xi \in Q_\pm \), let us denote
\[
F_\xi U_q \xi (g) = \bigoplus_{\eta \in Q_\pm} U_q \eta (g)_{\xi, \eta}.
\]

For \( \xi_\pm \in P_\pm \), we set
\[
F_{\xi_\pm} U_q (g) = F_{\xi_\pm} (U_q \xi (g)) F_{\xi_\pm} (U_q \eta (g)) \mathcal{F}
\]
\[
= F_{\xi_\pm} (U_q \eta (g)) F_{\xi_\pm} (U_q \xi (g)) \mathcal{F},
\]
where \( \mathcal{F} = \bigoplus_{h \in P^*} Q(q) q^h \). For \( \xi \in Q_+ \), we set \( F_{\xi} U_q (g) = F_{\xi - \xi} U_q (g) \). Then, by Proposition 2.2.2, \( F_{\xi_\pm} U_q (g) a_{\lambda + \mu} \) is generated by global bases. The purpose of this section is to prove the following proposition.

**Proposition 6.3.1.** Set \( L(V(\lambda) \otimes V(\mu)) = \Phi_{\lambda, \mu}(L(U_q(g) a_{\lambda + \mu})) \). Let \( \overline{\Phi}_{\lambda, \mu} \) be the induced morphism \( L(U_q(g) a_{\lambda + \mu})/qL(U_q(g) a_{\lambda + \mu}) \rightarrow L(V(\lambda) \otimes V(\mu))/qL(V(\lambda) \otimes V(\mu)) \). Set \( B(V(\lambda) \otimes V(\mu)) = \{ \overline{\Phi}_{\lambda, \mu}(b); b \in B(U_q(g) a_{\lambda + \mu}) \} \setminus \{0\} \). Then:

(i) \( L(V(\lambda) \otimes V(\mu)), B(V(\lambda) \otimes V(\mu)) \) is a crystal base of the integrable \( U_q(g) \)-module \( V(\lambda) \otimes V(\mu) \). Here \( U_q(g) \) is the subalgebra of \( U_q(g) \) generated by \( e_i, f_i, \) and \( q^h (h \in P^*) \).

(ii) For \( b \in B(U_q(g) a_{\lambda + \mu}) \), \( \overline{\Phi}_{\lambda, \mu}(b) \neq 0 \) if and only if \( \epsilon^\tau(b) < \langle h_i, \lambda \rangle \).

(iii) \( \{ b \in B(U_q(g) a_{\lambda + \mu}) \mid \overline{\Phi}_{\lambda, \mu}(b) \neq 0 \} \) is isomorphic to \( B(V(\lambda) \otimes V(\mu)) \).

(iv) \( \{ G(b)(u_{i,j} \otimes u_{i,j}) \} \) forms a base of \( V(\lambda) \otimes V(\mu) \) where \( b \) ranges over \( \{ b \in B(U_q(g) a_{\lambda + \mu}) \mid \overline{\Phi}_{\lambda, \mu}(b) \neq 0 \} \).

(v) \( \overline{\Phi}_{\lambda, \mu} \) is a morphism (in \( \mathfrak{g} \{ \langle 1 \rangle, P \} \)) from \( B(U_q(g) a_{\lambda + \mu}) \) to \( B(V(\lambda) \otimes V(\mu)) \).

**Proof.** Let us take \( \xi \in Q_+ \). Since \( V(\lambda) \otimes V(\mu) \) is integrable as \( U_q(g) \)-module, \( F_{\xi + N \alpha} U_q(g)(u_{i,j} \otimes u_{i,j}) \) is stationary when \( n \) increases. Take \( N \) such that \( F_{\xi + N \alpha} U_q(g)(u_{i,j} \otimes u_{i,j}) = F_{\xi + N \alpha} U_q(g)(u_{i,j} \otimes u_{i,j}) \) for \( n \geq N \). Then \( F_{\xi + N \alpha} U_q(g)(u_{i,j} \otimes u_{i,j}) \) is a finite-dimensional \( U_q(g) \)-module. Taking \( \eta \in P_+ \) such that \( \langle h_j, \eta \rangle \gg 0 \) for \( j \neq i \) and \( \langle h_i, \eta \rangle = 0 \),
\[
F_{\xi + N \alpha} U_q(g)(u_{i,j} \otimes u_{i,j}) \rightarrow V(\lambda + \eta) \otimes V(\mu - \eta)
\]
is injective. Now consider the chain of homomorphisms

\[ U_q(g)a_{\lambda+\mu} \xrightarrow{\Phi_{\lambda+\mu}} V(\lambda) \otimes V(\mu) \xrightarrow{p} V(\lambda + \eta) \otimes V(\mu - \eta). \]

Then \( p\Phi_{\lambda+\mu}(F_{\xi+N\eta}U_q(g)a_{\lambda+\mu} \cap L(U_q(g)a_{\lambda+\mu})) = F_{\xi+N\eta}U_q(g)(u_{\lambda+\eta} \otimes u_{\mu-\eta}) \cap L(\lambda + \eta) \otimes L(\mu - \eta). \) Hence \( F_{\xi+N\eta}U_q(g)(u_{i\lambda} \otimes u_{i\mu}) \cap L(V(\lambda) \otimes V(\mu)) \) is contained in \( p^{-1}(0) + \Phi_{\lambda+\mu}(F_{\xi+N\eta}U_q(g)a_{\lambda+\mu} \cap L(U_q(g)a_{\lambda+\mu})). \) Since \( p^{-1}(0) \cap F_{\xi+N\eta}U_q(g)(u_{i\lambda} \otimes u_{i\mu}) = 0, \) we obtain

\[ F_{\xi+N\eta}U_q(g)(u_{i\lambda} \otimes u_{i\mu}) \cap L(V(\lambda) \otimes V(\mu)) = \Phi_{\lambda+\mu}(F_{\xi+N\eta}U_q(g)a_{\lambda+\mu} \cap L(U_q(g)a_{\lambda+\mu})). \]

Thus, we obtain

\[ (F_{\xi+N\eta}U_q(g))a_{\lambda+\mu} \cap L(U_q(g)a_{\lambda+\mu}) \]

\[ \rightarrow (F_{\xi+N\eta}U_q(g))(u_{i\lambda} \otimes u_{i\mu}) \cap L(V(\lambda) \otimes V(\mu)) \]

\[ \simeq (F_{\xi+N\eta}U_q(g))(u_{\lambda+\eta} \otimes u_{\mu-\eta}) \cap L(\lambda + \eta) \otimes L(\mu - \eta). \]

