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THE CRYSTAL BASE AND LITTELMANN'S REFINED DEMAZURE CHARACTER FORMULA

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0. Introduction. Demazure's character formula describes the weight multiplicities of the $U(\mathfrak{n}^+)$-module generated by an extremal vector of the irreducible highest-weight $U(\mathfrak{g})$-module (cf. [D], [A], [RR], [J], [M], [SK]). More precisely, we define the operator D_i by

$$D_i(e^\lambda) = \frac{e^\lambda - e^{\lambda-(1+\langle h_i, \lambda\rangle)\alpha_i}}{1 - e^{-\alpha_i}},$$

and for an element w of the Weyl group W with a reduced decomposition $s_{i_1} \cdots s_{i_l}$, we define $D_w = D_{i_1} \cdots D_{i_l}$. If $u_{w\lambda}$ is the extremal vector of weight $w\lambda$ of the irreducible highest-weight module $V(\lambda)$ of weight λ over a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, then

$$ch(U(\mathfrak{n}^+))u_{w\lambda} = D_w(e^\lambda).$$

Littelmann gave the following conjecture of a generalization of the Demazure character formula and gave a proof in most cases when $\mathfrak{g}$ is finite-dimensional ([L]). Let $V(\lambda)$ be the irreducible $U_q(\mathfrak{g})$-module with highest weight λ, and $(L(\lambda), B(\lambda))$ its crystal base. His conjecture states that there is a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\frac{U_q^+(\mathfrak{g})u_{w\lambda} \cap L(\lambda)}{U_q^+(\mathfrak{g})u_{w\lambda} \cap qL(\lambda)} = \bigoplus_{b \in B_w(\lambda)} \mathbf{Q}b \tag{0.1}$$

and that it satisfies

$$\sum_{b \in B_w(\lambda)} b = \mathscr{D}_{i_1} \cdots \mathscr{D}_{i_l} u_\lambda. \tag{0.2}$$

Here $\mathscr{D}_i$ is the additive operator on $\mathbf{Z}^{\oplus B(\lambda)}$ given by

$$\mathscr{D}_i b = \begin{cases} \sum_{0 \leqslant k \leqslant \langle h_i, wt(b)\rangle} \tilde{f}_i^k b & \text{if } \langle h_i, wt(b)\rangle \geqslant 0 \\ -\sum_{1 \leqslant k < -\langle h_i, wt(b)\rangle} \tilde{e}_i^k b & \text{if } \langle h_i, wt(b)\rangle < 0. \end{cases}$$

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Then $\mathrm{ewt}(\mathscr{D}_i b) = D_i(\mathrm{ewt}(b))$. Here $\mathrm{ewt}(b) = e^{wt(b)}$. Hence Littelmann's conjecture implies the Damazure character formula. In this paper, we shall prove his conjecture for any symmetrizable case. In fact, we shall prove more precise statements. We prove first

$$U_q^+(\mathfrak{g})u_{w\lambda} = \bigoplus_{b \in B_w(\lambda)} \mathbf{Q}(q)G_\lambda(b). \tag{0.3}$$

Here $G_\lambda(b)$ is the lower global base. This fact is an easy consequence of the following statement of $U_q(\mathfrak{sl}_2)$-modules.

(0.4) Let M be a $U_q(\mathfrak{sl}_2)$-module with (lower) global bases, and N a sub-$U_q^+(\mathfrak{sl}_2)$-module generated by global bases. Then $U_q(\mathfrak{sl}_2)N$ is also generated by global bases.

In fact, (0.3) follows from (0.4) and $U_q^+(\mathfrak{g})u_{w\lambda} = U_q(\mathfrak{g}_i)U_q^+(\mathfrak{g})u_{s_iw\lambda}$ for $s_iw < w$. Here $U_q(\mathfrak{g}_i)$ is the copy of $U_q(\mathfrak{sl}_2)$ inside $U_q(\mathfrak{g})$ corresponding to i.

We shall then prove the following three properties of $B_w(\lambda)$.

(i) $\tilde{e}_i B_w(\lambda) \subset B_w(\lambda) \sqcup \{0\}$.

(ii) *If $s_iw < w$, then we have*

$$B_w(\lambda) = \{\tilde{f}_i^k b;\ k \geqslant 0,\ b \in B_{s_iw}(\lambda),\ \tilde{e}_i b = 0\} \setminus \{0\}.$$

(iii) *For any i-string S, $S \cap B_w(\lambda)$ is either empty or S or {the highest weight vector of S}.*

Here i-string means $\{\tilde{f}_i^k b;\ 0 \leqslant k \leqslant \varphi_i(b)\}$ for b with $\varepsilon_i(b) = 0$, and b is called the highest-weight vector of S. In these statements, the first two follow from the definition, and the last one is nontrivial. These three properties imply (0.2).

In fact, arguing by induction we may assume $s_iw < w$, and the formula (0.2) is reduced to

$$\sum_{b \in B_w(\lambda)} b = \mathscr{D}_i\Bigg(\sum_{b \in B_{s_iw}(\lambda)} b\Bigg).$$

Since $B(\lambda)$ is the disjoint union of i-strings, it is enough to show that for any i-string S

$$\sum_{b \in B_w(\lambda) \cap S} b = \mathscr{D}_i\Bigg(\sum_{b \in B_{s_iw}(\lambda) \cap S} b\Bigg). \tag{0.5}$$

By (i), (ii), and (iii), only the following three cases are possible.

(a) $S \cap B_w(\lambda) = S \cap B_{s_iw}(\lambda) = \emptyset$.

(b) $S \cap B_w(\lambda) = S \cap B_{s_iw}(\lambda) = S$.

(c) $S \cap B_w(\lambda) = S$ and $S \cap B_{s_iw}(\lambda)$ consists of the highest-weight vector.

In either case, (0.5) can be easily checked. Thus, we have a new proof of Demazure's character formula for symmetrizable Kac-Moody algebras.

1. Crystals.

1.1. Notation. We follow the notation in [K]. In particular, $\{\alpha_i\}_{i \in I}$ is the set of simple roots, $\{h_i\}_{i \in I}$ is the set of simple coroots, P is the weight lattice, $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra generated by e_i, f_i, $q^h (h \in P^*)$, A is the subring of $\mathbf{Q}(q)$ consisting of rational functions regular at $q = 0$, etc.

1.2. Definition of crystals. In this section, abstracting the properties of crystal bases, we will introduce the notion of crystals. They form a tensor category. Let us endow $\mathbf{Z} \sqcup \{-\infty\}$ with the linear order such that $-\infty$ is the smallest element. We define the addition on $\mathbf{Z} \sqcup \{-\infty\}$ by

(1.2.1) $$-\infty + x = -\infty, \qquad \text{for any } x \in \mathbf{Z} \sqcup \{-\infty\}.$$

Definition 1.2.1. A crystal B is a set with

(1.2.2) a map $wt: B \to P$, $\varepsilon_i: B \to \mathbf{Z} \sqcup \{-\infty\}$, and $\varphi_i: B \to \mathbf{Z} \sqcup \{-\infty\}$,

(1.2.3) $$\tilde{e}_i: B \to B \sqcup \{0\}$$
$$\tilde{f}_i: B \to B \sqcup \{0\} \qquad \text{for } i \in I.$$

Here 0 is the ideal element that does not belong to B. They are subject to the following axioms:

(C1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle$;

(C2) if $b \in B$ and $\tilde{e}_i b \in B$, then $wt(\tilde{e}_i b) = wt(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$;

(C2)′ if $b \in B$ and $\tilde{f}_i b \in B$, then $wt(\tilde{f}_i b) = wt(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$;

(C3) for $b, b' \in B$ and $i \in I$, $b' = \tilde{e}_i b$ if and only if $b = \tilde{f}_i b'$;

(C4) for $b \in B$, if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

For two crystals B_1 and B_2 a morphism ψ from B_1 to B_2 is a map $B_1 \to B_2 \sqcup \{0\}$ that satisfies the following conditions (1.2.4)–(1.2.6).

(1.2.4) If $b \in B_1$ and $\psi(b) \in B_2$, then $wt(\psi(b)) = wt(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\psi(b)) = \varphi_i(b)$.

(1.2.5) For $b \in B_1$, we have $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ provided $\psi(\tilde{e}_i b)$ and $\psi(b) \in B_2$.

(1.2.6) For $b \in B_1$, we have $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$ provided $\psi(\tilde{f}_i b)$ and $\psi(b) \in B_2$.

Here we understand $\psi(0) = 0$.

Then the crystals form a category, that is denoted by $\mathscr{C}$.

