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On Crystal Bases

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ABSTRACT. The crystal base is introduced by the investigation of the quantized universal enveloping algebra at $q = 0$. It carries a combinatorial structure, which permits us a combinatorial study of representations. We explain here this notion and its properties.

Contents

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0. Introduction

The notion of quantized universal enveloping algebra was introduced by Drinfeld [**23**] and Jimbo [**24**] around 1985 in order to explain trigonometric R-matrices in 2-dimensional solvable models in statistical mechanics. Since then, the quantized universal enveloping algebra has been one of the important tools to describe new symmetries in Mathematical Physics and other fields.

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ contains a parameter q, which is a parameter of temperature in the 2-dimensional solvable model, and $q = 0$ corresponds to the absolute temperature zero. This work on crystal bases was motivated by the belief that the phenomena must be simple at the absolute temperature zero. In fact, as we shall explain in this talk, the representations of $U_q(\mathfrak{g})$ have good bases at $q = 0$, which we call crystal bases.

The crystal bases have good properties, such as uniqueness, stability by tensor product, etc. Moreover the $U_q(\mathfrak{g})$-module structure induces a combinatorial structure on the crystal bases, called crystal graph. This permits us to reduce many problems in the representation theory to problems of the combinatorics. For example, the Littlewood-Richardson rule, describing the decomposition of the tensor product of two representations of $\mathfrak{gl}_n$ into irreducible components, may be clearly explained by the use of crystal bases.

Crystal base is a base at $q = 0$ but we can extend this base to the whole q-space to obtain a true base of the representation, which we call global base (see §12). Independently, G.Lusztig introduced the notion of canonical base inspired by the work of Ringel to describe $U_q^-(\mathfrak{g})$ by quivers (see [**27**]). It is shown that canonical base and global base coincide (I.Grojnowski and G.Lusztig [**25**]).

In this talk, we shall explain the notion of crystal bases and its application to the representation theory.

§1–4 treat the general theory of crystal bases. In §1 and §2, we review the quantized universal enveloping algebra and its representation theory. In §3, we introduce the notion of local base, which is a "base at $q = 0$". In §4, we define the crystal base and give its fundamental properties.

In the sections 5, 6 and 10, we give a concrete description of crystal graphs, by Young tableaux for the $\mathfrak{gl}_n$-case in §5, by sequences of the crystal bases of a finite-dimensional representation for the affine case in §6, and by paths in the weight vector space for the general case in §10.

In §7, we introduce the notion of crystal, by abstracting the combinatorial aspects of crystal bases. With this tool in hand, we shall give another way of describing crystal bases in §8. In §8, we also introduce the crystal base of $U_q^-(\mathfrak{g})$, the half of $U_q(\mathfrak{g})$. In §9, we introduce the crystal base of $\tilde{U}_q(\mathfrak{g})$, the algebra obtained from $U_q(\mathfrak{g})$ by replacing its Cartan part with the space of projectors onto the weight spaces. In §12, we extend the crystal base to the whole q-space to obtain the true base of the representation space.

Leitfaden

$$\begin{tikzcd}[column sep=small]
 & & & & 5 & & & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] \arrow[ur] \arrow[dr] & 7 \arrow[r] \arrow[dr] & 8 \arrow[r] & 9 \arrow[r] & 11 \arrow[r] & 12 \\
 & & & & 6 & 10 & & &
\end{tikzcd}$$

1. Representations of $U_q(\mathfrak{sl}_2)$

1.1. Definition. Let us begin with the $\mathfrak{sl}_2$ case. We fix a field K (with an arbitrary characteristic) and $q \in K$. We assume

$$q \neq 0 \text{ and } q^n \neq 1 \text{ for any integer } n \geq 1. \tag{1.1}$$

DEFINITION 1.1. *$U_q(\mathfrak{sl}_2)$ is the K-algebra generated by the symbols e,f,t and the inverse t^{-1} of t with the defining relations:*

$$\begin{aligned} tet^{-1} &= q^2 e, \\ tft^{-1} &= q^{-2} f \quad \textit{and} \\ [e,f] &= (t - t^{-1})/(q - q^{-1}). \end{aligned}$$

When we set $t = q^h$ and q tends to 1, this becomes the universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$, which is the algebra generated by the three elements e, f, h with the defining relations

$$\begin{aligned} [h,e] &= 2e, \\ [h,f] &= -2f \quad \text{and} \\ [e,f] &= h. \end{aligned}$$

1.2. Hopf algebra structure. For two left $U_q(\mathfrak{sl}_2)$-modules M_1 and M_2, $M_1 \otimes_K M_2$ has also a structure of $U_q(\mathfrak{sl}_2)$-module by the following action:

$$\begin{aligned} t(u_1 \otimes u_2) &= tu_1 \otimes tu_2, \\ e(u_1 \otimes u_2) &= eu_1 \otimes t^{-1}u_2 + u_1 \otimes eu_2, \\ f(u_1 \otimes u_2) &= fu_1 \otimes u_2 + tu_1 \otimes fu_2. \end{aligned}$$

This can be explained also by the coproduct Δ. Let us endow the ring structure on $U_q(\mathfrak{sl}_2) \otimes_K U_q(\mathfrak{sl}_2)$ by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. Then

$$\begin{aligned} \Delta t &= t \otimes t, \\ \Delta e &= e \otimes t^{-1} + 1 \otimes e, \\ \Delta f &= f \otimes 1 + t \otimes f \end{aligned}$$

extends to a ring homomorphism

$$\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes_K U_q(\mathfrak{sl}_2).$$

Since $M_1 \otimes M_2$ has a structure of left $U_q(\mathfrak{sl}_2) \otimes_K U_q(\mathfrak{sl}_2)$-module, this has also a structure of $U_q(\mathfrak{sl}_2)$-module via Δ. This $U_q(\mathfrak{sl}_2)$-module structure coincides with the one defined earlier.

For three $U_q(\mathfrak{sl}_2)$-modules M_1, M_2 and M_3, the K-linear isomorphism

$$(M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3)$$

given by $(u_1 \otimes u_2) \otimes u_3 \mapsto u_1 \otimes (u_2 \otimes u_3)$ is a $U_q(\mathfrak{sl}_2)$-linear isomorphism. This fact is equivalent to the commutativity of the diagram:

$$\begin{tikzcd}[column sep=small]
 & U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \arrow[dr, "1\otimes\Delta"] & \\
U_q(\mathfrak{sl}_2) \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)\,. \\
 & U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \arrow[ur, "\Delta\otimes 1"'] &
\end{tikzcd}$$

This property is refereed to the coassociativity of Δ.

Remark 1.1. The coproduct is not cocommutative. That is, the homomorphism $M_1 \otimes M_2 \to M_2 \otimes M_1$ given by $u_1 \otimes u_2 \mapsto u_2 \otimes u_1$ is not $U_q(\mathfrak{sl}_2)$-linear. However, for finite-dimensional representations, there is a non-trivial $U_q(\mathfrak{sl}_2)$-linear isomorphism $M_1 \otimes M_2 \xrightarrow{\sim} M_2 \otimes M_1$, called R-matrix, and it satisfies the Yang-Baxter equation. This is the motivation for Drinfeld and Jimbo introducing the quantized universal enveloping algebras. However, we don't go in this direction.

Remark 1.2. There are several ways of defining coproducts. They are transformed by exchanging the first and the second factors, or automorphisms of the algebra. We used this coproduct in order that the crystal base is stable by the tensor product.

1.3. Useful formulas. In the explicit calculations, we need formulas on q-integers, etc. We shall give some of those formulas.

Example 1.1. Let x, y be elements (of some algebra over K) satisfying the commutation relation:

$$xy = q^2 yx.$$

Then we have the following q-analogue of the binomial formula

$$(1.2) \qquad \begin{aligned} (x+y)^n &= \sum_{i+j=n} q^{-ij} \begin{bmatrix} n \\ i \end{bmatrix} x^i y^j \\ &= \sum_{i+j=n} q^{ij} \begin{bmatrix} n \\ i \end{bmatrix} y^j x^i. \end{aligned}$$

Here we use the notations:

(1.3)
$$\begin{aligned} [n] &= \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{for } n \in \mathbb{Z}\,, \\ [n]! &= \prod_{i=1}^{n} [i]! \quad \text{for } n \geq 0\,, \\ \begin{bmatrix} n \\ i \end{bmatrix} &= \frac{[n]!}{[i]![n-i]!} \quad \text{for } 0 \leq i \leq n\,. \end{aligned}$$

We understand $[0]! = 1$. They are called q-integer, q-factorial, q-binomial coefficient, respectively.

EXAMPLE 1.2. In $U_q(\mathfrak{sl}_2)$, we have

$$e^{(n)} f^{(m)} = \sum_{0 \leq k \leq n,m} f^{(m-k)} e^{(n-k)} \left\{ \begin{matrix} q^{n-m} t \\ k \end{matrix} \right\}.$$

Here $e^{(n)} = e^n/[n]!$, $f^{(n)} = f^n/[n]!$,

$$\begin{aligned} \left\{ \begin{matrix} x \\ k \end{matrix} \right\} &= \frac{\{x\}\{q^{-1}x\} \cdots \{q^{1-k}x\}}{[k]!}, \\ \{x\} &= (x - x^{-1})/(q - q^{-1}). \end{aligned}$$

Hence $\left\{ \begin{smallmatrix} q^n \\ k \end{smallmatrix} \right\} = \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ for $0 \leq k \leq n$.

EXAMPLE 1.3. For $m, m', n, n' \in \mathbb{Z}$, we have

$$[m][m'] - [n][n'] = [m-n] \left[\frac{mm' - nn'}{m-n} \right]$$

if $m - n = \pm(m' - n')$.

1.4. 1-dimensional representations. Let $V = Ku$ be a 1-dimensional left $U_q(\mathfrak{sl}_2)$-module. Then we can write $tu = cu$ for $c \in K \backslash \{0\}$. Then $tv = cv$ for any $v \in V$. Hence $ceu = teu$. Since $te = q^2 et$, we have $teu = q^2 etu = cq^2 eu$. Thus we obtain $c(q^2 - 1)eu = 0$. Since we assumed $q^2 \neq 1$, $eu = 0$. Similarly $fu = 0$. Then

$$0 = [e, f]u = \frac{t - t^{-1}}{q - q^{-1}} u = \frac{c - c^{-1}}{q - q^{-1}} u$$

implies $c^2 = 1$. Hence $c = 1$ or -1. In fact we have two kinds of 1-dimensional representations $K = K \cdot 1$ with $t \cdot 1 = 1$, $e \cdot 1 = f \cdot 1 = 0$, and $V_- = K \cdot 1_-$ with $t \cdot 1_- = -1_-$, $e \cdot 1_- = f \cdot 1_- = 0$. If K is of characteristic 2, $K \cong V_-$.

1.5. Finite-dimensional representations. For an integer $l \geq 0$, we can construct an $(l+1)$-dimensional representation $V(l)$ as follows. The vector space $V(l)$ has a base $\{u_k^{(l)}\}_{0\leq k\leq l}$ with the action of $U_q(\mathfrak{sl}_2)$ given by

$$\begin{cases} tu_k^{(l)} = q^{l-2k}u_k^{(l)}, \\ eu_k^{(l)} = [l-k+1]u_{k-1}^{(l)}, \\ fu_k^{(l)} = [k+1]u_{k+1}^{(l)}. \end{cases}$$

Here we understand $u_k^{(l)} = 0$ unless $0 \leq k \leq l$. We can easily check that this gives a $U_q(\mathfrak{sl}_2)$-module structure on $V(l)$. For example, let us check $[e,f] = \{t\}$. We have $efu_k^{(l)} = [l-k][k+1]u_k^{(l)}$, $feu_k^{(l)} = [l-k+1][k]u_k^{(l)}$ and hence, by applying Example 1.3, we obtain $[e,f]u_k^{(l)} = \big([l-k][k+1]-[l-k+1][k]\big)u_k^{(l)} = [l-2k]u_k^{(l)} = \{t\}u_k^{(l)}$.

The bases $\{u_k^{(l)}\}$ are chosen so that we have

$$\begin{aligned} u_k^{(l)} &= f^{(k)}u_0^{(l)} \\ &= e^{(l-k)}u_l^{(l)}. \end{aligned}$$

We visualize this as

$$u_0^{(l)} \to u_1^{(l)} \to \cdots \to u_l^{(l)}.$$

Arrows indicate that f (resp. e) sends the vector at the source (resp. the target) of the arrow to the vector at the target (resp. the source), up to constant multiple.

The following theorem is well-known.

THEOREM 1.1.

(1.4) *Any finite-dimensional $U_q(\mathfrak{sl}_2)$-module is completely reducible.*

(1.5) *Any irreducible $(l+1)$-dimensional $U_q(\mathfrak{sl}_2)$-module is isomorphic to $V(l)$ or $V(l)\otimes V_- \cong V_-\otimes V(l)$.*

REMARK 1.3. The eigenvalues of t on $V(l)$ have the form q^k while the ones on $V(l)\otimes V_-$ have the form $-q^k$.

By this theorem and the remark above, we have the following result.

COROLLARY 1.1. *Let M be a finite-dimensional $U_q(\mathfrak{sl}_2)$-module. Then the following two conditions are equivalent.*

(1.6) $M = \oplus_{k\in\mathbb{Z}}M_k$ *where* $M_k = \{u \in M; tu = q^k u\}$.

(1.7) M *is isomorphic to a direct sum of* $V(l)$*'s.*

For such a module M, we define its character $\mathrm{ch}(M)$ by

$$\mathrm{ch}(M)(x) = \sum_{k\in\mathbb{Z}} \dim M_k x^k \in \mathbb{Z}[x,x^{-1}].$$

Then $\operatorname{ch}\big(V(l)\big)=(x^{l+1}-x^{-l-1})/(x-x^{-1})$ and $\{\operatorname{ch}V(l)\}_{l\ge 0}$ is linearly independent. If $M\cong\oplus V(l_j)$, then $\operatorname{ch}(M)=\Sigma\operatorname{ch}\big(V(l_j)\big)$, and hence $\{l_j\}$ is uniquely determined by the character of M. Comparing the characters, we have the following q-analogue of the Clebsch-Gordan rule:

$$V(l_1)\otimes V(l_2)\cong \bigoplus_{\substack{|l_1-l_2|\le k\le l_1+l_2\\ l_1+l_2\equiv k \bmod 2}} V(k). \tag{1.8}$$

2. Quantized universal enveloping algebras

2.1. Definition. In the previous section, we defined $U_q(\mathfrak{sl}_2)$. In this section, we shall define $U_q(\mathfrak{g})$ for Lie algebras $\mathfrak{g}$ other than $\mathfrak{sl}_2$. We can in fact define $U_q(\mathfrak{g})$ for an arbitrary symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$.

Assume that we are given the following data.

P : a free $\mathbb{Z}$-module (called a weight lattice)

I : an index set (for simple roots)

$\alpha_i\in P$ for $i\in I$ (called a simple root)

$h_i\in P^*=\operatorname{Hom}_{\mathbb{Z}}(P,\mathbb{Z})$ (called a simple coroot)

$(\,\cdot\,,\,\cdot\,):P\times P\to\mathbb{Q}$ a bilinear symmetric form.

We shall denote by $\langle\,\cdot\,,\,\cdot\,\rangle : P^*\times P\to\mathbb{Z}$ the canonical pairing.

The data above are assumed to satisfy the following axioms.

$$(\alpha_i,\alpha_i)\in 2\mathbb{Z}_{>0}\quad\text{for } i\in I\quad\text{(cf. Def.2.1(iv))}. \tag{2.1}$$

$$\langle h_i,\lambda\rangle=\frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}\quad\text{for } i\in I \text{ and } \lambda\in P. \tag{2.2}$$

$$(\alpha_i,\alpha_j)\le 0\quad\text{for } i,j\in I \text{ with } i\ne j. \tag{2.3}$$

In particular, (α_i,α_j) is a non-positive integer for $i\ne j$.

Now as in §1, let K be a field and $q\in K$ a non-zero element with (1.1).