Now, let us prove Proposition 6.2.1. For example, let us show that \( L(V(\lambda) \otimes V(\mu)) \) is a crystal lattice. Since \( F_{\xi+N\eta}U_q(g)(u_{i\lambda} \otimes u_{i\mu}) \cap L(V(\lambda) \otimes V(\mu)) \) is a crystal lattice of the integrable \( U_q(g) \)-module \( F_{\xi+N\eta}U_q(g)(u_{\lambda+\eta} \otimes u_{\mu-\eta}), \)

\( F_{\xi+N\eta}U_q(g)(u_{i\lambda} \otimes u_{i\mu}) \cap L(V(\lambda) \otimes V(\mu)) \) is a crystal lattice of \( F_{\xi+N\eta}U_q(g)(u_{i\lambda} \otimes u_{i\mu}). \)

Since \( V(\lambda) \otimes V(\mu) \) is a union of \( F_{\xi+N\eta}U_q(g)(u_{i\lambda} \otimes u_{i\mu}), \) \( L(V(\lambda) \otimes V(\mu)) \) is a crystal lattice. The other statements can be proven similarly.

Q.E.D.

6.4. Interpretation of \( \varepsilon \) and \( \varphi. \) By using Proposition 6.3.1, we can sharpen Proposition 6.1.1. Set

\[ I'_N = U_q(g)e_i^N + \bar{U}_q(g)f_i^N. \]

Then the proof of the following proposition is similar to the one of Proposition 6.1.1.

PROPOSITION 6.4.1. Let \( n \) be a nonnegative integer and \( i \in I. \) Then \( u \in \bar{U}_q(g) \)

satisfies \( u \in f_i^n\bar{U}_q(g) + I'_N \) (resp. \( u \in e_i^n\bar{U}_q(g) + I'_N \)) for any \( N \) if and only if \( u \) is a linear combination of \( G(b) \) with \( \varepsilon_i(b) \geq n. \)

PROPOSITION 6.4.2. For \( a, c \in \mathbb{Z}_{\geq 0} \) and \( i \in I, \)

\[ e_i^a\bar{U}_q(g) + f_i^b\bar{U}_q(g) = \bigoplus_b \mathcal{Q}(q)G(b). \]

Here \( b \) ranges over \( \{ b \in B(\bar{U}_q(g)); \varphi_i(b) \geq a \text{ or } \varepsilon_i(b) \geq c \}. \)
Proof. Taking $\ast$, it is enough to show

\[(6.4.2) \quad U_q(\mathfrak{g})e_i^\ast a_\lambda + U_q(\mathfrak{g})f_i^\ast a_\lambda = \bigoplus_b Q(q)G(b).\]

Here $b$ ranges over $\{b \in B(\tilde{U}_q(\mathfrak{g})a_\lambda), \Phi_i^\ast(b) \geq a \text{ or } \varepsilon_i^\ast(b) \geq c\}$.

(Step I) The case where $\langle h_i, \lambda \rangle = c - a$. Let us take $\xi \in P_+$ and $\eta \in P_-$ such that

\[\lambda = \xi + \eta, \quad \langle h_i, \xi \rangle = 1 + c, \quad \langle h_i, \eta \rangle = -1 - a.\]

Then the left-hand side of (6.4.2) coincides with the kernel of $\Phi^i(\xi, \eta)$. Since the kernel of $\Phi^i(\xi, \eta)$ is generated by $G(b)$ with $b \in B(\tilde{U}_q(\mathfrak{g})a_\lambda), B(\xi) \otimes B(\eta) = \{b_1 \otimes \xi \otimes b_2; \varepsilon_i^\ast(b_1) \leq c, \Phi_i^\ast(b_2) \leq a\}$, we obtain the desired result.

(Step II) General case. We define $\tilde{a}, \tilde{c}$ by

\[\tilde{c} = c, \quad \tilde{a} = c - \langle h_i, \lambda \rangle \quad \text{if} \quad \langle h_i, \lambda \rangle \leq c - a,\]

\[\tilde{a} = a, \quad \tilde{c} = \langle h_i, \lambda \rangle + a \quad \text{if} \quad \langle h_i, \lambda \rangle \geq c - a.\]

Then $\tilde{c} \geq c$ and $\tilde{a} \geq a$. On the other hand, $\varepsilon_i^\ast(b) - \Phi_i^\ast(b) = \langle h_i, \lambda \rangle$ implies that the condition $\Phi_i^\ast(b) \geq a$ or $\varepsilon_i^\ast(b) \geq c$ is equivalent to $\Phi_i^\ast(b) \geq \tilde{a}$ or $\varepsilon_i^\ast(b) \geq \tilde{c}$. Hence the right-hand side of (6.4.2) is equal to $\tilde{U}_q(\mathfrak{g})e_i^\ast a_\lambda + \tilde{U}_q(\mathfrak{g})f_i^\ast a_\lambda$. Then it is enough to apply

\[U_q(\mathfrak{g})e_i^\ast a_\lambda + U_q(\mathfrak{g})f_i^\ast a_\lambda = U_q(\mathfrak{g})e_i^\ast a_\lambda + U_q(\mathfrak{g})f_i^\ast a_\lambda.\]

Q.E.D.

Let us remark that $\varepsilon_i(b) \geq n$ does not imply $G(b) \in f_i^n\tilde{U}_q(\mathfrak{g})$ in general. This is only true modulo $I_N^i$ or modulo $e_i^n\tilde{U}_q(\mathfrak{g})$.

Proposition 6.4.3. For $b \in B(\tilde{U}_q(\mathfrak{g}))$,

\[(6.4.3) \quad f_i^{(n)}G(b) = \left[\varepsilon_i(b) + n\right]_{I} G(\tilde{f}_i^n b) + \sum_{b'} F_{b,b'}(q)G(b').\]

Here $b'$ ranges over the set of $B(\tilde{U}_q(\mathfrak{g}))$ such that $\varepsilon_i(b') > \varepsilon_i(b) + n$ and $F_{b,b'}(q) \in Z[q, q^{-1}]$. In addition,

\[(6.4.4) \quad e_i^{(n)}G(B) = \left[\Phi_i(b) + n\right]_{I} G(\tilde{e}_i^n b) + \sum_{b'} E_{b,b'}(q)G(b').\]

Here $b'$ ranges over the set of $B(\tilde{U}_q(\mathfrak{g}))$ such that $\Phi_i(b') > \Phi_i(b) + n$.