A crystal B is called *upper normal* (resp. *lower normal*) if, for any $b \in B$, $\varepsilon_i(b) \in \mathbf{Z}$ and

$$\varepsilon_i(b) = \max\{k \geqslant 0; \tilde{e}_i^k b \in B\}$$

$$(\text{resp. } \varphi_i(b) = \max\{k \geqslant 0; \tilde{f}_i^k b \in B\}).$$

If a crystal is upper and lower normal, it is called *normal.*

A morphism $\psi\colon B_1 \to B_2$ is called *strict* if it commutes with all $\tilde{e}_i$ and $\tilde{f}_i$.

The following lemma is obvious.

LEMMA 1.2.2. *An isomorphism is strict.*

LEMMA 1.2.3. *If B_1 and B_2 are normal, then any morphism from B_1 to B_2 is strict.*

For two crystals B_1 and B_2, we define the *direct sum* $B_1 \oplus B_2$ whose underlying set is $B_1 \sqcup B_2$ with the obvious actions. We have

(1.2.7) $$\mathrm{Hom}_{\mathscr{C}}(B_1 \oplus B_2, B) \cong \mathrm{Hom}_{\mathscr{C}}(B_1, B) \times \mathrm{Hom}_{\mathscr{C}}(B_2, B).$$

A morphism $\psi\colon B_1 \to B_2$ is called an *embedding* if ψ induces the injective map $B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$. In this case, we call B_1 a *subcrystal* of B_2. An embedding ψ is called *full* when, if $b \in B_1$ satisfies $\tilde{e}_i\psi(b) \in B_2$, then $\tilde{e}_i b \in B_1$. In this case B_1 is called a *full subcrystal* of B_2. A strict embedding $\psi\colon B_1 \to B_2$ is full, and B_2 is isomorphic to the direct sum of B_1 and $B_2 \backslash \psi(B_1)$.

For any morphism $\psi\colon B_1 \to B_2$, B_1 is the direct sum of the subcrystals $\psi^{-1}(B_2)$ and $\psi^{-1}(0)$.

Example 1.2.4. For $\lambda \in P$, T_λ is the crystal consisting of a single element t_λ with $wt(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$. Of course, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$.

Example 1.2.5. The crystal $C = \{c\}$ is defined by $wt(c) = 0$, $\varepsilon_i(c) = \varphi_i(c) = 0$, $\tilde{e}_i c = \tilde{f}_i c = 0$.

Example 1.2.6. For $i \in I$, B_i is the crystal defined as follows:

$$B_i = \{b_i(n); n \in \mathbf{Z}\} \quad \text{and}$$

$$wt(b_i(n)) = n\alpha_i,$$

$$\varphi_i(b_i(n)) = n,\ \varepsilon_i(b_i(n)) = -n,$$

$$\varphi_j(b_i(n)) = \varepsilon_j(b_i(n)) = -\infty \qquad \text{for } j \neq i,$$

We define the action of $\tilde{e}_j$ and $\tilde{f}_j$ by

$$\tilde{e}_i(b_i(n)) = b_i(n+1),$$

$$\tilde{f}_i(b_i(n)) = b_i(n-1),$$

$$\tilde{e}_j(b_i(n)) = \tilde{f}_j(b_i(n)) = 0 \qquad \text{for } j \neq i.$$

We write b_i for $b_i(0)$.

Example 1.2.7. For $\lambda \in P_+$, $B(\lambda)$ is the normal crystal associated with the crystal base of the simple module with highest weight λ. The unique element of $B(\lambda)$ of weight λ is denoted by u_λ. Remark that $B(0)$ is isomorphic to C.

Example 1.2.8. $B(\infty)$ is the crystal associated with the crystal base of $U_q^-(\mathfrak{g})$ (cf. [K_2]). We set $\varepsilon_i(b) = \max\{k \geqslant 0; \tilde{e}_i^k b \neq 0\}$ and $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b)\rangle$. Then $B(\infty)$ is upper normal but not lower normal. The unique element of $B(\infty)$ of weight 0 is denoted by u_∞. By the result in [K_2], there exists a full embedding $B(\lambda) \to B(\infty) \otimes T_\lambda$ that sends u_λ to $u_\infty \otimes t_\lambda$. This commutes with all $\tilde{e}_i$.

For a crystal B, we define the crystal $B^\vee$ as follows. As a set, $B^\vee = \{b^\vee; b \in B\} \cong B$ and

$$wt(b^\vee) = -wt(b)$$

$$\varepsilon_i(b^\vee) = \varphi_i(b),\ \varphi_i(b^\vee) = \varepsilon_i(b)$$

$$\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee,\ \tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee .$$

Here $0^\vee$ is understood to be 0.

Then $B^{\vee\vee}$ is canonically isomorphic to B.

Example 1.2.9. $B(-\infty) = B(\infty)^\vee$. This may be regarded as a crystal base of $U_q^+(\mathfrak{g})$.

Example 1.2.10. $B^-(\lambda) = B(-\lambda)^\vee$ for $\lambda \in P_- = -P_+$. This is the crystal base of the irreducible module $V_-(\lambda)$ of lowest weight λ.

1.3. Tensor product. For two crystals B_1 and B_2, we define its tensor product $B_1 \otimes B_2$ as follows:

$$B_1 \otimes B_2 = \{b_1 \otimes b_2; b_1 \in B_1 \text{ and } b_2 \in B_2\},$$

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - wt_i(b_1)),$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + wt_i(b_2)),$$

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2).$$

Here $wt_i(b) = \langle h_i, wt(b)\rangle$. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0.

One can check easily that $B_1 \otimes B_2$ is again a crystal. Note that axiom (C4) is necessary for $B_1 \otimes B_2$ to satisfy axiom (C3).

Thus $\otimes$ is a functor from $\mathscr{C} \times \mathscr{C}$ to $\mathscr{C}$.

PROPOSITION 1.3.1 (associativity). *For three crystals B_1, B_2, and B_3, $(B_1 \otimes B_2) \otimes B_3$ is isomorphic to $B_1 \otimes (B_2 \otimes B_3)$ by $(b_1 \otimes b_2) \otimes b_3 \mapsto b_1 \otimes (b_2 \otimes b_3)$.*

Proof. We shall only prove $\tilde{e}_i((b_1 \otimes b_2) \otimes b_3) = \tilde{e}_i(b_1 \otimes (b_2 \otimes b_3))$. The other axioms can be checked similarly. We have

$$\tilde{e}_i((b_1 \otimes b_2) \otimes b_3)$$

$$= \begin{cases} \tilde{e}_i(b_1 \otimes b_2) \otimes b_3 & \text{if } \varphi_i(b_1 \otimes b_2) \geqslant \varepsilon_i(b_3) \\ (b_1 \otimes b_2) \otimes \tilde{e}_i b_3 & \text{if } \varphi_i(b_1 \otimes b_2) < \varepsilon_i(b_3) \end{cases}$$

$$= \begin{cases} (\tilde{e}_i b_1 \otimes b_2) \otimes b_3 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2) \text{ and } \varphi_i(b_1) + wt_i(b_2) \geqslant \varepsilon_i(b_3) \\ (b_1 \otimes \tilde{e}_i b_2) \otimes b_3 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \text{ and } \varphi_i(b_2) \geqslant \varepsilon_i(b_3) \\ (b_1 \otimes b_2) \otimes \tilde{e}_i b_3 & \text{if } \varphi_i(b_1) + wt_i(b_2) < \varepsilon_i(b_3) \text{ and } \varphi_i(b_2) < \varepsilon_i(b_3). \end{cases}$$

On the other hand, we have

$$\tilde{e}_i(b_1 \otimes (b_2 \otimes b_3))$$

$$= \begin{cases} \tilde{e}_i b_1 \otimes (b_2 \otimes b_3) & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2 \otimes b_3) \\ b_1 \otimes \tilde{e}_i(b_2 \otimes b_3) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2 \otimes b_3) \end{cases}$$

$$= \begin{cases} \tilde{e}_i b_1 \otimes (b_2 \otimes b_3) & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2) \text{ and } \varphi_i(b_1) \geqslant \varepsilon_i(b_3) - wt_i b_2 \\ b_1 \otimes (\tilde{e}_i b_2 \otimes b_3) & \text{if } \varphi_i(b_2) \geqslant \varepsilon_i(b_3) \text{ and } \varphi_i(b_1) < \varepsilon_i(b_2) \\ b_1 \otimes (b_2 \otimes \tilde{e}_i b_3) & \text{if } \varphi_i(b_2) < \varepsilon_i(b_3) \text{ and } \varphi_i(b_1) < \varepsilon(b_3) - wt_i b_2 . \end{cases}$$

Thus we have the desired result. Q.E.D.

Note that if B_1 and B_2 are normal, then so is $B_1 \otimes B_2$.