DEFINITION 2.1. *The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the algebra over K generated by the symbols e_i, f_i $(i\in I)$ and $q(h)$ $(h\in P^*)$ with the following defining relations.*

(i) $q(h)=1$ *for* $h=0$.

(ii) $q(h_1)q(h_2)=q(h_1+h_2)$ *for* $h_1,h_2\in P^*$.

(iii) *For any* $i\in I$ *and* $h\in P^*$,

$$\begin{aligned} q(h)e_iq(h)^{-1} &= q^{\langle h,\alpha_i\rangle}e_i \text{ and}\\ q(h)f_iq(h)^{-1} &= q^{-\langle h,\alpha_i\rangle}f_i \end{aligned}$$

(iv) $[e_i,f_j]=\delta_{ij}\frac{t_i-t_i^{-1}}{q_i-q_i^{-1}}$ *for* $i,j\in I$. *Here* $q_i=q^{(\alpha_i,\alpha_i)/2}$ *and* $t_i=q(\frac{(\alpha_i,\alpha_i)}{2}h_i)$.

(v) *(Serre relation) For $i \neq j$,*

$$\sum_{k=0}^{b}(-1)^k e_i^{(k)} e_j e_i^{(b-k)} = \sum_{k=0}^{b}(-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - \langle h_i, \alpha_j \rangle$ and

$$e_i^{(k)} = e_i^k/[k]_i! \ , \qquad f_i^{(k)} = f_i^k/[k]_i! \ ,$$
$$[k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1}) \ , \qquad [k]_i! = [1]_i \cdots [k]_i \, .$$

Remark 2.1. As is easily seen, we have

$$\begin{aligned} t_i e_j t_i^{-1} &= q^{(\alpha_i,\alpha_j)} e_j \quad \text{and} \\ t_i f_j t_i^{-1} &= q^{-(\alpha_i,\alpha_j)} f_j. \end{aligned}$$

Note that we have $q^{(\alpha_i,\alpha_j)} = q_i^{\langle h_i,\alpha_j \rangle}$.

Setting $q(h) = q^h$ and letting q tend to 1, $U_q(\mathfrak{g})$ becomes the universal enveloping algebra $U(\mathfrak{g})$ of the corresponding Kac-Moody Lie algebra $\mathfrak{g}$ generated by the abelian subalgebra $K \otimes_{\mathbb{Z}} P^*$ and $\{e_i, f_i; i \in I\}$ with the defining relations

$$\begin{aligned} [h, e_i] &= \langle h, \alpha_i \rangle e_i \\ [h, f_i] &= -\langle h, \alpha_i \rangle f_i \quad \text{for } h \in K \otimes_{\mathbb{Z}} P^* \text{ and} \\ [e_i, f_j] &= \delta_{ij} h_i \\ \mathrm{ad}(e_i)^{1-\langle h_i,\alpha_j \rangle} e_j &= \mathrm{ad}(f_i)^{1-\langle h_i,\alpha_j \rangle} f_j = 0 \, . \end{aligned}$$

The subalgebra of $U_q(\mathfrak{g})$ generated by $\{q(h); h \in P^*\}$ is isomorphic to the group algebra $K[P^*]$. Let us denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra generated by the e_i's (resp. the f_i's). Then we have an isomorphism of K-vector spaces

$$U_q(\mathfrak{g}) \xleftarrow{\sim} U_q^+(\mathfrak{g}) \otimes K[P^*] \otimes U_q^-(\mathfrak{g})$$

by $a \otimes q(h) \otimes b \mapsto a q(h) b$ for $a \in U_q^+(\mathfrak{g}), b \in U_q^-(\mathfrak{g}), h \in P^*$.

In the sequel, we assume for the sake of simplicity

(2.4) $\qquad \{\alpha_i\}_{i \in I}$ and $\{h_i\}_{i \in I}$ are linearly independent.

However almost all statements in this paper still hold without this assumption.

2.2. The Hopf algebra structure. As in the case of $U_q(\mathfrak{sl}_2), U_q(\mathfrak{g})$ has a Hopf algebra structure. We define the coproduct as the ring homomorphism

$$\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$$

by

$$\begin{aligned} \Delta q(h) &= q(h) \otimes q(h) \, , \\ \Delta e_i &= e_i \otimes t_i^{-1} + 1 \otimes e_i \, , \\ \Delta f_i &= f_i \otimes 1 + t_i \otimes f_i \, . \end{aligned}$$

We can easily see that Δ is well-defined. By Δ, we can define the $U_q(\mathfrak{g})$-module structure on the tensor product of two $U_q(\mathfrak{g})$-modules. This coproduct is coassociative. Hence the category of $U_q(\mathfrak{g})$-modules is a tensor category (see §7.1). See Remark 1.2.

2.3. Integrable modules. Let us denote by $U_q(\mathfrak{g})_i$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, t_i and t_i^{-1}. Then $U_q(\mathfrak{g})_i$ is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$. Hence we can say that $U_q(\mathfrak{g})$ is made up of several quantized $\mathfrak{sl}_2$.

DEFINITION 2.2. *A left $U_q(\mathfrak{g})$-module M is called integrable if it satisfies*

(2.5) $M = \bigoplus_{\lambda \in P} M_\lambda$ *where* $M_\lambda = \{u \in M; q(h)u = q^{\langle h,\lambda\rangle}u$ *for any* $h \in P^*\}$.

(2.6) *For any $i \in I, M$ is a union of finite-dimensional $U_q(\mathfrak{g})_i$-submodules.*

Any $U_q(\mathfrak{g})$-module M satisfying only (2.6) has a canonical decomposition

$$M = \bigoplus_{\chi} M^\chi \otimes V^\chi$$

provided that $\{\lambda \in P\,; \langle h_i, \lambda\rangle = 0$ for every $i \in I\} = 0$. Here $\{V^\chi\}$ is the set of 1-dimensional $U_q(\mathfrak{g})$-modules, and M^χ are integrable $U_q(\mathfrak{g})$-modules. Therefore, we shall treat only integrable modules in the sequel.

REMARK 2.2. Similar arguments to the $\mathfrak{sl}_2$-case show that the one dimensional representations of $U_q(\mathfrak{g})$ are parameterized by the characters $\chi : P^* \to K^\times$ such that $\chi((\alpha_i,\alpha_i)h_i) = 1$. The corresponding module $V^\chi = K1_\chi$ is given by :

$$e_i 1_\chi = f_i 1_\chi = 0, \quad q(h)1_\chi = \chi(h)1_\chi.$$

2.4. Representations of $U_q(\mathfrak{g})$. In general, an integrable $U_q(\mathfrak{g})$-module is not completely reducible. However, as in the Kac-Moody Lie algebra case, there exists a family of completely reducible modules.

DEFINITION 2.3. *Let $\mathcal{O}_{int}(\mathfrak{g})$ denote the category of integrable $U_q(\mathfrak{g})$-modules M such that*

$$\text{(2.7)}\qquad \begin{aligned}&\textit{For any } u \in M, \textit{ there exists } l \geq 1\\ &\textit{such that } e_{i_1}\cdots e_{i_l}u = 0 \textit{ for any } i_1.\cdots, i_l \in I.\end{aligned}$$

The condition (2.7) is (under the condition (2.4)) equivalent to

$$\text{(2.8)}\qquad \dim U_q^+(\mathfrak{g})u < \infty \quad \text{for any } u \in M\ .$$

Let us set

$$\text{(2.9)}\qquad P_+ = \{\lambda \in P; < h_i, \lambda > \geq 0 \text{ for any } i\}$$

and we call an element of P_+ a *dominant* integral weight. For $\lambda \in P_+$, let us denote by $V(\lambda)$ the $U_q(\mathfrak{g})$-module generated by u_λ with the defining relation:

$$\begin{aligned} &q(h)u_\lambda = q^{<h,\lambda>}u_\lambda\,,\\ &e_i u_\lambda = 0\,,\\ &f_i^{1+<h_i,\lambda>}u_\lambda = 0\,. \end{aligned}$$

THEOREM 2.1 (G.LUSZTIG[**26**]). *Assume that K is of characteristic* 0. *Let* $\lambda \in P_+$.

(i) $V(\lambda)_\lambda = \{u \in V(\lambda); e_i u = 0$ *for any* $i\} = Ku_\lambda \neq 0$.
(ii) $V(\lambda)$ *is an irreducible integrable* $U_q(\mathfrak{g})$*-module in* $\mathcal{O}_{int}(\mathfrak{g})$.
(iii) *Any* $U_q(\mathfrak{g})$*-module in* $\mathcal{O}_{int}(\mathfrak{g})$ *is completely reducible.*
(iv) *Any irreducible* $U_q(\mathfrak{g})$*-module in* $\mathcal{O}_{int}(\mathfrak{g})$ *is isomorphic to* $V(\lambda)$ *for some* $\lambda \in P_+$.

We conjecture that this theorem is true for an arbitrary characteristic (under the condition (1.1), cf. Problem 2 in the last section). I think that this is known for finite-dimensional $\mathfrak{g}$.

2.5. Motivation. Let M be an integrable $U_q(\mathfrak{g})$-module. We shall ask if M has a good base in the following sense. For any i, M is completely reducible as a $U_q(\mathfrak{g})_i$-module and there exists a (not necessarily unique) $U_q(\mathfrak{g})_i$-linear isomorphism

$$M \simeq \bigoplus_j V(l_j)\,. \tag{2.10}$$

Here $V(l_j)$ is the $(1 + l_j)$-dimensional irreducible $U_q(\mathfrak{g})_i$-module given in §1.5. Each $V(l_j)$ has a base $\{u_k^{(l_j)}\}_{0\le k\le l_j}$.

Question Is there a base B of the K-vector space M (independent of i) such that for any i, there exists a $U_q(\mathfrak{g})_i$-linear isomorphism $M \simeq \bigoplus_j V(l_j)$ by which B is sent to the base $\{u_k^{(l_j)}\}$ of $\bigoplus_j V(l_j)$?

Of course, it is not true even at the classical limit $q = 1$ (i.e. for a $U(\mathfrak{g})$-module). However we shall see that it is true at $q = 0$.

3. Local bases

3.1. Definition. In order to give a precise meaning to the question above at $q = 0$, we shall introduce the notion of local base.

Let us take a field k and let $K = k(q)$ be the field of rational functions in a variable q with coefficients in k. Let V be a K-vector space. For a subring C of K, a C-lattice of V is, by definition, a C-submodule L of V such that $V \cong K \otimes_C L$ (or equivalently, V is generated by L as a K-vector space provided that K is a quotient field of C).

Let us denote by A the subring of K consisting of rational functions $f(q)$ in K without a pole at $q = 0$.
Hence by the evaluation map $f(q) \mapsto f(0)$, we have an isomorphism

$$A/qA \xrightarrow{\sim} k.$$

DEFINITION 3.1. *Let V be a K-vector space. A local base of V at $q = 0$ is a pair (L, B) where*

(3.1) *L is an A-lattice of V that is a free A-module.*

(3.2) *B is a base of the k-vector space L/qL.*

Similarly to a local base at $q = 0$, we can define a notion of a local base at any point of $\mathbb{P}^1 = \mathrm{Spec}(k[q]) \cup \mathrm{Spec}(k[q^{-1}])$. Since we use only local bases at $q = 0$ in this paper, we simply say local base instead of saying local base at $q = 0$.

EXAMPLE 3.1. To a base B of the K-vector space V, we can associate a local base (L, B), where L is the A-module generated by B, and we identify B with the image of B by $L \to L/qL$. We call it the local base associated with B. Note that any local base is associated with some base of the K-vector space V.

Let B and B' be bases of the K-vector space V. Let us write $b = \sum_{b' \in B'} f_{bb'}(q) b'$ and $b' = \sum_{b \in B} g_{b'b}(q) b$. Then the local bases associated with B and B' are equal if and only if there exists a bijection $\varphi : B \to B'$ satisfying the following equivalent two conditions.

(i) $f_{bb'}(q) \in A$ and $f_{bb'}(0) = \delta_{\varphi(b), b'}$,

(ii) $g_{b'b}(q) \in A$ and $g_{b'b}(0) = \delta_{\varphi(b), b'}$.

Hence we may regard a local base as an equivalence class of bases with respect to the equivalence relation above.

3.2. Direct sums and tensor products. Let $\{V_j\}$ be a family of K-vector spaces and let (L_j, B_j) be a local base of V_j. Then $L = \oplus_j L_j \subset \oplus_j V_j$ is a free A-lattice of $\oplus_j V_j$ and $B = \bigsqcup_j B_j \subset \oplus (L_j/qL_j) \cong L/qL$ is a base of L/qL. Hence (L, B) is a local base of $\oplus_j V_j$. We call it the direct sum of $\{(L_j, B_j)\}_j$ and denote it by $\oplus_j (L_j, B_j)$.

Let V_1 and V_2 be two K-vector spaces and (L_j, B_j) a local base of V_j $(j = 1, 2)$. Then $L = L_1 \otimes_A L_2 \subset V_1 \otimes_K V_2$ is a free A-lattice of $V_1 \otimes_K V_2$ and

$$\begin{aligned} B = B_1 \otimes B_2 &= \{b_1 \otimes b_2; b_1 \in B_1, b_2 \in B_2\} \\ &\subset (L_1/qL_1) \otimes_k (L_2/qL_2) \cong L/qL \end{aligned}$$

is a base of L/qL. Hence (L, B) is a local base of $V_1 \otimes_K V_2$. We call it the tensor product of (L_1, B_1) and (L_2, B_2) and denote it by $(L_1, B_1) \otimes (L_2, B_2)$.

4. Crystal bases

4.1. Definition. In the sequel, we fix a field k of characteristic 0 (in order to have Theorem 2.1), and set $K = k(q)$. Hence, in our consideration, $U_q(\mathfrak{g})$ is a K-algebra.

Let us define crystal base as a solution to the question in §2.5 at $q = 0$. Let us take an integrable $U_q(\mathfrak{g})$-module M and let $M = \bigoplus_{\lambda \in P} M_\lambda$ be its weight space decomposition.

DEFINITION 4.1. *A crystal base of M is a local base (L, B) of the K-vector space M satisfying the following conditions.*

(4.1) *There is a local base (L_λ, B_λ) of M_λ such that*
$$(L, B) = \bigoplus_{\lambda \in P} (L_\lambda, B_\lambda).$$

(4.2) *For any $i \in I$, there exists a $U_q(\mathfrak{g})_i$-linear isomorphism*
$$\xi : M \xrightarrow{\sim} \bigoplus_j V(l_j)$$
by which (L, B) is sent to the local base $\{u_k^{(l_j)}\}_{0 \le k \le l_j}$ of $\bigoplus_j V(l_j)$.

4.2. Crystal graph. A crystal base carries a combinatorial structure induced by the $U_q(\mathfrak{g})$-module structure.

Let (L, B) be a crystal base of an integrable $U_q(\mathfrak{g})$-module M. Let us take $i \in I$. Then there is an isomorphism ξ as in (4.2). For $b \in B$, $\xi(b) = u_k^{(l_j)}$ for some j and $0 \le k \le l_j$. If $0 \le k < l_j$, let $\tilde{f}_i b$ be the element of B such that $\xi(\tilde{f}_i b) = u_{k+1}^{(l_j)}$. We set $\tilde{f}_i b = 0$ if $k = l_j$. Similarly, for $0 < k \le l_j$, let $\tilde{e}_i b$ be the element of B such that $\xi(\tilde{e}_i b) = u_{k-1}^{(l_j)}$. We set $\tilde{e}_i b = 0$ if $k = 0$. Thus we obtain a map $\tilde{f}_i : B \to B \sqcup \{0\}$ and $\tilde{e}_i : B \to B \sqcup \{0\}$. We can easily see that this definition does not depend on the choice of $\xi : M \xrightarrow{\sim} \oplus V(l_j)$.

For $i \in I$ we shall join $b, b' \in B$ by an arrow named by i if $b' = \tilde{f}_i b$ ($\Leftrightarrow b = \tilde{e}_i b'$).

$$b \xrightarrow{i} b'$$

Thus we obtain a colored (by I) oriented graph whose set of vertices is B. We shall call it the *crystal graph*. Note that $\tilde{e}_i$ and $\tilde{f}_i$ are recovered by this graph structure.