Formulas (6.4.3) and (6.4.4) follow immediately from the corresponding results on global bases of integrable modules (see [K2]).
6.5. Further property. The following results are used later.

**Lemma 6.5.1.** If \( \langle h_i, \lambda \rangle \geq k \), then

\[
L(u_q(g) a_{\lambda - k \alpha_i} f_i) = q_i^{k - \langle h_i, \lambda \rangle} L(u_q(g)) + u_q(g)e_i + \bar{u}_q(g)f_i + f_i + \langle h_i, \lambda \rangle.
\]

**Proof.** Taking \(*\) and changing \( \lambda \) with \(-\lambda\), it is enough to show

\[
a_i f_i L(\bar{u}_q(g)) = q_i^{k + \langle h_i, \lambda \rangle} L(\bar{u}(g)) + e_i \bar{u}_q(g) + f_i + f_i - \langle h_i, \lambda \rangle \bar{u}_q(g)
\]

when \( \langle h_i, \lambda \rangle + k \leq 0 \),

or for \( b \in B(\bar{u}_q(g)) \) of weight \( \lambda + k \alpha_i \), writing \( f_i G(b) = \sum c_{i'} G(b') \),

\[
(6.5.1) \quad \varphi_i(b') = 0 \quad \text{implies} \quad c_{i'} \in q_i^{k + \langle h_i, \lambda \rangle}.
\]

This reduces to the same statement for \( b \in B(V(\xi) \otimes V(\eta)) \) for \( \xi \in P_+ \) and \( \eta \in P_- \). Write \( G(b) = \sum u_{i,n} e_i u_n \) with \( e_i u_n = 0 \) and \( \text{wt}(u_n) = \lambda + (n + k) \alpha_i \). Then \( u_n \in L(\xi) \otimes L(\eta) \) and \( f_i G(b) = \sum_{n+k}^{n+k+n} f_i(u_{i,n}) u_n \). If \( k + n \neq \langle h_i, \lambda \rangle \), then \( f_i^{(k+n)} u_n \)

belongs to \( e_i(V(\xi) \otimes V(\eta)) \). If \( k + n = \langle h_i, \lambda \rangle \), then \( f_i^{(k+n)} u_n \)

belongs to \( q_i^{-n} L(\xi) \otimes L(\eta) = q_i^{k + \langle h_i, \lambda \rangle + k} L(\xi) \otimes L(\eta) \). Hence \( f_i G(b) = q_i^{k + \langle h_i, \lambda \rangle + k} L(\xi) \otimes L(\eta) + e_i(V(\xi) \otimes V(\eta)) \). This implies (6.5.1).

Q.E.D.

7. The Weyl group action on crystal bases.

7.1. Action of simple reflections. Let \( B \) be a normal crystal. We define the action of the Weyl group \( W \) on the underlying set \( B \). For \( i \in I \) and \( b \in B \), we set

\[
S_i b = \begin{cases} f_i^{\langle h_i, \text{wt}(b) \rangle} b & \text{if } \langle h_i, \text{wt}(b) \rangle \geq 0, \\ e_i^{\langle h_i, \text{wt}(b) \rangle} b & \text{if } \langle h_i, \text{wt}(b) \rangle \leq 0. \end{cases}
\]

Then we have the obvious relation

\[
S_i \text{wt}(b) = \text{wt}(S_i b)
\]

where \( s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i \) is the simple reflection,

\[
S_i^2 = \text{id},
\]

\[
S_i e_i S_i^{-1} = \tilde{f}_i.
\]

We show that this extends to the action of the Weyl group. In order to see this, it is enough to check the braid relation for a finite-dimensional \( g \) of rank 2.
7.2. **Braid relation.** Set \( I = \{1, 2\} \) and assume \( g \) is finite-dimensional. Let \( w_0 = s_{i_1} \cdots s_{i_t} \) be a reduced expression of the longest element of \( W \). There are two choices. We show that

\[
(7.2.1) \quad S_{i_1} \cdots S_{i_t} b \text{ does not depend on the choice of reduced expression.}
\]

In order to see this we may assume that \( wt(b) \) is dominant. If \( wt(b) \) is not regular, \((7.2.1)\) is trivial. Hence we may assume \( wt(b) \) is regular and dominant.

For any normal crystal \( B \), set \( \tilde{f}_{i_t}^{\max} b = \tilde{f}_{i_t}^{\rho_i(b)} b \). Then we have that

\[
(7.2.2) \quad \tilde{f}_{i_t}^{\max} \cdots \tilde{f}_{i_1}^{\max} b \text{ does not depend on the choice of reduced expression.}
\]

In fact this vector is the unique lowest-weight vector in the connected component containing \( b \) (cf. [K3]).

Now we remark

\[
(7.2.3) \quad \tilde{f}_{i_t}^{\max}(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_{i_t}^{\rho_{i_t}(b_1) - \rho_{i_t}(b_2)} b_1 \otimes \tilde{f}_{i_t}^{\max} b_2 & \text{if } \rho_{i_t}(b_1) \geq \rho_{i_t}(b_2) \\
\tilde{f}_{i_t}^{\max} b_1 \otimes \tilde{f}_{i_t}^{\max} b_2 & \text{if } \rho_{i_t}(b_1) \leq \rho_{i_t}(b_2).
\end{cases}
\]

**Lemma 7.2.1.** Let \( b \) be a vector with \( \langle h_i, wt(b) \rangle > 0 \). Then, for any \( b' \) in any crystal \( B' \), there exists an integer \( m \) and \( b'' \in B^\otimes m \otimes B' \) such that for any \( n \gg m \),

\[
\tilde{f}_{i_t}^{\max}(b^\otimes n \otimes b') = (S_t b)^\otimes (n-m) \otimes b''.
\]

**Proof.** We have for \( 1 \leq v \)

\[
\epsilon_i(b^\otimes v \otimes b') = \max \{ \epsilon_i(b) - k \langle h_i, wt(b) \rangle(0 \leq k < v), \epsilon_i(b') - v \langle h_i, wt(b) \rangle \}
\]

\[
= \max(\epsilon_i(b), \epsilon_i(b') - v \langle h_i, wt(b) \rangle).
\]

Hence if \( \epsilon_i(b') - v \langle h_i, wt(b) \rangle \leq \epsilon_i(b) \), \((7.2.3)\) and \( \varphi_i(b) - \epsilon_i(b) = \langle h_i, wt(b) \rangle \) imply \( \tilde{f}_{i_t}^{\max}(b^\otimes (v+1) \otimes b') = S_t b \otimes \tilde{f}_{i_t}^{\max}(b^\otimes v \otimes b') \). Thus we obtain the desired result.