Note also that for any crystals B_1 and B_2

(1.3.1) $$(B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee \qquad \text{by } (b_1 \otimes b_2)^\vee \leftrightarrow b_2^\vee \otimes b_1^\vee .$$

If $\varphi_i(b_1)$, $\varphi_i(b_2) \in \mathbf{Z}$, then the action of a power of $\tilde{f}_i$ on $b_1 \otimes b_2$ is given by

(1.3.2) $$\tilde{f}_i^a(b_1 \otimes b_2) = \tilde{f}_i^x b_1 \otimes \tilde{f}_i^y b_2$$

with $y = (a - (\varphi_i(b_1) - \varepsilon_i(b_2))_+)_+$ and $x = \min(a, (\varphi_i(b_1) - \varepsilon_i(b_2))_+)$.

Example 1.3.2. Let B be a crystal. Then

$$wt(b \otimes t_\lambda) = wt(b) + \lambda,$$

$$\varepsilon_i(b \otimes t_\lambda) = \varepsilon_i(b),$$

$$\varphi_i(b \otimes t_\lambda) = \varphi_i(b) + \langle h_i, \lambda \rangle,$$

and

$$wt(t_\lambda \otimes b) = wt(b) + \lambda,$$

$$\varepsilon_i(t_\lambda \otimes b) = \varepsilon_i(b) - \langle h_i, \lambda \rangle,$$

$$\varphi_i(t_\lambda \otimes b) = \varphi_i(b).$$

Example 1.3.3. $B_i \otimes T_\lambda \cong T_{s_i\lambda} \otimes B_i$ by $b_i(n) \otimes t_\lambda \leftrightarrow t_{s_i\lambda} \otimes b_i(n + \langle h_i, \lambda \rangle)$. Here $s_i\lambda = \lambda - \langle h_i, \lambda \rangle \alpha_i$.

Example 1.3.4. $B \otimes T_0 \cong T_0 \otimes B \cong B$, $T_\lambda \otimes T_\mu \cong T_{\lambda+\mu}$.

Hence T_0 is the neutral object with respect to the tensor product.

Example 1.3.5.

$$B_i \otimes B_i \cong \bigoplus_{k \in \mathbf{Z}} B_i \otimes T_{k\alpha_i}$$

by

$$b_i(n) \otimes b_i(m) \leftrightarrow b_i(x) \otimes t_{k\alpha_i}$$

with

$$x = \min(n, m + 2n), \qquad k = \max(m, -n),$$

$$n = \max(-k, x), \qquad m = \min(k, x + 2k).$$

Let us mention the following lemma, which can be proven directly.

LEMMA 1.3.6. *Let B_k $(1 \leqslant k \leqslant n)$ be a crystal and $b_k \in B_k$. We set*

$$a_k = \varepsilon_i(b_k) - \sum_{1 \leqslant \nu < k} wt_i(b_\nu).$$

Then we have

(i) $\tilde{e}_i(b_1 \otimes \cdots \otimes b_n) = b_1 \otimes \cdots \otimes \tilde{e}_i b_k \otimes \cdots \otimes b_n$

$$\text{if } a_k > a_\nu \text{ for } 1 \leqslant \nu < k \text{ and}$$

$$a_k \geqslant a_\nu \text{ for } k < \nu \leqslant n;$$

(ii) $\tilde{f}_i(b_1 \otimes \cdots \otimes b_n) = b_1 \otimes \cdots \otimes \tilde{f}_i b_k \otimes \cdots \otimes b_n$

$$\text{if } a_k \geqslant a_\nu \text{ for } 1 \leqslant \nu < k \text{ and}$$

$$a_k > a_\nu \text{ for } k < \nu \leqslant n.$$

2. $\tilde{e}_i^*$ and $\tilde{f}_i^*$.

2.1. Invariance by $*$. In this section, we shall use the notations in [K2]. In [K2] we proved that the crystal lattice $L(\infty)$ of $U_q^-(\mathfrak{g})$ is stable by the antiautomorphism $*$ of $U_q^-(\mathfrak{g})$ and $B(\infty) \sqcup (-B(\infty))$ is stable by $*$. We shall show the following statement in this section.

THEOREM 2.1.1. $B(\infty)^* = B(\infty)$.

We shall define the operators $\tilde{e}_i^*$ and $\tilde{f}_i^*$ of $U_q^-(\mathfrak{g})$ by

$$\tilde{e}_i^* = *\tilde{e}_i* \quad \text{and} \quad \tilde{f}_i^* = *\tilde{f}_i*. \tag{2.1.2}$$

Then $L(\infty)$ is stable by $\tilde{e}_i^*$ and $\tilde{f}_i^*$. By Lemma 5.2.3 in [K2], we have

$$*e_i'* = Ad(t_i)e_i''. \tag{2.1.3}$$

Hence for $P = \sum P_n f_i^{(n)}$ with $e_i'' P_n = 0$, we have

$$\tilde{f}_i^* P = \sum P_n f_i^{(n+1)}.$$

PROPOSITION 2.1.2. *Let* $m \geqslant 0$ *and* $P \in L(\infty)$. *Assume that* $e_i'' P = 0$ *and* $P \bmod qL(\infty)$ *belongs to* $B(\infty)$. *Then for* $\lambda \in P_+$ *such that* $\langle h_i, \lambda \rangle = 0$ *and* $\langle h_j, \lambda \rangle \gg 0$ *for any* $j \neq i$ *and for* $\mu \in P_+$ *such that* $\langle h_i, \mu \rangle \gg 0$, *we have*

$$P f_i^{(m)}(u_\lambda \otimes u_\mu) \equiv P u_\lambda \otimes f_i^{(m)} u_\mu \bmod qL(\lambda) \otimes L(\mu).$$

Proof. We have $f_i^{(m)}(u_\lambda \otimes u_\mu) = u_\lambda \otimes f_i^{(m)} u_\mu$. On the other hand, we have

$$\Delta P \equiv P \otimes 1 \qquad \text{modulo } \sum_{\xi \neq 0} U_q(\mathfrak{g}) \otimes U_q^-(\mathfrak{g})_\xi.$$

This implies

$$Pf_i^{(m)}(u_\lambda \otimes u_\mu) \equiv Pu_\lambda \otimes f_i^{(m)} u_\mu \qquad \text{modulo} \sum_{\xi \neq \mu - m\alpha_i} V(\lambda) \otimes V(\mu)_\xi .$$

By [K2], $Pf_i^{(m)}$ mod $qL(\infty)$ belongs to $B(\infty)$ or $-B(\infty)$. Accordingly, $Pf_i^{(m)}(u_\lambda \otimes u_\mu)$ belongs to $B(\lambda) \otimes B(\mu)$ or $-B(\lambda) \otimes B(\mu)$. Since $Pu_\lambda \otimes f_i^{(m)} u_\mu$ belongs to $B(\lambda) \otimes B(\mu)$, we obtain the desired result. Q.E.D.

Now let us show Theorem 2.1.1. By the induction of weight, it is enough to show that $b \in B(\infty)$, $\tilde{e}_i^* b = 0$ implies $\tilde{f}_i^{*m} b \in B(\infty)$. Take a representative $P \in L(\infty)$ of b with $e_i'' P = 0$. Then $\tilde{f}_i^{*m} b \equiv Pf_i^{(m)}$ mod $qL(\infty)$. Since $Pf_i^{(m)}(u_\lambda \otimes u_\mu)$ belongs to $B(\lambda) \otimes B(\mu)$, $Pf_i^{(m)}$ belongs to $B(\infty)$.

2.2. Description of $B(\infty)$. Proposition 2.1.2 implies the following theorem.

THEOREM 2.2.1. (i) *For any i there exists a unique strict embedding of crystals*

$$\Psi_i\colon B(\infty) \hookrightarrow B(\infty) \otimes B_i$$

that sends u_∞ *to* $u_\infty \otimes b_i$.

(ii) *If* $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$, *then* $\Psi_i(\tilde{f}_i^* b) = b_0 \otimes \tilde{f}_i^{m+1} b_i$ *and* $\varepsilon_i(b^*) = m$.

(iii) $\operatorname{Im} \Psi_i = \{b \otimes \tilde{f}_i^m b_i;\ \varepsilon_i(b^*) = 0,\ m \geqslant 0\}$.

Proof. Any element b of $B(\infty)$ can be uniquely written in the form $b = \tilde{f}_i^{*m} b_0$ with $\tilde{e}_i^* b_0 = 0$. Let us define Ψ_i by $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$. It is enough to show that Ψ_i is a strict embedding. In order to see this, we shall first check $\Psi_i(\tilde{e}_j b) = \tilde{e}_j \Psi_i(b)$. Note that this implies $\Psi_i(\tilde{f}_j b) = \tilde{f}_j \Psi_i(b)$.