We call an i-string a subset of B corresponding to $u_0^{(l_j)} \xrightarrow{i} \cdots \xrightarrow{i} u_{l_j}^{(l_j)}$ by $\xi : M \xrightarrow{\sim} \oplus V(l_j)$. Hence an i-string is a connected component of the graph only with the i-arrows. Therefore B is the disjoint union of i-strings. Hence the crystal graph has a simple structure if we consider only i-arrows. However combining all the i-arrows, the crystal graph has a rich structure (see Example 5.5).

For $b \in B$, let us set

$$\begin{aligned} \varepsilon_i(b) &= \max\{n \ge 0; \tilde{e}_i^n b \ne 0\}, \\ \varphi_i(b) &= \max\{n \ge 0; \tilde{f}_i^n b \ne 0\}. \end{aligned}$$

These numbers are visualized as follows:

$$\overbrace{\circ \xrightarrow{i} \cdots\cdots \xrightarrow{i}}^{\varepsilon_i(b)} \overset{b}{\circ} \overbrace{\xrightarrow{i} \cdots\cdots \xrightarrow{i}}^{\varphi_i(b)} \circ\,.$$

For $b \in B_\lambda$ ($\lambda \in P$), we set $\mathrm{wt}(b) = \lambda$ and call it the weight of b.
If $\xi(b) = u_k^{(l_j)}$, then $\varepsilon_i(b) = k$, $\psi_i(b) = l - k$ and $\langle h_i, \mathrm{wt}(b)\rangle = l - 2k$. Hence we have the following properties.

(4.3) $$\langle h_i, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b)\,.$$

(4.4) $\varphi_i(b) + \varepsilon_i(b)$ is the length of the i-string containing b.

REMARK 4.1. Although e_i and f_j commute if $i \neq j$, $\tilde{e}_i$ and $\tilde{f}_j$ do not commute in general (when $(\alpha_i, \alpha_j) = 0$, they commute).

4.3. Elementary Properties of Crystal bases. The following proposition is obvious from the definition.

PROPOSITION 4.1. *Let $\{M_j\}$ be a family of integrable $U_q(\mathfrak{g})$-modules and let (L_j, B_j) be a crystal base of M_j. Then $\oplus(L_j, B_j)$ is a crystal base of $\oplus M_j$.*

One of the most remarkable properties of crystal base is the following stability by tensor product.

THEOREM 4.1 ([**1**]). *Let M_j be an integrable $U_q(\mathfrak{g})$-module and (L_j, B_j) a crystal base of M_j ($j = 1, 2$).*

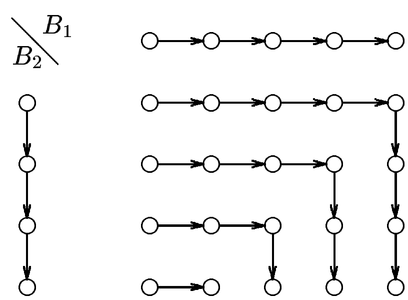
(i) *$(L_1, B_1) \otimes (L_2, B_2)$ is a crystal base of $M_1 \otimes_K M_2$.*

(ii) *For $b_j \in B_j$ ($j = 1, 2$) and $i \in I$ we have*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

We can easily reduce this theorem to the $U_q(\mathfrak{sl}_2)$-case, and then to the case $M_1 = V(l_1)$ and $M_2 = V(l_2)$. In this case, we can check the theorem by a direct calculation. The last property (ii) can be visualized as follows.



The rules in (ii) imply

$$\begin{aligned} \varepsilon_i(b_1 \otimes b_2) &= \max\left(\varepsilon_i(b_1)\,,\, \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle\right), \\ \varphi_i(b_1 \otimes b_2) &= \max\left(\varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle\,,\, \varphi_i(b_2)\right). \end{aligned} \tag{4.5}$$

4.4. Crystal bases in $\mathcal{O}_{int}$. For $\lambda \in P_+$, let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight vector u_λ of weight λ (see §2.4). Then we can show the following existence theorem.

THEOREM 4.2 ([**3**]). (i) *$V(\lambda)$ has a unique crystal base $(L(\lambda), B(\lambda))$ such that $(L(\lambda)_\lambda, B(\lambda)_\lambda) = \{u_\lambda\}$.*
(ii) $B(\lambda) = \{\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_l}^{a_l} u_\lambda : l \geq 0, i_1, \ldots, i_l \in I,\ a_1, \ldots, a_l > 0\} \backslash \{0\}$.

The proof of this theorem is rather involved. It is proved in fact simultaneously with Theorem 8.1 by the induction on the weight. Theorem 4.1 plays a key role in the course of the proof.

The following theorem guarantees a uniqueness of crystal base up to an isomorphism.

THEOREM 4.3 ([**3**]). *Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{int}(\mathfrak{g})$, and let (L, B) be a crystal base of M. Then there exists a $U_q(\mathfrak{g})$-linear isomorphism*

$$M \xrightarrow{\sim} \bigoplus_j V(\lambda_j)$$

by which (L, B) is isomorphic to $\bigoplus_j (L(\lambda_j), B(\lambda_j))$.

4.5. Irreducible decomposition. Let us investigate some consequences of the theorems in the previous section.

First remark that $B(\lambda)$ is connected for $\lambda \in P_+$. It means that the crystal graph is connected by forgetting colors and the directions of arrows. In fact, Theorem 4.2 (ii) shows that every element of $B(\lambda)$ is connected with u_λ. Hence if $(L, B) \cong \oplus(L(\lambda_j), B(\lambda_j))$, then $B \cong \sqcup B(\lambda_j)$ is the connected component decomposition of B.

We shall say that an element b of a crystal base B is a *highest weight vector* if $\tilde{e}_i b = 0$ for any i. The property (ii) of Theorem 4.2 shows that $B(\lambda)$ has a unique highest weight vector u_λ. Thus we have schematically

$$\begin{array}{c} \text{The decomposition of } M \text{ into irreducible components} \\ \updownarrow \\ \text{The decomposition of } B \text{ into connected components} \\ \updownarrow \\ \text{The highest weight vectors of } B. \end{array} \tag{4.6}$$

As an example, let us consider the decomposition of $V(\lambda) \otimes V(\mu)$ into irreducible components.

We remark that the following lemma follows from (4.5).

LEMMA 4.1. *Let b_j be an element of a crystal base B_j $(j = 1, 2)$. Then $\tilde{e}_i(b_1 \otimes b_2) = 0$ if and only if $\tilde{e}_i b_1 = 0$ and $\varepsilon_i(b_2) \le \varphi_i(b_1)$.*

Now, let us investigate the condition for $b \otimes b'$ to be a highest weight vector for $b \in B(\lambda)$ and $b' \in B(\mu)$. If $b \otimes b'$ is a highest weight vector, then b is a highest weight vector by the lemma above, and hence $b = u_\lambda$. The property (4.3) implies $\varphi_i(u_\lambda) = \varepsilon_i(u_\lambda) + \langle h_i, \mathrm{wt}(u_\lambda)\rangle = \langle h_i, \lambda\rangle$. Hence $b \otimes b'$ is a highest weight vector of $B(\lambda) \otimes B(\mu)$ if and only if $b = u_\lambda$ and $\varepsilon_i(b') \le \langle h_i, \lambda\rangle$. Since $\mathrm{wt}(b \otimes b') = \lambda + \mathrm{wt}(b')$, we obtain the following result.

PROPOSITION 4.2. *For $\lambda, \mu \in P_+$,*

$$V(\lambda) \otimes V(\mu) \cong \oplus V(\lambda + \mathrm{wt}(b)).$$

Here the direct sum ranges over $b \in B(\mu)$ such that $\varepsilon_i(b) \le \langle h_i, \lambda\rangle$ for every $i \in I$.

Therefore if we know the crystal graph of $B(\mu)$, we can calculate the decomposition of $V(\lambda) \otimes V(\mu)$ into irreducible components.

4.6. Restriction to subalgebras. By crystal bases, we can also describe the irreducible decomposition of $V(\lambda)$ regarded as a representation of a subalgebra of $U_q(\mathfrak{g})$.

Let J be a subset of I. Let $U_q(\mathfrak{g}_J)$ be the quantized universal enveloping algebra associated with the data $\{P, \{\alpha_i\}_{i\in J}, \{h_i\}_{i\in J}\}$. Then $U_q(\mathfrak{g}_J)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i, f_i; i \in J\}$ and $\{q(h); h \in P^*\}$. A crystal base of an integrable $U_q(\mathfrak{g})$-module may be regarded as a crystal base of the associated $U_q(\mathfrak{g}_J)$-module M. Let P_+^J denote the set $\{\lambda \in P; \langle h_i, \lambda\rangle \ge 0$ for any $i \in J\}$ of integral dominant weights with respect to $\mathfrak{g}_J$, and for any $\lambda \in P_+^J$ let us denote by $V_J(\lambda)$ the irreducible integrable $U_q(\mathfrak{g}_J)$-module with highest weight λ. Then (4.6) implies the following result.

PROPOSITION 4.3. *For $\lambda \in P_+$, we have a $U_q(\mathfrak{g}_J)$-linear isomorphism*

$$V(\lambda) \cong \oplus V_J(\mathrm{wt}(b))\,.$$

Here the direct sum ranges over $b \in B(\lambda)$ such that $\varepsilon_i(b) = 0$ for any $i \in J$.

REMARK 4.2. The above proposition and Proposition 4.2 still hold at the classical limit $q = 0$ (i.e. for $\mathfrak{g}$-modules).

5. Young tableaux and crystal bases

5.1. $U_q(\mathfrak{gl}_n)$**.** As seen in the preceding section, if we know the crystal graph of $B(\lambda)$, we can for example compute the decomposition of the tensor product into irreducible components. Of course, this program is not performed before we know the crystal graph. In the case of $\mathfrak{gl}_n$, we can describe the crystal graphs explicitly by using the Young tableaux. In this section we shall explain this relation of crystal base and Young tableaux.

Let us define first $U_q(\mathfrak{gl}_n)$. As a weight lattice, we take $P = \oplus_{i=1}^n \mathbb{Z}\varepsilon_i$. We set $I = \{1, 2, \cdots, n-1\}$ and take $\varepsilon_i - \varepsilon_{i+1}$ as a simple root α_i $(i \in I)$. Let us define the symmetric bilinear form on P by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. We give the coroots h_i $(i \in I)$ by $< h_i, \lambda >= (\varepsilon_i - \varepsilon_{i+1}, \lambda)$. Let us denote by $U_q(\mathfrak{gl}_n)$ the quantized universal enveloping algebra associated with these data.

Set $\Lambda_j = \varepsilon_1 + \cdots + \varepsilon_j$ $(1 \le j \le n)$. Then $< h_i, \Lambda_j >= \delta_{i,j}$ $(i \in I,\ 1 \le j \le n)$ and we have

$$P_+ = \{\lambda \in P\,;\, \langle h_i, \lambda\rangle \ge 0 \text{ for any } i \in I\} = \sum_{i=1}^{n-1} \mathbb{Z}_{\ge 0}\Lambda_i + \mathbb{Z}\Lambda_n\,.$$

Since $< h_i, \Lambda_n >= 0$ for every $i \in I$, $V(m\Lambda_n)$ is a 1-dimensional $U_q(\mathfrak{gl}_n)$-module. Since $V(\lambda) = V(\lambda + m\Lambda_n) \otimes V(-m\Lambda_n)$, we shall study only $V(\lambda)$ with $\lambda \in \tilde{P}_+ = \oplus_{i=1}^n \mathbb{Z}_{\ge 0}\Lambda_i$.

5.2. The vector representation. The representation $V(\Lambda_1)$ of $U_q(\mathfrak{gl}_n)$ is n-dimensional and called the vector representation. It is explicitly given as follows:

$$\begin{aligned} V(\Lambda_1) &= \bigoplus_{i=1}^{n} K\boxed{i}, \\ q(h)\boxed{i} &= q^{<h,\varepsilon_i>}\boxed{i}, \\ e_j\boxed{i} &= \delta_{j,i-1}\boxed{i-1}, \\ f_j\boxed{i} &= \delta_{j,i}\boxed{i+1}\,. \end{aligned}$$

The vector $\boxed{1}$ is a highest weight vector. The base $\{\boxed{i}\,; i = 1, \cdots, n\}$ forms a crystal base of $V(\Lambda_1)$ and its crystal graph is:

$$B(\Lambda_1) : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n}\,.$$

Remark that $\mathrm{wt}(\boxed{i}) = \varepsilon_i$.

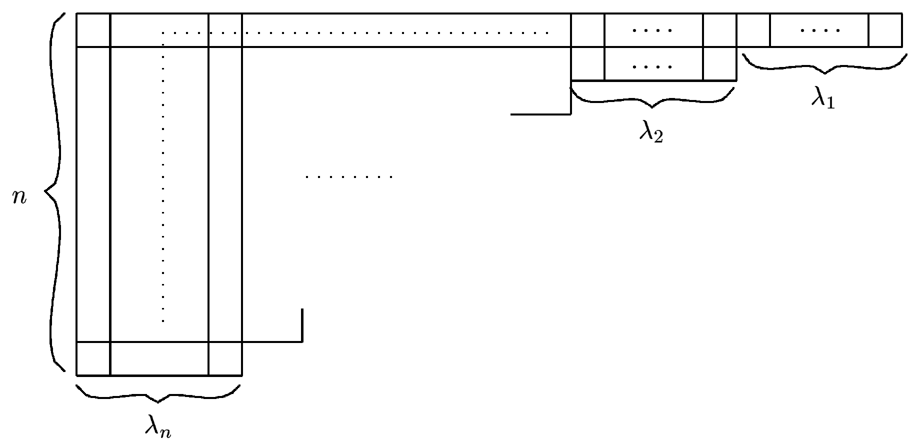
5.3. Fundamental representation. The representations $V(\Lambda_i)$ are called the fundamental representations. We shall describe $B(\Lambda_i)$ by embedding it into the tensor product of copies of $B(\Lambda_1)$. Let us consider the vector $\boxed{1} \otimes \cdots \otimes \boxed{i}$ in $B(\Lambda_1)^{\otimes i}$. Then it is a highest weight vector of weight Λ_i. Hence by Theorem 4.3, $B(\Lambda_i)$ is isomorphic to the connected component of $B(\Lambda_1)^{\otimes i}$ containing $\boxed{1} \otimes \cdots \otimes \boxed{i}$. This component can be described as follows.

PROPOSITION 5.1. *By the isomorphism above,*

$$B(\Lambda_i) \cong \{\boxed{a_1} \otimes \cdots \otimes \boxed{a_i} \in B(\Lambda_1)^{\otimes i}; 1 \le a_1 < a_2 < \cdots < a_i \le n\}\,.$$

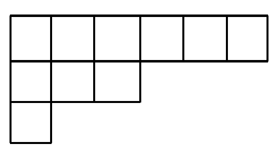
We shall write $\boxed{a_1} \otimes \cdots \otimes \boxed{a_i} \in B(\Lambda_i)$ by $\begin{array}{|c|}\hline a_1 \\ \hline \vdots \\ \hline a_i \\ \hline \end{array}$. Hence the highest weight vector u_{Λ_i} corresponds to $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline i \\ \hline \end{array}$.

5.4. General representations. We shall recall that $\lambda \in \tilde{P}_+ = \sum_{i=1}^n \mathbb{Z}_{\ge 0}\Lambda_i$ corresponds to the Young diagram as follows. For $\lambda = \sum_{i=1}^n \lambda_i\Lambda_i$, we associate the diagram $Y(\lambda)$:



This is a *Young diagram*, i.e. it consists of n rows of blocks whose lengths (≥ 0) are decreasing (in the generalized sense). The length of the i-th row of $Y(\lambda)$ is $\lambda_i + \cdots + \lambda_n$. Equivalently, $Y(\lambda)$ has λ_i columns of length i.

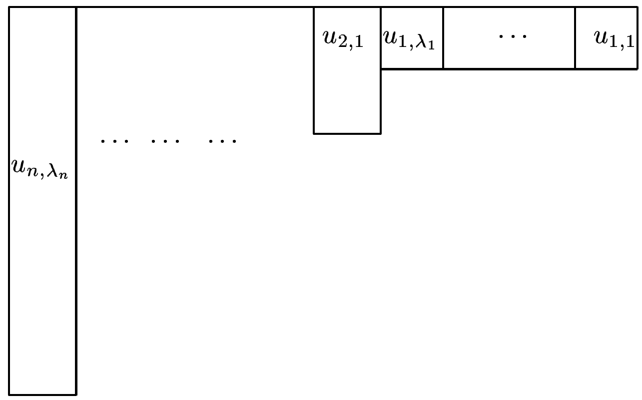
EXAMPLE 5.1. The Young diagram $Y(3\Lambda_1 + 2\Lambda_2 + \Lambda_3)$ is



We embed $B(\lambda)$ into $B(\Lambda_1)^{\otimes\lambda_1} \otimes \cdots \otimes B(\Lambda_n)^{\otimes\lambda_n}$ by $u_\lambda \mapsto u_{\Lambda_1}^{\otimes\lambda_1} \otimes \cdots \otimes u_{\Lambda_n}^{\otimes\lambda_n}$. Since $B(\Lambda_i)$ is embedded into $B(\Lambda_1)^{\otimes i}$, $B(\lambda)$ is embedded into $B(\Lambda_1)^{\otimes N}$ with $N = \sum i\lambda_i$. We denote the vector

$$u_{1,1} \otimes \cdots \otimes u_{1,\lambda_1} \otimes u_{2,1} \otimes \cdots \otimes u_{2,\lambda_2} \otimes \cdots \otimes u_{n,1} \otimes \cdots \otimes u_{n,\lambda_n} \\ \in B(\Lambda_1)^{\otimes \lambda_1} \otimes \cdots \otimes B(\Lambda_n)^{\otimes \lambda_n}$$

as (representing each $u_{i,j}$ by a column with numbers)



This is to translate the English writing order into the Japanese writing order (of course the Japanese order followed the Chinese order). This is a Young diagram $Y(\lambda)$ whose blocks are filled with positive integers at most n. We call such an object a *Young tableau* with shape $Y(\lambda)$.

EXAMPLE 5.2. For $\lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$, the vector $\boxed{1} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{4} \otimes \boxed{5} \otimes \boxed{6}$ is written in the following Young tableau with shape $Y(\lambda)$.

$$\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

We call a Young tableau *semi-standard* if the numbers at blocks are strictly increasing vertically and increasing horizontally (from the left to the right).

THEOREM 5.1 ([8]). *By the above embedding, $B(\lambda)$ is identified with the set of semi-standard Young tableaux with shape $Y(\lambda)$.*

The action of $\tilde{e}_i$ and $\tilde{f}_i$ is described as follows. Set $b = \boxed{a_1} \otimes \cdots \otimes \boxed{a_N}$. Then $\tilde{e}_i b$ and $\tilde{f}_i b$ are obtained by the following procedure.

(i) Neglect the a_k's other than i and $i+1$.

(ii) Neglect $\boxed{i} \otimes \boxed{i+1}$.

(iii) At the end of the procedures above, the vector is in the form $\boxed{i+1} \otimes \cdots \otimes \boxed{i+1} \otimes \boxed{i} \otimes \cdots \otimes \boxed{i}$. Then, $\tilde{e}_i$ changes the most right $i+1$ to i and $\tilde{f}_i$ changes the most left i to $i+1$. If such $i+1$ or i does not exist, then it changes the vector to 0, respectively.

Observe that $\boxed{i} \otimes \boxed{i+1}$ belongs to the trivial representation of $U_q(\mathfrak{gl}_n)_i$.

EXAMPLE 5.3. The following is a 1-string.

$$\begin{array}{l}
\boxed{2}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{2} \quad \xrightarrow{1} \\
\boxed{2}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{1} \quad \xrightarrow{1} \\
\boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{1} \quad \xrightarrow{1} \\
\boxed{2}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{1} \quad \xrightarrow{1} \\
\boxed{1}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{1} \quad .
\end{array}$$

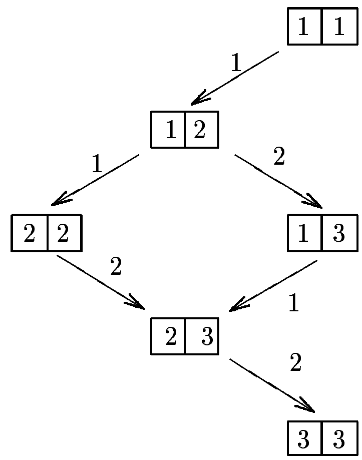
EXAMPLE 5.4. Take $b = \overbrace{\boxed{1}\otimes\boxed{2}}\otimes\boxed{2}\otimes\overbrace{\boxed{4}\otimes\boxed{3}}\otimes\boxed{2}$. Then, with respect to $i=1$, the parts under the braces may be neglected by the procedures (i) and (ii). Hence the 1-string containing this vector is

$$\begin{array}{l}
\boxed{1}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{4}\otimes\boxed{3}\otimes\boxed{1} \quad \xrightarrow{1} \\
\boxed{1}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{3}\otimes\boxed{1} \quad \xrightarrow{1} \\
\boxed{1}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{3}\otimes\boxed{2} \quad .
\end{array}$$

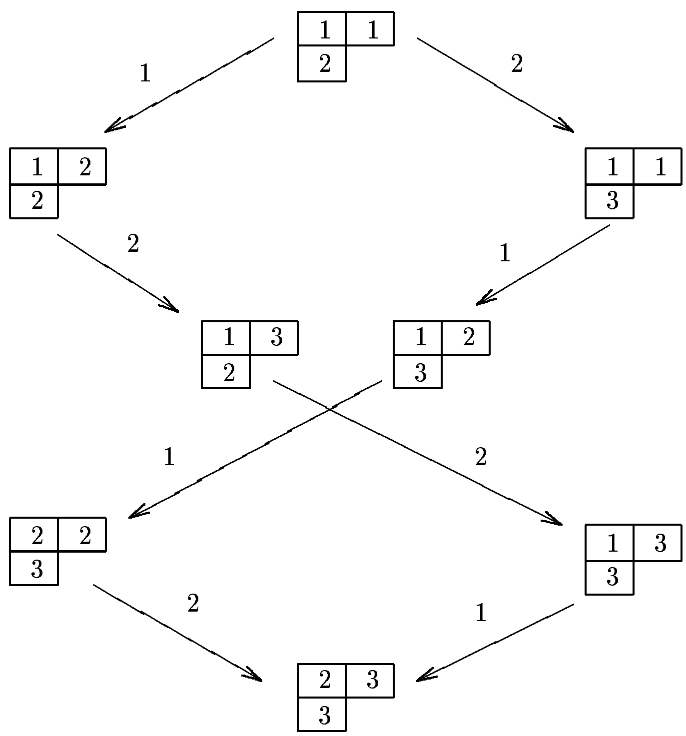
With respect to $i=2$, in $\overbrace{\boxed{1}}\otimes\boxed{2}\otimes\overbrace{\boxed{2}\otimes\boxed{4}\otimes\boxed{3}}\otimes\boxed{2}$, the parts under the braces are neglected and the 2-string containing b is

$$\begin{array}{l}
\boxed{1}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{3}\otimes\boxed{2} \quad \xrightarrow{2} \\
\boxed{1}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{3}\otimes\boxed{2} \quad \xrightarrow{2} \\
\boxed{1}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{3}\otimes\boxed{3} \quad .
\end{array}$$

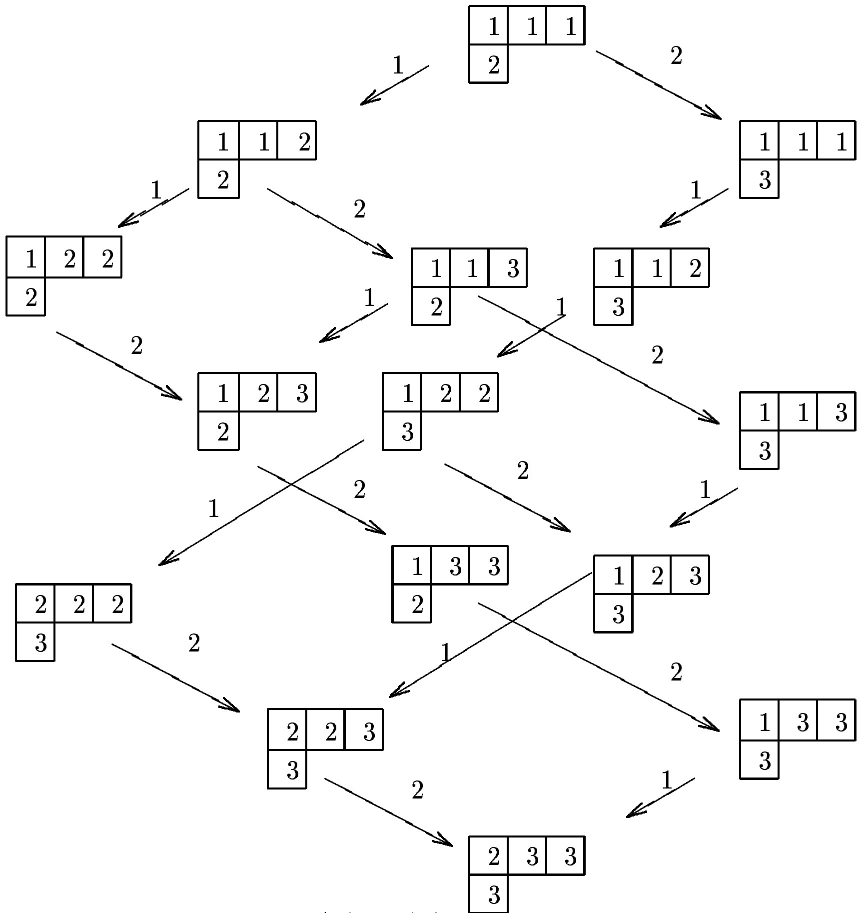
EXAMPLE 5.5. Let us take $\mathfrak{g} = \mathfrak{gl}_3$. The crystal graph of $B(\lambda)$ for $\lambda = 2\Lambda_1, \Lambda_1 + \Lambda_2, 2\Lambda_1 + \Lambda_2$ is as follows.



$B(2\Lambda_1)$



$B(\Lambda_1 + \Lambda_2)$



$B(2\Lambda_1 + \Lambda_2)$

5.5. Littlewood-Richardson rule. After describing the crystal base in terms of Young tableaux, we can calculate the decomposition of $V(\lambda) \otimes V(\mu)$ into the irreducible components and we obtain (an equivalent form of) the Littlewood-Richardson rule. First we observe the following two properties.

(5.1) $u_\lambda \otimes \boxed{i}$ is a highest weight vector if and only if the diagram $Y(\lambda) + i$ (the diagram obtained from $Y(\lambda)$ by adding one block at the i-th row) is again a Young diagram.

In this case, the weight of $u_\lambda \otimes \boxed{i}$ is $\lambda + \varepsilon_i$ and it is represented by the Young diagram $Y(\lambda) + i$

(5.2) $u_\lambda \otimes b_1 \otimes \cdots \otimes b_l$ is a highest weight vector if and only if $u_\lambda \otimes b_1 \otimes \cdots \otimes b_k$ are highest weight vectors for all $k \leq l$.

By those observations, we obtain

THEOREM 5.2 (T.NAKASHIMA[**9**]). *For $\lambda, \mu \in P_+$*

$$V(\lambda) \otimes V(\mu) \cong \oplus V(\lambda + \varepsilon_{a_{i_1}} + \cdots + \varepsilon_{a_{i_N}})\,.$$

Here $N = \sum_i i\mu_i$ and the direct sum ranges over $\boxed{a_{i_1}} \otimes \cdots \otimes \boxed{a_{i_N}} \in B(\mu) \subset B(\Lambda_1)^{\otimes N}$ such that all $Y(\lambda) + a_{i_1}, (Y(\lambda) + a_{i_1}) + a_{i_2}, \cdots$ are Young diagrams.

EXAMPLE 5.6. Consider the case $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = \Lambda_2$, $\mu = \Lambda_1 + \Lambda_2$. We have

$$B(\mu) = \{\ \ytableaushort{11,2},\ \ytableaushort{12,2},\ \ytableaushort{13,2},\ \ytableaushort{11,3},\ \ytableaushort{12,3},\ \ytableaushort{13,3},\ \ytableaushort{22,3},\ \ytableaushort{23,3}\,\}.$$

$$((\,\ydiagram{1,1} + 1) + 1) + 2 \;=\; (\,\ydiagram{2,1} + 1) + 2 = \ydiagram{3,1} + 2 = \ydiagram{3,2}\,.$$

$$((\,\ydiagram{1,1} + 2) + 1) + 2 \;=\; (\,\ydiagram{1,2} + 1) + 2 \quad \text{and} \quad \ydiagram{1,2}\ \text{is not a Young diagram.}$$

Continuing this procedure, $B(\mu) \ni \ytableaushort{11,2},\ \ytableaushort{13,2},\ \ytableaushort{11,3}$ give the contributions to the irreducible decomposition and we obtain

$$\ydiagram{1,1} \otimes \ydiagram{2,1} = \ydiagram{3,2} \oplus \ydiagram{2,2,1} \oplus \ydiagram{3,1,1}$$

Here we identified a Young diagram with the corresponding irreducible representation.

So far, we discussed the $\mathfrak{gl}_n$-case. The crystal bases in the other classical Lie algebras $\mathfrak{sp}_n, \mathfrak{so}_n$ are also described by variations of Young tableaux (see [**8**]), and the Littlewood-Richardson rule is also generalized to those cases (see [**9**]).

6. Path description of crystal bases for affine algebras

In the affine case, we can describe the crystal graph for modules in $\mathcal{O}_{int}$ by using paths. This construction is motivated by the study of solvable models, and in fact it enables us to calculate one-point functions of the solvable models as string functions of the representations. We shall explain here only the case of basic representations of $\widehat{\mathfrak{sl}}_2$. For other cases and one-point functions, see [**12, 13**].

6.1. $U_q(\widehat{\mathfrak{sl}}_2)$ and $U_q^{cl}(\widehat{\mathfrak{sl}}_2)$. We take the data as follows. Set $I = \{0, 1\}$, $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta$, and let $h_0, h_1, \vartheta \in P^*$ be the dual base of $\{\Lambda_0, \Lambda_1, \delta\}$. We set $\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$ and $\alpha_1 = 2\Lambda_1 - 2\Lambda_0$. We take a symmetric bilinear form $(\cdot,\cdot) : P \times P \to \mathbb{Q}$ so that $(\alpha_0, \alpha_0) = (\alpha_1, \alpha_1) = 2$, $(\alpha_0, \alpha_1) = -2$. Hence $(\alpha_0, \Lambda_0) = 1$, $(\alpha_1, \Lambda_0) = 0$ and we can choose (Λ_0, Λ_0) as we like. The associated quantized universal enveloping algebra is denoted by $U_q(\widehat{\mathfrak{sl}}_2)$. We denote by $U_q^{cl}(\widehat{\mathfrak{sl}}_2)$ the quantized universal enveloping algebra associated with $P_{cl} = P/\mathbb{Z}\delta$ (hence $P_{cl}^* = \mathbb{Z}h_0 + \mathbb{Z}h_1 \subset P^*$). Then we have

$$U_q(\widehat{\mathfrak{sl}}_2) = U_q^{cl}(\widehat{\mathfrak{sl}}_2) \otimes K[q(\vartheta), q(\vartheta)^{-1}]\,.$$

6.2. 2-dimensional representation of $U_q^{cl}(\widehat{\mathfrak{sl}}_2)$. The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ does not have finite-dimensional representations except trivial ones. However, $U_q^{cl}(\widehat{\mathfrak{sl}}_2)$ has many finite-dimensional representations. In fact we have a ring homomorphism

$$U_q^{cl}(\widehat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2)$$

by

$$\begin{aligned} e_0\,,\, f_1 &\mapsto f\,, \\ e_1\,,\, f_0 &\mapsto e\,, \\ t_0 &\mapsto t^{-1}\,, \\ t_1 &\mapsto t\,. \end{aligned}$$

Let V be the two-dimensional representation of $U_q^{cl}(\widehat{\mathfrak{sl}}_2)$ induced by the $U_q(\mathfrak{sl}_2)$-module $V(1)$. Then $V = Ku_0 \oplus Ku_1$ with

$$\begin{aligned} q(h)u_0 &= q^{\langle h\,,\,\Lambda_1-\Lambda_0\rangle}u_0\,, \\ q(h)u_1 &= q^{\langle h\,,\,\Lambda_0-\Lambda_1\rangle}u_0\,, \\ e_0u_k &= f_1u_k = u_{k+1}\,, \\ e_1u_k &= f_0u_k = u_{k-1}\,. \end{aligned}$$

Here we understand $u_{-1} = u_2 = 0$. Hence $B = \{u_0\,,\, u_1\}$ gives a crystal base of V and its crystal graph is

$$u_0 \overset{1}{\underset{0}{\rightleftarrows}} u_1$$

6.3. Path description. The key fact is the following proposition.

Proposition 6.1 (K.Misra–T.Miwa[**10**]). *There are unique isomorphisms of crystals (see §7)*

$$\Phi_0 \;:\; B(\Lambda_0) \xrightarrow{\sim} B(\Lambda_1) \otimes B \quad \textit{and} \tag{6.1}$$

$$\Phi_1 \;:\; B(\Lambda_1) \xrightarrow{\sim} B(\Lambda_0) \otimes B \tag{6.2}$$

sending u_{Λ_0} and u_{Λ_1} to $u_{\Lambda_1} \otimes u_1$ and $u_{\Lambda_0} \otimes u_0$, respectively.

Then for any integer n we have a chain of isomorphisms

$$\Psi_n : B(\Lambda_0) \underset{\Phi_0}{\xrightarrow{\sim}} B(\Lambda_1) \otimes B \underset{\Phi_1\otimes B}{\xrightarrow{\sim}} B(\Lambda_0) \otimes B \otimes B \xrightarrow{\sim} \cdots \xrightarrow{\sim} B(\Lambda_0) \otimes B^{\otimes 2n}\,.$$

We have $\Psi_n(u_{\Lambda_0}) = u_{\Lambda_0} \otimes (u_0 \otimes u_1)^{\otimes n}$. Set

$$p_g(n) = \begin{cases} u_0 & \text{if } n \text{ is even,} \\ u_1 & \text{if } n \text{ is odd.} \end{cases}$$

For $b \in B(\Lambda_0)$, if we take n large enough, then $\Psi_n(b)$ has the form $u_{\Lambda_0} \otimes b_{2n} \otimes \cdots \otimes b_1$. Then for $n' \geq n$ we have $\Psi_{n'}(b) = u_{\Lambda_0} \otimes (u_0 \otimes u_1)^{\otimes(n'-n)} \otimes b_{2n} \otimes \cdots \otimes b_1$.

Hence associating with b the sequence $\{b_1, \cdots, b_{2n}, p_g(2n+1), p_g(2n+2), \cdots\}$, we obtain a bijection

$$B(\Lambda_0) \xrightarrow{\sim} \mathcal{P} = \{\{p(n)\}_{n=1,2,\cdots}; \{p(n)\} \text{ is a sequence in } B \text{ such that } p(n) = p_g(n) \text{ for } n \gg 0\}.$$

Thus $B(\Lambda_0)$ is described by a space of paths in B.

REMARK 6.1. We have an isomorphism of crystals

$$B(\Lambda_0) \xrightarrow{\sim} B(\Lambda_1) \otimes B\,.$$

But we have no $U_q^{cl}(\widehat{\mathfrak{sl}}_2)$-linear homomorphism $V(\Lambda_0) \to V(\Lambda_1) \otimes V$. In order to get such a homomorphism (called vertex operator, see [**16**]) , we need a completion of $V(\Lambda_0)$ and $V(\Lambda_1)$.

7. Category of crystals

There is a way of describing crystal graphs for an arbitrary $U_q(\mathfrak{g})$. In order to explain this, we need to introduce the notion of crystals obtained by abstracting the properties of crystal graphs associated with integrable $U_q(\mathfrak{g})$-modules.

7.1. Tensor Category. Let us recall briefly the definition of a tensor category (see e.g. [**29**] for the details). Let C be a category with a bifunctor

$$C \times C \ni X \times Y \longmapsto X \otimes Y \in C.$$

Moreover we are given a functorial isomorphism

$$S(X, Y, Z) : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

in $X, Y, Z \in C$. We assume that the following diagram is commutative for any $X, Y, Z, W \in C$.

$$\begin{tikzcd}
((X\otimes Y)\otimes Z)\otimes W \arrow[r, "S(X,Y,Z)\otimes W"] \arrow[d, "S(X\otimes Y,Z,W)"] & (X\otimes(Y\otimes Z))\otimes W \arrow[r, "S(X,Y\otimes Z,W)"] & X\otimes((Y\otimes Z)\otimes W) \arrow[d, "X\otimes S(Y,Z,W)"'] \\
(X\otimes Y)\otimes(Z\otimes W) \arrow[rr, "S(X,Y,Z\otimes W)"] & & X\otimes(Y\otimes(Z\otimes W))
\end{tikzcd}$$

This axiom guarantees that $X_1 \otimes \cdots \otimes X_n$ is well defined for $X_1, \ldots, X_n \in C$ and $(X_1 \otimes \cdots \otimes X_k) \otimes (X_{k+1} \otimes \cdots \otimes X_n)$ is canonically isomorphic to $X_1 \otimes \cdots \otimes X_n$. An object $1 \in C$ is called a neutral object if we have functorial isomorphisms $\alpha(X) : 1 \otimes X \to X$ and $\beta(X) : X \otimes 1 \to X$ such that

$$\begin{aligned}
\alpha(X \otimes Y) &= \alpha(X) \otimes Y : 1 \otimes X \otimes Y \to X \otimes Y, \\
\beta(X \otimes Y) &= X \otimes \beta(Y) : X \otimes Y \otimes 1 \to X \otimes Y \quad \text{and} \\
\alpha(X) \otimes Y &= X \otimes \beta(Y) : X \otimes 1 \otimes Y \to X \otimes Y.
\end{aligned}$$

A neutral object is unique up to a canonical isomorphism.

7.2. Category of crystals. We keep the notations in §2.1. Let us construct the tensor category $\mathcal{C}$ of crystals. An object of $\mathcal{C}$, called a crystal, is a set B endowed with

$$\begin{aligned} \mathrm{wt} : B &\longrightarrow P \\ \varepsilon_i : B &\longrightarrow \mathbb{Z} \sqcup \{-\infty\} \\ \varphi_i : B &\longrightarrow \mathbb{Z} \sqcup \{-\infty\} \end{aligned}$$

and

$$\begin{aligned} \tilde{e}_i : B &\longrightarrow B \sqcup \{0\} \\ \tilde{f}_i : B &\longrightarrow B \sqcup \{0\}. \end{aligned}$$

Here $-\infty$ is the smallest element of $\mathbb{Z} \sqcup \{-\infty\}$ and 0 is a ghost element. We assume the following axioms.

(7.1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for any i (cf.(4.3)).

(7.2) If $b \in B$ satisfies $\tilde{e}_i b \neq 0$, then

$$\begin{aligned} &\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1\,, \\ &\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1\,, \\ &\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \quad \text{(cf.(4.3))}. \end{aligned}$$

(7.3) If $b \in B$ satisfies $\tilde{f}_i b \neq 0$, then

$$\begin{aligned} &\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1\,, \\ &\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1\,, \\ &\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i\,. \end{aligned}$$

(7.4) For $b_1, b_2 \in B$, $b_2 = \tilde{f}_i b_1$ if and only if $b_1 = \tilde{e}_i b_2$.

(7.5) If $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Here we understand $-\infty + n = -\infty$ for any integer n.

Note that a crystal base of an integral $U_q(\mathfrak{g})$-module satisfies these conditions.

Let B_1 and B_2 be two crystals. A morphism $\psi : B_1 \to B_2$ is a map $\psi : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ satisfying the following properties.

(7.6) $\psi(0) = 0\,.$

(7.7) If $\psi(b) \neq 0$ for $b \in B_1$, then

$$\begin{aligned} &\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \\ &\varepsilon_i(\psi(b)) = \varepsilon_i(b) \quad \text{and} \\ &\varphi_i(\psi(b)) = \psi_i(b). \end{aligned}$$

(7.8) For $b \in B_1$ such that $\psi(b) \neq 0$ and $\psi(\tilde{e}_i b) \neq 0$, we have

$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)\,.$$

(7.9) For $b \in B_1$ such that $\psi(b) \neq 0$ and $\psi(\tilde{f}_i b) \neq 0$, we have

$$\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b).$$

It is easy to check that a composition of two morphisms of crystals is again a morphism of crystals. Thus, we defined the category of crystals.

7.3. Tensor Product of crystals. As in Theorem 4.1(ii) and (4.5), we define the tensor product $B_1 \otimes B_2$ of two crystals B_1 and B_2. As a set, $B_1 \otimes B_2$ is the set $\{b_1 \otimes b_2; b_1 \in B_1, b_2 \in B_2\}$. We define

$$
\begin{aligned}
&\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)\,,\\
&\varepsilon_i(b_1 \otimes b_2) = \max\big(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle\big)\,,\\
&\varphi_i(b_1 \otimes b_2) = \max\big(\varphi_i(b_1) + \langle h_i, \mathrm{wt}\, b_2\rangle, \varphi_i(b_2)\big)\,,\\
&\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2)\,, \end{cases}\\
&\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2)\,. \end{cases}
\end{aligned}
$$

Here we understand $b_1 \otimes 0 = 0 \otimes b_2 = 0$. It is easily checked that $B_1 \otimes B_2$ satisfies the axiom of crystal, and $(B_1, B_2) \mapsto B_1 \otimes B_2$ defines a bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

The next lemma assures that $\mathcal{C}$ becomes a tensor category.

LEMMA 7.1. *Let B_1, B_2 and B_3 be three crystals. Then $(B_1 \otimes B_2) \otimes B_3 \to B_1 \otimes (B_2 \otimes B_3)$, given by $(b_1 \otimes b_2) \otimes b_3 \mapsto b_1 \otimes (b_2 \otimes b_3)$, is an isomorphism of crystals.*

7.4. Reversing arrows. For a crystal B, let us denote by $B^\vee$ the crystal obtained by reversing the direction of arrows. Namely, $B^\vee = \{b^\vee; b \in B\}$ with

$$
\begin{aligned}
\mathrm{wt}(b^\vee) &= -\mathrm{wt}(b),\\
\varepsilon_i(b^\vee) &= \varphi_i(b),\\
\varphi_i(b^\vee) &= \varepsilon_i(b),\\
\tilde{e}_i(b^\vee) &= (\tilde{f}_i b)^\vee,\\
\tilde{f}_i(b^\vee) &= (\tilde{e}_i b)^\vee.
\end{aligned}
$$

This corresponds to the following ring automorphism of $U_q(\mathfrak{g})$:

$$
\begin{aligned}
q(h) &\mapsto q(-h)\,,\\
e_i &\mapsto f_i\,,\\
f_i &\mapsto e_i\,.
\end{aligned}
$$

We have

$$
(B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee\,. \tag{7.10}
$$

7.5. Examples. Let us give several examples of crystals.

EXAMPLE 7.1. For $\lambda \in P_+$, $B(\lambda)$ is a crystal. More generally, the crystal base of an integrable $U_q(\mathfrak{g})$-module is a crystal.

EXAMPLE 7.2. For $\lambda \in P_+$, we set $B(-\lambda) = B(\lambda)^\vee$. Then $B(-\lambda)$ is regarded as the crystal base of the integrable $U_q(\mathfrak{g})$-module $V(-\lambda)$ of lowest weight $-\lambda$.

EXAMPLE 7.3. T_λ $(\lambda \in P)$.

$T_\lambda = \{t_\lambda\}$ with $\mathrm{wt}(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$. We have

(7.11) $T_\lambda \otimes T_\mu \cong T_{\lambda+\mu}$ for $\lambda, \mu \in P$.

(7.12) For any crystal B, $T_0 \otimes B \cong B \otimes T_0 \cong B$ by $t_0 \otimes b \leftrightarrow b \otimes t_0 \leftrightarrow b$.

Hence T_0 is a neutral object of the category of crystals.

EXAMPLE 7.4. B_i $(i \in I)$

$$
\begin{aligned}
&B_i = \{b_i(n); n \in \mathbb{Z}\} \quad \text{and} \\
&\mathrm{wt}(b_i(n)) = n\alpha_i\,, \\
&\varepsilon_j(b_i(n)) = \begin{cases} -n & j = i \\ -\infty & j \neq i\,, \end{cases} \\
&\varphi_j(b_i(n)) = \begin{cases} n & j = i \\ -\infty & j \neq i\,, \end{cases} \\
&\tilde{e}_j b_i(n) = \begin{cases} b_i(n+1) & j = i \\ 0 & j \neq i\,, \end{cases} \\
&\tilde{f}_j b_i(n) = \begin{cases} b_i(n-1) & j = i \\ 0 & j \neq i\,. \end{cases}
\end{aligned}
$$

We denote $b_i = b_i(0)$. The associated crystal graph is

$$\cdots \xrightarrow{i} b_i(1) \xrightarrow{i} b_i(0) \xrightarrow{i} b_i(-1) \xrightarrow{i} \cdots .$$

We have

$$T_\lambda \otimes B_i \cong B_i \otimes T_{s_i\lambda} \quad \text{for} \quad i \in I \quad \text{and} \quad \lambda \in P. \tag{7.13}$$

Here $s_i\lambda = \lambda - \langle h_i, \lambda\rangle\alpha_i$. The isomorphism is given by

$$t_\lambda \otimes b_i(n) \longleftrightarrow b_i(n + \langle h_i, \lambda\rangle) \otimes t_{s_i\lambda}\,.$$

7.6. Various Notions. We call a morphism $\psi : B_1 \to B_2$ *strict* if ψ commutes with the $\tilde{e}_i$'s and the $\tilde{f}_i$'s. If the associated map $B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ is injective, we say that ψ is an *embedding* and call B_1 a *subcrystal* of B_2.

We call a crystal B *semi-normal* if $\varepsilon_i(b) = \max\{n \geq 0; \tilde{e}_i^n b \neq 0\}$ and $\varphi_i(b) = \max\{n \geq 0; \tilde{f}_i^n b \neq 0\}$ for any $b \in B$ and $i \in I$. It means that B considered as a crystal over $U_q(\mathfrak{g})_i$, is isomorphic to the crystal associated with an integrable

$U_q(\mathfrak{g})_i$-module. The crystals $B(\lambda)$ and $B(-\lambda)$ are semi-normal but T_λ and B_i are not semi-normal.

If B_1 is a subcrystal of B_2 and if B_1 and B_2 are semi-normal, then $B_2 \cong B_1 \sqcup (B_2 \backslash B_1)$.

For a crystal B and a subset J of I, let us denote by $\Phi_J(B)$ the crystal B regarded as a crystal over $U_q(\mathfrak{g}_J)$ (see §4.6). Then we can show the following proposition.

PROPOSITION 7.1. *For any crystal B, the following two conditions are equivalent.*

(i) *For any finite subset J of I such that $\big((\alpha_i, \alpha_j)\big)_{i,j \in J}$ is a positive-definite symmetric matrix(i.e. $\mathfrak{g}_J$ is finite-dimensional), $\Phi_J(B)$ is isomorphic to the crystal associated with an integrable $U_q(\mathfrak{g}_J)$-module.*

(ii) *The condition above is satisfied for any $J \subset I$ with at most two elements.*

We call a crystal B *normal* if B satisfies the equivalent two conditions above.

8. Crystal base of $U_q^-(\mathfrak{g})$

8.1. Crystal base of $U_q^-(\mathfrak{g})$. Similarly to $V(\lambda)$, $U_q^-(\mathfrak{g})$ has a "crystal base". This is in fact derived from the crystal base of $V(\lambda)$ as its limit when λ tends to the infinity.

For $\lambda \in P_+$, there is a surjective $U_q^-(\mathfrak{g})$-linear homomorphism

$$\pi_\lambda : U_q^-(\mathfrak{g}) \to V(\lambda) \tag{8.1}$$

by $\pi_\lambda(P) = Pu_\lambda$.

For $\xi \in Q_- = \sum \mathbb{Z}_{\le 0}\alpha_i$, we set

$$U_q^-(\mathfrak{g})_\xi = \{P \in U_q^-(\mathfrak{g}); q(h)Pq(h)^{-1} = q^{\langle h, \xi\rangle}P\}.$$

Then $U_q^-(\mathfrak{g}) = \bigoplus_{\xi \in Q_-} U_q^-(\mathfrak{g})_\xi$ and $\pi_\lambda(U_q^-(\mathfrak{g})_\xi) \subset V(\lambda)_{\lambda+\xi}$. The kernel of π_λ is $\sum_i U_q^-(\mathfrak{g}) f_i^{1+\langle h_i, \lambda\rangle}$. Therefore, if $\langle h_i, \lambda\rangle \gg 0$, then $U_q^-(\mathfrak{g})_\xi \to V(\lambda)_{\lambda+\xi}$ is an isomorphism. Hence we may regard $U_q^-(\mathfrak{g})$ as the limit of $V(\lambda)$ when $\langle h_i, \lambda\rangle$ tend to the infinity. In order to see what happens when we change λ, let us take $\mu \in P_+$ and study the relation of π_λ and $\pi_{\lambda+\mu}$.

There is a unique $U_q(\mathfrak{g})$-linear map $S_{\lambda,\mu} : V(\lambda+\mu) \hookrightarrow V(\lambda) \otimes V(\mu)$ that sends $u_{\lambda+\mu} \mapsto u_\lambda \otimes u_\mu$. Let $p_\mu : V(\mu) \to K$ denote the K-linear map that sends u_μ to 1 and $V(\mu)_\eta$ $(\eta \neq \mu)$ to 0. Then the diagram

$$\begin{array}{ccc}
U_q^-(\mathfrak{g}) \xrightarrow{\pi_{\lambda+\mu}} & V(\lambda+\mu) \xrightarrow{S_{\lambda,\mu}} & V(\lambda) \otimes V(\mu) \\
\Big\downarrow \pi_\lambda & \swarrow p_{\lambda,\lambda+\mu} & \Big\downarrow V(\lambda) \otimes p_\mu \\
V(\lambda) & = \!= \!= \!= \!= & V(\lambda) \otimes K
\end{array}$$

commutes. In fact this follows from the fact that all the arrows are $U_q^-(\mathfrak{g})$-linear. Then $S_{\lambda,\mu}$ sends $L(\lambda+\mu)$ into $L(\lambda)\otimes L(\mu)$ and $V(\lambda)\otimes p_\mu$ sends $L(\lambda)\otimes L(\mu)$ to $L(\lambda)$. Moreover those homomorphisms send a crystal base to a crystal base (or 0). Moreover they commute with $\tilde{f}_i$ (in fact $\tilde{f}_i(b\otimes u_\mu)=\tilde{f}_i b\otimes u_\mu$ or $b\otimes\tilde{f}_i u_\mu$ and they are sent to $\tilde{f}_i b$ or 0 by $V(\lambda)\otimes p_\mu$ accordingly). Thus, $p_{\lambda,\lambda+\mu}: V(\lambda+\mu)\to V(\lambda)$ sends $L(\lambda+\mu)$ to $L(\lambda)$ and induces morphisms of crystals $B(\lambda+\mu)\to B(\lambda)\otimes B(\mu)\to B(\lambda)\otimes T_\mu$. This morphism commutes with the $\tilde{f}_i$'s but not $\tilde{e}_i$'s. Thus tending λ to the infinity we obtain

THEOREM 8.1 ([**3**]). *There is a unique local base $(L(\infty),B(\infty))$ of $U_q^-(\mathfrak{g})$ with the following properties.*

(8.2) $\pi_\lambda(L(\infty))\subset L(\lambda)$ *and* $\bar{\pi}_\lambda(B(\infty))\subset B(\lambda)\sqcup 0$ *for any* $\lambda\in P_+$,

(8.3) $B(\infty)$ *has a structure of crystal such that* $B(\infty)\to B(\lambda)\otimes T_{-\lambda}$ $(b\mapsto\pi_\lambda(b)\otimes t_{-\lambda})$ *is a morphism of crystals for any* $\lambda\in P_+$,

(8.4) $B(\infty)\to B(\lambda)\otimes T_{-\lambda}$ *commutes with* $\tilde{f}_i$ *for any* $\lambda\in P_+$,

(8.5) $\tilde{f}_i B(\infty)\subset B(\infty)$,

(8.6) $\{b\in B(\infty);\bar{\pi}_\lambda(b)\neq 0\}\xrightarrow{\bar{\pi}_\lambda}B(\lambda)$ *is bijective for any* $\lambda\in P_+$.

Hence $B(\lambda)$ may be regarded as a subcrystal of $B(\infty)\otimes T_\lambda$ by $B(\lambda)\ni\bar{\pi}_\lambda(b)\leftrightarrow b\otimes t_\lambda\in B(\infty)\otimes T_\lambda$. Then $B(\lambda)\hookrightarrow B(\infty)\otimes T_\lambda$ commutes with the $\tilde{e}_i$'s (but not $\tilde{f}_i$'s). Its image will be given in Proposition 8.2.

Let us denote by u_∞ the element of $B(\infty)$ corresponding to $1\in U_q^-(\mathfrak{g})$. Hence we have $\bar{\pi}_\lambda(u_\infty)=u_\lambda$.

REMARK 8.1. For $b\in B(\infty)$, we have

$$\varepsilon_i(b)=\max\{n\geq 0;\tilde{e}_i^n b\neq 0\}.$$

However, $\varphi_i(b)$ may be negative (e.g. $\varphi_i(\tilde{f}_i u_\infty)=-1$). Also note that

(8.7) $$\{b\in B(\infty)\,;\,\varepsilon_i(b)=0\text{ for every }i\in I\}=\{u_\infty\}.$$

8.2. Description of $B(\infty)$. In order to describe $B(\infty)$, let us first show that $B(\infty)$ is embedded into $B(\infty)\otimes B_i$ $(u_\infty\mapsto u_\infty\otimes b_i)$ for any i. This can be explained as follows.

Take $\Lambda_i\in\mathbb{Q}\otimes P$ such that $\langle h_j,\Lambda_i\rangle=\delta_{ij}$ for every $j\in I$. Take $b\in B(\infty)$. Take $\lambda\in P_+$ such that $\langle h_i,\lambda\rangle=0$ and $\langle h_j,\lambda\rangle\gg 0$ for $j\neq i$. Take $N\gg 0$ such that $N\Lambda_i\in P$. Let $b_1\otimes b_2\in B(\lambda)\otimes B(N\Lambda_i)$ be the image of b by the composition $B(\infty)\xrightarrow{\bar{\pi}_{\lambda+N\Lambda_i}}B(\lambda+N\Lambda_i)\hookrightarrow B(\lambda)\otimes B(N\Lambda_i)$. Take $b'\in B(\infty)$ such that $\bar{\pi}_\lambda(b')=b_1$. We can see easily that b_2 has the form $\tilde{f}_i^m u_{N\Lambda_i}$ and b' and m do not depend on the choice of N and λ. Then $b\mapsto b'\otimes\tilde{f}_i^m b_i$ is in fact defines a morphism of $B(\infty)\to B(\infty)\otimes B_i$.

THEOREM 8.2 ([**5**]). *There is a unique strict embedding of crystals*

$$\Psi_i: B(\infty)\to B(\infty)\otimes B_i$$

that sends u_∞ *to* $u_\infty \otimes b_i$.

We can see by the similar argument

$$B(\infty) \otimes B_i \cong \bigsqcup_{n \geq 0} B(\infty) \otimes T_{n\alpha_i} \tag{8.8}$$

where $B(\infty) \otimes T_{n\alpha_i} \to B(\infty) \otimes B_i$ is given by $u_\infty \otimes t_{n\alpha_i} \mapsto u_\infty \otimes \tilde{e}_i^n b_i$.

Now let us take a sequence $i_1, i_2, \cdots$ in I such that $\{n \in \mathbb{Z}_{>0}; i = i_n\}$ is an infinite set for any $i \in I$. Then we have

$$\begin{aligned} \Psi_n : B(\infty) \quad & \xrightarrow{\Psi_{i_1}} B(\infty) \otimes B_{i_1} \xrightarrow{\Psi_{i_2} \otimes B_{i_1}} B(\infty) \otimes B_{i_2} \otimes B_{i_1} \\ & \to \cdots \to B(\infty) \otimes B_{i_n} \otimes \cdots \otimes B_{i_1}. \end{aligned}$$

For any $b \in B(\infty)$, if we take $n \gg 0$ then $\Psi_n(b)$ has the form $u_\infty \otimes \tilde{f}_{i_n}^{a_n} b_{i_n} \otimes \cdots \otimes \tilde{f}_{i_1}^{a_1} b_{i_1}$. Set $a_m = 0$ $(m > n)$. Since $\Psi_{n'}(b) = u_\infty \otimes b_{i_{n'}} \otimes \cdots \otimes b_{i_n+1} \otimes \tilde{f}_{i_n}^{a_n} b_{i_n} \otimes \cdots \otimes \tilde{f}_{i_1}^{a_1} b_{i_1}$ for $n' \geq n$, the sequence $\{a_1, a_2 \cdots\}$ is independent of the choice of n. Thus we obtain an embedding

$$\begin{aligned} \Psi : B(\infty) \to \{(a_n)_{n>0}; (a_n) & \text{ is a sequence of non-negative integers} \\ & \text{such that } a_n = 0 \text{ for } n \gg 0\}. \end{aligned}$$

This gives a description of $B(\infty)$. This description depends on the choice of a sequence $\{i_1, i_2, \cdots\}$. The map Ψ is injective but we don't know the image of Ψ in general.

EXAMPLE 8.1. For $\mathfrak{g} = \mathfrak{sl}_3$, $I = \{1, 2\}$, if we take $i_1 = 1, i_2 = 2, i_3 = 1$, then the image of Ψ is the set of (a_n) with

$$a_2 \geq a_3 \text{ and } a_n = 0 \text{ for } n > 3.$$

8.3. $*$-operation. Let $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the anti-automorphism given by

$$\begin{aligned} q^* &= q \\ e_i^* &= e_i \\ f_i^* &= f_i \\ q(h)^* &= q(-h). \end{aligned}$$

Then $*$ sends $U_q^-(\mathfrak{g})$ to $U_q^-(\mathfrak{g})$.

THEOREM 8.3. $L(\infty)^* = L(\infty)$, $B(\infty)^* = B(\infty)$.

We shall set

$$\begin{aligned}\tilde{e}_i^*(b) &= (\tilde{e}_i(b^*))^*\\ \tilde{f}_i^*(b) &= (\tilde{f}_i(b^*))^*\\ \varepsilon_i^*(b) &= \varepsilon_i(b^*)\\ \varphi_i^*(b) &= \varphi_i(b^*).\end{aligned}$$

Then it gives another crystal structure on $B(\infty)$. This can be explained also by $\Psi_i : B(\infty) \to B(\infty) \otimes B_i$ as follows.

PROPOSITION 8.1. *For $b \in B(\infty)$, set $\Psi_i(b) = b' \otimes \tilde{f}_i^m b_i$. Then we have*

(8.9) $\varepsilon_i^*(b') = 0\,,$

(8.10) $\varepsilon_i^*(b) = m\,,$

(8.11) $\Psi_i(\tilde{f}_i^* b) = b' \otimes \tilde{f}_i^{m+1} b_i$ *and*

(8.12) $\Psi_i(\tilde{e}_i^* b) = \begin{cases} b' \otimes \tilde{f}_i^{m-1} b_i & \textit{if } m > 0\\ 0 & \textit{if } m = 0.\end{cases}$

Hence the image of $\Psi_i : B(\infty) \to B(\infty) \otimes B_i$ is $\{b \otimes \tilde{f}_i^n b_i \in B(\infty) \otimes B_i\,;\, n \geq 0,\, \varepsilon_i^*(b) = 0\}$. We have also

PROPOSITION 8.2. *For any $\lambda \in P_+$, the image of $B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda$ is*

$$\{b \otimes t_\lambda \in B(\infty) \otimes T_\lambda,;\, \varepsilon_i^*(b) \leq \langle h_i, \lambda\rangle \textit{ for any } i \in I\}\,.$$

REMARK 8.2. By Proposition 8.1, $\tilde{e}_i^*$ and $\tilde{f}_j$ commutes if $i \neq j$, but $\tilde{e}_i^*$ and $\tilde{f}_i$ do not commute (cf. §9.3).

REMARK 8.3. Similarly, $U_q^+(\mathfrak{g})$ has a "crystal base". We shall denote it by $\big(L(-\infty), B(-\infty)\big)$. We have $B(-\infty) \cong B(\infty)^\vee$ as a crystal. We set $u_{-\infty} = {u_\infty}^\vee$, which corresponds to $1 \in U_q^+(\mathfrak{g})$

9. Crystal Base of Modified quantized enveloping algebras

9.1. Modified quantized enveloping algebra. Let $U_q(\mathfrak{g})a_\lambda$ be the left $U_q(\mathfrak{g})$-module with the defining relation $q(h)a_\lambda = q^{\langle h,\lambda\rangle}a_\lambda$. Set

$$\tilde{U}_q(\mathfrak{g}) = \underset{\lambda\in P}{\oplus} U_q(\mathfrak{g})a_\lambda$$

and we endow the $U_q(\mathfrak{g})$-bimodule structure on $\tilde{U}_q(\mathfrak{g})$ as follows:

(9.1) For $\xi \in Q = \oplus\mathbb{Z}\alpha_i$ and
$R \in U_q(\mathfrak{g})_\xi = \{R \in U_q(\mathfrak{g}); q(h)Rq(-h) = q^{\langle h,\xi\rangle}R \text{ for any } h \in P^*\}\,,$
$a_\lambda R = R\, a_{\lambda-\xi}\,.$

Moreover $\tilde{U}_q(\mathfrak{g})$ has the ring structure satisfying

$$a_\lambda a_\mu = \delta_{\lambda,\mu} a_\lambda \,. \tag{9.2}$$

$$(uP)v = u(Pv) \quad \text{for } u, v \in \tilde{U}_q(\mathfrak{g}) \text{ and } P \in U_q(\mathfrak{g}) \,. \tag{9.3}$$

For any integrable $U_q(\mathfrak{g})$-module M, $\tilde{U}_q(\mathfrak{g})$ operates on M such a way that a_λ is the projection operator to the weight space M_λ.

We have

$$\tilde{U}_q(\mathfrak{g}) = \bigoplus_{\substack{\lambda,\mu \in P \\ \lambda-\mu \in Q}} a_\lambda U_q(\mathfrak{g}) a_\mu .$$

9.2. Crystal base of $\tilde{U}_q(\mathfrak{g})$. We shall show that $\tilde{U}_q(\mathfrak{g})$ has also a "crystal base". The reasoning is similar to the $U_q^-(\mathfrak{g})$-case. Take $\lambda \in P$ and choose $\zeta, \mu \in P_+$ such that $\lambda = \zeta - \mu$. Then we have a $U_q(\mathfrak{g})$-linear homomorphism

$$\pi_{\zeta,\mu} : U_q(\mathfrak{g}) a_\lambda \to V(\zeta) \otimes V(-\mu) \quad \text{by } a_\lambda \mapsto u_\zeta \otimes u_{-\mu}.$$

Here $V(-\mu)$ is the irreducible integrable $U_q(\mathfrak{g})$-module with lowest weight $-\mu$. Hence $V(-\mu) = U_q(\mathfrak{g}) u_{-\mu}$ with the defining relations: $q(h) u_{-\mu} = q^{\langle h, -\mu \rangle} u_{-\mu}$, $f_i u_{-\mu} = 0$, $e_i^{1+\langle h_i, \mu \rangle} u_{-\mu} = 0$. Then $\pi_{\zeta,\mu}$ is surjective. For $\xi \in P_+$ we have a commutative diagram.

$$\begin{tikzcd}
U_q(\mathfrak{g}) a_\lambda \arrow[r, "\pi_{\zeta+\xi,\mu+\xi}"] \arrow[ddr, "\pi_{\zeta,\mu}"'] & V(\zeta+\xi) \otimes V(-\mu-\xi) \arrow[d, "\alpha"] \\
 & V(\zeta) \otimes V(\xi) \otimes V(-\xi) \otimes V(-\mu) \arrow[d, "\beta"] \\
 & V(\zeta) \otimes V(-\mu)
\end{tikzcd}$$

Here α is given by the $U_q(\mathfrak{g})$-linear homomorphisms $V(\zeta + \xi) \to V(\zeta) \otimes V(\xi)$ $(u_{\zeta+\xi} \mapsto u_\zeta \otimes u_\xi)$ and $V(-\mu-\xi) \to V(-\xi) \otimes V(-\mu)$ $(u_{-\mu-\xi} \mapsto u_{-\xi} \otimes u_{-\mu})$. The morphism β is given by a unique $U_q(\mathfrak{g})$-linear homomorphism $V(\xi) \otimes V(-\xi) \to \mathbb{C}$ $(u_\xi \otimes u_{-\xi} \mapsto 1)$. In the level of crystals, α and β induces

$$\begin{aligned}
B(\lambda + \xi) \otimes B(-\mu - \xi) &\to B(\lambda) \otimes B(\xi) \otimes B(-\xi) \otimes B(-\mu) \\
&\to B(\lambda) \otimes B(0) \otimes B(-\mu), \\
&\cong B(\lambda) \otimes B(-\mu).
\end{aligned}$$

Here $B(\xi) \otimes B(-\xi) \to B(0) = \{u_0\}$ is given by

$$b_1 \otimes b_2 \mapsto \begin{cases} u_0 & \text{if } b_1 = u_\xi \text{ and } b_2 = u_{-\xi} \,, \\ 0 & \text{otherwise.} \end{cases}$$

We can easily see that they are strict morphisms of crystals. This gives the following

THEOREM 9.1 (G.LUSZTIG[**28**]). *For any $\lambda \in P$, there exists a unique local base $\big(L(U_q(\mathfrak{g})a_\lambda), B(U_q(\mathfrak{g})a_\lambda)\big)$ of $U_q(\mathfrak{g})a_\lambda$ that satisfies the following properties.*

(i) *For any $\zeta, \mu \in P_+$ with $\lambda = \zeta - \mu$,*

$$\pi_{\zeta,\mu}\big(L(U_q(\mathfrak{g})a_\lambda)\big) \subset L(\zeta) \otimes L(-\mu)$$

and the induced map $\bar{\pi}_{\zeta,\mu}$:

$$L(U_q(\mathfrak{g})a_\lambda)/qL(U_q(\mathfrak{g})a_\lambda) \to (L(\zeta)/qL(\zeta)) \otimes (L(-\mu)/qL(-\mu))$$

satisfies

$$\bar{\pi}_{\zeta,\mu}\big(B(U_q(\mathfrak{g})a_\lambda)\big) \subset B(\zeta) \otimes B(-\mu) \sqcup \{0\}.$$

(ii) *There is a structure of crystal on $B(U_q(\mathfrak{g})a_\lambda)$ such that $\bar{\pi}_{\zeta,\mu}$ gives a strict morphism of crystals from $B(U_q(\mathfrak{g})a_\lambda)$ to $B(\zeta) \otimes B(-\mu)$ for any $\zeta, \mu \in P_+$ with $\lambda = \zeta - \mu$.*

Set $\big(L(\tilde{U}_q(\mathfrak{g})), B(\tilde{U}_q(\mathfrak{g}))\big) = \bigoplus_{\lambda} \big(L(U_q(\mathfrak{g})a_\lambda), B(U_q(\mathfrak{g})a_\lambda)\big)$. Let us remark that $B(\tilde{U}_q(\mathfrak{g}))$ is normal (see §7.6).

For $\zeta, \mu \in P_+$, we have $B(\zeta) \hookrightarrow B(\infty) \otimes T_\zeta$ and $B(-\mu) \hookrightarrow T_{-\mu} \otimes B(-\infty)$. Hence $B(\zeta) \otimes B(-\mu) \hookrightarrow B(\infty) \otimes T_{\zeta-\mu} \otimes B(-\infty)$. Thus fixing $\lambda = \zeta - \mu \in P_+$ and letting ζ and μ tend to infinity, we obtain

THEOREM 9.2 ([**7**]). *$B(U_q(\mathfrak{g})a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty)$ as a crystal.*

This is compared with

$$U_q(\mathfrak{g})a_\lambda \cong U_q^-(\mathfrak{g}) \otimes K a_\lambda \otimes U_q^+(\mathfrak{g})\,. \tag{9.4}$$

9.3. Crystal structure over $\mathfrak{g} \oplus \mathfrak{g}$. As $\tilde{U}_q(\mathfrak{g})$ has a $U_q(\mathfrak{g})$-bimodule structure, it may be regarded as a $U_q(\mathfrak{g} \oplus \mathfrak{g})$-module. Similarly, we can extend the crystal structure on $B(\tilde{U}_q(\mathfrak{g}))$ to a crystal structure over $\mathfrak{g} \oplus \mathfrak{g}$.

Let $*$ be the anti-automorphism of $\tilde{U}_q(\mathfrak{g})$ given by

$$\begin{aligned} q^* &= q\,, \\ a_\lambda^* &= a_{-\lambda}\,, \\ (Pu)^* &= u^*P^*,\ (uP)^* = P^*u^* \quad \text{for } u \in \tilde{U}_q(\mathfrak{g}) \text{ and } P \in U_q(\mathfrak{g})\,. \end{aligned}$$

Then we can show

THEOREM 9.3. (i) $L(\tilde{U}_q(\mathfrak{g}))^* = L(\tilde{U}_q(\mathfrak{g}))$, $B(\tilde{U}_q(\mathfrak{g}))^* = B(\tilde{U}_q(\mathfrak{g}))$.

(ii) *By the identification $B(\tilde{U}_q(\mathfrak{g})) \cong \bigoplus_{\lambda \in P} B(\infty) \otimes T_\lambda \otimes B(-\infty)$, we have*

$$(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{-\lambda - \mathrm{wt} b_1 - \mathrm{wt} b_2} \otimes b_2^* \tag{9.5}$$

for $b_1 \in B(\infty)$ and $b_2 \in B(-\infty)$.

Now setting

$$\begin{aligned}
\varepsilon_i^*(b) &= \varepsilon_i(b^*)\\
\varphi_i^*(b) &= \varphi_i(b^*)\\
\tilde{e}_i^*(b) &= (\tilde{e}_i(b^*))^*\\
\tilde{f}_i^*(b) &= (\tilde{f}_i(b^*))^*\\
\mathrm{wt}^*(b) &= -\lambda \quad \text{for } b \in B(U_q(\mathfrak{g})a_\lambda)\,,
\end{aligned}$$

we obtain another structure of crystal on $B(\tilde{U}_q(\mathfrak{g}))$. These two crystal structures are compatible (i.e. $\tilde{e}_i^*, \tilde{f}_i^* : B(\tilde{U}_q(\mathfrak{g})) \to B(\tilde{U}_q(\mathfrak{g}))$ is a morphism of crystal), so that $B(\tilde{U}_q(\mathfrak{g}))$ may be regarded as a crystal over $\mathfrak{g} \oplus \mathfrak{g}$.

Let us denote by T the (integral) Tits cone, i.e.

$$\begin{aligned}
T &= W \cdot P_+\\
&= \{\lambda \in P\,;\, (\alpha, \lambda) < 0 \ \text{ only for a finite number of positive roots } \alpha\}\,.
\end{aligned}$$

Here W denotes the Weyl group. Set

$$B_T(\tilde{U}_q(\mathfrak{g})) = \{b \in B(\tilde{U}_q(\mathfrak{g}))\,;\, \mathrm{wt}(b),\ -\mathrm{wt}^*(b) \in T\}\,.$$

Then we have a Peter-Weyl type theorem (also see Problem 1 in the last section).

THEOREM 9.4 ([**7**]).

$$B_T(\tilde{U}_q(\mathfrak{g})) \cong \bigoplus_{\lambda \in P_+} B(\lambda) \otimes B(-\lambda) \tag{9.6}$$

as a crystal over $\mathfrak{g} \oplus \mathfrak{g}$.

Regarding $P \oplus P$ as the weight lattice of $\mathfrak{g} \oplus \mathfrak{g}$, $B(\lambda) \otimes B(-\lambda)$ means $B(\lambda \oplus 0) \otimes B(0 \oplus (-\lambda))$. Hence $B(\lambda) \otimes B(-\lambda)$ is identified with $\{b_1 \otimes b_2\,;\, b_1 \in B(\lambda)$ and $b_2 \in B(-\lambda)\}$, and $\tilde{e}_i$ and $\tilde{f}_i$ act on the first factor and $\tilde{e}_i^*$ and $\tilde{f}_i^*$ act on the second factor, and $\varepsilon_i(b_1 \otimes b_2) = \varepsilon_i(b_1)$, $\varepsilon_i^*(b_1 \otimes b_2) = \varepsilon_i(b_2)$, etc.

10. Littelmann's path realization

P. Littelmann ([**17, 18**]) gave a way of describing $B(\lambda)$ in a geometric way. Let us explain it briefly. Set $P_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} P$. A path is a piecewise-linear, continuous map from $[0,1]$ to $P_{\mathbb{R}}$. We consider it modulo parameterization (i.e. $\pi \equiv \pi \circ \varphi$ if $\varphi : [0,1] \to [0,1]$ is a piecewise-linear continuous increasing surjective map). Remark that π may be not injective. Let us denote by $\mathcal{P}$ the set of paths π such that $\pi(0) = 0$ and $\pi(1) \in P$. He defined a crystal structure on $\mathcal{P}$ as follows. For $\pi \in \mathcal{P}$, we set $\mathrm{wt}(\pi) = \pi(1)$. For $i \in I$, let us set

$$h = \inf\bigl(\mathbb{Z} \cap \{\langle h_i, \pi(t)\rangle; 0 \le t \le 1\}\bigr) \le 0.$$

We set $\varepsilon_i(\pi) = -h$. If $\varepsilon_i(\pi) = 0$ then $\tilde{e}_i b = 0$. If $\varepsilon_i(\pi) > 0$, let us take the smallest $t_1 > 0$ such that $\langle h_i, \pi(t_1)\rangle = h$. Let $t_0 < t_1$ be the largest number such that $\langle h_i, \pi(t_0)\rangle = h + 1$. Then we define

$$(\tilde{e}_i\pi)(t) = \begin{cases} \pi(t) & \text{for } 0 \le t \le t_0 \\ \pi(t) - \langle h_i, \pi(t) - \pi(t_0)\rangle\alpha_i & \text{for } t_0 \le t \le t_1 \\ \pi(t) + \alpha_i & \text{for } t_1 \le t \le 1. \end{cases}$$

Hence the part $t_0 \le t \le t_1$ of the path π is reflected along the hyperplane $\{\lambda \in P_{\mathbb{R}}; \langle h_i, \lambda\rangle = h + 1\}$ and the part $t_1 \le t \le 1$ is parallel translated. We set

$$\tilde{f}_i\pi = \left(\tilde{e}_i(\pi^\vee)\right)^\vee,$$
$$\varphi_i(\pi) = \varepsilon_i(\pi^\vee) = \langle h_i, \pi(1)\rangle - h\,.$$

Here $\pi^\vee$ is the reversed path $[0,1] \ni t \mapsto \pi(1-t) - \pi(1)$.

Then we can see easily

Lemma 10.1. *The data above define a structure of crystal on $\mathcal{P}$.*

Remark that $\mathcal{P}$ is a semi-normal crystal. Let $* : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ be the map given

$$(\pi_1 * \pi_2)(t) = \begin{cases} \pi_1(2t) & 0 \le t \le 1/2\,, \\ \pi_1(1) + \pi_2(2t-1) & 1/2 \le t \le 1\,. \end{cases}$$

The following is also easy to prove.

Lemma 10.2. *$\pi_1 \otimes \pi_2 \mapsto \pi_1 * \pi_2$ gives a morphism of crystals from $\mathcal{P} \otimes \mathcal{P}$ to $\mathcal{P}$.*

Now we can state the result. For $\lambda \in P$, let π_λ be the straight path joining 0 and λ.

Theorem 10.1. *For $\lambda \in P_+$, there is a unique morphism of crystal from $B(\lambda)$ to $\mathcal{P}$ sending u_λ to π_λ. By this morphism, $B(\lambda)$ is isomorphic to the connected component of the crystal $\mathcal{P}$ containing π_λ.*

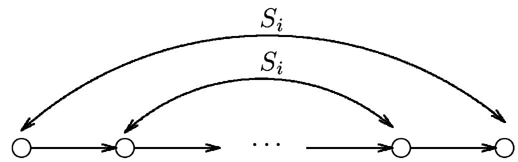
The image of $B(\lambda) \hookrightarrow \mathcal{P}$ is explicitly described (referred to as Lakshmibai-Seshadri paths) in [**17**].

11. Weyl group action on crystals

Let B be a semi-normal crystal (see §7.6). For $i \in I$, we define the automorphism S_i of the set B by

(11.1)
$$S_i b = \begin{cases} \tilde{f}_i^{\langle h_i, \mathrm{wt}\, b\rangle}\, b & \text{if } \langle h_i, \mathrm{wt}(b)\rangle \ge 0 \\ \tilde{e}_i^{-\langle h_i, \mathrm{wt}\, b\rangle}\, b & \text{if } \langle h_i, \mathrm{wt}(b)\rangle \le 0\,. \end{cases}$$

This is visualized as



Hence we have

$$S_i^2 = \mathrm{id}\,.$$

We have

$$\mathrm{wt}(S_i b) = s_i(\mathrm{wt}(b))\,. \tag{11.2}$$

Here s_i is the simple reflection $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle\alpha_i$. Let W be the Weyl group, i.e. the subgroup of $GL(P)$ generated by the s_i's.

THEOREM 11.1 ([**7**]). *If B is a normal crystal, then there exists a unique action $S : W \to Aut(B)$ of W on the set B such that $S_{s_i} = S_i$ for any $i \in I$.*

We have

$$\mathrm{wt}(S_w b) = w(\mathrm{wt}(b)) \quad \text{for any } w \in W, \text{ and } b \in B. \tag{11.3}$$

REMARK 11.1. Even if $w\alpha_i = \alpha_j$, $S_w \circ \tilde{e}_i = \tilde{e}_j \circ S_w$ does not hold in general.

12. Global bases

12.1. Introduction. So far, we discussed crystal bases, which is a local base at $q = 0$. In fact, starting from this we can construct a true base of the representation $V(\lambda)$.

This is based on the following observation.

12.2. Balanced triple. Let k be a field and $K = k(q)$. Let $A = \{f(q) \in K; f \text{ is regular at } q = 0\}$ and $A_\infty = \{f(q) \in K; f \text{ is regular at } q = \infty\}$.

Let V be a K-vector space. We regard V as the space of meromorphic sections of a vector bundle on $\mathbb{P}^1 = \mathrm{Spec}(k[q]) \cup \mathrm{Spec}(k[q^{-1}])$. Then a local base (L, B) at $q = 0$ defines the vector bundle on a neighborhood of $q = 0$ and a base of the stalk of the vector bundle at $q = 0$. If we can extend this vector bundle globally to $\mathbb{P}^1$ and if this vector bundle is trivial, then any section at $q = 0$ uniquely extends to a global section. This gives a base of V as a K-vector space.

More precisely, let L be an A-lattice of V, L_∞ an A_∞-lattice of V and $V_{k[q,q^{-1}]}$ a $k[q, q^{-1}]$-lattice of V.

PROPOSITION 12.1. *Set* $E = L \cap L_\infty \cap V_{k[q,q^{-1}]}$. *Then the following four conditions are equivalent.*

(12.1) $E \to L/qL$ *is an isomorphism.*

(12.2) $E \to L_\infty/q^{-1}L_\infty$ *is an isomorphism.*

(12.3) $(L \cap V_{k[q,q^{-1}]}) \oplus (q^{-1}L_\infty \cap V_{k[q,q^{-1}]}) \to V_{k[q,q^{-1}]}$ *is an isomorphism.*

(12.4) $A \underset{k}{\otimes} E \to L$, $A_\infty \underset{k}{\otimes} E \to L_\infty$, $k[q,q^{-1}] \underset{k}{\otimes} E \to V_{k[q,q^{-1}]}$ *and* $K \underset{k}{\otimes} E \to V$ *are isomorphisms.*

In a geometric language, $(L, L_\infty, V_{k[q,q^{-1}]})$ determines a quasi-coherent sheaf $\mathcal{V}$ on $\mathbb{P}^1$, and $V_{k[q,q^{-1}]}$ corresponds to the space of its sections over $\mathbb{P}^1 \backslash \{0,\infty\}$, E to the space of global sections, L/qL to the stalk at $q=0$ and $L_\infty/q^{-1}L_\infty$ to the stalk at $q=\infty$. The kernel and the cokernel of the homomorphism in (12.3) correspond to the 0-th and the first cohomology of $\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$. The above four conditions are equivalent to the triviality of $\mathcal{V}$.

DEFINITION 12.1. *If* $(L, L_\infty, V_{k[q,q^{-1}]})$ *satisfies the equivalent four conditions above, we call* $(L, L_\infty, V_{k[q,q^{-1}]})$ *balanced.*

In such a case, let us denote by $G : L/qL \to E$ the inverse of the isomorphism in (12.1). If B is a base of L/qL, then $G(B)$ becomes a base of V. We call it a *global base* of V.

12.3. Global bases of $V(\lambda)$**.** Let us apply the arguments in the preceding section to obtain a true base of $V(\lambda)$. Let $U_q(\mathfrak{g})_{k[q,q^{-1}]}$ denote the $k[q,q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$, $q(h)$ and $\left\{ {q(h) \atop n} \right\}$ $(n \geq 1, i \in I, h \in P^*)$. Let $U_q^+(\mathfrak{g})_{k[q,q^{-1}]}$ (resp. $U_q^-(\mathfrak{g})_{k[q,q^{-1}]}$) denote the $k[q,q^{-1}]$-subalgebra generated by the $e_i^{(n)}$'s (resp. the $f_i^{(n)}$'s). Then

$$U_q(\mathfrak{g})_{k[q,q^{-1}]} \cong U_q^-(\mathfrak{g})_{k[q,q^{-1}]} \underset{k[q,q^{-1}]}{\otimes} \mathcal{T}_{k[q,q^{-1}]} \otimes U_q^+(\mathfrak{g})_{k[q,q^{-1}]} .$$

Here $\mathcal{T}_{k[q,q^{-1}]}$ is the $k[q,q^{-1}]$-subalgebra of $K[P^*]$ generated by $q(h)$ and $\left\{ {q(h) \atop n} \right\}$.

Let $- : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the ring automorphism given by

$$\begin{aligned} q^- &= q^{-1} , \\ e_i^- &= e_i , \\ f_i^- &= f_i , \\ q(h)^- &= q(-h) . \end{aligned}$$

For $\lambda \in P_+$, let $- : V(\lambda) \to V(\lambda)$ be the map $Pu_\lambda \mapsto \bar{P}u_\lambda$ $(P \in U_q(\mathfrak{g}))$. This is well-defined. Then $L(\lambda)^-$ is an A_∞-module. Set

$$V(\lambda)_{k[q,q^{-1}]} = U_q(\mathfrak{g})_{k[q,q^{-1}]}u_\lambda = U_q^-(\mathfrak{g})_{k[q,q^{-1}]}u_\lambda .$$

THEOREM 12.1 ([**3**]). $\big(L(\lambda), L(\lambda)^-, V(\lambda)_{k[q,q^{-1}]}\big)$ *is a balanced triple.*

Hence if we denote by G_λ the inverse of the isomorphism $L(\lambda) \cap L(\lambda)^- \cap V(\lambda)_{k[q,q^{-1}]} \xrightarrow{\sim} L(\lambda)/qL(\lambda)$, then we have

$$V(\lambda) = \bigoplus_{b \in B(\lambda)} KG_\lambda(b). \tag{12.5}$$

We call $\{G_\lambda(b)\}_{b \in B(\lambda)}$ the *global base* of $V(\lambda)$.
We have

$$G_\lambda(b)^- = G_\lambda(b) \tag{12.6}$$

Similarly we have

THEOREM 12.2 ([**3**]). *$\big(L(\infty), L(\infty)^-, U_q^-(\mathfrak{g})_{k[q,q^{-1}]}\big)$ is a balanced triple.*

Hence if we denote by G_∞ the inverse of the isomorphism

$$L(\infty) \cap L(\infty)^- \cap U_q^-(\mathfrak{g})_{k[q,q^{-1}]} \xrightarrow{\sim} L(\infty)/qL(\infty),$$

we have

$$U_q^-(\mathfrak{g}) = \bigoplus_{b \in B(\infty)} KG_\infty(b). \tag{12.7}$$

We call $\{G_\infty(b)\}_{b \in B(\infty)}$ the global base of $U_q^-(\mathfrak{g})$.

If we denote $\bar{\pi}_\lambda : L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda)$ the map induced by $\pi_\lambda : U_q^-(\mathfrak{g}) \to V(\lambda)$ given by $\pi_\lambda(P) = Pu_\lambda$ (see §8.1), we have

$$G_\infty(b)u_\lambda = G_\lambda(\bar{\pi}_\lambda(b)) \quad \text{for } b \in B(\infty). \tag{12.8}$$

We have also

$$\begin{aligned} G_\infty(b)^- &= G_\infty(b)\,, \\ G_\infty(b)^* &= G_\infty(b^*)\,. \end{aligned} \tag{12.9}$$

Similarly we can define the global base of $\tilde{U}_q(\mathfrak{g})$. We define the ring automorphism $-$ of $\tilde{U}_q(\mathfrak{g})$ by $(Pa_\lambda)^- = \bar{P}a_\lambda$ and set

$$\tilde{U}_q(\mathfrak{g})_{k[q,q^{-1}]} = \bigoplus_{\lambda \in P} U_q(\mathfrak{g})_{k[q,q^{-1}]}a_\lambda\,.$$

THEOREM 12.3 (LUSZTIG[**28**]). *$\big(L(\tilde{U}_q(\mathfrak{g})), L(\tilde{U}_q(\mathfrak{g}))^-, \tilde{U}_q(\mathfrak{g})_{k[q,q^{-1}]}\big)$ is a balanced triple.*

Hence $\tilde{U}_q(\mathfrak{g})$ has also global bases.

EXAMPLE 12.1. If $\mathfrak{g} = \mathfrak{sl}_2$, the global bases of $U_q^-(\mathfrak{g})$ is $\{f^{(n)}; n \geq 0\}$.

EXAMPLE 12.2. If $\mathfrak{g} = \mathfrak{sl}_3$, $I = \{1, 2\}$, the global bases of $U_q^-(\mathfrak{g})$ is

$$\{f_1^{(l)} f_2^{(m)} f_1^{(n)}; m \geq l + n\} \cup \{f_2^{(l)} f_1^{(m)} f_2^{(n)}; m \geq l + n\}.$$

Note that $f_1^{(l)} f_2^{(n+l)} f_1^{(n)} = f_2^{(n)} f_1^{(n+l)} f_2^{(l)}$ and those bases are mutually distinct except those cases.

EXAMPLE 12.3. The global base of $\tilde{U}_q(\mathfrak{sl}_2)$ is given by

$$f^{(m)}e^{(n)}a_\lambda \quad \text{where } \langle h, \lambda\rangle \geq m-n \text{ and} \tag{12.10}$$

$$e^{(n)}f^{(m)}a_\lambda \quad \text{where } \langle h, \lambda\rangle < m-n \tag{12.11}$$

Remark that $f^{(m)}e^{(n)}a_\lambda = e^{(n)}f^{(m)}a_\lambda$ if $\langle h, \lambda\rangle = m-n$.

12.4. Properties of global bases. The global bases of $V(\lambda)$ have good properties.

LEMMA 12.1.

$$\begin{aligned} f_iG(b) &= [\varepsilon_i(b)+1]G(\tilde{f}_ib) + \sum_{\varepsilon_j(b')\geq\varepsilon_j(b)+\langle h_j,\alpha_i\rangle} F^i_{b,b'}G(b')\,, \\ e_iG(b) &= [\varphi_i(b)+1]G(\tilde{e}_ib) + \sum_{\varphi_j(b')\geq\varphi_j(b)+\langle h_j,\alpha_i\rangle} E^i_{bb'}G(b')\,. \end{aligned}$$

Remark that, as for the remainder terms $G(b')$ $(\varepsilon_j(b') \geq \varepsilon_j(b) + \langle h_j, \alpha_i\rangle)$, b' belongs to an i-string whose length is greater than the length of the i-string containing b.

LEMMA 12.2.

$$\begin{aligned} f_i^m V(\lambda) &= \bigoplus_{\varepsilon_i(b)\geq m} KG_\lambda(b) \\ e_i^m V(\lambda) &= \bigoplus_{\varphi_i(b)\geq m} KG_\lambda(b) \end{aligned}$$

Let $w \in W$ be an element of the Weyl group. Then $\dim V(\lambda)_{w\lambda} = 1$ and let $u_{w\lambda} \in V(\lambda)_{w\lambda}$ be a global basis. Then this is explicitly given by

$$u_{w\lambda} = f_i^{(\langle h_i, w'\lambda\rangle)}u_{w'\lambda}$$

if $w = s_iw'$, $l(w) = 1 + l(w')$. Here l is the length function. This follows from Lemma 12.1.

Now let $w = s_{i_1}\cdots s_{i_l}$ be a reduced expression. Then we can easily see

$$U_q^+(\mathfrak{g})u_{w\lambda} = \sum_{a_1,\dots,a_l\geq 0} Kf_{i_1}^{a_1}\cdots f_{i_l}^{a_l}u_\lambda.$$

Set

$$B_w(\lambda) = \{\tilde{f}_{i_1}^{a_1}\cdots\tilde{f}_{i_l}^{a_l}u_\lambda; a_1,\dots a_l \in \mathbb{Z}_{\geq 0}\}\backslash\{0\} \subset B(\lambda).$$

Then we have ([**5**])

THEOREM 12.4. $U_q^+(\mathfrak{g})u_{w\lambda} = \bigoplus_{b\in B_w(\lambda)} KG_\lambda(b)$.

Let $\leqq$ denote the Bruhat order on W. Then for $w' \leqq w$, we have

$$B_{w'}(\lambda) \subset B_w(\lambda)\,. \tag{12.12}$$

Set $\bar{B}_w(\lambda) = B_w(\lambda) \setminus \bigcup_{w' \lneqq w} B_{w'}(\lambda)$.

Then we have the decomposition (Littelmann[**17**])

$$B(\lambda) = \bigsqcup_{w \in W} \bar{B}_w(\lambda). \tag{12.13}$$

The set $B_w(\lambda)$ has the following property. For any $i \in I$ and any i-string $S \subset B(\lambda)$, we have one of the following three cases :

$$B_w(\lambda) \supset S\,. \tag{12.14}$$

$$B_w(\lambda) \cap S = \{\varnothing\}\,. \tag{12.15}$$

$$B_w(\lambda) \cap S \quad \text{consists of the highest weight vector of } S. \tag{12.16}$$

The Demazure character formula is obtained easily from this fact (see [**5**]).

13. Problems

PROBLEM 1. *Describe explicitly the crystal graph of* $B\big(\widetilde{U}_q(\mathfrak{g})\big)$, *when* $\mathfrak{g}$ *is affine.*

By Theorem 9.4, if $\mathfrak{g}$ is finite-dimensional, we have an analogue of Peter-Weyl theorem:

$$B\big(\tilde{U}_q(\mathfrak{g})\big) \cong \bigoplus_{\lambda \in P_+} B(\lambda) \otimes B(-\lambda)$$

as a crystal over $\mathfrak{g} \oplus \mathfrak{g}$. In the affine case, we have a similar result for non-zero level. More precisely, let $c \in \sum_i \mathbb{Z}_{>0} h_i$ be a center of $\mathfrak{g}$ (i.e.$\langle c, \alpha_i \rangle = 0$). Then Theorem 9.4 implies

$$B\big(\tilde{U}_q(\mathfrak{g})\big) = B\big(\tilde{U}_q(\mathfrak{g})\big)_+ \oplus B\big(\tilde{U}_q(\mathfrak{g})\big)_0 \oplus B\big(\tilde{U}_q(\mathfrak{g})\big)_-$$

where $B\big(\tilde{U}_q(\mathfrak{g})\big)_*$ consists of vectors b such that $\langle c, \mathrm{wt}(b) \rangle > 0, = 0, < 0$ according to $* = +, 0, -$. Then

$$B\big(\tilde{U}_q(\mathfrak{g})\big)_\pm \cong \bigoplus_{\substack{\lambda \in P_\pm \\ \langle c, \lambda \rangle \gtrless 0}} B(\lambda) \otimes B(-\lambda)$$

as a crystal over $\mathfrak{g} \oplus \mathfrak{g}$. However, we know very little about the structure of $B\big(\tilde{U}_q(\mathfrak{g})\big)_0$. (cf. [**15**])

PROBLEM 2. *For* $\lambda \in P_+$ *and* $\xi \in P$, *calculate* $\det\big(G_\lambda(b), G_\lambda(b')\big)_{b, b' \in B(\lambda)_\xi}$.

There is a unique symmetric bilinear form $(\cdot,\cdot)$ on $V(\lambda)$ such that

$$\begin{cases} (u_\lambda, u_\lambda) = 1, \\ (e_i u, v) = (u, f_i v), \\ (q(h)u, v) = (u, q(h)v). \end{cases}$$

Then for $b, b' \in B(\lambda)_\xi$, $\varphi_{bb'}(q) = \big(G_\lambda(b), G_\lambda(b')\big)$ belongs to $\mathbb{Z}[q+q^{-1}]$. Furthermore we have

$$\varphi_{bb'}(q) \in q^{((\xi,\xi)-(\lambda,\lambda))/2}(\delta_{bb'} + q\mathbb{Z}[q]).$$

We consider q as a non-zero element of an arbitrary field K and $U_q(\mathfrak{g})$ as an algebra over K. Then the $U_q(\mathfrak{g})$-module $V(\lambda)$ is not irreducible if and only if q is a root of $\det\big(G_\lambda(b), G_\lambda(b')\big)_{b,b' \in B(\lambda)_\xi} = 0$ for some ξ. On the other hand, $V(\lambda)$ is (believed to be) irreducible unless q is a root of unity. Hence $\det\big(G_\lambda(b), G_\lambda(b')\big)_{b,b'}$ must be a product of cyclotomic polynomials in q (and a power of q).

The similar and easier problem may be asked for the global base of $U_q^-(\mathfrak{g})$ and the Verma modules.

PROBLEM 3. *Has a crystal graph for non-symmetrizable $\mathfrak{g}$ a meaning ?*

The procedure in §8.2 is applied also to non-symmetrizable case. Namely there exists a crystal $B(\infty)$ with an embedding $B(\infty) \hookrightarrow B(\infty) \otimes B_i$ for any i. Hence $B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda$ can be also defined. However $U_q(\mathfrak{g})$ is not known for non-symmetrizable $\mathfrak{g}$.

PROBLEM 4. *Is there an analogue of crystal base and global base for the q-analogue of other algebra cases ?*

The Hecke algebra case is already known (Kazhdan-Lusztig polynomials). Other cases such as generalized Kac-Moody Lie algebras of Borcherds are not known, as far as I know.

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