Q.E.D.

This lemma implies that \( \tilde{f}_{i_t}^{\max} \cdots \tilde{f}_{i_1}^{\max}(b^\otimes n) = S_{i_t} \cdots S_{i_1} b \otimes b' \) for some \( b' \in B^\otimes (n-1) \) if \( n \gg 0 \). Thus \((7.2.2)\) implies the following result.

**Theorem 7.2.2.** \( \{ S_t \} \) satisfies the braid relation.

7.3. **Application.** For \( \lambda \in P \) and \( w \in W \), take a reduced expression \( w = s_{i_t} \cdots s_{i_1} \) of \( w \). Then the condition \( \langle h_{i_t}, s_{i_t-1} \cdots s_{i_1} \lambda \rangle \gg 0, \ldots, \langle h_{i_1}, \lambda \rangle \gg 0 \) does not depend on the choice of reduced expression. If this condition is satisfied, we say that \( \lambda \) is \( w \)-dominant. If \( \lambda \) is \( w \)-dominant, we set \( \tilde{f}_{w, \lambda} = \tilde{f}_{i_t}^{(h_{i_t}; s_{i_t-1} \cdots s_{i_1}; \lambda)} \cdots \tilde{f}_{i_1}^{(h_{i_1}; \lambda)} \).

**Proposition 7.4.1.** If \( \lambda \in P \) is \( w \)-dominant, the definition of \( \tilde{f}_{w, \lambda} \) does not depend on the choice of reduced expression of \( w \). Here we regard \( \tilde{f}_{w, \lambda} \) as an operator on normal crystals, \( B(\infty) \) or \( B(-\infty) \).
Proof. It is enough to show this on $B(\infty)$. Then it follows from
\begin{equation}
S_{i_1} \cdots S_{i_n} (b \otimes t_{k-w(b)} \otimes u_{-w}) = \tilde{f}_{w,i} b \otimes t_{k-w(b)} \otimes u_{-w}.
\end{equation}
Q.E.D

Remark 7.4.2. $\omega_i = \alpha_j$ does not imply $S_w \tilde{e}_i S_w^{-1} = \tilde{e}_j$ even in the $A_2$-case.

8. Extremal vectors.

8.1. Definition of extremal vectors. Let $M$ be an integral $U_q(g)$-module. A weight vector $u$ of weight $\lambda \in P$ of $M$ is called $i$-extremal if $e_i u = 0$ or $f_i u = 0$. In this case, we set $S_i u = f_i^{1/2 \langle h_i, \lambda \rangle} u$ or $e_i^{1/2 \langle h_i, \lambda \rangle} u$, respectively.

Definition 8.1.1. A weight vector $u$ is called extremal if, for any $l \geq 0$, $S_{i_1} \cdots S_{i_l} u$ is $i$-extremal for any $i, i_1, \ldots, i_l \in I$.

This notion generalizes that of highest-weight vector. A similar definition is possible for an element of a normal crystal. An element $b$ of a normal crystal $B$ is called $i$-extremal if $\tilde{e}_i b = 0$ or $\tilde{f}_i b = 0$.

Definition 8.1.2. An element $b$ of $B$ is called extremal if, for any $l \geq 0$, $S_{i_1} \cdots S_{i_l} b$ is $i$-extremal for any $i, i_1, \ldots, i_l \in I$.

8.2. Modules generated by extremal vectors. Let $\lambda \in P$ be an integral weight. Set $\lambda_+ = \sum_{\langle h_i, \lambda \rangle \geq 0} \langle h_i, \lambda \rangle \Lambda_i$ and $\lambda_- = \lambda_+ - \lambda \in P_+$. Then $V'(\lambda) = V(\lambda_+) \otimes V(-\lambda_-)$ is isomorphic to
\begin{equation}
U_q(g) a_{\lambda} / \sum_{\langle h_i, \lambda \rangle \geq 0} U_q(g) e_i^{1/2 \langle h_i, \lambda \rangle} + \sum_{\langle h_i, \lambda \rangle \leq 0} U_q(g) f_i^{1/2 \langle h_i, \lambda \rangle},
\end{equation}
and $V'(\lambda)$ has a global base: $V'(\lambda) = \bigoplus_{\langle h_i, \lambda \rangle = 0} a_{\lambda}$. Let $p_{\lambda}: U_q(g) a_{\lambda} \to V'(\lambda)$ be the projection.

Lemma 8.2.1. For $i \in I$ and $\lambda \in P$, consider the commutative diagram
\begin{equation}
\begin{array}{ccc}
U_q(g) a_{\lambda} & \xrightarrow{\psi_i} & U_q(g) a_{s_{i}\lambda} \\
\downarrow{\psi} & & \downarrow{p_{s_{i}\lambda}} \\
V'(s_{i}\lambda) & & \\
\end{array}
\end{equation}
Here $\psi_i$ is given by
\begin{equation}
\psi_i(a_{\lambda}) = \begin{cases} e_i^{1/2 \langle h_i, \lambda \rangle} a_{s_{i}\lambda} & \text{if } \langle h_i, \lambda \rangle \geq 0, \\
f_i^{1/2 \langle h_i, \lambda \rangle} a_{s_{i}\lambda} & \text{if } \langle h_i, \lambda \rangle \leq 0.
\end{cases}
\end{equation}
Then for $b \in B(U_q(g) a_{\lambda})$ we have $\psi(G(b)) = p_{s_{i}\lambda} G(S_{i}^* b)$. Here $S_{i}^* b = (S_{i}(b^*))^*$. 

Proof. The other case being similarly proved, let us assume $\langle h_i, \lambda \rangle \geq 0$. Then $\psi(G(b)) = p_{s_{i}\lambda} G(b) e_i^{1/2 \langle h_i, \lambda \rangle} a_{s_{i}\lambda}$. If $\tilde{e}_i^* (b) > 0$, then $G(b) \in \tilde{U}_q(g) e_i + I_N$ for any $N \geq 0$
and hence \( p_{\lambda,} (e_i^{(h_{\lambda})+1} a_{\lambda}) = 0 \) implies \( \psi(G(b)) = 0 \). If \( e_i^*(b) = 0 \) then

\[
G(b) e_i^{(h_{\lambda})} \equiv G(\tilde{e}_i^{(h_{\lambda})} b) \mod \tilde{U}_q(b) e_i^{(h_{\lambda})+1} + I_N.
\]

Therefore we have

\[
\psi(G(b)) = p_{\lambda,} (G(\tilde{e}_i^{(h_{\lambda})} b) a_{\lambda}). \quad \text{Q.E.D.}
\]

Thus, repeating this procedure, we obtain the following.