Take $P \in L(\infty) \cap \operatorname{Ker} e_i''$ such that $b_0 \equiv P$. Then $b \equiv Pf_i^{(m)}$. Here and in the sequel, $\equiv$ means modulo $qL(\infty)$ or $qL(\lambda)$, etc. Now we write $\tilde{e}_j b \equiv Qf_i^{(k)}$ with $Q \in L(\infty) \cap \operatorname{Ker} e_i''$. Then we have $\tilde{e}_j(Pf_i^{(m)}) \equiv Qf_i^{(k)}$. Hence taking λ and μ as in Proposition 2.1.2, we have

$$\begin{aligned} (2.2.1) \qquad Qu_\lambda \otimes f_i^{(k)} u_\mu &\equiv Qf_i^{(k)}(u_\lambda \otimes u_\mu) \\ &\equiv (\tilde{e}_j(Pf_i^{(m)}))(u_\lambda \otimes u_\mu) \\ &\equiv \tilde{e}_j((Pf_i^{(m)})(u_\lambda \otimes u_\mu)) \\ &\equiv \tilde{e}_j(Pu_\lambda \otimes f_i^{(m)} u_\mu). \end{aligned}$$

On the other hand, we have

$$(2.2.2) \qquad \Psi_i(\tilde{e}_j b) = Q \otimes \tilde{f}_i^k b_i$$

and

$$\tilde{e}_j \Psi_i(b) = \tilde{e}_j(b_0 \otimes \tilde{f}_i^m b_i).$$

Assume first $j \neq i$. Then (2.2.1) and $\varepsilon_j(f_i^{(m)}u_\mu) = 0 \leqslant \varphi_j(Pu_\lambda)$ implies

$$Qu_\lambda \otimes f_i^{(k)}u_\mu \equiv \tilde{e}_j Pu_\lambda \otimes f_i^{(m)}u_\mu .$$

Hence $Q \equiv \tilde{e}_j P \equiv \tilde{e}_j b_0$ and $m = k$. Thus we obtain $\tilde{e}_j \Psi_i(b) = \tilde{e}_j(b_0 \otimes \tilde{f}_i^m b_i) = \tilde{e}_j b_0 \otimes \tilde{f}_i^m b_i = \Psi_i(\tilde{e}_j b)$.

Now consider the case $j = i$. In this case, $\varphi_i(Pu_\lambda) = \varphi_i(b_0) + \langle h_i, \lambda\rangle = \varphi_i(b_0) \geqslant 0$ and $\varepsilon_i(f_i^{(m)}u_\mu) = m$. Hence we have

$$\tilde{e}_i(Pu_\lambda \otimes f_i^{(m)}u_\mu) \equiv \begin{cases} \tilde{e}_i Pu_\lambda \otimes f_i^{(m)}u_\mu & \varphi_i(b_0) \geqslant m \\ Pu_\lambda \otimes f_i^{(m-1)}u_\mu & \varphi_i(b_0) < m . \end{cases}$$

Therefore (2.2.1) implies

$$Q \equiv \tilde{e}_i P, \ m = k \qquad \text{if } \varphi_i(b_0) \geqslant m,$$

$$Q \equiv P, \ k = m - 1 \qquad \text{if } \varphi_i(b_0) < m.$$

Note that $\varphi_i(b_0) < m = 0$ cannot occur because $\varphi_i(b_0) \geqslant 0$. Accordingly, we have

$$\Psi_i(\tilde{e}_i b) \equiv \tilde{e}_i b_0 \otimes \tilde{f}_i^m b_i \quad \text{or} \quad b_0 \otimes \tilde{f}_i^{m-1} b_i .$$

This equals $\tilde{e}_i(b_0 \otimes \tilde{f}_i^m b_i)$.

Now let us prove $\varepsilon_j(b) = \varepsilon_j(\Psi_i(b))$. Take b_0, m, and P as above, and take λ and μ as in Proposition 2.1.1. Then

$$\begin{aligned} \varepsilon_j(b) &= \varepsilon_j(Pf_i^{(m)}(u_\lambda \otimes u_\mu)) = \varepsilon_j(Pu_\lambda \otimes f_i^{(m)}u_\mu) \\ &= \max(\varepsilon_j(Pu_\lambda), \varepsilon_j(f_i^{(m)}u_\mu) - \langle h_j, wt(Pu_\lambda)\rangle) \\ &= \max(\varepsilon_j(b_0), \varepsilon_j(f_i^{(m)}) - \langle h_j, \lambda\rangle - \langle h_j, wt b_0\rangle). \end{aligned}$$

If $j \neq i$, then $\langle h_j, \lambda\rangle \gg 0$ implies $\varepsilon_j(b) = \varepsilon_j(b_0) = \varepsilon_j(b_0 \otimes \tilde{f}_i^m b_i) = \varepsilon_j(\Psi_i(b))$. If $j = i$, then $\varepsilon_i(b) = \max(\varepsilon_i(b_0), m - \langle h_i, wt b_0\rangle) = \varepsilon_i(b_0 \otimes \tilde{f}_i^m b_i) = \varepsilon_i(\Psi_i(b))$.

This completes the proof. Q.E.D.

This theorem immediately implies the following result.

COROLLARY 2.2.2. *If $j \neq i$, then $\tilde{f}_j$ and $\tilde{e}_i^*$ (resp. $\tilde{f}_j$ and $\tilde{f}_i^*$, $\tilde{e}_j$ and $\tilde{e}_i^*$, $\tilde{e}_j$ and $\tilde{f}_i^*$) commute. Moreover, $\varepsilon_j(\tilde{e}_i^* b) = \varepsilon_j(b)$ if $\tilde{e}_i^* b \neq 0$, etc.*

Proof. Let us show $\tilde{f}_i^* \tilde{f}_j b = \tilde{f}_j \tilde{f}_i^* b$ for $i \neq j$. Write $\Psi_i(b) = b_0 \otimes \tilde{f}_i^k b_i$. Then we have $\Psi_i(\tilde{f}_j b) = \tilde{f}_j \Psi_i(b) = \tilde{f}_j b_0 \otimes \tilde{f}_i^k b_i$ and hence $\Psi_i(\tilde{f}_i^* \tilde{f}_j b) = \tilde{f}_j b_0 \otimes \tilde{f}_i^{k+1} b_i = \tilde{f}_j(b_0 \otimes \tilde{f}_i^{k+1} b_i) = \tilde{f}_j \Psi_i(\tilde{f}_i^* b) = \Psi_i(\tilde{f}_j \tilde{f}_i^* b)$. This implies $\tilde{f}_i^* \tilde{f}_j b = \tilde{f}_j \tilde{f}_i^* b$. The proof of the other statements is similar. Q.E.D.

This gives a procedure to determine the crystal $B(\infty)$. For $i_1, \dots, i_l \in I$, we define $\Psi_{i_1,\dots,i_l}\colon B(\infty) \hookrightarrow B(\infty) \otimes B_{i_1} \otimes \cdots \otimes B_{i_l}$ by $\Psi_{i_1} \circ \cdots \circ \Psi_{i_l}$. Then for any $b \in B(\infty)$, we can choose $i_1, \dots, i_l$, so that

$$\Psi_{i_1,\dots,i_l}(b) \in u_\infty \otimes B_{i_1} \otimes \cdots \otimes B_{i_l}.$$

Hence $B(\infty)$ may be considered as a subcrystal of the limit of $B_{i_1} \otimes \cdots \otimes B_{i_l}$.

We shall investigate them in the rank-2 case. Set $I = \{1, 2\}$ and write $c_1 = -\langle h_1, \alpha_2 \rangle$, $c_2 = -\langle h_2, \alpha_1 \rangle$. Set $t = c_1 c_2$ and define $\{z_n\}_{n \geqslant 1}$ by

$$z_1 = 1 \quad \text{and} \quad z_n = 1 - \frac{1}{t z_{n-1}}.$$

We understand that, if $z_n = 0$, then $z_m = 0$ for $m \geqslant n$. This happens when $t = 1, 2, 3$. If $t = 0$, then set $z_n = 0$ for $n > 1$. We have $1 \geqslant z_n \geqslant z_{n+1} \geqslant 0$.

PROPOSITION 2.2.3. *$u_\infty \otimes \cdots \otimes \tilde{f}_2^{y_2} b_2 \otimes \tilde{f}_1^{x_1} b_1 \otimes \tilde{f}_2^{y_1} b_2 \otimes \tilde{f}_1^{x_0} b_1$ belongs to $\Psi_{\cdots 21}(B(\infty))$ if and only if x_ν and y_ν satisfies*

(2.2.3)
$$0 \leqslant x_0,\ 0 \leqslant x_\nu \leqslant c_1 z_{2\nu-1} y_\nu \quad \textit{for } 1 \leqslant \nu \textit{ and}$$
$$0 \leqslant y_\nu \leqslant c_2 z_{2\nu-2} x_{\nu-1} \quad \textit{for } 2 \leqslant \nu.$$

Proof. Let S be the set of $u_\infty \otimes \cdots \otimes \tilde{f}_2^{y_1} b_2 \otimes \tilde{f}_1^{x_0} b_1$ with (2.2.1). It is enough to show the following properties on S.

(2.2.4) If $b \in S$ satisfies $\tilde{e}_i b = 0$ for $i = 1, 2$ then all x_ν and y_ν are zero.

(2.2.5) S is stable by $\tilde{f}_i$.

(2.2.6) $S \sqcup \{0\}$ is stable by $\tilde{e}_i$.

The case $t = 0$ is easily proved; we shall assume $t > 0$. Let us first prove (2.2.4). Set $b = u_\infty \otimes b'$ with $b' = \cdots \otimes \tilde{f}_1^{x_0} b_1$ and assume $\tilde{e}_i b = 0$. Since $\tilde{e}_i b' \neq 0$, we have $\tilde{e}_i(u_\infty \otimes b') = \tilde{e}_i u_\infty \otimes b'$. Thus $0 = \varphi_i(u_\infty) \geqslant \varepsilon_i(b')$. Hence if $b' = \tilde{f}_i^x b_i \otimes b''$, then $0 \geqslant \varepsilon_i(b') \geqslant x \geqslant 0$. Thus $x = 0$.

Let us prove (2.2.5). Since the proof of the case $i = 2$ is similar, we shall only prove the case $i = 1$. Set

$$b = u_\infty \otimes \cdots \otimes \tilde{f}_1^{x_0} b_1$$

and $\tilde{f}_1 b = u_\infty \otimes \cdots \otimes \tilde{f}_1^{1+x_k} b_1 \otimes \cdots$. It is enough to show that, when $k > 0$,

$1 + x_k \leqslant c_1 z_{2k-1} y_k$. By Lemma 1.3.6,

$$x_k + 2 \sum_{\nu > k} x_\nu - c_1 \sum_{\nu > k} y_\nu > x_{k-1} + 2 \sum_{\nu \geqslant k} x_\nu - c_1 \sum_{\nu \geqslant k} y_\nu, \text{ or equivalently}$$

$$c_1 y_k \geqslant x_{k-1} + 1 + x_k. \tag{2.2.7}$$

Hence if $k = 1$, then $1 + x_k \leqslant c_1 z_{2k-1} y_k$. Assume $k > 1$. Then we have

$$y_k \leqslant c_2 z_{2k-2} x_{k-1}. \tag{2.2.8}$$

If $c_2 z_{2k-2} = 0$, then $y_k = 0$ and (2.2.7) cannot happen. Hence $z_{2k-2} > 0$ and $c_2 > 0$. Then (2.2.7) and (2.2.8) imply

$$c_1 z_{2k-1} y_k = c_1 \left(1 - \frac{1}{t z_{2k-2}}\right) y_k \geqslant c_1 y_k - x_{k-1} \geqslant 1 + x_k.$$

The proof of (2.2.4) is similar. Q.E.D.

By taking $*$, we obtain the following result.

PROPOSITION 2.2.4. *Set $b = \tilde{f}_1^{x_0} \tilde{f}_2^{y_1} \tilde{f}_1^{x_1} \tilde{f}_2^{y_2} \cdots u_\infty$. Then the following conditions are equivalent.*

$$\begin{aligned} &\tilde{e}_1(\tilde{f}_2^{y_1} \tilde{f}_1^{x_1} \tilde{f}_2^{y_2} \cdots u_\infty) = 0, \\ &\tilde{e}_2(\tilde{f}_1^{x_1} \tilde{f}_2^{y_2} \cdots u_\infty) = 0, \\ &\tilde{e}_1(\tilde{f}_2^{y_2} \cdots u_\infty) = 0, \\ &\cdots. \end{aligned} \tag{2.2.9}$$

(2.2.10) *$\{x_\nu\}$ and $\{y_\nu\}$ satisfy (2.2.3).*

Example 2.2.5. $\mathfrak{g} = A_2, I = \{1, 2\}, \langle h_1, \alpha_2 \rangle = \langle h_2, \alpha_1 \rangle = -1$. In this case $B(\infty)$ is fully embedded into $B_1 \otimes B_2 \otimes B_1$ by $u_\infty \mapsto b_1 \otimes b_2 \otimes b_1$. The image is

$$\{\tilde{f}_1^n b_1 \otimes \tilde{f}_2^m b_2 \otimes \tilde{f}_1^l b_1; 0 \leqslant n \leqslant m, 0 \leqslant l\}.$$

Example 2.2.6. $\mathfrak{g} = B_2, I = \{1, 2\}, \langle h_1, \alpha_2 \rangle = -2, \langle h_2, \alpha_1 \rangle = -1$. In this case, $B(\infty)$ is fully embedded into $B_1 \otimes B_2 \otimes B_1 \otimes B_2$ and also into $B_2 \otimes B_1 \otimes B_2 \otimes B_1$. The images are

$$\{\tilde{f}_1^a b_1 \otimes \tilde{f}_2^b b_2 \otimes \tilde{f}_1^c b_1 \otimes \tilde{f}_2^d b_2; 0 \leqslant d, 0 \leqslant a \leqslant b \leqslant c\}$$

and

$$\{\tilde{f}_2^a b_2 \otimes \tilde{f}_1^b b_1 \otimes \tilde{f}_2^c b_2 \otimes \tilde{f}_1^d b_1;\, d \geqslant 0, 0 \leqslant b \leqslant 2c, 0 \leqslant 2a \leqslant b\}.$$

Example 2.2.7. $\mathfrak{g} = G_2$, $I = \{1, 2\}$, $\langle h_1, \alpha_2 \rangle = -3$, $\langle h_2, \alpha_1 \rangle = -1$. In this case, $B(\infty)$ is fully embedded into $B_1 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1 \otimes B_2$ and into $B_2 \otimes B_1 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1$. The images are

$$\{\tilde{f}_1^{a_1} b_1 \otimes \tilde{f}_2^{a_2} b_2 \otimes \tilde{f}_1^{a_3} b_1 \otimes \tilde{f}_2^{a_4} b_2 \otimes \tilde{f}_1^{a_5} b_1 \otimes \tilde{f}_2^{a_6} b_2;$$

$$0 \leqslant a_6, 0 \leqslant a_4 \leqslant a_5, 0 \leqslant a_3 \leqslant 2a_4, 0 \leqslant a_2 \leqslant \tfrac{1}{2}a_3, 0 \leqslant a_1 \leqslant a_2\}$$

and

$$\{\tilde{f}_2^{a_1} b_2 \otimes \tilde{f}_1^{a_2} b_1 \otimes \tilde{f}_2^{a_3} b_2 \otimes \tilde{f}_1^{a_4} b_1 \otimes \tilde{f}_2^{a_5} b_2 \otimes \tilde{f}_1^{a_6} b_1;$$

$$0 \leqslant a_6, 0 \leqslant a_4 \leqslant 3a_5, 0 \leqslant a_3 \leqslant \tfrac{2}{3}a_4, 0 \leqslant a_2 \leqslant \tfrac{3}{2}a_3, 0 \leqslant a_1 \leqslant \tfrac{1}{3}a_2\}.$$

PROPOSITION 2.2.8. *$B(\infty) \otimes B_i \cong \bigoplus_{k \geqslant 0} B(\infty) \otimes T_{k\alpha_i}$ where $u_\infty \otimes \tilde{e}_i^k b_i \leftrightarrow u_\infty \otimes t_{k\alpha_i}$.*

Proof. By Example 1.3.4, we have $B_i \otimes B_i \cong \bigoplus_{k \in \mathbf{Z}} B_i \otimes T_{k\alpha_i}$.
Hence we have for $k \geqslant 0$

$$\xi_i\colon B(\infty) \otimes T_{k\alpha_i} \to B(\infty) \otimes B_i \otimes T_{k\alpha_i}$$

$$\to B(\infty) \otimes B_i \otimes B_i,$$

where $u_\infty \otimes t_{k\alpha_i}$ is sent to $u_\infty \otimes b_i \otimes \tilde{e}_i^k b_i$. By Theorem 2.2.1(iii), its image is contained in $\operatorname{Im} \psi_i \otimes B_i$. Therefore ξ_i splits $B(\infty) \otimes T_{k\alpha_i} \to B(\infty) \otimes B_i \xrightarrow{\psi_i \otimes B_i} B(\infty) \otimes B_i \otimes B_i$. Thus, we obtain

$$\bigsqcup_{k \geqslant 0} B(\infty) \otimes T_{k\alpha_i} \xrightarrow{\psi} B(\infty) \otimes B_i.$$

We shall show that it is an isomorphism. Since $\tilde{f}_j$ is injective on $B(\infty) \otimes B_i$, $\psi(B(\infty) \otimes T_{k\alpha_i}) \not\ni 0$. Hence ψ commutes with $\tilde{f}_j$. Since $B(\infty) \otimes T_{k\alpha_i}$ and $B(\infty) \otimes B_i$ are upper normal, ψ commutes with $\tilde{e}_j$. Note that for any object b in $B(\infty) \otimes B_i$, there exists $i_1, \ldots, i_l$ such that $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\infty \otimes b_i(k))$. If $k < 0$, then $u_\infty \otimes b_i(k) = \tilde{f}_i^{-k}(u_\infty \otimes b_i)$, and we may assume $k \geqslant 0$. Hence ψ is surjective. Injectivity follows from the fact that ψ sends bijectively the set of highest-weight vectors of $\sqcup B(\infty) \otimes T_{k\alpha_i}$ to the one of $B(\infty) \otimes B_i$. Q.E.D.