**Proposition 8.2.2.** For \( \lambda \in P \), set \( B^\text{max}(\lambda) = \{ b \in B(U_q(\mathfrak{g})a_\lambda); \; b^* \text{ is extremal} \} \) and

\[
I_\lambda = \bigoplus_{b \in B(U_q(\mathfrak{g})a_\lambda) \setminus B^\text{max}(\lambda)} \mathbb{Q}(q) G(b).
\]

Then we have:

(i) \( I_\lambda \) is a left \( U_q(\mathfrak{g}) \)-submodule of \( U_q(\mathfrak{g})a_\lambda \).

(ii) \( V^\text{max}(\lambda) = U_q(\mathfrak{g})a_\lambda / I_\lambda \) is an integrable \( U_q(\mathfrak{g}) \)-module.

(iii) Let \( p_\lambda \colon U_q(\mathfrak{g}) a_\lambda \rightarrow V^\text{max}(\lambda) \) be the projection and \( u_\lambda = p_\lambda(a_\lambda) \). Then \( u_\lambda \) is an extremal vector of weight \( \lambda \).

(iv) For any \( i \in I \), we have an isomorphism

\[
V^\text{max}(\lambda) \cong V^\text{max}(s_i \lambda)
\]

by \( u_\lambda \mapsto s_i u_{s_i \lambda} \).

Moreover this isomorphism sends the global base to the global base.

(v) For any \( i \), \( S_i^\# \) gives an isomorphism \( B^\text{max}(\lambda) \cong B^\text{max}(s_i \lambda) \).

We have

\[
(8.2.2) \quad V^\text{max}(\lambda) \cong \bigoplus_{b \in B^\text{max}(\lambda)} \mathbb{Q}(q) G(b).
\]

Thus \( B^\text{max}(\lambda) \) is a crystal base of \( V^\text{max}(\lambda) \).

If \( \lambda \) is dominant (resp. antidominant), then \( V^\text{max}(\lambda) \) is the irreducible \( U_q(\mathfrak{g}) \)-module with highest- (resp. lowest-) weight \( \lambda \).

**8.3. A proof of the Parthasarathy-Varadarajan-Rao conjecture.** For \( \lambda \in P \), let us denote by \( B(\lambda) \) the connected component of \( B(U_q(\mathfrak{g})a_\lambda) \) containing \( a_\lambda \). Let us write \( u_\lambda \in B(\lambda) \) for \( a_\lambda \). Then \( S_i^\# = *S_{u_i} \) gives an isomorphism

\[
(8.3.1) \quad B(\lambda) \cong B(w\lambda) \quad \text{for any} \; w \in W.
\]

If \( \lambda \) is dominant (resp. antidominant), \( B(\lambda) \) and \( B^\text{max}(\lambda) \) coincide with the sixth example in Example 1.5.3. (cf. the proof of Lemma 10.2.1.).
LEMMA 8.3.1. For $\lambda \in P_+$ and $\mu \in P$, $B(\lambda + \mu)$ is isomorphic to the connected component of $B(\lambda) \otimes B(\mu)$ containing $u_\lambda \otimes u_\mu$.

Proof. This follows immediately from the following chain of morphisms

$B(\lambda + \mu) \subsetneq B(\lambda + \mu_+ \otimes B(-\mu_-) \subsetneq B(\lambda) \otimes B(\mu_+) \otimes B(-\mu_-) \subsetneq B(\lambda) \otimes B(\mu)$.

Q.E.D.

As an application, we can obtain a new proof of the Parthasarathy-Varadarajan-Rao conjecture:

(8.3.2) For $\lambda, \mu \in P_+, w, w' \in W$, if $w'(\lambda + w\mu) \in P_+$, then $V(w'(\lambda + w\mu))$ appears in $V(\lambda) \otimes V(\mu)$.

In fact, it is enough to show that $B(w'(\lambda + w\mu))$ appears in $B(\lambda) \otimes B(\mu)$. However this follows from $B(w'(\lambda + w\mu)) \cong B(\lambda + w\mu) \subset B(\lambda) \otimes B(w(\mu) \cong B(\lambda) \otimes B(\mu)$.

9. A property of $L(\tilde{U}_q(g))$.

9.1. Property of $L(\tilde{U}_q(g))u$. Let us investigate properties of $L(\tilde{U}_q(g))u$ for an element $u$ of an integrable $\tilde{U}_q(g)$-module.

PROPOSITION 9.1.1. Let $M$ be an integrable $\tilde{U}_q(g)$-module, $\lambda \in P$, and $u$ an element of $M_\lambda$. Then $L(\tilde{U}_q(g)a_\lambda)u$ is invariant by $\tilde{e}_i$ and $\tilde{f}_i$.

Proof. Take $N$ such that $e_i^N u = f_i^N u = 0$ for any $i$. Then take $\xi \in P_+ \eta \in P_-$ such that $\lambda = \xi + \eta$ and $\langle h_i, \xi \rangle > N$, $-\langle h_i, \eta \rangle > N$. Then the morphism $\tilde{U}_q(g)a_\lambda \to M$ splits through $V(\xi) \otimes V(\eta)$. The result follows from the fact that any $\tilde{U}_q(g)$-linear homomorphism commutes with $\tilde{e}_i$ and $\tilde{f}_i$.

Q.E.D.

PROPOSITION 9.1.2. Let $L$ be a crystal base of an integrable $\tilde{U}_q(g)$-module $M$. For $\lambda \in P$, set

$$N = \{ u \in M ; L(\tilde{U}_q(g))u \subset q^cL \text{ for some } c \}.$$ 

Then $N$ is a $\tilde{U}_q(g)$-module.

Proof. It is trivial that $N$ is a $\mathbb{Q}(q)$-vector space. Hence it is enough to show that $N$ is invariant by $e_i$ and $f_i$. Thus the proposition is reduced to the following statement.

(9.1.1) If $u \in M_\lambda$ satisfies $L(\tilde{U}_q(g)a_\lambda)u \subset L$, then $L(\tilde{U}_q(g)a_{\lambda - \alpha_i})f_i u \subset q^cL$ for some $c$.

Let us take $N$ such that $f_i^{N+1} u = 0$. We shall show

(9.1.2) $L(\infty)u \subset q^{-N}L$.