3. Global bases of $U_q^+(\mathfrak{g})$-modules.

3.1. A general thoerem. In this section, we assume $I = \{i\}$. Let M be a finite-dimensional integrable $U_q(\mathfrak{g})$-module and assume that M has a lower global crystal base $(M_{\mathbf{Q}}, L_0, L_\infty, B)$ (see [K3]). Here L_0 and L_∞ is a lower crystal lattice at $q = 0$ and $q = \infty$ respectively, (L_0, B) is a lower crystal base at $q = 0$, and $M_{\mathbf{Q}}$ is a submodule of M over the $\mathbf{Q}[q, q^{-1}]$-algebra $U_q^-(\mathfrak{g})_{\mathbf{Q}}$ generated by $f_i^{(n)}$. We assume therefore

$$L_0 \cap L_\infty \cap M_{\mathbf{Q}} \xrightarrow{\sim} L_0/qL_0 .$$

Let us denote by G the inverse map. Then we have $M_{\mathbf{Q}} = \bigoplus_{b \in B} \mathbf{Q}[q, q^{-1}]G(b)$, $L_0 = \bigoplus_{b \in B} AG(b)$, etc. We set $I^\ell(B) = \{b \in B; \varepsilon_i(b) + \varphi_i(b) = \ell\}$ and $W^\ell(B) = \{b \in B; \varepsilon_i(b) + \varphi_i(b) \geqslant \ell\}$. Then B is the direct sum of $I^\ell(B)$. Let us denote by $I^\ell(M)$ the sum of all $(\ell + 1)$-dimensional irreducible $U_q(\mathfrak{g})$-submodules of M and $W^\ell(M) = \bigoplus_{\ell \leqslant \ell'} I^{\ell'}(M)$. Then by [K3],

$$W^\ell(M) = \bigoplus_{b \in W^\ell(B)} \mathbf{Q}(q)G(b). \tag{3.1.1}$$

Moreover, if $b \in I^\ell(B)$, then

$$f_i^{(k)}G(b) \equiv \begin{bmatrix} \varepsilon_i(b) + k \\ k \end{bmatrix}_i G(\tilde{f}_i^k b) \bmod W^{\ell+1}(M) \tag{3.1.2}$$

and

$$e_i^{(k)}G(b) \equiv \begin{bmatrix} \varphi_i(b) + k \\ k \end{bmatrix}_i G(\tilde{e}_i^k b) \bmod W^{\ell+1}(M).$$

Here $[\ ]_i$ is the q-analogue of the binomial coefficients (see [K2]). Note that $\begin{bmatrix} n \\ k \end{bmatrix}_i$ does not vanish for $0 \leqslant k \leqslant n$. Now let N be a sub-$U_q^+(\mathfrak{g})$-module. We assume that there is a subset B_N of B such that

$$N = \bigoplus_{b \in B_N} \mathbf{Q}(q)G(b). \tag{3.1.3}$$

Set $\tilde{N} = U_q(\mathfrak{g})N = \sum_{k \geqslant 0} f_i^k N$.

THEOREM 3.1.1. *Set $B_{\tilde{N}} = \bigcup_{k \geqslant 0} \tilde{f}_i^k B_N \setminus \{0\}$. Then we have*

$$\tilde{N} = \bigoplus_{b \in B_{\tilde{N}}} \mathbf{Q}(q)G(b) \quad \textit{and} \quad \tilde{N} \cap M_{\mathbf{Q}} = U^{\mathbf{Q}}(\mathfrak{g})(N \cap M_{\mathbf{Q}}).$$

Before proving this, we shall study the properties of B_N.

LEMMA 3.1.2. $\tilde{e}_i B_N \subset B_N \sqcup \{0\}$.

Proof. Assume $b \in B_N$ and $\tilde{e}_i b \in B_N$. Then $e_i G(b)$ has a nonzero coefficient of $G(\tilde{e}_i b)$ if we write it as a linear combination of $G(B)$. Hence $G(\tilde{e}_i b) \in N$. Q.E.D.

LEMMA 3.1.3. *$W^\ell(N) = \bigoplus_{b \in W^\ell(B_N)} \mathbf{Q}(q)G(b)$. Here $W^\ell(B_N) = B_N \cap W^\ell(B)$ and $W^\ell(N) = W^\ell(M) \cap N$.*

This is obvious by (3.1.1).

LEMMA 3.1.4. *$W^\ell(\tilde{N}) = U_q(\mathfrak{g})W^\ell(N)$.*

Proof. It is obvious that $W^\ell(\tilde{N}) \supset U_q(\mathfrak{g})W^\ell(N)$. In order to prove the converse, it is enough to show that if $u \in W^\ell(\tilde{N})$ satisfies $e_i u = 0$, then $u \in U_q(\mathfrak{g})W^\ell(N)$. Write $u = \sum f_i^k u_k$ with $wt(u_k) = wt(u) + k\alpha_i$ and $u_k \in N$. Then $\langle h_i, wt(u_k)\rangle \geqslant \ell$ implies $u_k \in W^\ell(M)$. Hence $u \in W^\ell(\tilde{N})$. Q.E.D.

Proof of Theorem 3.1.1. We shall prove

$$W^\ell(\tilde{N}) = \bigoplus_{b \in W^\ell(B_{\tilde{N}})} \mathbf{Q}(q)G(b) \tag{3.1.4$_\ell$}$$

by the descending induction on ℓ. It is obvious for $\ell \gg 0$ since both sides of (3.1.4)$_\ell$ are equal to 0. Assuming that (3.1.4)$_{\ell+1}$, we shall prove (3.1.4)$_\ell$. Since both sides of (3.1.4)$_\ell$ contain $W^{\ell+1}(\tilde{N})$, we may assume $W^{\ell+1}(\tilde{N}) = 0$ by replacing M with $M/W^{\ell+1}(\tilde{N})$. Replacing N with $W^\ell(N)$, we may assume $N = W^\ell(N)$. Thus we have $B_N \subset I^\ell(B)$. For $b \in B_N$ with $\tilde{e}_i b = 0$, $\langle h_i, wt(b)\rangle = \ell$ implies $e_i G(b) \in W^{\ell+1}(\tilde{N})$, and hence $e_i G(b) = 0$. Hence $\tilde{f}_i^k G(b) = f_i^{(k)} G(b) \in L_0 \cap L_\infty \cap M_{\mathbf{Q}}$. This implies $G(\tilde{f}_i^k b) = f_i^{(k)} G(b)$. Since $B_{\tilde{N}} = \{\tilde{f}_i^k b;\ b \in B_N,\ \tilde{e}_i b = 0\}$, we obtain

$$\bigoplus_{b \in B_{\tilde{N}}} \mathbf{Q}(q)G(b) = \bigoplus_{b \in B_N,\, \tilde{e}_i b = 0} U_q(\mathfrak{g})G(\tilde{f}_i^{(k)} b) = \tilde{N}$$

and

$$\bigoplus_{b \in B_{\tilde{N}}} \mathbf{Q}[q, q^{-1}]G(b) = \bigoplus_{b \in B_N,\, \tilde{e}_i b = 0} \mathbf{Q}[q, q^{-1}]f_i^{(k)} G(b) = \tilde{N} \cap M_{\mathbf{Q}} = U_q^{\mathbf{Q}}(q)(N \cap M_{\mathbf{Q}}).$$

Q.E.D.