For any $P \in L(\infty)$, there exists $Q, \in L(\infty) \wedge \text{Ker } e_i^n$ such that $P = \sum Q, f_i^n(\text{cf. [K1]}).$
Then

\[ Pu = \sum \mathcal{Q}_n f_i^{(n)} f_i u \]

\[ = \sum [n + 1] \mathcal{Q}_n f_i^{(n+1)} u. \]

Since \( f_i^{N+1} u = 0 \), we may assume that \( n \) ranges over the integers \( n \leq N \). In this case \( [n + 1] \mathcal{Q}_n f_i^{(n+1)} u \subset q_i^{-n} L(\mathfrak{g})a_\lambda u \subset q_i^{-N} L. \) Thus we obtain (9.1.2). To complete the proof, it is enough to apply the following proposition.

Q.E.D.

**Proposition 9.1.3.** Let \( M \) be an integrable \( U_q(\mathfrak{g}) \)-module and \( L \) an \( A \)-submodule of \( M \) invariant by \( \hat{\varepsilon}_i \) and \( \hat{f}_i \). Let \( \psi : U_q(\mathfrak{g})a_\lambda \to M \) be a \( U_q(\mathfrak{g}) \)-linear homomorphism such that \( \psi(L(\infty) a_\lambda) \subset L \). Then \( \psi(U_q(\mathfrak{g})a_\lambda) \subset L \).

**Proof.** Since \( \psi \) splits \( U_q(\mathfrak{g})a_\lambda \to V(\xi) \otimes V(\eta) \) for some \( \xi, -\eta \in P^+ \) with \( \lambda = \xi + \eta \), the result follows from Lemma 9.1.4. below.

**Lemma 9.1.4.** Let \( \xi, \eta \in P^+ \) and \( \xi, \eta \in P^- \). Then \( L(\xi) \otimes L(\eta) \) is the smallest \( A \)-module of \( V(\xi) \otimes V(\eta) \) that is invariant by \( \hat{\varepsilon}_i \) and \( \hat{f}_i \) and that contains \( L(\lambda) \otimes u_n \).

**Proof.** Let \( L \) be an \( A \)-submodule of \( V(\xi) \otimes V(\eta) \) invariant by \( \hat{\varepsilon}_i \) and \( \hat{f}_i \). For \( w \in W \), with reduced expression \( w = s_{i_1} \cdots s_{i_k} \), set \( V_w(\lambda) = \sum \mathcal{Q}(q)i_i^{(n)} \cdots f_i^{(n)} u_\lambda \) and \( V_w(\eta) = \sum \mathcal{Q}(q)l_i^{(n)} \cdots e_i^{(n)} u_\eta \). Then \( V_w(\lambda) \) and \( V_w(\eta) \) are finite-dimensional submodules. Set \( L_w(\lambda) = L(\lambda) \cap V_w(\lambda) \) and \( L_w(\eta) = L(\mu) \cap V_w(\lambda) \).

In order to prove the lemma it is enough to show that,

\[ (9.1.3) \text{ if } s_i w > w, s_i w' < w' \text{ and if } L' \supset L_w(\xi) \otimes (L(\eta) \cap V_w(\eta)), \text{ then } L' \supset L_w(\xi) \otimes (L(\eta) \cap V_w(\eta)). \]

Let us take \( b_1 \in B_w(\xi) \) and \( b_2 \in B_{s_i w}(\eta) \). Then \( \tilde{f}_i^{\text{max}} b_2 \in B_w(\eta) \), and there is \( c \) such that

\[ \tilde{f}_i^{\text{max}} b_1 \otimes b_2 = \tilde{f}_i^{\text{max}} b_1 \otimes \tilde{f}_i^{\text{max}} b_2. \]

Hence \( \tilde{f}_i^{\text{max}} b_1 \otimes \tilde{f}_i^{\text{max}} b_2 \) belongs to \( L_w(\xi) \otimes L_w(\eta) \) mod \( qL_w(\xi) \otimes L_{s_i w}(\eta) \). Hence \( b_1 \otimes b_2 \) belongs to \( L \) mod \( qL_w(\xi) \otimes L_{s_i w}(\eta) \). This shows that

\[ L_w(\xi) \otimes L_{s_i w}(\eta) \subset L + qL_w(\xi) \otimes L_{s_i w}(\eta). \]

Nakayama's lemma implies the desired result: \( L_w(\xi) \otimes L_{s_i w}(\eta) \subset L \). Q.E.D.

**Theorem 9.2.1.** Let \( M \) be an integrable \( U_q(\mathfrak{g}) \)-module and \( L \) a lower crystal lattice of \( M \). Then

\[ L' = \bigoplus \{ u \in M_\lambda; L(U_q(\mathfrak{g})a_\lambda) u \subset L \} \]

is invariant by \( \tilde{\varepsilon}_i^{\text{up}} \) and \( \tilde{f}_i^{\text{up}} \). (For \( \tilde{\varepsilon}_i^{\text{up}} \) and \( \tilde{f}_i^{\text{up}} \) see [K2]).
We remark that, by the relation of upper and lower crystal bases, the statement above is equivalent to the statement that

\[ \bigoplus_{\hat{\lambda}} \{ u \in M_{\hat{\lambda}} ; \ L(U_q(g)a_{\hat{\lambda}})u \leq q^{1/2(\langle \hat{\lambda} \rangle^2 - c)}L \} \]

is invariant by \( \tilde{e}_i \) and \( \tilde{f}_i \). Here \( c \) is a number such that \( \| \hat{\lambda} \| \|^2 - c \in 2\mathbb{Z} \) for any weight \( \hat{\lambda} \) of \( M \). Here \( \| \hat{\lambda} \| \|^2 = (\lambda, \lambda) \). Note that, if \( \lambda, \mu \in P \) satisfy \( \lambda - \mu \in \mathbb{Z}\alpha_i \), then \( \| \lambda \| \|^2 - \| \mu \| \|^2 \in 2\mathbb{Z} \).