3.2. Global base of $U_q^+(\mathfrak{g})u_{w\lambda}$. Let $\lambda \in P_+$ and let $V(\lambda)$ be the irreducible $U_q(\mathfrak{g})$-module generated by the highest-weight vector u_λ of highest weight λ. Let $-$ be the involution of $V(\lambda)$ defined by $\overline{Pu_\lambda} = \bar{P}u_\lambda$. Let $(L(\lambda), B(\lambda))$ be the lower crystal base of $V(\lambda)$. Set $V_{\mathbf{Q}}(\lambda) = U_q^-(\mathfrak{g})_{\mathbf{Q}} u_\lambda$. Then $L(\lambda) \cap L(\lambda)^- \cap M_{\mathbf{Q}} \xrightarrow{\sim} L(\lambda)/qL(\lambda)$. If we denote by G_λ the inverse of this ismorphism, then $V(\lambda) = \bigoplus_{b \in B(\lambda)} \mathbf{Q}(q)G_\lambda(b)$. For $w \in W$, let us denote by $u_{w\lambda}$ the lower global crystal base of weight $w\lambda$. Then we have

$$u_{w\lambda} = u_\lambda \qquad \text{if } w = 1, \tag{3.2.1}$$

$$u_{s_i w\lambda} = f_i^{(m)} u_{w\lambda} \qquad \text{if } m = \langle h_i, w\lambda\rangle \geqslant 0.$$

Set

$$V_w(\lambda) = U_q^+(\mathfrak{g})u_{w\lambda}. \tag{3.2.2}$$

Let us remark that

$$U_q^+(\mathfrak{g})U_q(\mathfrak{g}_i) = U_q(\mathfrak{g}_i)U_q^+(\mathfrak{g}). \tag{3.2.3}$$

LEMMA 3.2.1. (i) *If* $s_iw < w$, *then*

$$f_iV_w(\lambda) \subset V_w(\lambda).$$

(ii) *If* $s_iw < w$, *then*

$$V_w(\lambda) = U_q(\mathfrak{g}_i)V_{s_iw}(\lambda).$$

Proof. (i) follows from (ii), and (ii) follows from (3.2.3) and $U_q^+(\mathfrak{g}_i)u_{w\lambda} = U_q(\mathfrak{g}_i)u_{s_iw\lambda}$.

COROLLARY 3.2.2. *If* $w = s_{i_1}\cdots s_{i_\ell}$ *is a reduced expression, then*

$$V_w(\lambda) = \sum_{k_1,\dots,k_\ell \geqslant 0} \mathbf{Q}(q)f_{i_1}^{k_1}\cdots f_{i_\ell}^{k_\ell}u_\lambda.$$

PROPOSITION 3.2.3. (i) *There exists a subset* $B_w(\lambda)$ *of* $B(\lambda)$ *such that*

$$V_w(\lambda) = \bigoplus_{b\in B_w(\lambda)} \mathbf{Q}(q)G_\lambda(b).$$

(ii) $\tilde{e}_iB_w(\lambda) \subset B_w(\lambda)\sqcup\{0\}$.
(iii) *If* $s_iw < w$, *then*

$$B_w(\lambda) = \bigcup_{k\geqslant 0} \tilde{f}_i^kB_{s_iw}(\lambda)\setminus\{0\}.$$

This follows immediately from Lemma 3.2.1 and the results in §3.1 by induction on $\ell(w)$.

PROPOSITION 3.2.4. *If* $w \geqslant w'$ *by the Bruhat order, then*

$$B_w(\lambda) \supset B_{w'}(\lambda).$$

This follows immediately from $V_w(\lambda) \supset V_{w'}(\lambda)$, which is a consequence of Corollary 3.2.2.

Tending λ to the infinity, the results of $V(\lambda)$ imply the following results on $U_q^-(\mathfrak{g})$.

PROPOSITION 3.2.5. *For any* $w \in W$ *there exists a unique subset* $B_w(\infty)$ *of* $B(\infty)$ *satisfying the following properties*:

(i) $B_w(\infty) = \{u_\infty\}$ *if* $w = 1$;
(ii) *if* $s_iw < w$, *then*

$$B_w(\infty) = \bigcup_{k\geqslant 0} \tilde{f}_i^kB_{s_iw}(\infty);$$

Moreover, they satisfy

(iii) $\tilde{e}_i B_w(\infty) \subset B_w(\infty) \sqcup \{0\}$;
(iv) *if* $w \geqslant w'$, *then*

$$B_w(\infty) \supset B_{w'}(\infty);$$

(v) *if* $w = s_{i_1} \cdots s_{i_\ell}$ *is a minimal expression, then* $\bigoplus_{b \in B_w(\infty)} \mathbf{Q}(q)G(b) = \sum \mathbf{Q}(q) f_{i_1}^{k_1} \cdots f_{i_\ell}^{k_\ell}$;
(vi) *for* $\lambda \in P_+$, $B_w(\lambda) = \bar{\pi}_\lambda B_w(\infty) \setminus \{0\}$.

Remark 3.2.6. A slightly more precise argument shows that

$$U_q^{\mathbf{Z}}(\mathfrak{g}) u_{w\lambda} = \bigoplus_{b \in B_w(\lambda)} \mathbf{Z}[q, q^{-1}] G_\lambda(b),$$

$$\bigoplus_{b \in B_w(\infty)} \mathbf{Z}[q, q^{-1}] G(b) = \sum \mathbf{Z}[q, q^{-1}] f_{i_1}^{(k_1)} \cdots f_{i_\ell}^{(k_\ell)}$$

for a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$.

3.3. *Further properties of* $B_w(\infty)$. We have the following result by (v) of Proposition 3.2.5.

PROPOSITION 3.3.1. $B_w(\infty)^* = B_{w^{-1}}(\infty)$.

The following proposition is a crucial property of $B_w(\infty)$.

THEOREM 3.3.2. *If* $b \in B(\infty)$ *and* $w \in W$ *satisfy* $\tilde{f}_i b \in B_w(\infty)$, *then* $\tilde{f}_i^k b \in B_w(\infty)$ *for any* $k \geqslant 0$.

In order to prove this let us remark the following lemma that follows immediately from Proposition 3.2.3(ii) and (iii).

LEMMA 3.3.3. *Let* $w \in W$ *and* $i \in I$ *satisfy* $s_i w < w$. *If* $b \in B(\infty)$ *satisfies* $\tilde{e}_i b = 0$ *and* $\tilde{f}_i^t b \in B_w(\infty)$ *for some* $t \geqslant 0$, *then* $b \in B_{s_i w}(\infty)$.

Now let us prove Theorem 3.3.2. Replacing b with $\tilde{e}_i b$ if necessary, we may assume $\varepsilon_i(b) = 0$. Hence replacing b with b^*, it is enough to show the following.

(3.3.1) *If* $\varepsilon_i(b^*) = 0$ *and* $\tilde{f}_i^* b \in B_w(\infty)$, *then* $\tilde{f}_i^{*k} b \in B_w(\infty)$ *for any* $k \geqslant 0$.

We shall prove this by induction on the length of w.
We have

$$\Psi_i(b) = b \otimes b_i \quad \text{and} \quad \Psi_i(\tilde{f}_i^* b) = b \otimes \tilde{f}_i b_i. \tag{3.3.2}$$

If $\ell(w) = 0$, then (3.3.1) is obvious. Hence we may assume $\ell(w) > 0$. Let us take $j \in I$ such that $s_j w < w$.

(a) Case $j \neq i$. Write $b = \tilde{f}_j^t b'$ with $\tilde{e}_j b' = 0$. Then $\tilde{f}_i^* b' \in B_{s_j w}(\infty)$ by Lemma 3.3.3. Since $\varepsilon_i(b'^*) = 0$, $\tilde{f}_i^{*k} b' \in B_{s_j w}(\infty)$ by the hypothesis of induction. Hence $\tilde{f}_i^{*k} b = \tilde{f}_j^t(\tilde{f}_i^{*k} b')$ belongs to $B_w(\infty)$.

(b) Case $j = i$. If $\varphi_i(b) \leqslant \varepsilon_i(\tilde{f}_i b_i) = 1$, then $\Psi_i(\tilde{f}_i^{k-1} \tilde{f}_i^* b) = \tilde{f}_i^{k-1}(b \otimes \tilde{f}_i b_i) = b \otimes \tilde{f}_i^k b_i = \Psi_i(\tilde{f}_i^{*k} b)$, and hence $\tilde{f}_i^{*k} b$ belongs to $B_w(\infty)$. Hence we may assume $\varphi_i(b) > 1$. Write $b = \tilde{f}_i^t b'$ with $\varepsilon_i b' = 0$. Then $\varphi_i(b') = \varphi_i(b) + t > 1$, and hence $\varepsilon_i(b' \otimes \tilde{f}_i b_i) = 0$ and $b \otimes \tilde{f}_i b_i = \tilde{f}_i^t(b' \otimes \tilde{f}_i b_i)$ (see (1.3.2)). Hence $b' \otimes \tilde{f}_i b_i$ belongs to $B_{s_i w}(\infty)$. Therefore, by the hypothesis of induction $b' \otimes \tilde{f}_i^k b_i$ belongs to $B_{s_i w}(\infty)$ for any $k \geqslant 0$. On the other hand, we have

$$\tilde{f}_i^{*k} b = b \otimes \tilde{f}_i^k b_i = \tilde{f}_i^t b' \otimes \tilde{f}_i^k b_i = \tilde{f}_i^p(b' \otimes \tilde{f}_i^s b_i)$$

with $s = k$, $p = t$ if $\varphi_i(b) > k$ and $s = \varphi_i(b)$, $p = t + k - \varphi_i(b)$ if $\varphi_i(b) \leqslant k$ (see (1.3.2)). Hence $\tilde{f}_i^{*k} b$ belongs to $B_w(\infty)$. Q.E.D.

For a crystal B and $i \in I$, let us call *i-string* a subset of the form

$$S = \{\tilde{e}_i^k b;\ k \geqslant 0\} \cup \{\tilde{f}_i^k b;\ k \geqslant 0\} \setminus \{0\}$$

for some $b \in B$.

Then B decomposes into the disjoint union of i-strings. For an i-string S, an element $b \in S$ with $\tilde{e}_i b = 0$ is called the highest-weight vector of S. If S has a highest-weight vector b (e.g., when B is upper normal), $S = \{\tilde{f}_i^k b;\ k \geqslant 0\} \setminus \{0\}$, Theorem 3.3.2 implies the following result.

PROPOSITION 3.3.4. *For any i-string S of $B(\infty)$ with highest-weight vector b, $B_w(\infty) \cap S$ is either empty, S, or $\{b\}$.*

For $\lambda \in P_+$, $B_w(\lambda)$ is the inverse image of $B_w(\infty) \otimes T_\lambda$ by the embedding $B(\lambda) \to B(\infty) \otimes T_\lambda$. Thus we obtain the following result.

PROPOSITION 3.3.5. *For any i-string S of $B(\lambda)$ with a highest-weight vector b, $B_w(\lambda) \cap S$ is either empty, S, or $\{b\}$.*

As shown in the introduction, this proposition implies Littelmann's refined Demazure character formula and therefore Demazure's character formula.

Notice that $\{\mathscr{D}_i\}$ do not satisfy the braid relation in general (e.g., when $\mathfrak{g} = A_2$ and $\lambda = 2\Lambda_1 + \Lambda_2$, $\mathscr{D}_1 \mathscr{D}_2 \mathscr{D}_1(\tilde{f}_1 u_\lambda) \neq \mathscr{D}_2 \mathscr{D}_1 \mathscr{D}_2(\tilde{f}_1 u_\lambda)$).

4. Global bases of $U_q^-(\mathfrak{g})$ and $V(\lambda)$. We shall prove in this section that $U_q^-(\mathfrak{g}) u_{w\lambda}$ is also generated by global bases for arbitrary symmetrizable $\mathfrak{g}$. When $\mathfrak{g}$ is finite-dimensional, this follows from the fact that $U_q^+(\mathfrak{g}) u_{w\lambda}$ is generated by global bases. We shall prove here a more precise statement.

PROPOSITION 4.1. *Let $\lambda \in P_+$ and $w \in W$.*

(i) *For any $b \in B(\infty)$, $G(b) u_{w\lambda}$ belongs to $G_\lambda(B(\lambda)) \sqcup \{0\}$.*
(ii) *If $b, b' \in B(\infty)$ satisfy $G(b) u_{w\lambda} = G(b') u_{w\lambda} \neq 0$, then $b = b'$.*

Proof. We shall prove them by induction on $\ell(w)$. The case $\ell(w) = 0$ is already known ([K2]). Assuming $\ell(w) > 0$, let us take i such that $s_i w < w$. Then $f_i u_{w\lambda} = 0$. If $\varepsilon_i(b^*) > 0$, then $G(b) \in U_q^-(\mathfrak{g})f_i$, and hence $G(b)u_{w\lambda} = 0$. Thus we may assume $\varepsilon_i(b^*) = 0$. Set $w' = s_i w$ and $m = \langle h_i, w'\lambda \rangle$. Then $u_{w\lambda} = f_i^{(m)} u_{w'\lambda}$. On the other hand, we have $f_i^{(m)} G(b^*) \equiv G(\tilde{f}_i^m b^*) \bmod f_i^{m+1} U_q^-(\mathfrak{g})$ by [K2], and hence $G(b) f_i^{(m)} \equiv G(\tilde{f}_i^{*m} b) \bmod U_q^-(\mathfrak{g}) f_i^{m+1}$. Since $f_i^{m+1} u_{w'\lambda} = 0$, we obtain

$$G(b)u_{w\lambda} = G(b) f_i^{(m)} u_{w'\lambda} = G(\tilde{f}_i^{*m} b) u_{w'\lambda}.$$

Thus the induction proceeds. Q.E.D.

Let $B^w(\lambda)$ be the set of $b \in B(\lambda)$ such that $G_\lambda(b) \in U_q^-(\mathfrak{g})u_{w\lambda}$. Then we have

(4.1)
$$U_q^-(\mathfrak{g})u_{w\lambda} = \bigoplus_{b \in B^w(\lambda)} \mathbf{Q}(q) G_\lambda(b).$$

The results in §3.1 give the following results on $B^w(\lambda)$.

PROPOSITION 4.2. (i) $\tilde{f}_i B^w(\lambda) \subset B^w(\lambda) \sqcup \{0\}$.
(ii) *If* $s_i w > w$, *then* $B^w(\lambda) = \{\tilde{e}_i^k b; 0 \leqslant k, b \in B^{s_i w}(\lambda)\} \setminus \{0\}$.

The following results are consequences of the results in §3.1

PROPOSITION 4.3.
(i) $\tilde{f}_i B^w(\lambda) \subset B^w(\lambda) \cup \{0\}$.
(ii) *If* $s_i w < w$, *then* $B^{s_i w}(\lambda) = \{\tilde{e}_i^k b; b \in B^w(\lambda), k \geqslant 0\} \setminus \{0\}$.

Although $B^w(\lambda)$ shares other properties of $B_w(\lambda)$, we shall not state them here.

PROPOSITION 4.4. *For* $w_1, w_2 \in W$, *the following conditions are equivalent*:

(i) $B^{w_1}(\lambda) \cap B_{w_2}(\lambda) \neq \phi$;
(ii) $B^{w_1}(\lambda) \ni u_{w_2\lambda}$;
(iii) $B_{w_2}(\lambda) \ni u_{w_1\lambda}$;
(iv) $w_1 \leqslant w_2$.

Proof. By Corollary 3.2.2, (iv) implies $U_q^+(\mathfrak{g})u_{w_2\lambda} \ni u_{w_1\lambda}$, and hence (iv) implies (iii). Let (,) be a nondegenerate symmetric form on $V(\lambda)$ such that ${}^t q^h = q^h$, ${}^t e_i = f_i$. Then (ii) $\Leftrightarrow U_q^-(\mathfrak{g})u_{w_1\lambda} \ni u_{w_2\lambda} \Leftrightarrow (U_q^-(\mathfrak{g})u_{w_1\lambda}, u_{w_2\lambda}) \neq 0 \Leftrightarrow (u_{w_1\lambda}, U_q^+(\mathfrak{g})u_{w_2\lambda}) \neq 0 \Leftrightarrow U_q^+(\mathfrak{g})u_{w_2\lambda} \ni u_{w_1\lambda} \Leftrightarrow$ (iii).

Thus it remains to prove (i) $\Rightarrow$ (iv). Let us prove this by the induction on $\ell(w_2)$. If $w_2 = id$, then it is obvious. Otherwise, take i such that $s_i w_2 < w_2$. For $b \in B^{w_1}(\lambda) \cap B_{w_2}(\lambda)$, write $b = \tilde{f}_i^k b'$ with $\tilde{e}_i b' = 0$. Then $b' \in B_{s_i w_2}(\lambda)$.

(a) Case $s_i w_1 < w_1$. Then $b' \in B^{s_i w_1}(\lambda) \cap B_{s_i w_2}(\lambda)$, and the hypothesis of induction implies $s_i w_1 \leqslant s_i w_2$. Therefore $w_1 \leqslant w_2$.

(b) Case $s_i w_1 > w_1$. Then $b' \in B^{w_1}(\lambda) \cap B_{s_i w_2}(\lambda)$, and the hypothesis of the induction implies $w_1 \leqslant s_i w_2$. Q.E.D.

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