We shall prove the theorem under the last form. Set \( L' = \bigoplus_{\hat{\lambda}} \{ u \in M_{\hat{\lambda}} ; \ L(U_q(g)a_{\hat{\lambda}})u \leq q^{1/2(\langle \hat{\lambda} \rangle^2 - c)}L \} \)

**Lemma 9.2.2.** If \( e_iu = 0 \) and if \( u \in L' \) then \( f_i^{(k)}u \in L' \).

**Proof.** We may assume \( k < \langle h_i, \lambda \rangle \). Since \( \| \lambda - k\alpha_i \|^2 = \| \lambda \|^2 + k(\alpha_i, \alpha_i) \) \((k < \langle h_i, \lambda \rangle)\), it is enough to show

\[ L(U_q(g)a_{\lambda - k\alpha_i})f_i^{(k)}u \leq q_i^{k(\langle h_i, \lambda \rangle)}L(U_q(g)a_{\lambda})u. \]

This is reduced to the statement

\[ (9.2.1) \quad L(U_q(g)a_{\lambda - k\alpha_i})f_i^{(k)} \leq q_i^{k(\langle h_i, \lambda \rangle)}L(U_q(g)a_{\lambda}) + \tilde{U}_q(g)e_i + \tilde{U}_q(g)f_i^{1+\langle h_i, \lambda \rangle} \]

This follows from Lemma 6.5.1. Q.E.D.

**Proof of Theorem 9.2.1.** Let us take an element \( u \) in \( L' \) of weight \( \lambda \). We write \( u = \sum_{n \in \mathbb{N}} f_i^{(n)}u_n \) with \( e_iu_n = 0 \) and \( wt(u_n) = \lambda + n\alpha_i \). By the preceding lemma, it is enough to show \( u_n \in L' \). We begin with induction on \( N \). We have

\[ e_i^{(N)}u = e_i^{(N)}f_i^{(N)}u_N = \left[ \langle h_i, \lambda + N\alpha_i \rangle \right]_N u_N \]

and \( e_i^{N+1}u = 0 \).

The crystal lattice of \( \tilde{U}(g) \) has the property

\[ (9.2.2) \quad L(U_q(g)a_{\lambda + N\alpha_i})e_i^{(N)} \leq L(U_q(g)a_{\lambda}) + \tilde{U}_q(g)e_i^{N+1} + I_m \quad \text{for any} \ m. \]

Admitting this, let us finish the proof of Theorem 9.2.1. We have

\[ L(U_q(g)a_{\lambda + N\alpha_i})u_N = q_i^{N(\langle h_i, \lambda \rangle + N)}L(U_q(g)a_{\lambda + N\alpha_i})e_i^{(N)}u \]

\[ \leq q_i^{N(\langle h_i, \lambda \rangle + N)}L(U_q(g)a_{\lambda})u \leq q_i^{N(\langle h_i, \lambda \rangle + N)}q^{1/2(\langle \lambda \rangle^2 - c)}L \]

\[ = q^{1/2((\lambda + N\alpha_i)^2 - c)}L. \]
Thus $u_N$ belongs to $L'$. This implies $f_i^{(N)}u_N \in L'$ and hence $\sum_{n<N} f_i^{(n)}u_n$ also belongs to $L'$. Thus the induction proceeds.

Now it remains to prove (9.2.2). For $b \in B(\tilde{U}_q(g))$, if $\psi_i^+(b) > 0$, then $G(b)e_i^{(N)} = \tilde{U}_q(g)e_i^{N+1} + I_m$ for any $m$ by Proposition 6.4.2. If $\psi_i^+(b) = 0$, then Proposition 6.4.4 implies $G(b)e_i^{(N)} \equiv G(\tilde{e}_i^{N}b) \mod \tilde{U}_q(g)e_i^{N+1} + I_m$. This completes the proof of Theorem 9.2.1.

9.3. Applications. Let us give applications of Theorem 9.2.1.

PROPOSITION 9.3.1. For $\lambda \in P_+$, let $L^{up}(\lambda)$ be the upper crystal lattice of $V(\lambda)$. Then $L(\tilde{U}_q(g))L^{up}(\lambda) \subset L(\lambda)$.

Proof. Set $L' = \{u \in V(\lambda); L(\tilde{U}_q(g))u \subset L(\lambda)\}$. Then $L'_\lambda = A\mu_\lambda$, and $L'$ is invariant by $\tilde{e}_i^{up}$ and $\tilde{f}_i^{up}$. Hence $L' = L^{up}(\lambda)$. Q.E.D.

PROPOSITION 9.3.2. For any connected component $B'$ of $B(\tilde{U}_q(g))$, $\|wt(b)\|^2; b \in B'$ is bounded from above.

Proof. Let us take $\xi \in P_+$ and $\eta \in P_-$ such that $B' \subset B(\xi) \otimes B(\eta)$. Let us take $b_0 \in B'$ and let $\lambda_0$ be the weight of $b_0$. Then by Lemma 9.1.2 there exists $c \in \|\lambda_0\|^2 + 2\mathbb{Z}$ such that

$$L(\tilde{U}_q(g))G(b_0) \subset q^{1/2(1\|\lambda_0\|^2-c)}L(\xi) \otimes L(\eta).$$

Set

$$L' = \bigoplus_{\lambda} \{u \in (L(\xi) \otimes L(\eta))_\lambda; L(\tilde{U}_q(g)a_\lambda)u \subset q^{1/2(1\|\lambda\|^2-c)}L(\xi) \otimes L(\eta)\}.$$

Then $L'$ is invariant by $\tilde{e}_i$ and $\tilde{f}_i$ by Theorem 9.2.1. Let $\psi$ be the map $L' \to L(\lambda) \otimes L(\eta)/qL(\xi) \otimes L(\eta)$. Then $\psi$ is invariant by $\tilde{e}_i$ and $\tilde{f}_i$, and hence the image of $\psi$ contains $B'$. For any $b \in B'$, let us take $v \in L'$ such that $\psi(v) = b$. Then $v \notin qL(\xi) \otimes L(\eta)$ and $v \in L(\tilde{U}_q(g))v \subset q^{1/2(1\|wt(b')\|^2-c)}L(\xi) \otimes L(\eta)$. They imply $\|wt(b')\|^2 \leq c$. Q.E.D.

For a connected component $B'$ of $B(\tilde{U}_q(g))$, an element $b \in B'$ is an extremal vector if $\|wt(b)\|^2$ is maximal. Hence we obtain the following corollary.

COROLLARY 9.3.3. Any connected component of $B(\tilde{U}_q(g))$ contains an extremal vector.

We obtain Corollary 9.3.4 by applying this to $b^*$.

COROLLARY 9.3.4. Any connected component of $B(\tilde{U}_q(g))$ can be embedded into some $B^\text{max}(\lambda)$.

In the course of the proof of Proposition 9.3.2, if $B'$ is the connected component $B(\lambda)$ of $B(\tilde{U}_q(g))$ containing $a_\lambda$, we can take $b_0 = a_\lambda$ and $c = 0$. Thus we obtain the following.

PROPOSITION 9.3.5. For any $b \in B(\lambda)$, $\|wt(b)\|^2 \leq \|\lambda\|^2$. 

Remark. The result of Proposition 9.3.2 gives a strong constraint on the crystal structure of $B(U_q(g))$. For example, for $\lambda \in P_+$ and $\mu \in P_+$, the connected component $B$ of $B(\lambda) \otimes B(\mu)$ containing $u_\lambda \otimes u_\mu$ does not satisfy the bounded condition in Proposition 9.3.2. In fact, taking $\epsilon_i^{\text{max}}$ successively, $B$ contains $u_{w\lambda} \otimes u_\mu$ for any $w \in W$. However $\{||w\lambda + \mu||^2; w \in W\}$ is not bounded from above even in the affine case (if $\lambda, \mu$ are regular).

10. Comparison with the result of [K2].

10.1. Relation of $A_q(g)$ and $\tilde{U}_q(g)$. In [K2], we define the crystal base of $A_q(g)$. Let us recall that

$$A_q(g) = \bigoplus_{\lambda \in P_+} \{ u \in (U_q(g)^*) \lambda \mid \text{there exists } l \geq 0 \text{ such that } e_i^{l} \cdots e_i u = u f_i \cdots f_i u = 0 \text{ for any } i_1, \ldots, i_l \in I \}.$$ 

$A_q(g) \cong \bigoplus_{\lambda \in P_+} V(\lambda) \otimes V(\lambda)$, and $A_q(g)$ has an upper global base. Here $V(\lambda)$ is the irreducible right highest-weight $U_q(g)$-module generated by the highest-weight vector $v_\lambda$. We have $B(A_q(g)) = \bigoplus_{\lambda} B'(\lambda) \otimes B(\lambda)$. There exists a canonical coupling

(10.1.1) $\langle , \rangle : A_q(g) \otimes \tilde{U}_q(g) \to \mathbb{Q}(q).$

Set $A_q^Z(g) = \{ u \in A_q(g); \langle u, U_q^Z(g) \rangle \subset \mathbb{Z}[q, q^{-1}] \}$. Then we can see easily

$$\langle A_q^Z(g), \tilde{U}_q(g) \rangle \subset \mathbb{Q}(q, q^{-1}).$$

Theorem 10.1.1. There exists a unique embedding $\psi : B(A_q(g)) \subset B(\tilde{U}_q(g))$ such that $(G(b), G(b')) = \delta_{\psi(b), b'}$ for any $b \in B(A_q(g))$ and $b' \in B(\tilde{U}_q(g))$.

Proof. There exists a unique embedding of crystals over $g \otimes g$

$$B'(\lambda) \otimes B(\lambda) \to B(\tilde{U}_q(g))$$

that sends $v_\lambda \otimes u_\lambda$ to $a_\lambda$ for $\lambda \in P_+$. This gives an embedding $B(A_q(g)) \subset B(\tilde{U}_q(g))$. In order to see that this satisfies the required property, let us remark the following lemma.

Lemma 10.1.2. $\langle L(A_q(g)), L(\tilde{U}_q(g)) \rangle \subset A$.

Proof. By the definition, it is enough to show $\langle L^{up}(V(\lambda)), L(\tilde{U}_q(g))L^{up}(\lambda) \rangle \subset A$. Since $\{ u \in V(\lambda); \langle L^{up}(\lambda), u \rangle \subset A \} = L^{\text{low}}(\lambda)$, this follows from Proposition 9.3.1.

Q.E.D.

Thus we obtain $\langle G(b'), G(b) \rangle \subset A \cap \tilde{A} \cap \mathbb{Q}[q, q^{-1}] = \mathbb{Q}$ for $b \in B(A_q(g)), b' \in B(\tilde{U}_q(g))$. Since $\langle , \rangle$ is invariant by $\tilde{e}_i, \tilde{f}_i, \tilde{e}_i^* \text{ and } \tilde{f}_i^*$, we obtain Theorem 10.1.1, because $\langle G(v_\lambda \otimes u_\lambda), a_\lambda \rangle = \langle v_\lambda, u_\lambda \rangle = 1$. 

10.2. Finite-dimensional and affine case. Let us regard $B(A_q(g))$ as a subset of $B(\widetilde{U}_q(g))$. Then $B(A_q(g))$ is the smallest subcrystal (with respect $g \oplus g$) of $B(\widetilde{U}_q(g))$ that contains all $a_\lambda (\lambda \in P)$.

Let us denote by $T$ the Tits cone; i.e., $T = \bigcup_{w \in W} wP_+$.

**Lemma 10.2.1.** If $\lambda \in T$ then $B^{\text{max}}(\lambda) \subseteq B(A_q(g))$.

**Proof.** For $w \in W$, $S_w^*$ sends $B^{\text{max}}(\lambda)$ onto $B^{\text{max}}(w\lambda)$. Hence we may assume $\lambda \in P_+$ from the beginning. If $b = b_1 \otimes t_\lambda \otimes b_2 \in B^{\text{max}}(\lambda)$, then $\varphi^*_i(b) = \max(\varphi^*_i(b_1), \varphi^*_i(b_2)), \varepsilon^*_i(b_1) - \langle h_i, \lambda \rangle = 0$. Hence $\varphi^*_i(b_2) = 0$ for all $i$ and hence $b_2 = u_{-\infty}$ and $\varepsilon^*_i(b_1) \leq \langle h_i, \lambda \rangle$. This shows $B^{\text{max}}(\lambda) = B(\lambda) \otimes I_\lambda \otimes u_{-\infty}$. Then the desired result follows from the connectedness of $B(\lambda)$. Combining this with Corollary 9.3.3 and 9.3.4 we obtain the following result.

**Proposition 10.2.2.** (i) If $g$ is finite dimensional, $B(A_q(g)) = B(\widetilde{U}_q(g))$.

(ii) If $g$ is affine, let $B(\widetilde{U}_q(g)_+) = B(\widetilde{U}_q(g))$ consisting of vectors with positive level. Then $B(A_q(g)) = B(\widetilde{U}_q(g)_+) \bigcup \{a_0\}$.

Recall that when $g$ is affine we take $c = \sum Z_{i > 0} h_i$ with $\langle c, \alpha_i \rangle = 0$ for any $i$. Then $\langle c, \lambda \rangle$ is called the level of $\lambda \in P$.

**References**


Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan