RIMS

Perfect Crystals of Quantum affine Lie algebras

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0. Introduction

In 1985, while studying the solutions of the quantum Yang-Baxter equation, Drinfeld [D1] and Jimbo [J1] independently have discovered a fundamental algebraic object known as *quantized universal enveloping algebra* or *quantum group* $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ which may be thought of as a q-analogue or q-deformation of the universal enveloping algebra of $\mathfrak{g}$. The quantized universal enveloping algebra has a Hopf algebra structure and thus allows the tensor product structure on their representations. The quantized universal enveloping algebra associated with an affine Lie algebra is also known as a *quantum affine Lie algebra*. In [L] (also see [R]), it has been shown that for generic q (i.e., q is not a root of unity) the integrable representations of a Kac-Moody Lie algebra can be deformed consistently to those of the corresponding quantized universal enveloping algebra. In particular, the internal structure of the integrable highest weight representations of an affine Lie algebra is essentially the same as that of the corresponding quantum affine Lie algebra. However, working in the larger context of a quantum affine Lie algebra often it becomes easier to extract more informations about the representations of the corresponding affine Lie algebra by using the power of abstraction in representation theory.

The eminent role of the quantized universal enveloping algebras in two dimensional solvable lattice models is widely known. The R-matrices, which are the intertwiners of tensor product representations, give the Boltzmann weights of the lattice models with commuting transfer matrices ([J2]). The quantum parameter q corresponds to temperature in the lattice model. In particular, $q = 0$ corresponds to the absolute temperature zero in the lattice model. So one can expect that the quantized universal enveloping algebra has a simpler structure in that case. Motivated by this, Kashiwara introduced the notion of *crystal base* and proved the existence and uniqueness of this base for all integrable representation of $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is any symmetrizable Kac-Moody Lie algebra ([K1-K4]).

We first recall the basic concepts of crystal base theory. A *crystal* is a set B endowed with the maps $\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\}$ $(i \in I)$ which satisfy (i) $\tilde{e}_i 0 = \tilde{f}_i 0 = 0$, (ii) for any b and i, there is $n > 0$ such that $\tilde{e}_i^n b = \tilde{f}_i^n b = 0$, (iii)

[◇] supported in part by NSA/MSP Grant ♯ MDA 90-H-4039 for $b, b' \in B$ and $i \in I$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$. A crystal may be regarded as a colored (by I) oriented graph (also known as *crystal graph*) by defining arrows for $b, b' \in B$, $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$. For an element b of a crystal B, we set $\varepsilon_i(b) = \max\{n \geq 0 |\ \tilde{e}_i^n b \in B\}$ and $\varphi_i(b) = \max\{n \geq 0 |\ \tilde{f}_i^n b \in B\}$. Let P be a weight lattice. A crystal B is called a *P-weighted crystal* if it has a decomposition $B = \sqcup_{\lambda \in P} B_\lambda$ such that $\tilde{e}_i B_\lambda \subset B_{\lambda + \alpha_i} \sqcup \{0\}$, $\tilde{f}_i B_\lambda \subset B_{\lambda - \alpha_i} \sqcup \{0\}$, and for any $i \in I$ and $b \in B_\lambda$ the equality $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \lambda \rangle$ holds.

Let B_1 and B_2 be two crystals. A *morphism* $\phi : B_1 \to B_2$ of crystals is defined to be a map ϕ from B_1 to B_2 that commutes with the action of $\tilde{e}_i$ and $\tilde{f}_i$. Here we understand $\phi(0) = 0$. Then the crystals and their morphisms form a category. For two crystals B_1 and B_2, we define their tensor product as follows. The underlying set is $B_1 \times B_2$. We write $b_1 \otimes b_2$ for (b_1, b_2). We understand $b_1 \otimes 0 = 0 \otimes b_2 = 0$. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are given by

$$\begin{aligned} \tilde{f}_i(b_1 \otimes b_2) &= \tilde{f}_i b_1 \otimes b_2 \ \text{ if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ &= b_1 \otimes \tilde{f}_i b_2 \ \text{ if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \\ \tilde{e}_i(b_1 \otimes b_2) &= \tilde{e}_i b_1 \otimes b_2 \ \text{ if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ &= b_1 \otimes \tilde{e}_i b_2 \ \text{ if } \varphi_i(b_1) < \varepsilon_i(b_2). \end{aligned}$$

Then $B_1 \otimes B_2$ is a crystal and the category of crystals is endowed with the structure of tensor category. If both B_1 and B_2 are P-weighted crystals, then so is $B_1 \otimes B_2$.

Let M be an integrable $U_q(\mathfrak{g})$-module. Then we have $M = \bigoplus_{\lambda \in P} M_\lambda$, with $\dim M_\lambda < \infty$. For each $i \in I$, any weight vector $u \in M_\lambda$ can be written uniquely as $u = \sum f_i^{(n)} u_n$, where $u_n \in M_{\lambda + n\alpha_i} \cap \ker e_i$ and n ranges over integers such that $n \geq 0$ and $< h_i, \lambda > + n \geq 0$. Define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ by

$$\tilde{e}_i u = \sum f_i^{(n-1)} u_n, \quad \tilde{f}_i u = \sum f_i^{(n+1)} u_n.$$

Let A be the subring of $\mathbf{Q}(q)$ consisting of $f \in \mathbf{Q}(q)$ that is regular at $q = 0$.

A *crystal lattice* L of an integrable $U_q(\mathfrak{g})$-module M is a free A-submodule of M such that $M \cong \mathbf{Q}(q) \otimes_A L$, $L = \oplus_{\lambda \in P} L_\lambda$ where $L_\lambda = L \cap M_\lambda$, and $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$. A *crystal base* of the integrable $U_q(\mathfrak{g})$-module M is a pair (L, B) such that (i) L is a crystal lattice of M, (ii) B is a $\mathbf{Q}$-base of L/qL, (iii) $B = \sqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda / qL_\lambda)$, (iv) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B \sqcup \{0\}$, and (v) for $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$ for $i \in I$. We sometimes replace the condition (ii) by : $B_{\mathrm{ps}} = B' \sqcup (-B')$ where B' is a $\mathbf{Q}$-base of L/qL. We call (L, B_{ps}) a *crystal pseudo-base* and $B_{\mathrm{ps}}/\{\pm 1\}$ the *associated crystal* of (L, B_{ps}).

Let P^+ be a set of dominant integral weights and $V(\Lambda)$ be an irreducible integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\Lambda \in P^+$ and highest weight vector v_Λ. Define $L(\Lambda)$ to be the smallest A-module containing v_Λ and stable under $\tilde{f}_i$'s. Set $B(\Lambda) = \{b \in L(\Lambda)/qL(\Lambda) |\ b = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} v_\Lambda \bmod qL(\Lambda)\} \setminus \{0\}$. Then the pair $(L(\Lambda), B(\Lambda))$ is a crystal base for $V(\Lambda)$ ([K2]). A crystal is called a *crystal with highest weight* if it is isomorphic to $B(\Lambda)$ for some $\Lambda \in P^+$.

The theory of crystal base provides a remarkably powerful combinatorial tool to study the internal structure of the integrable highest weight representations of symmetrizable Kac-Moody Lie algebras. In [MM], using the Fock space representations of $U_q(\hat{sl}(n))$, Misra and Miwa gave an explicit description of the crystal base for the level one representations of the quantum affine Lie algebra $U_q(\hat{sl}(n))$ in terms of certain infinite Young diagrams which are parametrized by certain paths that arise naturally in solvable lattice models. In [JMMO], this result was generalized to integrable highest weight representations of arbitrary level for $U_q(\hat{sl}(n))$. In [KN], Kashiwara and Nakashima gave an explicit combinatorial description of crystal bases of finite dimensional irreducible representations of $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a finite dimensional classical simple Lie algebras of A, B, C, D type.

Recently, we have developed the theory of affine crystals which has enabled us to study the integrable highest weight representations of arbitrary level for any quantum affine Lie algebra $U_q(\mathfrak{g})$ ([KMN^2]). We briefly summarize the main results of [KMN^2]. From now on we will assume $\mathfrak{g}$ to be an indecomposable affine Lie algebra over $\mathbf{Q}$ generated by $\{e_i, f_i \mid i \in I\}$ $(I = \{0, 1, \cdots, n\})$ and the Cartan subalgebra $\mathfrak{t}$. Note that $\dim\mathfrak{t} = n + 2$ and $\mathfrak{g}$ has a one-dimensional center spanned by the canonical central element c. Recall that $\{\alpha_i \mid i \in I\} \subset \mathfrak{t}^*$ denotes the set of simple roots, and $\{h_i \mid i \in I\} \subset \mathfrak{t}$ the set of simple coroots. Also P (resp. Q) denotes the weight (resp. root) lattice. Let $\delta \in Q^+$ be the generator of null roots (see [Kac]). Set $\mathfrak{t}_{cl} = \oplus_{i\in I}\mathbf{Q}h_i \subset \mathfrak{t}$ and $\mathfrak{t}^*_{cl} = (\oplus_{i\in I}\mathbf{Q}h_i \subset \mathfrak{t})^*$. Let $cl : \mathfrak{t}^* \longrightarrow \mathfrak{t}^*_{cl}$ denote the canonical morphism. We have an exact sequence:

$$0 \longrightarrow \mathbf{Q}\delta \longrightarrow \mathfrak{t}^* \longrightarrow \mathfrak{t}^*_{cl} \longrightarrow 0.$$

Then $\dim\mathfrak{t}^*_{cl} = n + 1$ and $\{\lambda \in \mathfrak{t}^*_{cl} \mid \lambda(c) = 0\} = \sum_{i\in I}\mathbf{Q}cl(\alpha_i)$. Note that $\delta - \alpha_0 \in \sum_{i=1}^{n}\mathbf{Z}\alpha_i$. We define a map $af : \mathfrak{t}^*_{cl} \to \mathfrak{t}^*$ satisfying: $cl \circ af = id$ and $af \circ cl(\alpha_i) = \alpha_i$ for $i \neq 0$. Observe that for $i \in I$ the fundamental weight $\Lambda_i \in af(\mathfrak{t}^*_{cl}) \subset \mathfrak{t}^*$ and the weight lattice $P = \sum_{i\in I}\mathbf{Z}\Lambda_i + \mathbf{Z}\delta \subset \mathfrak{t}^*$. We set $P_{cl} = cl(P) = \sum_{i\in I}\mathbf{Z}cl(\Lambda_i) \subset \mathfrak{t}^*_{cl}$. We call an element of P_{cl} a *classical weight* and an element of P an *affine weight*. Note that $af \circ cl(\Lambda_i) = \Lambda_i$. Recall that the quantum affine Lie algebra $U_q(\mathfrak{g})$ is a $\mathbf{Q}(q)$-algebra generated by $\{e_i, f_i \mid i \in I\} \cup \{q^h \mid h \in \mathfrak{t}\}$. Let $U'_q(\mathfrak{g})$ be a $\mathbf{Q}(q)$-algebra generated by $\{e_i, f_i \mid i \in I\} \cup \{q^h \mid h \in \mathfrak{t}_{cl}\}$. Then $U'_q(\mathfrak{g})$ is also a quantized universal enveloping algebra with P_{cl} as the weight lattice. A P_{cl}-weighted crystal is called a *classical crystal* and a P-weighted crystal is called an *affine crystal*.

Let B be a classical crystal. For $b \in B$, we set $\varepsilon(b) = \sum \varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \sum \varphi_i(b)\Lambda_i$. Note that $wt(b) = cl(\varphi(b) - \varepsilon(b))$. A $\mathbf{Z}$-valued function H on $B \otimes B$ is called an *energy function* on B if for any $i \in I$ and $b \otimes b' \in B \otimes B$ such that $\tilde{e}_i(b \otimes b') \neq 0$ we have

$$\begin{aligned} H(\tilde{e}_i(b \otimes b')) &= H(b \otimes b') \quad \text{if } i \neq 0, \\ &= H(b \otimes b') + 1 \quad \text{if } i = 0 \text{ and } \varphi_0(b) \geq \varepsilon_0(b'), \\ &= H(b \otimes b') - 1 \quad \text{if } i = 0 \text{ and } \varphi_0(b) < \varepsilon_0(b'). \end{aligned}$$

For a subset J of I, we denote by $U_q(\mathfrak{g}_J)$ the $\mathbf{Q}(q)$-algebra generated by $\{e_k, f_k \mid k \in J\} \cup \{q^h \mid h \in \mathfrak{t}_{cl}\}$. We call a classical crystal *virtual* if for any $i, j \in I$, regarded as an $\{i, j\}$-crystal, B is a disjoint union of the crystals of finite-dimensional integrable $U_q(\mathfrak{g}_{\{i,j\}})$-modules. Let $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ be the category of finite dimensional $U'_q(\mathfrak{g})$-modules which have weight decompositions with weights in P_{cl}. Let B be an associated crystal of a crystal pseudo-base of an object of $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$. Then $\langle c, \varepsilon(b)\rangle = \langle c, \varphi(b)\rangle$ for any element $b \in B$. Set $P^+_{cl} = \sum \mathbf{Z}\Lambda_i$ and $(P^+_{cl})_l = \{\lambda \in P^+_{cl} \mid \lambda(c) = l\}$ for $l \in \mathbf{Z}$. A classical crystal B is a *perfect crystal of level l* if B satisfies:

(i) $B \otimes B$ is connected.
(ii) There exists $\lambda_0 \in P_{cl}$ such that $wt(B) \subset \lambda_0 + \sum_{i\neq 0}\mathbf{Z}_{\leq 0}\alpha_i$ and $\sharp(B_{\lambda_0}) = 1$.

(iii) There is an object of $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ with a crystal pseudo-base of which B is an associated crystal.
(iv) For any $b \in B$, we have $\langle c, \varepsilon(b)\rangle \geq l$.
(v) The maps ε and φ from $B_l = \{b | \langle c, \varepsilon(b)\rangle = l\}$ to $(P_{cl}^+)_l$ are bijective.

Let $B(\Lambda)$ be the crystal with dominant integral highest weight Λ and denote by u_Λ the highest weight element of $B(\Lambda)$. Let B be a perfect crystal of level l with an energy function H, and let $\Lambda \in (P_{cl}^+)_l$ be a dominant integral weight of level l. In [KMN^2], we proved the following isomorphism of classical crystals

$$B(\Lambda) \otimes B \cong B(\Lambda + wt(b_0)),$$

where b_0 is the unique element in B such that $\varepsilon(b_0) = \Lambda$.

For $\Lambda \in (P_{cl}^+)_l$, let $b(\Lambda)$ be the unique element of B such that $\varphi(b(\Lambda)) = \Lambda$. Then by the above isomorphism, we have an isomorphism of classical crystals $B(\Lambda) \cong B(\varepsilon(b(\Lambda))) \otimes B$. We define the sequence $(b_k)_{k\geq 1}$ and $(\lambda_k)_{k\geq 1}$ inductively as follows: let $b_1 = b(\Lambda)$, $\lambda_1 = \varepsilon(b(\Lambda))$. For $k \geq 2$, define $b_k = b(\lambda_{k-1})$, $\lambda_k = \varepsilon(b_k)$. Then by repeating the above procedure, we obtain an isomorphism of classical crystals

$$\psi_k : B(\Lambda) \cong B(\lambda_k) \otimes B^{\otimes k}$$

given by $u_\Lambda \longmapsto u_{\lambda_k} \otimes b_k \otimes \cdots \otimes b_1$. Moreover, it can be shown that for any $b \in B(\Lambda)$, there exists $k > 0$ such that $\psi_k(b) \in u_{\lambda_k} \otimes B^{\otimes k}$. The sequence $(b_1, b_2, \cdots)$ is called the *ground-state path of weight* Λ. A Λ-*path in* B is, by definition, a sequence $p = (p(n))_{n\geq 1}$ in B such that $p(n) = b_n$ for all $n \gg 0$. Let $\mathcal{P}(\Lambda, B)$ denote the set of Λ- paths. Then we have the following realization of the crystal $B(\Lambda)$ as the set $\mathcal{P}(\Lambda, B)$ of Λ-paths :

$B(\Lambda)$ is isomorphic to $\mathcal{P}(\Lambda, B)$ by $B(\Lambda) \ni b \longmapsto p \in \mathcal{P}(\Lambda, B)$ where $\psi_k(b) = u_{\Lambda_k} \otimes p(k) \otimes \cdots \otimes p(1)$ for $k \gg 0$.

The weight of a path $p = (p(n))_{n\geq 1}$ in B is given by the following formula:

$$\begin{aligned} wt(p) = \Lambda &+ \sum_{k=1}^{\infty} (af(wtp(k)) - af(wtb_k)) \\ &- \left(\sum_{k=1}^{\infty} k\,(H(p(k+1) \otimes p(k)) - H(b_{k+1} \otimes b_k)) \right) \delta. \end{aligned}$$

Hence we have

$$\mathrm{ch}V(\Lambda) = \sum_{\mu \in \mathfrak{t}^*} \mathrm{dim}V(\Lambda)_\mu e^\mu = \sum_{p \in \mathcal{P}(\Lambda, B)} e^{wt(p)}.$$

The one point functions are the basic macroscopic quantities that describe the multi-phase structure of a given lattice model of statistical mechanics. For the 2 dimensional solvable lattice models, a method of computing the one point functions is known as the *corner transfer matrix method* ([B]), which reduces the 2 dimensional statistical sums of the one point functions to the 1 dimensional statistical sums over certain paths ([ABF]). To apply Baxter's corner transfer matrix method, it is required that the second inversion relations hold. Suppose that a $U_q'(\mathfrak{g})$-module V has a crystal pseudo-base and its associated crystal B is perfect of level l. In

[KMN^2], we showed that the second inversion relations hold for V. Thus for $a \in P_{cl}$ and $\Lambda \in (P_{cl})_{+l}$, the one point function $P(a|\Lambda)$ can be written in the form

$$P(a|\Lambda) = \frac{G(a)}{Z},$$

where

$$\begin{aligned} G(a) &= q^{-4(\rho,\Lambda - af(a))} \sum_{p \in \mathcal{P}(\Lambda,B)(a)} q^{4(\rho,\delta)\omega(p)}, \\ \mathcal{P}(\Lambda,B)(a) &= \{(p(k))_{k=1}^{\infty} \in \mathcal{P}(\Lambda,B) | a + \sum_{i=1}^{k} \mathrm{wt}\, p(i) = \sum_{i=1}^{k} \mathrm{wt}\, b_i \quad \text{for} \quad k \gg 0\}, \\ \omega(p) &= \sum_{k=1}^{\infty} k(H(p(k+1) \otimes p(k)) - H(b_{k+1} \otimes b_k)), \\ Z &= \sum_{a \in P_{cl}} G(a). \end{aligned}$$

Here $(b_k)_{k=1}^{\infty}$ is the ground state path of weight Λ. Let $V(\Lambda)$ be the irreducible highest weight module over $U_q(\mathfrak{g})$ with highest weight Λ. Using the realization of the crystal base $B(\Lambda)$ as the set $\mathcal{P}(\Lambda, B)$ of Λ-paths, we obtain the closed expression of the one point function $P(a|\Lambda)$ in terms of string functions for $U_q(\mathfrak{g})$:

$$P(a|\Lambda) = \frac{\sum_i \dim V(\Lambda)_{\lambda_a - i\delta} q^{-4(\rho,\lambda_a - i\delta)}}{\sum_{\mu \in \mathrm{wt}(V(\Lambda))} \dim V(\Lambda)_\mu q^{-4(\rho,\mu)}},$$

where $\lambda_a = \Lambda - af(a)$.

As we have seen so far, the perfect crystals play a crucial role in realizing the crystal bases of integrable irreducible representations of quantum affine Lie algebras and in computing the one point functions of vertex models. In this paper, we undertake an extensive study of perfect crystals for quantum affine Lie algebras. Let $U_q(\mathfrak{g})$ be a quantum affine Lie algebra of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, or $D_{n+1}^{(2)}$. For a given level l, we construct a finite dimensional irreducible representation V_l of $U'_q(\mathfrak{g})$ with a crystal pseudo-base such that its associated crystal is perfect of level l. The main idea of the construction can be explained as follows.

We first start with a suitable crystal B for $U_q(\mathfrak{g}_{I \setminus \{0\}})$ whose internal structure is explicitly described in [KN]. We define the 0-arrows in such a way that B becomes a virtual crystal for $U'_q(\mathfrak{g})$, and prove in a purely combinatorial way that B is a perfect crystal of level l.

On the other hand, let V be a certain finite dimensional module over $U_q(\mathfrak{g}_{I \setminus \{0\}})$ with a crystal base which is characterized by a polarization on V. Let us denote by B_1 the associated crystal. The explicit description of B_1 is given in [KN]. We define the actions of e_0, f_0, and q^{h_0} on V to make it an irreducible module for $U'_q(\mathfrak{g})$, and verify that the polarization on V for $U_q(\mathfrak{g}_{I \setminus \{0\}})$ is also a polarization for $U'_q(\mathfrak{g})$. The action of f_0 induces the 0-arrows on B_1 extending it to a crystal for $U'_q(\mathfrak{g})$. We show that B_1 is a perfect crystal of level 1. Using the global base ([K4]), we compute the R-matrix for V explicitly, and by the fusion construction, we obtain a certain finite dimensional submodule V_l of $V^{\otimes l}$ for $U'_q(\mathfrak{g})$. The polarization on V

for $U_q'(\mathfrak{g})$ induces a polarization on V_l. Our results on polarization show that V_l has a crystal pseudo-base and that its associated crystal is isomorphic to B as crystals for $U_q(\mathfrak{g}_J)$ for some subsets J of I. Now we prove that these isomorphisms can be uniquely extended to that of crystals for $U_q'(\mathfrak{g})$, and thus conclude that V_l has a crystal pseudo-base whose associated crystal is perfect of level l. The fundamental results on polarization and fusion construction used here are developed in Sections 2 and 3.

We illustrate our construction for the case of $U_q(C_2^{(1)})$ of level 2. Let $U_q(\mathfrak{g})$ be the quantum affine Lie algebra of type $C_2^{(1)}$, and let $I = \{0, 1, 2\}$ be the index set of simple roots for $U_q(\mathfrak{g})$. We start with the crystal $B = B(2\Lambda_2)$ of the finite dimensional irreducible representation $V(2\Lambda_2)$ with highest weight $2\Lambda_2$ for $U_q(C_2)$. The crystal graph structure of B is given in [KN].

Figure 1

We define the 0-arrows on B as shown in the following picture.

Figure 2

Then it is easy to verify that B is a virtual crystal for $U_q'(\mathfrak{g})$. By letting $\lambda_0 = 2(\Lambda_2 - \Lambda_0)$, we show that B is perfect of level 2. By [KN], we have $B \cong B(2(\Lambda_2 - \Lambda_0))$ as crystals for $U_q(\mathfrak{g}_{\{1,2\}})$ and $B \cong B(2(\Lambda_0 - \Lambda_1))$ as crystals for $U_q(\mathfrak{g}_{\{0,1\}})$.

On the other hand, let $V = V(\Lambda_2)$ be the finite dimensional irreducible representation of $U_q(C_2)$ with highest weight Λ_2. It has a crystal base and the structure of the associated crystal $B_1 = B(\Lambda_2)$ is given in [KN].

Figure 3

Since the dimension of each weight space is one, we may identify the elements of the global base for $V(\Lambda_2)$ with those of the crystal base. So we will use the same symbol for the global base as for the crystal base. We define the actions of e_0 and f_0 by

$$e_0 \begin{array}{|c|}\hline 1\\ 2\\ \hline\end{array} = \begin{array}{|c|}\hline 2\\ \bar{1}\\ \hline\end{array}, \quad e_0 \begin{array}{|c|}\hline 1\\ \bar{2}\\ \hline\end{array} = \begin{array}{|c|}\hline \bar{2}\\ \bar{1}\\ \hline\end{array}, \qquad \text{and if } u \neq \begin{array}{|c|}\hline 1\\ 2\\ \hline\end{array}, \begin{array}{|c|}\hline 1\\ \bar{2}\\ \hline\end{array}, \text{ then } e_0 u = 0.$$

$$f_0 \begin{array}{|c|}\hline 2\\ \bar{1}\\ \hline\end{array} = \begin{array}{|c|}\hline 1\\ 2\\ \hline\end{array}, \quad f_0 \begin{array}{|c|}\hline \bar{2}\\ \bar{1}\\ \hline\end{array} = \begin{array}{|c|}\hline 1\\ \bar{2}\\ \hline\end{array}, \qquad \text{and if } u \neq \begin{array}{|c|}\hline 2\\ \bar{1}\\ \hline\end{array}, \begin{array}{|c|}\hline \bar{2}\\ \bar{1}\\ \hline\end{array}, \text{ then } f_0 u = 0.$$

The action of q^{h_0} is given by the relation $q^{h_0} = q^{-h_1-h_2}$. It is straightforward to verify that V is a well-defined module for $U_q'(\mathfrak{g})$ with the actions given above. Let $(\ ,\ )$ be the polarization on the $U_q(C_2)$-module V. Then one can directly check that it is also a polarization for the $U_q'(\mathfrak{g})$-module V. The action of f_0 defines the 0-arrows on B_1 making it a crystal for $U_q'(\mathfrak{g})$.

Figure 4

With $\lambda_0 = \Lambda_2 - \Lambda_0$, it is easy to check that B_1 is a perfect crystal of level 1.

Let us denote by V_x the $U_q'(\mathfrak{g})$-module $\mathbf{Q}[x, x^{-1}] \otimes V$ with the actions of e_i,

f_i, and t_i given by $x^{\delta_{i0}}e_i$, $x^{-\delta_{i0}}f_i$, and t_i, respectively. Then there is a $U'_q(\mathfrak{g})$-linear map $R(x,y) : V_x \otimes V_y \to V_y \otimes V_x$ satisfying the Yang-Baxter equation. The map $R(x,y)$ depends only on the ratio x/y, and is called the *R-matrix* for V. As a $U_q(C_2)$-module, we have

$$V(\Lambda_2) \otimes V(\Lambda_2) \cong V(2\Lambda_2) \oplus V(2\Lambda_1) \oplus V(0).$$

Let $z = xy^{-1}$. Then up to a multiple of an element of $\mathbf{Q}(q)(z)$, the R-matrix $R(x,y) = R(z)$ can be computed explicitly:

$$R(z) = (1-q^4z)(1-q^6z)P_{2\Lambda_2} + (z-q^4)(1-q^4z)P_{2\Lambda_1} + (z-q^4)(z-q^6)P_0,$$

where $P_{2\Lambda_2}$, $P_{2\Lambda_1}$, and P_0 are the projections of $V(\Lambda_2)\otimes V(\Lambda_2)$ onto $V(2\Lambda_2)$, $V(2\Lambda_1)$, and $V(0)$, respectively. We take the $U'_q(\mathfrak{g})$-module V_2 to be the image of $V_{q^2} \otimes V_{q^{-2}}$ under $R(q^2, q^{-2}) = R(q^4)$ in $V_{q^2} \otimes V_{q^{-2}}$. Since $R(q^4) = (1-q^8)(1-q^{10})P_{2\Lambda_2}$, as a $U_q(\mathfrak{g}_{\{1,2\}})$-module, V_2 is isomorphic to $V(2(\Lambda_2 - \Lambda_0))$. Since the Weyl group of C_2 contains -1, it can be shown that V_2 is isomorphic to $V(2(\Lambda_0 - \Lambda_2))$ as a $U_q(\mathfrak{g}_{\{0,1\}})$-module. The polarization on V induces a polarization on V_2, and our results on the polarization show that V_2 admits a crystal pseudo-base. Hence by the above observation, its crystal B_2 is isomorphic to $B(2(\Lambda_2 - \Lambda_0))$ as a crystal for $U_q(\mathfrak{g}_{\{1,2\}})$, and is isomorphic to $B(2(\Lambda_0 - \Lambda_2))$ as a crystal for $U_q(\mathfrak{g}_{\{0,1\}})$. Thus we have two isomorphisms $\psi_0 : B \to B_2$ as crystals for $U_q(\mathfrak{g}_{\{1,2\}})$ and $\psi_2 : B \to B_2$ as crystals for $U_q(\mathfrak{g}_{\{0,1\}})$. Then both ψ_0 and ψ_2 are isomorphisms of B onto B_2 regarded as crystals for $U_q(\mathfrak{g}_{\{1\}})$. As a crystal for $U_q(\mathfrak{g}_{\{1\}})$, B splits into a direct sum of crystals with highest weight for $U_q(\mathfrak{g}_{\{1\}})$:

$$\begin{aligned} B \cong\ & B(2\Lambda_2 - 2\Lambda_0) \oplus B(2\Lambda_1 - 2\Lambda_0) \oplus B(-2\Lambda_0 + 4\Lambda_1 - 2\Lambda_2) \\ & \oplus B(2\Lambda_1 - 2\Lambda_2) \oplus B(2\Lambda_0 - 2\Lambda_2). \end{aligned}$$

Since the highest weights for $U_q(\mathfrak{g}_{\{1\}})$ are all distinct, ψ_0 and ψ_2 must coincide for highest weight elements for $U_q(\mathfrak{g}_{\{1\}})$, and hence for all the elements of B, which defines a unique isomorphism of B onto B_2 as crystals for $U'_q(\mathfrak{g})$. Thus we conclude that V_2 has a crystal pseudo-base whose associated crystal is perfect of level 2.

1. Results

In this section, we summarize the results of this article, which give explicit forms of perfect crystals of an arbitrary level for $\mathfrak{g} = A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n+1}^{(2)}$ and $D_{n+1}^{(2)}$.

1.1. *Perfect crystal* (see [KMN^2]). We assume that the rank of $\mathfrak{g}$ is greater than 2. We set $P_{cl}^+ = \{\lambda \in P_{cl} \mid \langle h_i, \lambda\rangle \geq 0 \text{ for any } i\} \cong \sum \mathbf{Z}_{\geq 0}\Lambda_i$ and $(P_{cl}^+)_l = \{\lambda \in P_{cl}^+ \mid \langle c, \lambda\rangle = l\} \cong \{\lambda \in \sum \mathbf{Z}\Lambda_i \mid \langle c, \lambda\rangle = l\}$ for $l \in \mathbf{Z}_{\geq 0}$.

Let B be a classical crystal. For $b \in B$, we set $\varepsilon(b) = \sum \varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \sum \varphi_i(b)\Lambda_i$. Note that $\mathrm{wt}(b) = cl(\varphi(b) - \varepsilon(b))$.

Definition 1.1.1. *For $l \in \mathbf{Z}_{>0}$, we say that B is a perfect crystal of level l if B satisfies the following conditions.*

(1.1.1) *$B \otimes B$ is connected.*

(1.1.2) *There exists $\lambda_0 \in P_{cl}$ such that $\mathrm{wt}(B) \subset \lambda_0 + \sum_{i \neq i_0} \mathbf{Z}_{\leq 0}\alpha_i$ and that $\sharp(B_{\lambda_0}) = 1$.*

(1.1.3) *There is a $U_q'(\mathfrak{g})$-module in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ with a crystal pseudo-base (L, B') such that B is isomorphic to $B'/\{\pm 1\}$.*

(1.1.4) *For any $b \in B$, we have $\langle c, \varepsilon(b)\rangle \geq l$.*

(1.1.5) *The maps ε and φ from $B_l = \{b \mid \langle c, \varepsilon(b)\rangle = l\}$ to $(P_{cl}^+)_l$ are bijective.*

We call an element of B_l minimal.

Let B be a perfect crystal of level l. For $\lambda \in (P_{cl}^+)_l$, let $b(\lambda) \in B$ be the element such that $\varphi(b(\lambda)) = \lambda$. Let σ be the automorphism of $(P_{cl}^+)_l$ given by $\sigma\lambda = \varepsilon(b(\lambda))$. Then the conditions (4.5.1) and (4.5.2) in [KMN^2] are satisfied by taking $b_\nu = b(\sigma^{\nu-1}\lambda)$ and $\lambda_\nu = \sigma^\nu\lambda$. Hence by Theorem 4.5.2 in [KMN^2] we have the following result.

Proposition 1.1.2. *([KMN^2]) For $\lambda \in (P_{cl}^+)_l$, let $\mathcal{P}(\lambda, B)$ be the set of sequences $\{p(n)\}_{n\geq 1}$ in B such that $p(n) = b(\sigma^{n-1}\lambda)$ for $n > 0$. Then $B(\lambda)$ is isomorphic to $\mathcal{P}(\lambda, B)$ by $B(\lambda) \ni b \longmapsto u_{\sigma^k\lambda} \otimes p(k) \otimes \cdots \otimes p(1)$ for $k > 0$.*

1.2. $(A_n^{(1)}, B(l\Lambda_k))$ $(n \geq 2, 1 \leq k \leq n)$

Let $I = \mathbf{Z}/(n+1)\mathbf{Z}$ be the index set of the simple roots for the affine quantized enveloping algebra of type $A_n^{(1)}$ and let $J = \{1, \ldots, n\}$ be that of type A_n. For $i \in I$ we define $\iota^{(i)} : J \longrightarrow I$ by $\iota^{(i)}(j) = i + j \bmod n+1$, and $\bar{\iota}^{(i)} : J \longrightarrow I$ by $\bar{\iota}^{(i)}(j) = i - j \bmod n+1$. We also define $\iota^{(0,k)} : J\backslash\{k\} \longrightarrow I$ by $\iota^{(0,k)}(j) = j \bmod n+1$ and $\bar{\iota}^{(0,k)} : J\backslash\{k\} \longrightarrow I$ by $\bar{\iota}^{(0,k)}(j) = k - j \bmod n+1$.

Proposition 1.2.1. *For any integers k, l such that $1 \leq k \leq n$, $l \geq 1$, there exists a unique crystal $B^{k,l}$ of type $A_n^{(1)}$ such that $\iota^{(i)*}(B^{k,l}) = B(l\Lambda_k)$ and $\bar{\iota}^{(i)*}(B^{k,l}) = B(l\Lambda_{k'})$ for all i, where $k' = n + 1 - k$.*

Theorem 1.2.2. *$B^{k,l}$ is perfect of level l.*

Let $B = B(l\Lambda_k)$ $(1 \leq k \leq n)$ be the crystal of type A_n as described in [KN]. Set $K = \{1, 2, \cdots, n, n+1\}$. With each $b \in B$, we associate a table $(m_{j,j'})_{\{1\leq j\leq k, 1\leq j'\leq l\}} = m(b)$ where $m_{j,j'} \in K$, $m_{j,j'} \leq m_{j,j'+1}$ and $m_{j,j'} < m_{j+1,j'}$. Furthermore, we associate another table $(x_{j\,i})$ with $m(b)$. Set

$$\begin{aligned}&x_{0\,0} = l, \quad x_{0\,1} = x_{0\,2} = \cdots x_{0\,n+1} = 0,\\ &x_{j\,i} = \sharp\{j' | m_{j,j'} = i\} \quad (1 \leq j \leq k,\, 0 \leq i \leq n+1),\\ &y_{j\,i} = \sum_{j \leq j' \leq i} x_{j\,j'} \quad (0 \leq j \leq k,\, 0 \leq i \leq n+1).\end{aligned}$$

Then $(x_{j\,i}) = x(b)$ $(b \in B)$ satisfies

$$\begin{aligned}&x_{j\,i} = 0 \quad \text{unless } 0 \leq j \leq i \leq n+1+j-k \quad ,\\ &y_{j\,n+1+j-k} = l \quad (0 \leq j \leq k),\\ &y_{j\,i} \geq y_{j+1\,i+1} \quad (0 \leq j \leq i \leq n+j-k).\end{aligned}$$

Theorem 1.2.3. *An element $b \in B^{k,l}$ is minimal if and only if*

$$x_{ji} = x_{j-1\,i-1} \quad \textit{for } 2 \leq j \leq k \textit{ and } j+1 \leq i \leq j+k'-1.$$

The proofs of Proposition 1.2.1, Theorem 1.2.2 and Theorem 1.2.3 are given in 6.3.

Remark 1.2.4. By proposition 1.2.1, we may take $l(\Lambda_k - \Lambda_0)$ for $B^{k,l}$ as λ_0 in (1.1.2).

Now, we give the description of the bijection $\sigma\,:\,(P_{cl}^{+})_l \longrightarrow (P_{cl}^{+})_l$ for $A_n^{(1)}$. When $\lambda \in (P_{cl}^{+})_l$ can be written $\lambda = \sum_{i=0}^{n} m_i\Lambda_i$, we shall use the notation $\lambda = (m_0, m_1, \cdots, m_n)$.

Proposition 1.2.5. *For a perfect crystal $B^{k,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^{+})_l$,*

$$\sigma\,:\,(m_0, m_1, \cdots, m_{k'-1}, m_{k'}, \cdots, m_n) \longmapsto (m_k, m_{k+1}, \cdots, m_n, m_0, \cdots, m_{k-1}).$$

Proof. Let b be a minimal element in $B^{k,l}$. By Theorem 1.2.3, we obtain

$$\begin{aligned}\varphi(b) &= x_{1\,k'+1}\Lambda_0 + \sum_{i=1}^{k-1}(x_{ii} - x_{i+1\,i+1})\Lambda_i + \sum_{i=k}^{n} x_{ki}\Lambda_i,\\ \varepsilon(b) &= x_{kk}\Lambda_0 + \sum_{i=1}^{k'} x_{1\,i+1}\Lambda_i + \sum_{i=k'+1}^{n}(x_{i-k'+1\,i+1} - x_{i-k'\,i})\Lambda_i.\end{aligned}$$

By the restrictions on x_{ji}'s and Theorem 1.2.3, $x_{1\,i+1} = x_{k\,k+i}$ $(1 \leq i \leq k'-1)$ and $x_{ii} - x_{i+1\,i+1} = x_{i+1\,i+k'+1} - x_{i\,i+k'}$ $(i \leq i \leq k-1)$. Hence, we can get the desired result. □

1.3. $(C_n^{(1)}, B(l\Lambda_n))$ $(n \geq 2)$

Let $I = \{0, 1, \ldots, n\}$ be the index set of the simple roots for the quantized universal enveloping algebra of type $C_n^{(1)}$, and let $J = \{1, \ldots, n\}$ be that of type C_n. We define $\iota, \bar{\iota} : J \longrightarrow I$ by $\iota(j) = j$, $\bar{\iota}(j) = n - j$.

Proposition 1.3.1. *For any integer $l \geq 1$, there exists a unique crystal $B^{n,l}$ of type $C_n^{(1)}$ such that $\iota^*(B^{n,l}) = B(l\Lambda_n)$ and $\bar{\iota}^*(B^{n,l}) = B(l\Lambda_n)$.*

Theorem 1.3.2. *$B^{n,l}$ is perfect of level l.*

Let $B = B(l\Lambda_n)$ be the crystal of type C_n as described is [KN]. Set $K = \{1, 2, \ldots, n, \bar{n}, \ldots, \bar{2}, \bar{1}\}$. Sometimes it is convenient to identify $\bar{i}$ with $2n+1-i$. With this identification the natural order of K reads as $1 < 2 \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}$. With each $b \in B$ we associate a table $(m_{jj'}) = m(b)$ where $m_{jj'} \in K, 1 \leq j \leq n, 1 \leq j' \leq l$. The restriction on $(m_{jj'})$ given in [KN] is translated as follows. Set

$$\begin{aligned}
&x_{00} = l, \quad x_{01} = \ldots = x_{0\,2n} = 0,\\
&x_{jk} = \sharp\{j' \mid m_{jj'} = k\} \quad (1 \leq j \leq n, 0 \leq k \leq 2n),\\
&y_{jk} = \sum_{j \leq k' \leq k} x_{jk'} \quad (0 \leq j \leq n, 0 \leq k \leq 2n).
\end{aligned}$$

Then $(x_{jk}) = x(b) \quad (b \in B)$ satisfies

$$\begin{aligned}
&x_{jk} = 0 \quad \text{unless} \quad 0 \leq j \leq k \leq j+n \leq 2n,\\
&y_{j\,n+j} = l \quad (0 \leq j \leq n),\\
&y_{j^*\,\overline{i+1}} - y_{ji} = 0 \quad \text{or} \quad 1 \quad (j^* = n+j-i \quad \text{and} \quad 1 \leq j \leq i \leq n),\\
&y_{jk} \geq y_{j+1\,k+1} \quad (0 \leq j \leq n-1, 0 \leq k \leq 2n-1).
\end{aligned}$$

We also use

$$\begin{aligned}
&l_0 = l, \quad l_j = l - l_{j+n} = y_{jn} \quad (1 \leq j \leq n),\\
&z_{jj^*} = y_{ji} + y_{j^*\,\overline{i+1}} \quad (0 \leq j \leq j^* \leq n),\\
&z_{j\,j-1} = l_j + l_{j-1} \quad (1 \leq j \leq n),\\
&\omega_{ji} = z_{j-1\,j^*} + z_{j\,j^*-1} - z_{j-1\,j^*-1} - z_{j\,j^*} \quad (1 \leq j \leq i \leq n).
\end{aligned}$$

Note that $z_{0j} = 2l$ and $z_{jj} = 2l_j \quad (0 \leq j \leq n)$ and also that $\omega_{1i} = z_{1\,n-i} - z_{1\,n+1-i} \geq 0$.

Theorem 1.3.3. *For $b \in B^{n,l}$ the equality $\langle c, \varepsilon(b)\rangle = \sum_{i=0}^{n} \varepsilon_i(b) = l$ holds if and only if $z_{j\,j^*} = l_j + l_{j^*}$.*

The proofs of Proposition 1.3.1, Theorem 1.3.2 and Theorem 1.3.3 are given in 6.4.

Remark 1.3.4. By Proposition 1.3.1, we may take $l(\Lambda_n - \Lambda_0)$ for $B^{n,l}$ as λ_0 in (1.1.2).

Proposition 1.3.5. *For a perfect crystal $B^{n,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^+)_l$,*

$$\sigma : (m_0, m_1, \cdots, m_{n-1}, m_n) \longmapsto (m_n, m_{n-1}, \cdots, m_1, m_0).$$

The proof is similar to that of Proposition 1.2.5, so we omit it.

1.4. $(D_{n+1}^{(2)}, B(l\Lambda_n)) \quad (n \geq 2)$

Let $I = \{0, 1, \ldots, n\}$ be the index set of the simple roots for the quantized universal enveloping algebra of type $D_{n+1}^{(2)}$, and let $J = \{1, \ldots, n\}$ be that of type B_n. We define $\iota, \bar{\iota} : J \longrightarrow I$ by $\iota(j) = j, \bar{\iota}(j) = n - j$.

Proposition 1.4.1. *For any integer $l \geq 1$, there exists a unique crystal $B^{n,l}$ of type $D_{n+1}^{(2)}$ such that $\iota^*(B^{n,l}) = B(l\Lambda_n)$ and $\bar{\iota}^*(B^{n,l}) = B(l\Lambda_n)$.*

Theorem 1.4.2. *$B^{n,l}$ is perfect of level l.*

Let $B = B(l\Lambda_n)$ be the crystal of type B_n. The description of this has been given in [KN], however we shall introduce simpler description of $B(l\Lambda_n)$. Set $K = \{1, 2, \ldots, n, \bar{n}, \ldots, \bar{2}, \bar{1}\}$. Sometimes it is convenient to identify $\bar{i}$ with $2n+1-i$. With this identification the natural order of K reads as $1 < 2 \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}$. With each $b \in B$ we associate a table $(m_{jj'}) = m(b)$ where $m_{jj'} \in K, 1 \leq j \leq n, 1 \leq j' \leq l$. The restriction on $(m_{jj'})$ is as follows. Set

$$\begin{aligned}
&x_{00} = l, \quad x_{01} = \ldots = x_{0\,2n} = 0, \\
&x_{jk} = \sharp\{j' \mid m_{jj'} = k\} \quad (1 \leq j \leq n, k \in K), \\
&y_{jk} = \sum_{j \leq k' \leq k} x_{jk'} \quad (0 \leq j \leq n, k \in K).
\end{aligned}$$

Then $(x_{jk}) = x(b) \quad (b \in B)$ satisfies

$$\begin{aligned}
&x_{jk} = 0 \quad \text{unless} \quad 0 \leq j \leq k \leq \overline{n-j+1}, \\
&y_{j\,\overline{n-j+1}} = l \quad (0 \leq j \leq n), \\
&y_{n+j-i\,\overline{i+1}} - y_{ji} = 0 \quad (1 \leq j \leq i < n), \\
&y_{jk} \geq y_{j+1\,k+1} \quad (0 \leq j \leq n-1, 0 \leq k \leq 2n-1).
\end{aligned}$$

Note that the condition $y_{n+j-i\,\overline{i+1}} - y_{ji} = 0 \quad (1 \leq j \leq i < n)$ implies that if $m_{j,i} = k$ (resp. $\bar{k}$), then there is no j' such that $m_{j',i} = \bar{k}$ (resp. k).

Theorem 1.4.3. *For $b \in B^{n,l}$ the equality*

$$\langle c, \varepsilon(b) \rangle = \varepsilon_0(b) + 2\sum_{i=1}^{n-1} \varepsilon_i(b) + \varepsilon_n(b) = l$$

holds if and only if $x_{j,i} = x_{j+1,i+1}$ for $1 \leq j < i < n$ and $x_{j,n} = x_{j+1,\bar{n}}$ for $1 \leq j < n$.

The proofs of Proposition 1.4.1, Theorem 1.4.2 and Theorem 1.4.3 are given in 6.5.

Remark 1.4.4. By proposition 1.4.1, we may take $l(\Lambda_n - \Lambda_0)$ for $B^{n,l}$ as λ_0 in (1.1.2).

Proposition 1.4.5. *For a perfect crystal $B^{n,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^+)_l$,*

$$\sigma : (m_0, m_1, \cdots, m_{n-1}, m_n) \longmapsto (m_n, m_{n-1}, \cdots, m_1, m_0).$$

The proof is similar to that of Proposition 1.2.5 , so we omit it.

1.5. *($D_n^{(1)}$,$B(l\Lambda_{n-1})$ and $B(l\Lambda_n)$)* $(n \geq 4)$

Let $I = \{0, 1, \cdots, n\}$ be the index set of the simple roots for the quantized universal enveloping algebra of type $D_n^{(1)}$, and let $J = \{1, 2, \cdots, n\}$ be that of type D_n. We define ι, $\bar{\iota}; J \longrightarrow I$ by $\iota(j) = j$, $\bar{\iota}(j) = n - j$.

Proposition 1.5.1. *For any integer $l \geq 1$, there exists a unique crystal $B^{n,l}$ (resp. $\overline{B}^{n,l}$) of type $D_n^{(1)}$ such that*
(i) if n is even, then $\iota^*(B^{n,l}) = B(l\Lambda_n)$ (resp. $\iota^*(\overline{B}^{n,l}) = B(l\Lambda_{n-1})$) and $\overline{\iota}^*(B^{n,l}) = B(l\Lambda_n)$ (resp. $\overline{\iota}^*(\overline{B}^{n,l}) = B(l\Lambda_{n-1})$),
(ii) if n is odd, then $\iota^*(B^{n,l}) = B(l\Lambda_n)$ (resp. $\iota^*(\overline{B}^{n,l}) = B(l\Lambda_{n-1})$) and $\overline{\iota}^*(B^{n,l}) = B(l\Lambda_{n-1})$ (resp. $\overline{\iota}^*(\overline{B}^{n,l}) = B(l\Lambda_n)$).

Theorem 1.5.2. *$B^{n,l}$ (resp. $\overline{B}^{n,l}$) is perfect of level l.*

Let $B = B(l\Lambda_n)$ and $B' = B(l\Lambda_{n-1})$ be the crystals of type D_n. The description of these have been given in [KN], however similarly to the preceding case, we shall introduce simpler descriptions. Set $K = \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\}$. We give the order on K as follows;

$$1 < 2 < \cdots n-1 < \begin{matrix} n \\ \overline{n} \end{matrix} < \overline{n-1} < \cdots < \overline{2} < \overline{1}.$$

Note that there is no order between n and $\overline{n}$. With each $b \in B$ (resp. B'), we associate a table $(m_{j\,j'}) = m(b)$ where $m_{j\,j'} \in K$ for $1 \leq j \leq n$ and $1 \leq j' \leq l$. The restriction on $(m_{j\,j'})$ is as follows. Set

$$\begin{aligned} &x_{0\,0} = l, \quad x_{0\,1} = \cdots = x_{0\,\overline{1}} = 0, \\ &x_{j\,k} = \sharp\{j' | m_{j\,j'} = k\} \quad (1 \leq j \leq n, \quad k \in K), \\ &y_{j\,k} = \sum_{j \leq k' \leq k} x_{j\,k'} \quad (0 \leq j \leq n, \quad k \in K). \end{aligned}$$

Then $(x_{j\,k}) = x(b) \quad (b \in B\,(\text{resp. } B'))$ satisfies
(1) $x_{j\,n} = 0$ if $n-j$ is odd (resp. even).
(2) $x_{j\,\overline{n}} = 0$ if $n-j$ is even (resp. odd).
(3) $x_{j\,k} = 0$ unless $0 \leq j \leq k \leq \overline{n-j+1}$.
(4) $y_{j\,\overline{n-j+1}} = l$ $(0 \leq j \leq n)$.
(5) $y_{n+j-i\,\overline{i+1}} = y_{j\,i}$ $(1 \leq j \leq i < n-1)$.
(6) $y_{j\,n} = y_{j-1\,n-1}$ if $n-j$ is even (resp. odd).
(7) $y_{j\,\overline{n}} = y_{j-1\,n-1}$ if $n-j$ is odd (resp. even).
(8) $y_{j\,k} \geq y_{j\,k'}$ $(0 \leq j \leq n-1, \ k \leq k')$.

Note that the conditions (1) and (2) imply that n and $\overline{n}$ cannot appear simultaneously in one row and the conditions (5),(6) and (7) imply that k and $\overline{k}$ cannot appear simultaneously in one column.

Theorem 1.5.3. *For $b \in B^{n,l}$ the equality*

$$\langle c, \varepsilon(b) \rangle = \varepsilon_0(b) + \varepsilon_1(b) + 2\sum_{k=2}^{n-2} \varepsilon_k(b) + \varepsilon_{n-1}(b) + \varepsilon_n(b) = l. \tag{1.5.1}$$

holds if and only if $x_{1\,i} = x_{2\,i+1} = \cdots = x_{n-i\,n-1} = x_{n-i+3\,\overline{n-1}}$ for $2 \leq i < n$.

The proofs of Proposition 1.5.1, Theorem 1.5.2 and Theorem 1.5.3 are given in 6.6.

Remark 1.5.4. By Proposition 1.5.1, we may take $l(\Lambda_n - \Lambda_0)$ (resp. $l(\Lambda_{n-1} - \Lambda_0)$) for $B^{n,l}$ (resp. $\overline{B}^{n-1,l}$) as λ_0 in (1.1.2).

Proposition 1.5.5. *For a perfect crystal $B^{n,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^+)_l$,*
(1) if n is even,

$$\sigma : (m_0, m_1, m_2, \cdots, m_{n-2}, m_{n-1}, m_n) \longmapsto (m_{n-1}, m_n, m_{n-2}, \cdots, m_2, m_0, m_1),$$

(2) if n is odd,

$$\sigma : (m_0, m_1, m_2, \cdots, m_{n-2}, m_{n-1}, m_n) \longmapsto (m_n, m_{n-1}, m_{n-2}, \cdots, m_2, m_0, m_1).$$

The proof is similar to that of Proposition 1.2.5 , so we omit it.

1.6. *$(A_{2n-1}^{(2)}, B(l\Lambda_1) \quad (n \geq 3))$*

Let $I = \{0, 1, \cdots, n\}$ be the index set of the simple roots for the quantized universal enveloping algebra of type $U_q(A_{2n-1}^{(2)})$ and let $J = \{1, \cdots, n\}$ be that of type $U_q(C_n)$. We define the maps $\iota_0, \iota_1 : J \to I$ by $\iota_0(j) = j$ for all $j \in J$ and $\iota_1(1) = 0$, $\iota_1(j) = j$ for $j \neq 1$.

Let $B(l\Lambda_1)$ be the crystal for $U_q(C_n)$ with highest weight $l\Lambda_1$. Set $K = \{1, \cdots, n, \bar{n}, \cdots, \bar{1}\}$ and consider the ordering on K given by

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

Then the elements of $B(l\Lambda_1)$ are labeled by $b = (b_k)_{k=1}^l$, where $b_k \in K$, $b_k \leq b_{k+1}$ for all k. Let $x_i(b) = \sharp\{k|\ b_k = i\}$, $\bar{x}_i(b) = \sharp\{k|\ b_k = \bar{i}\}$ for $i = 1, \cdots, n$. It is clear that $\sum x_i(b) + \sum \bar{x}_i(b) = l$.

Proposition 1.6.1. *For any integer $l \geq 1$, there exists a unique crystal $B^{1,l}$ for $U_q(A_{2n-1}^{(2)})$ such that $\iota_0^*(B^{1,l}) \cong B(l\Lambda_1)$ and $\iota_1^*(B^{1,l}) \cong B(l\Lambda_1)$ as crystals for $U_q(C_n)$.*

Theorem 1.6.2. *The crystal $B^{1,l}$ is perfect of level l.*

Theorem 1.6.3. *For $b \in B^{1,l}$, the equality*

$$\langle c, \varepsilon(b) \rangle = \varepsilon_0(b) + \varepsilon_1(b) + 2\sum_{i=2}^{n} \varepsilon_i(b) = l$$

holds if and only if $x_i(b) = \overline{x}_i(b)$ for $i = 2, \cdots, n$.

The proofs of Proposition 1.6.1, Theorem 1.6.2 and Theorem 1.6.3 are given in 6.7.

Remark 1.6.4. By proposition 1.6.1, we may take $l(\Lambda_1 - \Lambda_0)$ for $B^{1,l}$ as λ_0 in (1.1.2).

Proposition 1.6.5. *For a perfect crystal $B^{1,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^+)_l$,*

$$\sigma : (m_0, m_1, m_2, \cdots, m_{n-1}, m_n) \longmapsto (m_1, m_0, m_2, \cdots, m_{n-1}, m_n).$$

The proof is easily obtained by (6.7.1), (6.7.3) and Theorem 1.6.3.

1.7. *$(B_n^{(1)}, B(l\Lambda_1)\,(n \geq 3))$*

Let $I = \{0, 1, \cdots, n\}$ be the index set of the simple roots for the quantized universal enveloping algebra of type $U_q(B_n^{(1)})$ and let $J = \{1, \cdots, n\}$ be that of type $U_q(B_n)$. We define the maps $\iota_0, \iota_1 : J \to I$ by $\iota_0(j) = j$ for all $j \in J$ and $\iota_1(1) = 0$, $\iota_1(j) = j$ for $j \neq 1$.

Let $B(l\Lambda_1)$ be the crystal for $U_q(B_n)$ with highest weight $l\Lambda_1$. Set $K = \{1, \cdots, n, 0, \bar{n}, \cdots, \bar{1}\}$ and consider the ordering on K given by

$$1 < 2 < \cdots < n < 0 < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

Then the elements of $B(l\Lambda_1)$ are labeled by $b = (b_k)_{k=1}^l$, where $b_k \in K$, $b_k \le b_{k+1}$ for all k. Let $x_i(b) = \sharp\{k|\ b_k = i\}$, $\bar{x}_i(b) = \sharp\{k|\ b_k = \bar{i}\}$ for $k = 1, \cdots, n$, and set $x_0(b) = \sharp\{k|\ b_k = 0\}$. Note that $x_0(b) = 0$ or 1 by [KN]. It is clear that $x_0(b) + \sum x_i(b) + \sum \bar{x}_i(b) = l$.

Proposition 1.7.1. *For any integer $l \ge 1$, there exists a unique crystal $B^{1,l}$ for $U_q(B_n^{(1)})$ such that $\iota_0^*(B^{1,l}) \cong B(l\Lambda_1)$ and $\iota_1^*(B^{1,l}) \cong B(l\Lambda_1)$ as crystals for $U_q(B_n)$.*

Theorem 1.7.2. *The crystal $B^{1,l}$ is perfect of level l.*

Theorem 1.7.3. *For $b \in B^{1,l}$, the equality*

$$\langle c, \varepsilon(b)\rangle = \varepsilon_0(b) + \varepsilon_1(b) + 2\sum_{i=2}^{n-1} \varepsilon_i(b) + \varepsilon_n(b) = l$$

holds if and only if

$$x_i(b) = \overline{x}_i(b)\ (i = 1, \cdots, n) \text{ and } x_0(b) = \begin{cases} 0 & l \text{ is even,} \\ 1 & l \text{ is odd.} \end{cases}$$

The proofs of Proposition 1.7.1, Theorem 1.7.2 and Theorem 1.7.3 are given in 6.8.

Remark 1.7.4. By proposition 1.7.1, we may take $l(\Lambda_1 - \Lambda_0)$ for $B^{1,l}$ as λ_0 in (1.1.2).

Proposition 1.7.5. *For a perfect crystal $B^{n,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^+)_l$,*

$$\sigma : (m_0, m_1, m_2, \cdots, m_{n-1}, m_n) \longmapsto (m_1, m_0, m_2, \cdots, m_{n-1}, m_n).$$

The proof is easily obtained by (6.8.1), (6.8.3) and Theorem 1.7.3.

1.8. $(D_n^{(1)}, B(l\Lambda_1)\,(n \ge 4))$

Let $I = \{0, 1, \cdots, n\}$ be the index set of the simple roots for the quantized universal enveloping algebra of type $U_q(D_n^{(1)})$ and let $J = \{1, \cdots, n\}$ be that of type $U_q(D_n)$. We define the maps $\iota_0, \iota_1 : J \to I$ by $\iota_0(j) = j$ for all $j \in J$ and $\iota_1(1) = 0$, $\iota_1(j) = j$ for $j \neq 1$.

Let $B(l\Lambda_1)$ be the crystal for $U_q(D_n)$ with highest weight $l\Lambda_1$. Set $K = \{1,\cdots,n,\bar{n},\cdots,\bar{1}\}$ and consider the ordering on K given by

$$1 < 2 < \cdots < n, \bar{n} < \cdots < \bar{2} < \bar{1}.$$

Then the elements of $B(l\Lambda_1)$ are labeled by $b = (b_k)_{k=1}^{l}$, where $b_k \in K$, $b_k \leq b_{k+1}$ for all k. Let $x_i(b) = \sharp\{k|\ b_k = i\}$, $\bar{x}_i(b) = \sharp\{k|\ b_k = \bar{i}\}$ for $k = 1,\cdots,n$. Note that we have either $x_n(b) = 0$ or $\bar{x}_n(b) = 0$. It is clear that $\sum x_i(b) + \sum \bar{x}_i(b) = l$.

Proposition 1.8.1. *For any integer $l \geq 1$, there exists a unique crystal $B^{1,l}$ for $U_q(D_n^{(1)})$ such that $\iota_0^*(B^{1,l}) \cong B(l\Lambda_1)$ and $\iota_1^*(B^{1,l}) \cong B(l\Lambda_1)$ as crystals for $U_q(D_n)$.*

Theorem 1.8.2. *The crystal $B^{1,l}$ is perfect of level l.*

Theorem 1.8.3. *For $b \in B^{1,l}$, the equality*

$$\langle c, \varepsilon(b)\rangle = \varepsilon_0(b) + \varepsilon_1(b) + 2\sum_{i=2}^{n-2} \varepsilon_i(b) + \varepsilon_{n-1}(b) + \varepsilon_n(b) = l$$

holds if and only if $x_i(b) = \overline{x}_i(b)$ for $i = 2,\cdots,n-1$.

The proofs of Proposition 1.8.1, Theorem 1.8.2 and Theorem 1.8.3 are given in 6.9.

Remark 1.8.4. By proposition 1.8.1, we may take $l(\Lambda_1 - \Lambda_0)$ for $B^{1,l}$ as λ_0 in (1.1.2).

Proposition 1.8.5. *For a perfect crystal $B^{n,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^+)_l$,*

$$\sigma : (m_0, m_1, m_2, \cdots, m_{n-2}, m_{n-1}, m_n) \longmapsto (m_1, m_0, m_2, \cdots, m_{n-2}, m_n, m_{n-1}).$$

The proof is easily obtained by (6.9.1), (6.9.3) and Theorem 1.8.3.

1.9. *$(D_{n+1}^{(2)}, B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)\,(n \geq 2))$*

Let $I = \{0,1,\cdots,n\}$ be the index set of the simple roots for the quantized universal enveloping algebra of type $U_q(D_{n+1}^{(2)})$ and let $J = \{1,\cdots,n\}$ be that of type $U_q(B_n)$. We define the maps $\iota_0, \iota_n : J \to I$ by $\iota_0(j) = j$ and $\iota_n(j) = n - j$ for $j \in J$.

Let $\tilde{B} = B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$ be the direct sum of crystals with highest weight for $U_q(B_n)$. Set $K = \{1,\cdots,n,0,\bar{n},\cdots,\bar{1}\}$ and consider the ordering on K given by

$$1 < 2 < \cdots < n < 0 < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

Then the elements of $\tilde{B}$ are labeled by $b = (b_k)_{k=1}^{j}$, where $b_k \in K$, $b_k \leq b_{k+1}$ for all k, and $0 \leq j \leq l$. Here we write $b = \phi$ when $j = 0$. Let $x_0(b) = \sharp\{k|\ b_k = 0\}$, $x_i(b) = \sharp\{k|\ b_k = i\}$, $\bar{x}_i(b) = \sharp\{k|\ b_k = \bar{i}\}$, for $i = 1,\cdots,n$, and let $s(b) =$

$\sum x_i(b) + x_0(b) + \sum \bar{x}_i(b)$. Note that $x_0(b) = 0$ or 1 by [KN], and for $b = (b_k)_{k=1}^{j}$, $s(b) = j$.

Proposition 1.9.1. *For any integer $l \geq 1$, there exists a unique crystal $B^{1,l}$ for $U_q(D_{n+1}^{(2)})$ such that*

$$\iota_0^*(B^{1,l}) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

and

$$\iota_n^*(B^{1,l}) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

as crystals for $U_q(B_n)$.

Theorem 1.9.2. *The crystal $B^{1,l}$ is perfect of level l.*

Theorem 1.9.3. *For $b \in B^{1,l}$, the equality*

$$\langle c, \varepsilon(b) \rangle = \varepsilon_0(b) + 2\sum_{i=1}^{n-1} \varepsilon_i(b) + \varepsilon_n(b) = l$$

holds if and only if

$$\iota^*(b) \in B(l\Lambda_1), \quad x_i(b) = \bar{x}_i(b)\,(i = 1, \cdots, n) \text{ and } x_0(b) = \begin{cases} 0 & l \text{ is even,} \\ 1 & l \text{ is odd.} \end{cases}$$

The proofs of Proposition 1.9.1, Theorem 1.9.2 and Theorem 1.9.3 are given in 6.10.

Remark 1.9.4. By Proposition 1.9.1, we may take $l(\Lambda_1 - \Lambda_0)$ for $B^{1,l}$ as λ_0 in (1.1.2).

Proposition 1.9.5. *For a perfect crystal $B^{1,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^+)_l$,*

$$\sigma : (m_0, m_1, \cdots, m_{n-1}, m_n) \longmapsto (m_0, m_1, \cdots, m_{n-1}, m_n).$$

The proof is easily obtained by (6.10.5), (6.10.6), (6.10.13) and Theorem 1.9.3.

1.10. $(A_{2n}^{(2)}, B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)\,(n \geq 2))$

Let $I = \{0, 1, \cdots, n\}$ be the index set of the simple roots for the quantized universal enveloping algebra of type $U_q(A_{2n}^{(2)})$. Let $J_0 = \{1, \cdots, n\}$ be that of type $U_q(C_n)$ and let $J_n = \{1, \cdots, n\}$ be that of type $U_q(B_n)$. We define the maps $\iota_0 : J_0 \to I$ by $\iota_0(j) = j$ for $j \in J_0$ and $\iota_n : J_n \to I$ by $\iota_n(j) = n - j$ for $j \in J_n$.

Let $\tilde{B} = B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$ be the direct sum of crystals with highest weight for $U_q(C_n)$. Set $K = \{1, \cdots, n, \bar{n}, \cdots, \bar{1}\}$ and consider the ordering on K given by

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

Then the elements of $\tilde{B}$ are labeled by $b = (b_k)_{k=1}^{j}$, where $b_k \in K$, $b_k \leq b_{k+1}$ for all k, and $0 \leq j \leq l$. Here we write $b = \phi$ when $j = 0$. Let $x_i(b) = \sharp\{k|\ b_k = i\}$, $\bar{x}_i(b) = \sharp\{k|\ b_k = \bar{i}\}$ for $i = 1, \cdots, n$, and let $s(b) = \sum x_i(b) + \sum \bar{x}_i(b)$.

Proposition 1.10.1. *For any integer $l \geq 1$, there exists a unique crystal $B^{1,l}$ for $U_q(A_{2n}^{(2)})$ such that*

$$\iota_0^*(B^{1,l}) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

as crystals for $U_q(C_n)$ and

$$\iota_n^*(B^{1,l}) \cong B((l-2\left[\frac{l}{2}\right])\Lambda_1) \oplus \cdots \oplus B((l-2)\Lambda_1) \oplus B(l\Lambda_1)$$

as crystals for $U_q(B_n)$.

Theorem 1.10.2. *The crystal $B^{1,l}$ is perfect of level l.*

Theorem 1.10.3. *For $b \in B^{1,l}$, the equality $\langle c, \varepsilon(b)\rangle = \varepsilon_0(b) + 2\sum_{i=1}^{n} \varepsilon_i(b) = l$ holds if and only if $x_i(b) = \overline{x}_i(b)$ for $i = 1, \cdots, n$.*

The proofs of Proposition 1.10.1, Theorem 1.10.2 and Theorem 1.10.3 are given in 6.11.

Remark 1.10.4. By Proposition 1.10.1, we may take $l(\Lambda_1 - \Lambda_0)$ for $B^{1,l}$ as λ_0 in (1.1.2).

Proposition 1.10.5. *For a perfect crystal $B^{1,l}$ and $\lambda = (m_0, m_1, \cdots, m_n)$ in $(P_{cl}^{+})_l$,*

$$\sigma\ :\ (m_0, m_1, \cdots, m_{n-1}, m_n) \longmapsto (m_0, m_1, \cdots, m_{n-1}, m_n).$$

The proof is easily obtained by (6.11.1), (6.11.2), (6.11.9) and Theorem 1.10.3.

2. Polarization

2.1. *Order on* $\mathbf{Q}(q)$ We shall give a total order on $\mathbf{Q}(q)$ as follows.

Set $\mathbf{Q}(q)_+ = \bigsqcup_{n\in\mathbf{Z}}\{q^n(c+qA) \mid c>0\}$ and $f \geq g$ if and only if $f-g \in \mathbf{Q}(q)_+$. Hence $f \geq g$ if and only if there exists $\varepsilon > 0$ such that $f(q) \geq g(q)$ for $0<q<\varepsilon$.

2.2. *Positive definite form on* V Let V be a $\mathbf{Q}(q)$-vector space and $(\ ,\ )$ a $\mathbf{Q}(q)$-valued symmetric bilinear form on V. We say that $(\ ,\ )$ is *positive semi-definite* if

$$(v,v) \geq 0 \quad \text{for any} \quad v \in V. \tag{2.2.1}$$

We say that $(\ ,\ )$ is *positive definite* if and only if

$$(v,v) > 0 \quad \text{for any non-zero} \quad v \in V. \tag{2.2.2}$$

Note that $(\ ,\ )$ is positive definite (resp. semi-definite) if and only if, for any finite-dimensional $\mathbf{Q}$-subspace W of V, $(\ ,\ )|_W$ is positive definite (resp. semi-definite) for $0 < q \ll 1$.

If $(\ ,\ )$ is positive semi-definite, we have

$$(u+v,u+v) \leq 2(u,u)+2(v,v), \tag{2.2.3}$$

$$(u,v)^2 \leq (u,u)(v,v). \tag{2.2.4}$$

Lemma 2.2.1. *Let* $(\ ,\ )$ *be a positive definite symmetric bilinear form on* V *and* $L = \{u \in V \mid (u,u) \in A\}$. *Then we have*

(i) L *is an* A*-module,*
(ii) $(L,L) \subset A$,
(iii) *if* $\dim V < \infty$, *then* L *is a free* A*-module of finite rank.*

Proof. It is obvious that L is stable by the multiplication of elements of A. Hence the first assertion follows from the fact that L is stable by summation, which is an easy consequence of (2.2.3). The assertion (ii) follows from (2.2.4). In order to prove (iii), let us take a free A-module L_0 of finite rank such that $\mathbf{Q}(q)\otimes_A L_0 = V$ and $L_0 \subset L$. Then $L \subset L_0^{\perp} = \{u \in V \mid (u, L_0) \subset A\}$ from (ii). Since $L_0^{\perp}$ is a finitely generated A-module, L is also finitely generated over A. □

Remark. Under the assumption of Lemma 2.2.1 the $\mathbf{Q}$-valued symmetric bilinear form on L/qL induced by $(\ ,\ )$ is positive semi-definite but not positive definite in general, as seen by the example $V = \mathbf{Q}(q)u$, $(u,u) = q$ and $L = Au$.

The following lemma gives a sufficient condition for a given L to be equal to $\{u \in V \mid (u,u) \in A\}$.

Lemma 2.2.2. *Let* $(\ ,\ )$ *be a bilinear symmetric form on a* $\mathbf{Q}(q)$*-vector space* V *and* L *a free* A*-submodule of* V *such that* $V \cong \mathbf{Q}(q)\otimes_A L$. *Assume that*

$$(L,L) \subset A, \tag{2.2.5}$$

$$(\ ,\ )_0 \text{ is positive-definite,} \tag{2.2.6}$$

where $(\ ,\ )_0$ *is the* $\mathbf{Q}$*-valued symmetric form on* L/qL *induced by* $(\ ,\ )$. *Then* $(\ ,\ )$ *is positive-definite and*

$$L = \{v \in V \mid (v,v) \in A\}, \tag{2.2.7}$$

$$L = \{v \in V \mid (v,L) \subset A\}. \tag{2.2.8}$$

Proof. For a non-zero $v \in V$, let us take n such that $v \in q^n L$ and $v \notin q^{n+1}L$. Then $(v,v) \in q^{2n}((q^{-n}v, q^{-n}v)_0 + qA)$ and $(q^{-n}v, q^{-n}v)_0 > 0$ by (2.2.6). Hence $(v,v) > 0$. Therefore $(\ ,\ )$ is positive definite. Now, we shall show (2.2.7). Assume that $v \in V$ satisfies $(v,v) \in A$. Let us take the smallest $n \geq 0$ such that $v \in q^{-n}L$. If $n > 0$ then $q^n v \notin qL$ and hence by (2.2.6) $(q^n v, q^n v) \notin qA$, which is a contradiction. Hence $n = 0$ and $v \in L$. This proves (2.2.7). Finally we shall show (2.2.8). For $v \in V$ such that $(v, L) \subset A$, let us take the smallest $n \geq 0$ such that $q^n v \in L$. If $n > 0$, then $(q^n v, L/qL)_0 = 0$ and hence by (2.2.6) $q^n v \equiv 0 \bmod qL$. This is a contradiction. □

2.3. **Z**-*form* Let us introduce the subalgebras $A_{\mathbf{Z}}$ and $K_{\mathbf{Z}}$ of $\mathbf{Q}(q)$ as follows:

$$A_{\mathbf{Z}} = \{f(q)/g(q) \mid f(q), g(q) \in \mathbf{Z}[q], g(0) = 1\},\qquad (2.3.1)$$
$$K_{\mathbf{Z}} = A_{\mathbf{Z}}[q^{-1}].$$

Then we have

$$K_{\mathbf{Z}} \cap A = A_{\mathbf{Z}}.\qquad (2.3.2)$$

The following lemma is immediate.

Lemma 2.3.1.
(i) *$qA_{\mathbf{Z}}$ is the Jacobson radical of $A_{\mathbf{Z}}$ (i.e., any element of $1 + qA_{\mathbf{Z}}$ is invertible).*
(ii) *$A_{\mathbf{Z}}/qA_{\mathbf{Z}} \simeq \mathbf{Z}$.*

Let V be a $\mathbf{Q}(q)$-vector space and $(\ ,\ )$ a positive-definite symmetric bilinear form on V. Let L be a free A-submodule of V such that $V = \mathbf{Q}(q) \otimes_A L$ and $V_{K_{\mathbf{Z}}}$ a $K_{\mathbf{Z}}$-submodule of V such that $V = \mathbf{Q}(q) \otimes_{K_{\mathbf{Z}}} V_{K_{\mathbf{Z}}}$. Assume

$$(V_{K_{\mathbf{Z}}}, V_{K_{\mathbf{Z}}}) \subset K_{\mathbf{Z}},\qquad (2.3.3)$$
$$(L, L) \subset A.\qquad (2.3.4)$$

Let $(\ ,\ )_0$ be the induced $\mathbf{Q}$-valued symmetric form on L/qL. By (2.3.3) and (2.3.4), $(\ ,\ )_0$ is $\mathbf{Z}$-valued on $V_{K_{\mathbf{Z}}} \cap L/V_{K_{\mathbf{Z}}} \cap qL$. Assume further that

(2.3.5) $(\ ,\ )_0$ is positive definite,

(2.3.6) $B = \{b \in V_{K_{\mathbf{Z}}} \cap L/V_{K_{\mathbf{Z}}} \cap qL \mid (b,b)_0 = 1\}$ generates L/qL over $\mathbf{Q}$.

Lemma 2.3.2. *Assume (2.3.3-6). Then we have*
(i) *B is a pseudo-base of L/qL.*
(ii) *$V_{K_{\mathbf{Z}}} \cap L/V_{K_{\mathbf{Z}}} \cap qL = \sum_{b \in B} \mathbf{Z}b$.*

Proof. We shall take $B' \subset B$ such that $B = B' \cup (-B')$ and $B' \cap (-B') = \phi$. Then B' also generates L/qL. For $b_1, b_2 \in B$, we have $(b_1, b_2)_0^2 \leq (b_1, b_1)_0 (b_2, b_2)_0 = 1$. If $(b_1, b_2)_0^2 = 1$ then $(b_1, b_2)_0 = \pm 1$ and hence $(b_1 \mp b_2, b_1 \mp b_2)_0 = 0$, which implies $b_1 = \pm b_2$. Thus B' is an orthonormal base of L/qL, which proves (i). For $u \in V_{K_{\mathbf{Z}}} \cap L/V_{K_{\mathbf{Z}}} \cap qL$, let us write $u = \sum_{b \in B'} a_b b$. Then $a_b = (u, b)_0$ is an integer. Thus we obtain (ii). □

2.4. *Polarization* Let M and N be a $U_q(\mathfrak{g})$-module. A bilinear form $(\ ,\ ) : M \otimes_{\mathbf{Q}(q)} N \longrightarrow \mathbf{Q}(q)$ is called an *admissible* pairing if it satisfies

$$(q^h u, v) = (u, q^h v),\qquad (2.4.1)$$
$$(e_i u, v) = (u, q_i^{-1} t_i^{-1} f_i v),$$
$$(f_i u, v) = (u, q_i^{-1} t_i e_i v),$$

for all $u \in M$ and $v \in N$.
Let M be a $U_q(\mathfrak{g})$-module. A symmetric bilinear form $(\ ,\ )$ on M is called a *pre-polarization* of M if it satisfies (2.4.1) for $u, v \in M$.

Let ψ be the antiautomorphism of $U_q(\mathfrak{g})$ given by

$$\psi(q^h) = q^h, \psi(e_i) = q_i^{-1}t_i^{-1}f_i, \psi(f_i) = q_i^{-1}t_ie_i. \tag{2.4.2}$$

Then we have

$$(Pu, v) = (u, \psi(P)v) \quad \text{for any} \quad P \in U_q(\mathfrak{g}). \tag{2.4.3}$$

In particular, we have

$$\begin{aligned} (e_i^{(n)}u, v) &= (u, q_i^{-n^2}t_i^{-n}f_i^{(n)}v), \\ (f_i^{(n)}u, v) &= (u, q_i^{-n^2}t_i^{n}e_i^{(n)}v). \end{aligned} \tag{2.4.4}$$

A pre-polarization is called a *polarization* if it is positive definite. The following proposition is proved in [K4].

Proposition 2.4.1. *For any $\lambda \in P_+$, $V(\lambda)$ has a polarization $(\ ,\ )$ such that the crystal lattice $L(\lambda)$ is characterized by $L(\lambda) = \{u \in V(\lambda) \mid (u, u) \in A\}$. Moreover $B(\lambda)$ is an orthonormal base of $L(\lambda)/qL(\lambda)$ with respect to the induced symmetric bilinear form on it.*

Corollary 2.4.2. *Let M be an integrable $U_q(\mathfrak{g})$-module in $O_{int}(\mathfrak{g})$ and $(\ ,\ )$ a pre-polarization on M. If $(\ ,\ )$ is positive definite on $H = \{u \in M \mid e_iu = 0 \text{ for all } i\}$, then $(\ ,\ )$ is a polarization. If $(\ ,\ )_0$ is positive definite on $H \cap L/H \cap qL$, then $(\ ,\ )_0$ is positive definite on L/qL.*

Proof. We have the orthogonal decomposition

$$M = \bigoplus_{\lambda \in P_+} H_\lambda \otimes V(\lambda).$$

If we denote by $(\ ,\ )_\lambda$ the polarization on $V(\lambda)$ such that $(u_\lambda, u_\lambda) = 1$, then for $v \in H_\lambda$ and $u \in V(\lambda)$,

$$(v \otimes u, v \otimes u) = (v, v)(u, u)_\lambda.$$

Then the assertion follows from the fact that the tensor product of positive definite forms is positive definite. The last statement is similarly proved. □

Lemma 2.4.3. *Let $u, u' \in M_\lambda$ and $e_iu = e_iu' = 0$. Then*

$$(f_i^{(k)}u, f_i^{(k)}u') = q_i^{k(\langle h_i, \lambda\rangle - k)} \begin{bmatrix} \langle h_i, \lambda\rangle \\ k \end{bmatrix}_i (u, u').$$

Proof. By (2.4.4), we have

$$\begin{aligned} (f_i^{(k)}u, f_i^{(k)}u') &= (u, q_i^{-k^2}t_i^k e_i^{(k)} f_i^{(k)}u') \\ &= q_i^{k(\langle h_i, \lambda\rangle - k)}(u, e_i^{(k)} f_i^{(k)}u'). \end{aligned}$$

On the other hand, $e_i^{(k)}f_i^{(k)} = \sum_l f_i^{(k-l)}e_i^{(k-l)}\left\{ {t_i \atop l} \right\}_i$ implies $e_i^{(k)}f_i^{(k)}u' = \left[{\langle h_i, \lambda\rangle \atop k} \right]_i u'$. Thus we obtained the desired result. □

Proposition 2.4.4. *Assume that M is an integrable $U_q(\mathfrak{g})$-module and $\dim M_\lambda < \infty$ for any λ. Let $(\ ,\ )$ be a polarization on M. Then we have*

(i) $(\tilde{e}_i u, \tilde{e}_i u) \leq (1+q)(u,u)$ *and* $(\tilde{f}_i u, \tilde{f}_i u) \leq (1+q)(u,u)$ *for any u and i.*
(ii) $L = \{u \in M \mid (u,u) \in A\}$ *is a crystal lattice of M.*

Proof. (i) We may assume $u \in M_\lambda$. Set $u = \sum f_i^{(k)} u_k$ where $u_k \in M_{\lambda+k\alpha_i} \cap \operatorname{Ker} e_i$. Then $\tilde{f}_i u = \sum f_i^{(k+1)} u_k$. By Lemma 2.4.3, we have

$$(u,u) = \sum_k q_i^{k(\langle h_i,\lambda\rangle+k)} \begin{bmatrix} \langle h_i,\lambda\rangle + 2k \\ k \end{bmatrix}_i (u_k,u_k) \tag{2.4.5}$$

and

$$(\tilde{f}_i u, \tilde{f}_i u) = \sum_k q_i^{(k+1)(\langle h_i,\lambda\rangle+k-1)} \begin{bmatrix} \langle h_i,\lambda\rangle + 2k \\ k+1 \end{bmatrix}_i (u_k,u_k). \tag{2.4.6}$$

Hence, $(\tilde{f}_i u, \tilde{f}_i u) \leq (1+q)(u,u)$ follows from $(u_k,u_k) \geq 0$ and

$$q_i^{(k+1)(n-1)} \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_i \leq (1+q) q_i^{kn} \begin{bmatrix} n+k \\ k \end{bmatrix}_i \quad \text{for} \quad k,n \geq 0. \tag{2.4.7}$$

The statement on $\tilde{e}_i$ is similarly proved.

(ii) By Lemma 2.2.1, L_λ is a free A-module and L is stable by $\tilde{e}_i$ and $\tilde{f}_i$ by (i). □

Lemma 2.4.5. *Let $(\ ,\ )$ be a polarization on an integrable $U_q(\mathfrak{g})$-module M. Then for $\lambda \in P$ and $u \in M_\lambda$, we have*

$$(\tilde{f}_i u, \tilde{f}_i u) \leq q_i^{2(1-\langle h_i,\lambda\rangle)} (f_i u, f_i u) \quad \textit{and} \quad (\tilde{e}_i u, \tilde{e}_i u) \leq q_i^{2(1+\langle h_i,\lambda\rangle)} (e_i u, e_i u).$$

Proof. Set $u = \sum f_i^{(k)} u_k$ where $u_k \in M_{\lambda+k\alpha_i} \cap \operatorname{Ker} e_i$. We have

$$(\tilde{f}_i u, \tilde{f}_i u) = \sum (f_i^{(k+1)} u_k, f_i^{(k+1)} u_k)$$

and

$$(f_i u, f_i u) = \sum [k+1]_i^2 (f_i^{(k+1)} u_k, f_i^{(k+1)} u_k).$$

If $k+1 > \langle h_i, \lambda + k\alpha_i\rangle$ then $f_i^{(k+1)} u_k = 0$. Hence we may assume $k \geq 1 - \langle h_i,\lambda\rangle$ which implies $q_i^{2(1-\langle h_i,\lambda\rangle)}[k+1]_i^2 \geq 1$. This shows the first inequality. The second inequality follows from the involution of of $U_q(\mathfrak{g})$, $f_i \mapsto e_i, e_i \mapsto f_i, q^h \mapsto q^{-h}$. □

Lemma 2.4.6. *Let L be a crystal lattice of an integrable $U_q(\mathfrak{g})$-module M, $(\ ,\ )$ a pre-polarization on M such that $(L,L) \subset A$ and $(\ ,\ )_0$ the induced symmetric bilinear form on L/qL. Then $(\tilde{e}_i u, v)_0 = (u, \tilde{f}_i v)_0$ for any u, $v \in L/qL$.*

Proof. We may assume $u = f_i^{(k)} u'$ and $v = f_i^{(j)} v'$ where u',$v' \in L$ and $e_i u' = e_i v' = 0$. Then Lemma 2.4.3 implies

$$(\tilde{e}_i u, v) = (f_i^{(k-1)} u', f_i^{(j)} v') = \delta_{k-1,j} (f_i^{(j)} u', f_i^{(j)} v') \in \delta_{k-1,j}(1+qA)(u',v').$$

Similarly, we have $(u, \tilde{f}_i v) = (f_i^{(k)}u', f_i^{(j+1)}v') \in \delta_{k,j+1}(1+qA)(u',v')$. □

2.5. *The complete reducibility of $U_q(\gamma)_{K_{\mathbf{Z}}}$-modules* Let us denote by $U_q(\mathfrak{g})_{\mathbf{Z}}$ the $\mathbf{Z}[q,q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}, f_i^{(n)}, q^h$ and $\left\{ {q^h \atop n} \right\}$. Let us denote by $U_q^+(\mathfrak{g})_{\mathbf{Z}}$(resp. $U_q^-(\mathfrak{g})_{\mathbf{Z}}$) the $\mathbf{Z}[q,q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$(resp. $f_i^{(n)}$). We set $U_q(\gamma)_{K_{\mathbf{Z}}} = K_{\mathbf{Z}} \otimes U_q(\gamma)_{\mathbf{Z}}, U_q^{\pm}(\gamma)_{K_{\mathbf{Z}}} = K_{\mathbf{Z}} \otimes U_q^{\pm}(\gamma)_{\mathbf{Z}}$
Let M be a $U_q(\mathfrak{g})$-module M in $O_{int}(\gamma)$. For $\lambda \in P_+$ we denote by $I_\lambda(M)$ the isotypic component of M of type $V(\lambda)$. Hence

$$M \simeq \bigoplus_{\lambda \in P_+} I_\lambda(M) \tag{2.5.1}$$

and

$$I_\lambda(M) \simeq \mathrm{Hom}_{U_q(\mathfrak{g})}(V(\lambda), M) \otimes V(\lambda). \tag{2.5.2}$$

The purpose of this section is to prove the following proposition.

Proposition 2.5.1. *Let M be an $U_q(\mathfrak{g})$-module in $O_{int}(\mathfrak{g})$ such that $I_\lambda(M) = 0$ except for finitely many $\lambda \in P_+$. Let $M_{K_{\mathbf{Z}}}$ be a $U_q(\mathfrak{g})_{K_{\mathbf{Z}}}$-submodule of M. Then $M_{K_{\mathbf{Z}}} \simeq \bigoplus_{\lambda \in P_+} (M_{K_{\mathbf{Z}}} \cap I_\lambda(M))$ and $M_{K_{\mathbf{Z}}} \cap I_\lambda(M) \simeq (I_\lambda(M)_\lambda \cap M_{K_{\mathbf{Z}}}) \underset{K_{\mathbf{Z}}}{\otimes} V(\lambda)_{K_{\mathbf{Z}}}$. Here $V(\lambda)_{K_{\mathbf{Z}}}$ is the $U_q(\mathfrak{g})_{K_{\mathbf{Z}}}$-submodule of $V(\lambda)$ generated by the highest weight vector u_λ.*

In order to prove this, we shall prove the following lemma.

Lemma 2.5.2. *For $\lambda \in P_+$ and $\mu \in \lambda - Q_+$, there exist finitely many $P_k \in (U_q^+(\mathfrak{g})_{K_{\mathbf{Z}}})_{\lambda-\mu}$ and $Q_k \in (U_q^-(\mathfrak{g})_{K_{\mathbf{Z}}})_{\mu-\lambda}$ such that $u = \sum_k Q_k P_k u$ for any $u \in V(\lambda)_\mu$.*

Proof. Set $V(\lambda)_{\mathbf{Z}} = U_q(\mathfrak{g})_{\mathbf{Z}} u_\lambda$. Then $V(\lambda)_{\mathbf{Z}} = \sum_{b \in B(\lambda)} \mathbf{Z}[q,q^{-1}]G_\lambda(b)$ where $G_\lambda(b)$ is the (lower) global base (cf.[K4]). Then we have $(G_\lambda(b), G_\lambda(b')) \in \delta_{bb'} + qA_{\mathbf{Z}}$. Hence, $\det((G_\lambda(b), G_\lambda(b')))_{b,b' \in B(\lambda)_\mu}$ is invertible in $A_{\mathbf{Z}}$ (cf. Lemma 2.3.1). Therefore there exists $G_\lambda^*(b) \in K_{\mathbf{Z}} \otimes V_{\mathbf{Z}}(\lambda)_\mu$ such that $(G_\lambda^*(b), G_\lambda(b')) = \delta_{bb'}$. Let us write $G_\lambda(b) = Q(b)u_\lambda$ and $G_\lambda^*(b) = R(b)u_\lambda$ for $Q(b), R(b) \in (U_q^-(\mathfrak{g})_{K_{\mathbf{Z}}})_{\mu-\lambda}$.

Now, we shall show

$$\sum_{b \in B(\lambda)_\mu} Q(b)\psi(R(b))u = u \quad \text{for any} \quad u \in V(\lambda)_\mu. \tag{2.5.3}$$

For any $b_0 \in B(\lambda)_\mu$, we have

$$\begin{aligned}(\psi(R(b))G_\lambda(b_0), u_\lambda) &= (G_\lambda(b_0), R(b)u_\lambda) \\ &= \delta_{bb_0}.\end{aligned}$$

Since $\psi(R(b))G_\lambda(b_0)$ is a constant multiple of u_λ, we obtain

$$\psi(R(b))G_\lambda(b_0) = \delta_{bb_0}u_\lambda.$$

Thus we obtain

$$\sum_b Q(b)\psi(R(b))G_\lambda(b_0) = \sum_b \delta_{bb_0}Q(b)u_\lambda = G_\lambda(b_0).$$

This shows (2.5.3). Now it is enough to note that there exists $P(b) \in \left(U_q^+(\mathfrak{g}othg)\right)_{K_{\mathbf{Z}}}$ such that $P(b)u = \psi(R(b))u$ for all $u \in V(\lambda)_\mu$. □

Corollary 2.5.3. *Let $\lambda \in P_+$ and let M be a direct sum of copies of $V(\lambda)$, and let $M_{K_{\mathbf{Z}}}$ be a $U_q(\mathfrak{g})_{K_{\mathbf{Z}}}$-submodule. Then*

$$M_{K_{\mathbf{Z}}} \simeq (M_\lambda \cap M_{K_{\mathbf{Z}}}) \otimes V(\lambda)_{K_{\mathbf{Z}}}.$$

Proof. It is obvious that $M_{K_{\mathbf{Z}}} \supset (M_\lambda \cap M_{K_{\mathbf{Z}}}) \otimes V(\lambda)_{K_{\mathbf{Z}}}$. Let us show the other inclusion. For $\mu \in \lambda - Q_+$ and $u \in (M_{K_{\mathbf{Z}}})_\mu$, we have $u = \sum Q_k P_k u$ where Q_k and P_k are as in Lemma 2.5.2. Then $P_k u \in M_\lambda \cap M_{K_{\mathbf{Z}}}$ and hence $Q_k P_k u \in (M_\lambda \cap M_{K_{\mathbf{Z}}}) \otimes V(\lambda)_{K_{\mathbf{Z}}}$. □

Proof of Proposition 2.5.1. Set $S = \{\lambda \in P_+ \mid I_\lambda(M) \neq 0\}$ and $H = \{u \in M \mid e_i u = 0 \text{ for all } i\}$. By the induction it is enough to show that for any $\lambda \in S$ such that $(\lambda + Q_+) \cap S = \{\lambda\}$, $M_{K_{\mathbf{Z}}} = (N \cap M_{K_{\mathbf{Z}}}) \oplus (H_\lambda \cap M_{K_{\mathbf{Z}}}) \otimes V(\lambda)_{K_{\mathbf{Z}}}$, where $N = \oplus_{\lambda' \neq \lambda} I_{\lambda'}(M)$.

By the assumption on λ, we have $M_\lambda = I_\lambda(M)_\lambda = H_\lambda$. Let us consider the exact sequence:

$$0 \longrightarrow N \longrightarrow M \xrightarrow{\pi} H_\lambda \otimes V(\lambda) \longrightarrow 0. \tag{2.5.4}$$

Then $\pi(M_{K_{\mathbf{Z}}}) = (H_\lambda \cap M_{K_{\mathbf{Z}}}) \otimes V(\lambda)_{K_{\mathbf{Z}}}$ by Corollary 2.5.3. On the other hand $M_{K_{\mathbf{Z}}} \supset (H_\lambda \cap M_{K_{\mathbf{Z}}}) \otimes V(\lambda)_{K_{\mathbf{Z}}}$. Thus $M_{K_{\mathbf{Z}}} = (N \cap M_{K_{\mathbf{Z}}}) \oplus \pi(M_{K_{\mathbf{Z}}})$. □

2.6. *Criterion for the existence of crystal pseudo-base*

Proposition 2.6.1. *Let M be an integrable $U_q(\mathfrak{g})$-module such that $\dim M_\lambda < \infty$ for all $\lambda \in P$, let $M_{K_{\mathbf{Z}}}$ be a $U_q(\mathfrak{g})_{K_{\mathbf{Z}}}$-submodule of M and let $(\ ,\ )$ be a polarization on M such that $(M_{K_{\mathbf{Z}}}, M_{K_{\mathbf{Z}}}) \subset K_{\mathbf{Z}}$. Let L be a free A-submodule of M such that $(L, L) \subset A$ and $\mathbf{Q}(q) \otimes_A L = M$. Assume that*

(2.6.1) *the induced bilinear from $(\ ,\ )_0$ on L/qL is positive definite,*

(2.6.2) *$B = \{b \in M_{K_{\mathbf{Z}}} \cap L / M_{K_{\mathbf{Z}}} \cap qL \mid (b, b)_0 = 1\}$ generates L/qL.*

Then (L, B) is a crystal pseudo-base.

Proof. By Lemma 2.2.2 and Proposition 2.4.4, L is a crystal lattice of M. Lemma 2.3.2 implies that B is a pseudo-base of L/qL. By Proposition 2.4.4, $(\ ,\ )_0$ satisfies

$$(\tilde{e}_i u, \tilde{e}_i u)_0 \leq (u, u)_0 \quad \text{and} \quad (\tilde{f}_i u, \tilde{f}_i u)_0 \leq (u, u)_0 \quad \text{for} \quad u \in L/qL.$$

Hence $B \cup \{0\}$ is stable by $\tilde{e}_i$ and $\tilde{f}_i$. Let us show that if $b \in B$ and $\tilde{e}_i b \in B$ then $\tilde{f}_i \tilde{e}_i b = b$. Lemma 2.4.6 implies $(\tilde{f}_i \tilde{e}_i b, b)_0 = (\tilde{e}_i b, \tilde{e}_i b)_0 = 1$, $(\tilde{f}_i \tilde{e}_i b, \tilde{f}_i \tilde{e}_i b)_0 = (\tilde{e}_i b, \tilde{e}_i \tilde{f}_i \tilde{e}_i b)_0 = (\tilde{e}_i b, \tilde{e}_i b)_0 = 1$. Hence $(\tilde{f}_i \tilde{e}_i b - b, \tilde{f}_i \tilde{e}_i b - b)_0 = 0$, which implies $b = \tilde{f}_i \tilde{e}_i b$. Similarly if $b \in B$ and $\tilde{f}_i b \in B$, then $\tilde{e}_i \tilde{f}_i b = b$. □

Proposition 2.6.2. *Assume that $\mathfrak{g}$ is finite-dimensional, and let M be a finite-dimensional integrable $U_q(\mathfrak{g})$-module. Let $(\ ,\ )$ be a pre-polarization on M, and $M_{K_{\mathbf{Z}}}$ a $U_q(\mathfrak{g})_{K_{\mathbf{Z}}}$-submodule of M such that $(M_{K_{\mathbf{Z}}}, M_{K_{\mathbf{Z}}}) \subset K_{\mathbf{Z}}$. Let $\lambda_1, \cdots, \lambda_m \in P_+$ and we assume the following conditions.*

(2.6.3) $\dim M_{\lambda_k} \leq \sum_{j=1}^{m} \dim V(\lambda_j)_{\lambda_k}$ *for $k = 1, \cdots, m$.*

(2.6.4) *There exist $u_j \in (M_{K_{\mathbf{Z}}})_{\lambda_j} (j = 1, \ldots, m)$ such that $(u_j, u_k) \in \delta_{jk} + qA$, and $(e_i u_j, e_i u_j) \in q\, q_i^{-2(1 + \langle h_i, \lambda_j \rangle)} A$ for all $i \in I$.*

Set $L = \{u \in M \mid (u, u) \in A\}$, and set $B = \{b \in M_{K_{\mathbf{Z}}} \cap L / M_{K_{\mathbf{Z}}} \cap qL \mid (b, b)_0 = 1\}$. Then we have the following.

(i) *$(\ ,\ )$ is a polarization on M.*

(ii) *$M \simeq \oplus V(\lambda_j)$.*

(iii) $(\ ,\ )_0$ *is positive-definite and* (L, B) *is a crystal pseudo-base of* M.

Proof. Let $Q_+ = \sum \mathbf{Z}_{\geq 0}\alpha_i$. For $\lambda \in P_+$ let $I_\lambda(M)$ be the isotypic component of M of type $V(\lambda)$. Then $M = \oplus I_\lambda(M)$ is an orthogonal decomposition with respect to $(\ ,\ )$. We shall prove the following for each $\lambda \in P_+$.

$(2.6.5)_\lambda$ $(\ ,\ )_0|_{I_\lambda(M)\cap L/I_\lambda(M)\cap qL}$ is positive definite.

$(2.6.6)_\lambda$ $I_\lambda(M) = V(\lambda)^{\oplus s}$ where $s = \sharp\{j \mid \lambda_j = \lambda\}$.

$(2.6.7)_\lambda$ There exist $v_j \in I_\lambda(M)_\lambda \cap M_{K_{\mathbf{Z}}}$ for j such that $\lambda_j = \lambda$ satisfying $(v_j, v_{j'}) \in \delta_{jj'} + qA$.

By the induction on λ, it is enough to show $(2.6.5\text{-}7)_\lambda$, under the assumptions that $(2.6.5\text{-}7)_{\lambda'}$ hold for $\lambda' \in P_+ \cap (\lambda + Q_+) \setminus \{\lambda\}$. Set $N = \bigoplus_{\lambda' \in P_+\cap(\lambda+Q_+)\setminus\{\lambda\}} I_{\lambda'}(M)$. By Corollary 2.4.2, $(\ ,\ )|_N$ is a polarization. Therefore, by Proposition 2.4.4, $L \cap N$ is a crystal lattice of N. Then, by Corollary 2.4.2, $(\ ,\ )_0|_{N\cap L/N\cap qL}$ is positive definite.

Set $D = \{j \mid \lambda_j = \lambda\}$. For $j \in D$, we write

$$u_j = v_j + u'_j$$

with $v_j \in I_\lambda(M)$ and $u'_j \in N$. Proposition 2.5.1 implies that v_j and u'_j belong to $M_{K_{\mathbf{Z}}}$. Then $(e_iu_j, e_iu_j) = (e_iu'_j, e_iu'_j) \in qq_i^{-2(1+\langle h_i,\lambda\rangle)}A$ by (2.6.4). Hence Lemma 2.4.5 implies that $(\tilde{e}_iu'_j, \tilde{e}_iu'_j) \in qA$, and by (2.6.5) $\tilde{e}_iu'_j \in qL$ for any i. Since N has no highest weight vector of weight λ, we have $\{v \in N_\lambda \cap L/N_\lambda \cap qL \mid \tilde{e}_iv = 0 \text{ for all } i \in I\} = 0$. This implies $u'_j \in qL$. Thus we obtain $(u'_j, u'_{j'}) \in qA$. Therefore $(v_j, v_{j'}) = (u_j, u_{j'}) - (u'_j, u'_{j'}) \in \delta_{jj'} + qA$. Thus, we obtain $(2.6.7)_\lambda$. Moreover, M contains $V(\lambda)$ at least $\sharp D$-times. On the other hand (2.6.3) implies that M contains $V(\lambda)$ at most $\sharp D$-times. Thus we have $(2.6.6)_\lambda$. Finally $(2.6.5)_\lambda$ is a consequence of $(2.6.7)_\lambda$ and Corollary 2.4.2. Thus we obtain $(2.6.5\text{-}7)_\lambda$ for any $\lambda \in P_+$. Then (i) and (ii) are consequences of $(2.6.5)_\lambda$ and $(2.6.6)_\lambda$. By Proposition 2.4.4, $B \sqcup \{0\}$ is invariant by $\tilde{e}_i$ and $\tilde{f}_i$. By $(2.6.7)_\lambda$ we can show that B generates $H \cap L/H \cap qL$ where $H = \{u \in M \mid e_iu = 0 \text{ for all } i \in I\}$. Thus B generates L/qL. Hence by Proposition 2.6.1 (L, B) is a crystal pseudo-base of M. □

3. Fusion Construction

3.1. *Elementary representations* We follow the notations in §2. In this section we construct $U'_q(\mathfrak{g})$ -modules with perfect crystal pseudo-base. We employ the fusion construction. Namely, we construct first a $U'_q(\mathfrak{g})$ -module whose crystal base is perfect of level 1 and then construct general ones by using its tensor products and R -matrix.

Let V be a finite-dimensional integrable $U'_q(\mathfrak{g})$ -module. We assume that there is a $U'_q(\mathfrak{g})_{K_{\mathbf{Z}}}$ -submodule $V_{K_{\mathbf{Z}}}$ of V such that $V_{K_{\mathbf{Z}}}$ is a finitely generated $K_{\mathbf{Z}}$ -module and $V = \mathbf{Q}(q) \otimes_{K_{\mathbf{Z}}} V_{K_{\mathbf{Z}}}$. Let $(\ , \ )$ be a polarization of V.

Let (L, B) be a crystal base of V satisfying the following properties.

(3.1.1) B is perfect of level $l = 1$.

In particular, V is an irreducible $U'_q(\mathfrak{g})$-module by (4.6.1) and Lemma 3.4.4 in [KMN^2]. Let λ_0 be an element of P_{cl} as in (4.6.2) in [KMN^2]. We shall take $u_0 \in L_{\lambda_0}$ such that $B_{\lambda_0} = \{u_0 \bmod qL\}$. We may assume, by replacing $V_{K_{\mathbf{Z}}}$,

$$V_{K_{\mathbf{Z}}} = U'_q(\mathfrak{g})_{K_{\mathbf{Z}}} u_0. \tag{3.1.2}$$

In particular we have

$$V_{K_{\mathbf{Z}}} \cap \mathbf{Q}(q) u_0 = K_{\mathbf{Z}} u_0 \tag{3.1.3}$$

In fact if $V_{K_{\mathbf{Z}}} \cap \mathbf{Q}(q) u_0 \ni \varphi u_0$ for $\varphi \in \mathbf{Q}(q)$, then $V_{K_{\mathbf{Z}}} \supset U'_q(\mathfrak{g})_{K_{\mathbf{Z}}} \varphi u_0 = \varphi V_{K_{\mathbf{Z}}}$. Hence $\varphi \in K_{\mathbf{Z}}$.

By (4.6.2) in [KMN^2], we have $\sharp(B \otimes B)_{2\lambda_0} = 1$. Hence using (4.6.1) and Lemma 3.4.4 in [KMN^2], $V \otimes V$ is an irreducible $U'_q(\mathfrak{g})$-module. Then we can apply Theorem 3.4.1 in [KMN^2]. Hence there exists a $U_q(\mathfrak{g})$ -linear endomorphism R of $\mathrm{Aff}(V) \otimes \mathrm{Aff}(V)$ satisfying

$$(1 \otimes T) \circ R = R \circ (T \otimes 1) \quad \text{and} \quad (T \otimes 1) \circ R = R \circ (1 \otimes T), \tag{3.1.4}$$

$$(R \otimes 1) \circ (1 \otimes R) \circ (R \otimes 1) = (1 \otimes R) \circ (R \otimes 1) \circ (1 \otimes R), \tag{3.1.5}$$

$$R\big(af(u_0) \otimes af(u_0)\big) = \varphi(T^{-1} \otimes T)\big(af(u_0) \otimes af(u_0)\big) \tag{3.1.6}$$

for a non-zero $\varphi(T^{-1} \otimes T) \in \mathbf{Z}[q, q^{-1}, T^{-1} \otimes T, T \otimes T^{-1}]$.

Since R^2 is in $\mathbf{Q}(q)(T \otimes T^{-1})$, we obtain

$$R^2 = \varphi(T \otimes T^{-1}) \varphi(T^{-1} \otimes T). \tag{3.1.7}$$

By normalizing the bilinear form $(\ , \ )$ we may assume that

$$(u_0, u_0) = 1. \tag{3.1.8}$$

This implies that

$$(V_{K_{\mathbf{Z}}}, V_{K_{\mathbf{Z}}}) \subset K_{\mathbf{Z}}. \tag{3.1.9}$$

In fact, $(V_{K_{\mathbf{Z}}}, V_{K_{\mathbf{Z}}}) = (V_{K_{\mathbf{Z}}}, u_0) \subset \big((V_{K_{\mathbf{Z}}})_{\lambda_0}, u_0\big) \subset (K_{\mathbf{Z}} u_0, u_0) \subset K_{\mathbf{Z}}$ by (3.1.2) and (3.1.3).

3.2. *R-matrix for multiple tensor products* For the sake of simplicity, let us denote $V_x = \Phi_x(\mathbf{Q}(q)[x, x^{-1}] \otimes_{\mathbf{Q}(q)} V)$. Then the R-matrix R gives a $U'_q(\mathfrak{g})$-linear map $V_x \otimes V_y \longrightarrow V_y \otimes V_x$. We denote it by $R(x,y)$. Note that $R(x,y)$ depends only on x/y. Let l be a positive integer and $\mathfrak{S}_l$ the l -th symmetric group. Let s_i be the simple reflection (the permutation of i and $i+1$) and let $l(w)$ denote the length of $w \in \mathfrak{S}_l$. Then for any $w \in \mathfrak{S}_l$, we can define $R_w(x_1, \dots, x_l) : V_{x_1} \otimes \cdots \otimes V_{x_l} \longrightarrow V_{x_{w(1)}} \otimes \cdots \otimes V_{x_{w(l)}}$ as follows:

(3.2.1) $R_1(x_1, \dots, x_l) = 1$.

(3.2.2) $R_{s_i}(x_1, \dots, x_l) = \left(\otimes_{j<i}\mathrm{id}_{V_{x_j}}\right) \otimes R(x_i, x_{i+1}) \otimes \left(\otimes_{j>i+1}\mathrm{id}_{V_{x_j}}\right)$.

(3.2.3) For w, w' with $l(ww') = l(w) + l(w')$,
$R_{ww'}(x_1, \dots, x_l) = R_{w'}(x_{w(1)}, \dots, x_{w(l)}) \circ R_w(x_1, \dots, x_l)$.

3.3. *Construction of V_l and $(V_l)_{K_{\mathbf{Z}}}$* Fix $r \in \mathbf{Z}_{>0}$. For each $l \in \mathbf{Z}_{>0}$ we set

$$R_l = R_{w_0}(q^{r(l-1)}, q^{r(l-3)}, \dots, q^{-r(l-1)}) :$$
$$V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}} \longrightarrow V_{q^{-r(l-1)}} \otimes V_{q^{-r(l-3)}} \otimes \cdots \otimes V_{q^{r(l-1)}}$$

where $w_0 \in \mathfrak{S}_l$ is the permutation given by $i \mapsto l+1-i$. Then R_l is a $U'_q(\mathfrak{g})$ -linear homomorphism. We define

$$V_l = \operatorname{Im} R_l. \tag{3.3.1}$$

Then V_l is an integrable $U'_q(\mathfrak{g})$ -module. We have

$$R_l\left(u_0^{\otimes l}\right) = \psi_l(q) u_0^{\otimes l}. \tag{3.3.2}$$

where

$$\psi_l(q) = \prod_{1 \le i < j \le l} \varphi\left(q^{2r(j-i)}\right) \tag{3.3.3}$$

Here φ is given by (3.1.6). Now we assume that
(3.3.4) $\varphi(q^{2kr})$ does not vanish for any $k > 0$.
We set

$$\widetilde{R} = \psi_l(q)^{-1} R_l. \tag{3.3.5}$$

We define the $K_{\mathbf{Z}}$ -form of V_l by

$$\left(V_l\right)_{K_{\mathbf{Z}}} = \widetilde{R}\left((V_{K_{\mathbf{Z}}})^{\otimes l}\right) \cap (V_{K_{\mathbf{Z}}})^{\otimes l}. \tag{3.3.6}$$

Then $\left(V_l\right)_{K_{\mathbf{Z}}}$ is a $U'_q(\mathfrak{g})_{K_{\mathbf{Z}}}$ -submodule of V_l such that $\mathbf{Q}(q) \otimes \left(V_l\right)_{K_{\mathbf{Z}}} = V_l$. If we set $u_l = u_0^{\otimes l}$, then $\left(V_l\right)_{K_{\mathbf{Z}}} \ni u_l$. We have

(3.3.7) $\left(V_l\right)_{l\lambda_0} = \mathbf{Q}(q) u_l$,

(3.3.8) the weights of V_l are contained in $l\lambda_0 + \sum_{i \neq i_0} \mathbf{Z}_{\le 0}\alpha_i$.

Let us denote by W the image of

$$R(q^r, q^{-r}) : V_{q^r} \otimes V_{q^{-r}} \longrightarrow V_{q^{-r}} \otimes V_{q^r}$$

and by K its kernel. For each i, R_l decomposes as

$$\begin{aligned} & V_{q^{r(l-1)}} \otimes \cdots \otimes V_{q^{-r(l-1)}} \\ & \qquad \longrightarrow \cdots V_{q^{-r(l-2i+3)}} \otimes V_{q^{-r(l-2i-1)}} \otimes V_{q^{-r(l-2i+1)}} \otimes V_{q^{-r(l-2i-3)}} \cdots \\ & \qquad \xrightarrow{\cdots id \otimes R(q^{-r(l-2i-1)}, q^{-r(l-2i+1)}) \otimes id \cdots} V_{q^{r(l-1)}} \otimes \cdots \otimes V_{q^{-r(l-1)}}. \end{aligned}$$

Therefore
(3.3.9) V_l considered as a submodule of $V^{\otimes l} = V_{q^{-r(l-1)}} \otimes \cdots \otimes V_{q^{r(l-1)}}$ is contained in $\bigcap_{i=0}^{l-2} V^{\otimes i} \otimes W \otimes V^{\otimes(l-2-i)}$.

Similarly, we have
(3.3.10) V_l is a quotient of $V^{\otimes l} / \sum_{i=0}^{l-2} V^{\otimes i} \otimes K \otimes V^{\otimes(l-2-i)}$.

3.4. *Polarization on V_l* Now we shall define the polarization on V_l. The following is immediate.

Lemma 3.4.1. *Let M_j and N_j be $U'_q(\mathfrak{g})$ -modules and let $(\ ,\ )_j$ be an admissible pairing between M_j and $N_j (j = 1, 2)$. Then the pairing $(\ ,\ )$ between $M_1 \otimes M_2$ and $N_1 \otimes N_2$ defined by $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_1 (u_2, v_2)_2$ for all $u_j \in M_j$ and $v_j \in N_j$ is admissible.*

The polarization on V gives an admissible pairing between V_x and $V_{x^{-1}}$. Hence it induces an admissible pairing between $V_{x_1} \otimes \cdots \otimes V_{x_l}$ and $V_{x_1^{-1}} \otimes \cdots \otimes V_{x_l^{-1}}$.

Lemma 3.4.2. *If $x_j = x_{l+1-j}^{-1}$ for $j = 1, \ldots, l$, then for any $u, u' \in V_{x_1} \otimes \cdots \otimes V_{x_l}$, we have*

$$\big(u, R_{w_0}(x_1, \ldots, x_l) u'\big) = \big(u', R_{w_0}(x_1, \ldots, x_l) u\big).$$

Proof. If $x_j = 1$ for all j, $V_{x_1} \otimes \cdots \otimes V_{x_l}$ is an irreducible $U'_q(\mathfrak{g})$-module by Lemma 3.4.4 and Corollary 4.6.3 in [KMN^2]. Hence Lemma 3.4.2 in [KMN^2] implies that $V_{x_1} \otimes \cdots \otimes V_{x_l}$ is an irreducible $U'_q(\mathfrak{g})$-module for generic $x_1, \ldots, x_l$. Hence it is enough to check it for $u = u' = u_l$. This is obvious. □

By taking $x_1 = q^{r(l-1)}, x_2 = q^{r(l-3)}$, etc., we obtain the admissible pairing $(\ ,\ )$ between $W = V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}}$ and $W' = V_{q^{-r(l-1)}} \otimes V_{q^{-r(l-3)}} \otimes \cdots \otimes V_{q^{r(l-1)}}$ that satisfies

$$(3.4.1) \qquad \big(w, \widetilde{R} w'\big) = \big(w', \widetilde{R} w\big) \quad \text{for any } w, w' \in W.$$

This allows us to define a pre-polarization $(\ ,\ )_l$ on V_l by

$$(3.4.2) \qquad (\widetilde{R} u, \widetilde{R} u')_l = (u, \widetilde{R} u')$$

for $u, u' \in V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}}$.

Since the pairing $(\ ,\ )$ between W and W' is non-degenerate, we obtain

Proposition 3.4.3.
(i) *$(\ ,\ )_l$ is a non-degenerate pre-polarization on V_l.*
(ii) $\big(\widetilde{R}(u_l), \widetilde{R}(u_l)\big)_l = 1$.
(iii) $\big((V_l)_{K_{\mathbf{Z}}}, (V_l)_{K_{\mathbf{Z}}}\big)_l \subset K_{\mathbf{Z}}$.

Then, by applying Proposition 2.6.2 (with $m = 1$) and Proposition 2.6.1, we obtain the following result.

Proposition 3.4.4. *Set $I_0 = I\backslash\{i_0\}$. If V_l is an irreducible $U_q(\mathfrak{g}_{I_0})$ -module, then the $U_q'(\mathfrak{g})$ -module V_l admits a crystal pseudo-base.*

Similarly, we have

Proposition 3.4.5. *Let m be a positive integer and assume the following conditions:*

(i) $\langle h_i, l\lambda_0 + j\alpha_{i_0}\rangle \geq 0$ *for* $i \neq i_0$ *and* $0 \leq j \leq m$.

(ii) $\dim(V_l)_{l\lambda_0+k\alpha_{i_0}} \leq \sum_{j=0}^{m} \dim V(l\lambda_0 + j\alpha_{i_0})_{l\lambda_0+k\alpha_{i_0}}$ *for* $0 \leq k \leq m$, *where $V(\lambda)$ is an irreducible $U_q(\mathfrak{g}_{I_0})$ -module with highest weight λ.*

(iii) *There exists $i_1 \in I$ such that $\{i \in I \mid \langle h_{i_0}, \alpha_i\rangle < 0\} = \{i_1\}$.*

(iv) $-\langle h_{i_0}, l\lambda_0 - \alpha_{i_1}\rangle \geq 0$.

Then, we have

$$V_l \cong \oplus_{j=0}^{m} V(l\lambda_0 + j\alpha_{i_0}) \quad \text{as a } U_q(\mathfrak{g}_{I_0})\text{-module,}$$

and V_l admits a crystal pseudo-base as a $U_q'(\mathfrak{g})$-module.

Proof. It is enough to show

$$\left(e_{i_0}^{(k)}u_l, e_{i_0}^{(k)}u_l\right)_l \in 1 + qA \quad \text{for} \quad 0 \leq k \leq m, \tag{3.4.3}$$

$$\left(e_i e_{i_0}^{(k)}u_l, e_i e_{i_0}^{(k)}u_l\right)_l \in q q_i^{-2(1+\langle h_i, l\lambda_0+k\alpha_{i_0}\rangle)} A \tag{3.4.4}$$

for $0 \leq k \leq m$ and $i \in I_0$. In fact, by applying Proposition 2.6.2 to the $U_q(\mathfrak{g}_{I_0})$-module V_l, we can show that it is isomorphic to $\oplus_{j=0}^{m} V(l\lambda_0 + j\alpha_{i_0})$. Moreover, if we define L and B as in Proposition 2.6.2, then (L, B) is a crystal pseudo-base of the $U_q(\mathfrak{g}_{I_0})$-module V_l and the induced symmetric bilinear form on L/qL is positive definite. Then Proposition 2.6.1 implies the desired result.

In the following we use

$$\begin{bmatrix} a \\ b \end{bmatrix} \in q^{-b(a-b)}(1 + qA), \quad [a] \in q^{1-a}A \quad \text{for} \quad a, b \geq 0.$$

Since $cl(-\alpha_{i_0}) = cl(\delta - \alpha_{i_0}) \in \sum_{i\neq i_0} \mathbb{Z}_{\geq 0} cl(\alpha_i)$, and the weights of V_l are contained in $l\lambda_0 + \sum_{i\neq i_0} \mathbb{Z}_{\leq 0} cl(\alpha_i)$, we obtain $f_{i_0}u_l = 0$. Therefore we have

$$\left(e_{i_0}^{(k)}u_l, e_{i_0}^{(k)}u_l\right)_l = q_{i_0}^{k(-\langle h_{i_0}, l\lambda_0\rangle - k)} \begin{bmatrix} -\langle h_{i_0}, l\lambda_0\rangle \\ k \end{bmatrix}_{i_0}. \tag{3.4.5}$$

From this follows (3.4.3).

Let us prove (3.4.4). For notational simplicity we shall write 0 or 1 instead of i_0 or i_1. If $i \neq 0, 1$, we have

$$e_i e_0^{(k)} u_l = e_0^{(k)} e_i u_l = 0 \tag{3.4.6}$$

If $i = 1$, we have

$$\left(e_1 e_0^{(k)} u_l, e_1 e_0^{(k)} u_l\right)_l = \left(v, u_l\right)_l \tag{3.4.7}$$

where

$$v = q_0^{-k^2} t_0^{-k} f_0^{(k)} q_1^{-1} t_1^{-1} f_1 e_1 e_0^{(k)} u_l$$

Now, by setting $\mu = l\lambda_0$, we have

$$v = q_0^{-k(\langle h_0,\mu\rangle+k)} q_1^{-1-\langle h_1,\mu+k\alpha_0\rangle} f_0^{(k)} f_1 e_1 e_0^{(k)} u_l \tag{3.4.8}$$

On the other hand, $f_1e_1 = e_1f_1 - \{t_1\}_1$ implies

$$\begin{aligned} f_0^{(k)} f_1 e_1 e_0^{(k)} u_l &= f_0^{(k)} \big(e_1f_1 - \{t_1\}_1\big) e_0^{(k)} u_l \\ &= e_1 f_0^{(k)} e_0^{(k)} f_1 u_l - \big[\langle h_1, \mu + k\alpha_0\rangle\big]_1 f_0^{(k)} e_0^{(k)} u_l. \end{aligned} \tag{3.4.9}$$

Similarly to $f_0u_l = 0$, we have $f_0f_1u_l = 0$. Hence

$$f_0^{(k)} e_0^{(k)} f_1 u_l = \begin{bmatrix} -\langle h_0, \mu - \alpha_1\rangle \\ k \end{bmatrix}_0 f_1 u_l.$$

Thus we obtain

$$\begin{aligned} e_1 f_0^{(k)} e_0^{(k)} f_1 u_l &= \begin{bmatrix} -\langle h_0, \mu - \alpha_1\rangle \\ k \end{bmatrix}_0 e_1 f_1 u_l \\ &= \big[\langle h_1, \mu\rangle\big]_1 \begin{bmatrix} -\langle h_0, \mu - \alpha_1\rangle \\ k \end{bmatrix}_0 u_l. \end{aligned} \tag{3.4.10}$$

Comparing (3.4.7-10), we have

$$\begin{aligned} (e_1e_0^{(k)}u_l, e_1e_0^{(k)}u_l)_l &= q_0^{k(-\langle h_0,\mu\rangle-k)} q_1^{-1-\langle h_1,\mu+k\alpha_0\rangle} \times \\ &\left(\big[\langle h_1,\mu\rangle\big]_1 \begin{bmatrix} -\langle h_0, \mu - \alpha_1\rangle \\ k \end{bmatrix}_0 - \big[\langle h_1, \mu + k\alpha_0\rangle\big]_1 \begin{bmatrix} -\langle h_0, \mu\rangle \\ k \end{bmatrix}_0 \right) \end{aligned}$$

Since $(\alpha_0,\alpha_0)\langle h_0,\alpha_1\rangle = (\alpha_1,\alpha_1)\langle h_1,\alpha_0\rangle$, we have

$$\begin{aligned} &(\alpha_1,\alpha_1)(1 - \langle h_1,\mu\rangle) - (\alpha_0,\alpha_0)k(-\langle h_0,\mu-\alpha_1\rangle - k) \\ &= (\alpha_1,\alpha_1)(1 - \langle h_1,\mu+k\alpha_0\rangle) - (\alpha_0,\alpha_0)k(-\langle h_0,\mu\rangle - k) \end{aligned}$$

and the sum of this and

$$(\alpha_0,\alpha_0)k(-\langle h_0,\mu\rangle - k) + (\alpha_1,\alpha_1)\big(-1 - \langle h_1,\mu+k\alpha_0\rangle\big)$$

becomes $-2(\alpha_1,\alpha_1)\langle h_1,\mu+k\alpha_0\rangle$. Hence $\big(e_1e_0^{(k)}u_l, e_1e_0^{(k)}u_l\big)_l \in q_1^{-2\langle h_1,\mu+k\alpha_0\rangle}A$.
□

4. Constructions of Level One Representations

In the following we calculate the explicit forms of R-matrices. Except the case of $(\mathrm{D}^{(2)}_{n+1}, V^1)$, all non-zero weight spaces of the representations which we treat here are one dimensional. So we denote by b the lower global base corresponding to an element b of the crystal of a representation except the case of $(\mathrm{D}^{(2)}_{n+1}, V^1)$. In §4.1-§4.9 we denote by $\mathfrak{t}$ the Cartan subalgebra of $\mathfrak{g}$, by $\{\alpha_i | \iota \in I\} \subset \mathfrak{t}^*$ the set of simple roots, by $\{h_i | i \in I\} \subset \mathfrak{t}$ the set of simple coroots and by $\{\Lambda_i | i \in I\}$ the set of fundamental weights of the corresponding Lie algebras, where $I = \{0, 1, \cdots, n\}$. We assume that the norm of a short root is equal to one. For any finite dimensional $U'_q(\mathfrak{g})$-module W and the choice of i_0, we set $W_x = \Phi_x(\mathbf{Q}(q)[x, x^{-1}] \otimes_{\mathbf{Q}(q)} W)$. The calculations of the R-matrices here are carried out in the following manner. For a finite dimensional $U'_q(\mathfrak{g})$-module W and the choice of i_0, we first decompose $W \otimes W$ into the direct sum $\oplus_{j \in J} W_j$ of irreducible $U'_q(\mathfrak{g}_{I \setminus \{i_0\}})$-modules. Then the R-matrix $R(x/y) : W_x \otimes W_y \to W_y \otimes W_x$ can be written as $R(x/y) = \oplus_j \gamma_j(x/y) P_{W_j}$ by Schur's lemma, where P_{W_j} is the $U'_q(\mathfrak{g}_{I \setminus \{i_0\}})$-linear projection from $W_x \otimes W_y$ to W_i. Let w_i be the highest weight vector of W_i. Except the case of §4.8 we find an element P_i of $U'_q(\mathfrak{g})$and $\beta_i(x, y)$ of $\mathbf{Q}(q)[x, x^{-1}, y, y^{-1}]$ which satisfy $P_i u_i = \beta_i(x, y) u_{i+1}$ in $W_x \otimes W_y$ for an appropriate order of J. Then we have a recursion relations

$$\gamma_i(x/y)\beta_{i+1}(y, x) = \beta_i(x, y)\gamma_{i+1}(x/y)$$

From those relations we can determine $\gamma_j(x/y)$ for all $j \in J$ up to a multiple of an element of $\mathbf{Q}(q)(x/y)$.

4.1. $(A_n^{(1)}, V^k$) Let $\mathfrak{g} = \hat{\mathfrak{sl}}(n+1)$ be the affine Lie algebra of type $A_n^{(1)}$. Define $\tilde{\Lambda}_i (i \in \mathbf{Z})$ by

$$\tilde{\Lambda}_i = \begin{cases} \Lambda_i & \text{for } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We assume that $(\langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is the Cartan matrix of type A_n. We set $i_0 = 0$. Let $U_q(\mathfrak{sl}(n+1))$be the subalgebra of $U_q(\hat{\mathfrak{sl}}(n+1))$associated with $\{h_i, \alpha_j | 1 \leq i, j \leq n\}$.

4.1.1. *Decomposition of tensor product*

Let $V(\Lambda_k)(1 \leq k \leq n)$ be the irreducible highest weight $U_q(\mathfrak{sl}(n+1))$-module with highest weight Λ_k and $(L(\Lambda_k), B(\Lambda_k))$ its crystal base. By [KN] the elements of $B(\Lambda_k)$ are labeled in the following way.

$$B(\Lambda_k) = \{(m_i)_{i=1}^k | 1 \leq m_1 < \cdots < m_k \leq n+1\}.$$

Then

$$V(\Lambda_k) \otimes V(\Lambda_k) \simeq \oplus_{i=0}^{\min(n+1-k,k)} V(\tilde{\Lambda}_{k-i} + \tilde{\Lambda}_{k+i}). \tag{4.1.1}$$

The highest weight elements for the corresponding crystals are given by

$$(i)_{i=1}^k \otimes (1, \cdots, k-i, k+1, \cdots, k+i) \quad \text{for} \quad B(\tilde{\Lambda}_{k-i} + \tilde{\Lambda}_{k+i}).$$

4.1.2. *Construction of the Representation of* $U'_q(\hat{\mathfrak{sl}}(n+1))$

Define the actions of e_0 and f_0 on $V(\Lambda_k)$ by

$$f_0 b = \begin{cases} (1, i_1, \cdots, i_{k-1}) & \text{if } b = (i_1, \cdots, i_{k-1}, n+1) \\ 0 & \text{otherwise} \end{cases},$$
$$e_0 b = \begin{cases} (j_1, \cdots, j_{k-1}, n+1) & \text{if } b = (1, j_1, \cdots, j_{k-1}) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that $V(\Lambda_k)$ is a well-defined $U_q'(\hat{\mathfrak{sl}}(n+1))$-module with the actions of e_0, f_0 given above and $q^{h_0} = q^{-(h_1+\cdots+h_n)}$. We denote this $U_q'(\hat{\mathfrak{sl}}(n+1))$-module by V^k. By the construction it is obvious that V^k has a crystal base.

4.1.3. *Construction of a Polarization of* V^k Let (,) be the polarization of the $U_q(\mathfrak{sl}(n+1))$-module $V(\Lambda_k)$. We shall show

$$(q^{h_0}u, v) = (u, q^{h_0}v), \tag{4.1.2}$$
$$(e_0 u, v) = (u, q_0^{-1} t_0^{-1} f_0 v), \tag{4.1.3}$$
$$(f_0 u, v) = (u, q_0^{-1} t_0 e_0 v), \tag{4.1.4}$$

for any $u, v \in V^k$. It is sufficient to prove (4.1.2)-(4.1.4) for the lower global bases u and v. The equality (4.1.2) is obvious by the definition of the action of q^{h_0}. By direct calculations we have the following lemma.

Lemma 4.1.2.
(1) *Let b be a global base which satisfies $e_i b = 0$ and $\langle h_i, wt(b)\rangle = 1$. Then $(b,b) = (f_i b, f_i b)$.*
(2) *Let b be a global base which satisfies $e_i b = 0$ and $\langle h_i, wt(b)\rangle = 2$. Then $(b,b) = (f_i^{(2)}b, f_i^{(2)}b) = q_i^{-1}[2]_i^{-1}(f_i b, f_i b)$.*

Note that for any $b \in B(\Lambda_k)$ and i $(1 \le i \le n)$, $\varphi_i(b) + \varepsilon_i(b) \le 1$. Hence, by using Lemma 4.1.2, (4.1.2)-(4.1.4) are verified for lower global bases u and v. Consequently (,) is a polarization of $U_q'(\hat{\mathfrak{sl}}(n+1))$-module V^k.

4.1.4. *Calculation of R-matrix* By the decomposition (4.1.1.), $\mathrm{R}(x/y)$ can be written as $\mathrm{R}(x/y) = \oplus_{i=0}^{\min(n+1-k,k)} \gamma_i P_{(\tilde\Lambda_{k-i}+\tilde\Lambda_{k+i})}$, where $P_{\tilde\Lambda_{k-i}+\tilde\Lambda_{k+i}}$ is the projection $P_{(\tilde\Lambda_{k-i}+\tilde\Lambda_{k+i})} : V(\Lambda_k) \otimes V(\Lambda_k) \to V(\tilde\Lambda_{k-i} + \tilde\Lambda_{k+i})$. Let $u_i (0 \le i \le \min(n+1-k, k))$ be the highest weight vector in the $U_q(\mathfrak{sl}(n+1))$-module $V(\Lambda_k) \otimes V(\Lambda_k)$ with highest weight $\tilde\Lambda_{k-i} + \tilde\Lambda_{k+i}$. We set $P_i = f_0 f_n \cdots f_{k+i} f_1 \cdots f_{k-i}$.

Lemma 4.1.3. *Let $v_{i-1} = P_i u_i$ $(1 \le i \le min(n+1-k, k))$. Then v_{i-1} is non-zero and is proportional to u_{i-1}.*

Proof. If $x = y = 1$, $v_{i-1} \neq 0$ in L/qL. Hence $v_{i-1} \neq 0$. By a direct calculation the weight of v_{i-1} is $\tilde\Lambda_{k-i} + \tilde\Lambda_{k+i}$. We must check that v_{i-1} is a highest weight vector. Since $[e_r, f_l] = 0$ for $r \neq l$, $e_r v_{i-1} = 0$ for $k - i < r < k + i$. As is immediately seen, the following set of vectors is a base of the weight space of $V(\Lambda_k) \otimes V(\Lambda_k)$ with weight $\tilde\Lambda_{k-i} + \tilde\Lambda_{k+i}$.

$$\{(1, \cdots, k-i, m_1, \cdots, m_i) \otimes (1, \cdots, k-i, l_1, \cdots, l_i) \mid \\ \{m_1, \cdots, m_i\} \cup \{l_1, \cdots, l_i\} = \{k-i+1, \cdots, k+i\}\}.$$

It follows that $e_r v_{i-1} = 0$ for $r < k - i$ and $r > k + i$. □

Let us define $b_1^{(i)}$ and $b_2^{(i)}$ by

$$\begin{aligned} b_1^{(i)} &= (j)_{j=1}^{k} \otimes (1, 2, \cdots, k-i, k+1, \cdots, k+i), \\ b_2^{(i)} &= (1, \cdots, k-i, k-i+2, \cdots, k, k+i) \otimes (1, \cdots, k-i+1, k+1, \cdots, k+i-1). \end{aligned}$$

For any element v of $V^k \otimes V^k$ we write $P_i v = \sum_{b'} F_v^{b'} b'$, where b' runs over the set of tensor products of global bases of V^k. In the following subsections we use these notations in a similar way. The following lemma is by direct calculations and we leave it to the reader.

Lemma 4.1.4. *Let b be an element of $V^k \otimes V^k$ which is a tensor product of two global bases of V^k and has the weight $\tilde{\Lambda}_{k-i} + \tilde{\Lambda}_{k+i}$. Then $F_b^{b_1^{(i-1)}} \neq 0$ if and only if $b = b_1^{(i)}$ or $b_2^{(i)}$. Moreover $P_i b_1^{(i)} = q^{-1}y^{-1}b_1^{(i-1)}$ and $P_i b_2^{(i)} = x^{-1}b_1^{(i+1)}$.*

Lemma 4.1.5. *If we write $u_i = b_1^{(i)} + \sum_b a_b b$, then $a_{b_2^{(i)}} = -q^{2i-1}$.*

Proof. Let $b_{m_1,\cdots,m_i} = (1, \cdots, k-i, m_1, \cdots, m_i) \otimes (1, \cdots, k-i, l_1, \cdots, l_i)$, where $\{m_1, \cdots, m_i\} \cup \{l_1, \cdots, l_i\} = \{j | k-i < j \leq k+i\}$. Note that $b_1^{(i)} = b_{k-i+1,\cdots,k}$ and $b_2^{(i)} = b_{k-i+2,\cdots,k,k+i}$. We write a_α instead of a_{b_α} for $\alpha = (m_1, \cdots, m_i)$. There are relations

$$\begin{aligned} e_k(b_{k-i+1,\cdots,k} - q b_{k-i+1,\cdots,k-1,k+1}) &= 0, \\ e_{k-j}(b_{k-i+1,\cdots,k-j,k-j+2,\cdots,k+1} - q b_{k-i+1,\cdots,k-j-1,k-j+1,\cdots,k+1}) &= 0 \quad \text{for} \quad 1 \leq j \leq i-2, \\ e_{k-i+1}(b_{k-i+1,k-i+3,\cdots,k+1} - q b_{k-i+2,\cdots,k+1}) &= 0, \\ e_{k+j}(b_{k-i+2,\cdots,k,k+j} - q b_{k-i+2,\cdots,k,k+j+1}) &= 0 \quad \text{for} \quad 1 \leq j \leq i-1. \end{aligned}$$

Since all the weight spaces of V^k are one dimensional and each length of j-strings $(1 \leq j \leq n)$ is at most 1, $e_r u_i = 0$ for all r implies

$$\begin{aligned} a_{k-i+1,\cdots,k-1,k+1} &= -q, \\ a_{k-i+1,\cdots,k-j,k-j+2,\cdots,k+1} : a_{k-i+1,\cdots,k-j-1,k-j+1,\cdots,k+1} &= 1 : -q \quad \text{for} \quad 1 \leq j \leq i-2, \\ a_{k-i+1,k-i+3,\cdots,k+1} : a_{k-i+2,\cdots,k+1} &= 1 : -q, \\ a_{k-i+2,\cdots,k,k+j} : a_{k-i+2,\cdots,k,k+j+1} &= 1 : -q \quad \text{for} \quad 1 \leq j \leq i-1. \end{aligned}$$

It follows that $a_{b_2^{(i)}} = -q^{2i-1}$. □

By these lemmas we have in $V_x^k \otimes V_y^k$

$$P_i u_i = q^{-1}x^{-1}y^{-1}(x - q^{2i}y)u_{i-1}.$$

From this

$$\frac{\gamma_i}{\gamma_{i-1}} = \frac{x - q^{2i}y}{y - q^{2i}x} \quad \text{for} \quad 1 \leq i \leq \min(n+1-k, k).$$

So we have proved

Proposition 4.1.6. *Let $z = xy^{-1}$. The R-matrix, up to a multiple of an element of $\mathbf{Q}(q)(z)$, is of the following form*

$$R(z) = \bigoplus_{r=0}^{min(n+1-k,k)} \prod_{i=1}^{r}(z-q^{2i}) \prod_{j=r+1}^{min(n+1-k,k)} (1-q^{2j}z)P_{\tilde{\Lambda}_{k-r}+\tilde{\Lambda}_{k+r}}.$$

4.2. $(C_n^{(1)}, V^n\quad)$ Let $\mathfrak{g} = \hat{\mathfrak{sp}}(n)$ be the affine Lie algebra of type $C_n^{(1)}$. We assume that $(\langle h_i, \alpha_j\rangle)_{1\le i,j\le n}$ is the Cartan matrix of type C_n. We set $i_0 = 0$. Let $U_q(\mathfrak{sp}(n))$be the subalgebra of $U_q(\hat{\mathfrak{sp}}(n))$associated with $\{h_i, \alpha_j | 1 \le i, j \le n\}$. In this case $q_0 = q_n = q^2$ and $q_i = q$ $(i \neq 0, n)$.

4.2.1. *Decomposition of tensor product*
Let $V(\Lambda_n)$ be the irreducible highest weight $U_q(\mathfrak{sp}(n))$-module with highest weight Λ_n and $(L(\Lambda_n), B(\Lambda_n))$ its crystal base. By [KN] the elements of $B(\Lambda_n)$ are labeled in the following way.

$$\begin{aligned} B(\Lambda_n) = &\{(m_i)_{i=1}^n | m_1 < \cdots < m_n, m_i \in \{1, \cdots, n, \bar{n}, \cdots, \bar{1}\}, \\ & i + (n-j+1) \le m_i \quad \text{if} \quad m_i = \overline{m_j}(i<j)\}, \end{aligned}$$

where the ordering of $\{1, \cdots, n, \bar{n}, \cdots, \bar{1}\}$ is given by

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{1}.$$

Then

$$V(\Lambda_n) \otimes V(\Lambda_n) \simeq V(2\Lambda_n) \oplus V(2\Lambda_{n-1}) \oplus \cdots \oplus V(2\Lambda_1) \oplus V(0), \tag{4.2.1}$$

where $V(0)$ is the trivial representation. The highest weight elements for the corresponding crystals are given by

$$\begin{aligned} (j)_{j=1}^n \otimes (1, \cdots, i, \overline{n}, \cdots, \overline{i+1}) &\quad \text{for} \quad B(2\Lambda_i), \\ (j)_{j=1}^n \otimes (\overline{n}, \overline{n-1}, \cdots, \bar{1}) &\quad \text{for} \quad B(0). \end{aligned}$$

4.2.2. *Construction of the Representation of $U_q'(\hat{\mathfrak{sp}}(n))$*
First we prove

Lemma 4.2.1. *Let $V(\Lambda_n)_\lambda$ denote the weight space of $V(\Lambda_n)$ of the weight λ. Then $dimV(\Lambda_n)_\lambda = 1$ for any λ.*

Proof. Suppose that $V(\Lambda_n)_\lambda \neq \{0\}$. Let $\lambda = \sum_{i=1}^n r_i\epsilon_i$ and $\mathrm{S} = \{i | r_i \neq 0\}$. Since there is an element $b \in B(\Lambda_n)_\lambda$, $\sharp\mathrm{S} = n - 2k$ for some integer $k \ge 0$. For $k = 0$, $\dim V(\Lambda_n)_\lambda = 1$ is obvious. So we assume $k \ge 1$. Take any $b \in \mathrm{B}(\Lambda_n)_\lambda$. Then there is a set of integers $(j_1, \cdots, j_k)$ which satisfies the following two conditions.

(1) $1 < j_1 < j_2 < \cdots \le j_k \le n$.

(2) b contains j_i and $\overline{j_i}$ for $1 \le i \le k$.

Take any $(j_1, \cdots, j_k)$ which satisfies (1) and (2). Set $l = \sharp\{p \in \mathrm{S} | p < j_k\}$. Since $j_1 \ge 2$, $j_k \ge 2k + l$ and hence $n - j_k \le n - 2k - l$. By the definition of l, $n - j_k$ must be equal to $n - 2k - l$. Then the following properties must be hold.

(3) If $l_p = \sharp\{i \in \mathrm{S} | j_p < i < j_{p+1}\}$, then $j_{p+1} = j_p + 2 + l_p$.

(4) If $l_0 = \sharp\{i \in \mathrm{S} | i < j_1\}$, then $j_1 = l_0 + 2$.
(5) If $\mathrm{S} \cup \{j_1, \cdots, j_k\} = \{r_i (1 \le i \le n-k) | r_i < r_{i+1} (1 \le i \le n-k-1)\}$, then $\{i | r_{i+1} - r_i > 2, \quad r_{i+1} \le j_k\} = \phi$.

It follows that the set $\{j_1, \cdots, j_k\}$ is uniquely determined by S and hence by λ. □

Define the actions of e_0 and f_0 on $V(\Lambda_n)$ by

$$f_0 b = \begin{cases} (1, i_1, \cdots, i_{n-1}) & \text{if } b = (i_1, \cdots, i_{n-1}, \overline{1}) \\ 0 & \text{otherwise} \end{cases},$$

$$e_0 b = \begin{cases} (j_1, \cdots, j_{n-1}, \overline{1}) & \text{if } b = (1, j_1, \cdots, j_{n-1}) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that $V(\Lambda_n)$ is a well-defined $U'_q(\hat{\mathfrak{sp}}(n))$-module with these actions of e_0, f_0 given above and $q^{h_0} = q^{-(h_1 + \cdots + h_n)}$. We denote this $U'_q(\hat{\mathfrak{sp}}(n))$-module by V^n.

4.2.3. *Construction of a Polarization of* V^n Let (,) be the polarization of the $U_q(\mathfrak{sp}(n))$-module $V(\Lambda_n)$. Let us prove (4.1.3). Set $b_{\Lambda_n} = (1, 2, \cdots, n)$. Then

$$f_1^{(2)} \cdots f_{n-1}^{(2)} f_n b_{\Lambda_n} = (2, \cdots, n, \overline{1}) = e_0 b_{\Lambda_n}. \tag{4.2.2}$$

Let $b = (1, j_1, \cdots, j_{n-1})$. Then b and $e_0 b$ can be written as

$$b = f_{i_k}^{(n_k)} \cdots f_{i_1}^{(n_1)} b_{\Lambda_n}, \tag{4.2.3}$$

$$e_0 b = f_{i_k}^{(n_k)} \cdots f_{i_1}^{(n_1)} e_0 b_{\Lambda_n}, \tag{4.2.4}$$

where $i_1, \cdots, i_k \in \{2, \cdots, n\}$ and $n_j \in \{1, 2\}$ $(1 \le j \le k)$. By Lemma 4.1.2 and (4.2.2), $(b_{\Lambda_n}, b_{\Lambda_n}) = (e_0 b_{\Lambda_n}, e_0 b_{\Lambda_n})$. Hence it follows from Lemma 4.1.2, (4.2.3),(4.2.4) and the commutativity of e_0 with $e_i (i \ne 1)$, $f_j (j \ne 0)$ that $(b, b) = (e_0 b, e_0 b)$. Since

$$(b, q_0^{-1} t_0^{-1} f_0 e_0 b) = (b, b),$$

we have proved (4.1.3). The equality (4.1.4) is similarly proved. Consequently (,) is a polarization of the $U'_q(\hat{\mathfrak{sp}}(n))$-module V^n. □

4.2.4. *Calculation of R-matrix* By the decomposition (4.2.1), $\mathrm{R}(x/y)$ can be written as $\mathrm{R}(x/y) = \oplus_{i=1}^n \gamma_i P_{2\Lambda_i} \oplus \gamma_0 P_0$, where $P_{2\Lambda_i}$ and P_0 is the projections $P_{2\Lambda_i} : V(\Lambda_n) \otimes V(\Lambda_n) \to V(2\Lambda_i)$ and $P_0 : V(\Lambda_n) \otimes V(\Lambda_n) \to V(0)$ respectively. Let $u_{2\Lambda_i} (1 \le i \le n)$ and u_0 be the highest weight vectors in the $U_q(\mathfrak{sp}(n))$-module $V(\Lambda_n) \otimes V(\Lambda_n)$ with weights $2\Lambda_i$ and 0 respectively.

We set $f_0 f_1^{(2)} \cdots f_i^{(2)}$. Let us define $b_1^{(i)}$ and $b_2^{(i)}$ by

$$b_1^{(i)} = (j)_{j=1}^n \otimes (1, \cdots, i, \overline{n}, \cdots, \overline{i+1}),$$
$$b_2^{(i)} = (1, \cdots, i, i+2, \cdots, n, \overline{i+1}) \otimes (1, \cdots, i+1, \overline{n}, \cdots, \overline{i+2}).$$

The proofs of the following lemmas are similar to that of Lemma 4.1.3 and Lemma 4.1.4.

Lemma 4.2.3. *Let $u_{i+1} = P_i u_{2\Lambda_i}$. Then u_{i+1} is non-zero and is proportional to $u_{2\Lambda_{i+1}}$.*

Lemma 4.2.4. *Let b be an element of $V^n \otimes V^n$ which is a tensor product of global bases and has the weight $2\Lambda_i$. Set $P_i b = \sum_{b'} F_b^{b'} b'$. Then $F_b^{b_1^{(i+1)}} \neq 0$ if and only if $b = b_1^{(i)}$ or $b_2^{(i)}$. Moreover $P_i b_1^{(i)} = q^{-2} y^{-1} b_1^{(i+1)}$ and $P_i b_2^{(i)} = x^{-1} b_1^{(i+1)}$.*

Lemma 4.2.5. *If we write $u_{2\Lambda_i} = b_1^{(i)} + \sum_{b \neq b_1^{(i)}} a_b b$, then $a_{b_2^{(i)}} = -q^{2(n-i)}$.*

Proof. Let $b_{m_1,\cdots,m_{n-i}} = (1, \cdots, i, m_1, \cdots, m_{n-i}) \otimes (1, \cdots, i, \overline{m_{n-i}}, \cdots, \overline{m_1})$ with $i < m_1 < \cdots < m_{n-i} < \bar{i}$. Note that $b_1^{(i)} = b_{i+1,\cdots,n}$ and $b_2^{(i)} = b_{i+2,\cdots,n,\overline{i+1}}$. We write a_α instead of a_{b_α} for $\alpha = (m_1, \cdots, m_{n-i})$. Using the fact that all the weight spaces of V^n are one dimensional and each length of j-strings $(1 \leq j \leq n)$ is at most 2, in a similar manner as in the proof of Lemma 4.1.4, we have

$$a_{i+1,\cdots,n-1,\bar{n}} = -q^2,$$

$$a_{i+1,\cdots,j,j+2,\cdots,n,\overline{j+1}} : a_{i+1,\cdots,j-1,j+1,\cdots,n,\overline{j+1}} : a_{i+1,\cdots,j-1,j+1,\cdots,n,\bar{j}} = 1 : -[2]_j^{-1} : q^2.$$

It follows that $b_2^{(i)} = -q^{2(n-i)}$. □

By Lemma 4.2.3- 4.2.5 we have in $V_x^1 \otimes V_y^1$

$$P_i u_{2\Lambda_i} = q^{-2} x^{-1} y^{-1} (x - q^{2(n-i+1)} y) u_{2\Lambda_{i+1}},$$

where $u_{2\Lambda_i} (1 \leq i \leq n)$ and u_0 are supposed to be normalized as in Lemma 4.2.5. From this

$$\frac{\gamma_i}{\gamma_{i+1}} = \frac{x - q^{2(n-i+1)} y}{y - q^{2(n-i+1)} x} \quad \text{for} \quad 0 \leq i \leq n-1.$$

Consequently we have

Proposition 4.2.6. *Let $z = xy^{-1}$.The R-matrix, up to a multiple of an element of $\mathbf{Q}(q)(z)$, is of the following form*

$$R(z) = \bigoplus_{j=0}^{n-1} \prod_{k=1}^{j} (z - q^{2(k+1)}) \prod_{i=j+1}^{n} (1 - q^{2(i+1)} z) P_{2\Lambda_{n-j}} \bigoplus \prod_{k=1}^{n} (z - q^{2(k+1)}) P_0.$$

4.3. $(D_n^{(1)}, V^n\quad)$ Let $\mathfrak{g} = \hat{\mathfrak{so}}(2n)$ be the affine Lie algebra of type $D_n^{(1)}$. Define $\tilde{\Lambda}_i (i \in \mathbf{Z})$ by

$$\tilde{\Lambda}_i = \begin{cases} \Lambda_i & \text{for } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We assume that $(\langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is the Cartan matrix of type D_n. We set $i_0 = 0$. Let $U_q(\mathfrak{so}(2n))$be the subalgebra of $U_q(\hat{\mathfrak{so}}(2n))$associated with $\{h_i, \alpha_j | 1 \leq i, j \leq n\}$.

4.3.1. *Decomposition of tensor product*

Let $V(\Lambda_n)$ be the irreducible highest weight $U_q(\mathfrak{so}(2n))$-module with highest weight Λ_n and $(L(\Lambda_n), B(\Lambda_n))$ its crystal base. By [KN] the elements of $B(\Lambda_n)$ are labeled in the following way.

$$B(\Lambda_n) = \{(m_i)_{i=1}^n | m_i = + \quad \text{or} \quad -, \quad \prod_{i=1}^n m_i = +\}.$$

Let N_n be the largest integer which does not exceed $\frac{n}{2}$. Then

$$V(\Lambda_n) \otimes V(\Lambda_n) \simeq \oplus_{i=0}^{N_n} V(\delta_{i0}\tilde{\Lambda}_n + \tilde{\Lambda}_{n-2i}). \tag{4.3.1}$$

The highest weight elements for the corresponding crystals are given by

$$(+,\cdots,+) \otimes (+,\cdots, \overset{n-2i}{\check{+}}, -,\cdots,-) \quad \text{for} \quad B(\delta_{i0}\tilde{\Lambda}_n + \tilde{\Lambda}_{n-2i}).$$

4.3.2. *Construction of the Representation of* $U_q'(\widehat{\mathfrak{so}}(2n))$
Define the actions of e_0 and f_0 on $V(\Lambda_n)$ by

$$f_0 b = \begin{cases} (+,+,i_1,\cdots,i_{n-2}) & \text{if } b = (-,-,i_1,\cdots,i_{n-2}) \\ 0 & \text{otherwise} \end{cases},$$
$$e_0 b = \begin{cases} (-,-,j_1,\cdots,j_{n-2}) & \text{if } b = (+,+,j_1,\cdots,j_{n-2}) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that $V(\Lambda_n)$ is a well-defined $U_q'(\widehat{\mathfrak{so}}(2n))$-module with the actions of e_0, f_0 given above and $q^{h_0} = q^{-h_1-2(h_2+\cdots+h_{n-1})-h_n}$. We denote this $U_q'(\widehat{\mathfrak{so}}(2n))$-module by V^n.

4.3.3. *Construction of a Polarization of* V^n Let (,) be the polarization of the $U_q(\mathfrak{so}(2n))$-module $V(\Lambda_n)$. Note that for any $b \in B(\Lambda_n)$ and i $(1 \leq i \leq n)$, $\varphi_i(b) + \varepsilon_i(b) \leq 1$. Hence, as in 4.1.3, (,) is a polarization of $U_q'(\widehat{\mathfrak{so}}(2n))$-module V^n.

4.3.4. *Calculation of R-matrix* By the decomposition (4.3.1), $\mathrm{R}(x/y)$ can be written as $\mathrm{R}(x/y) = \oplus_{i=1}^{N_n} \gamma_{n-2i} P_{(\delta_{i0}\tilde{\Lambda}_n+\tilde{\Lambda}_{n-2i})}$, where $P_{(\delta_{i0}\tilde{\Lambda}_n+\tilde{\Lambda}_{n-2i})}$ is the projection $P_{(\delta_{i0}\tilde{\Lambda}_n+\tilde{\Lambda}_{n-2i})} : V(\Lambda_n) \otimes V(\Lambda_n) \to V(\delta_{i0}\tilde{\Lambda}_n + \tilde{\Lambda}_{n-2i})$. Let $u_{n-2i} (1 \leq i \leq N_n)$ be the highest weight vector in the $U_q(\mathfrak{so}(2n))$-module $V(\Lambda_n) \otimes V(\Lambda_n)$ with the weight $\delta_{i0}\tilde{\Lambda}_n + \tilde{\Lambda}_{n-2i}$. We set $P_i = f_0 f_2 f_3 \cdots f_{n-2i+1} f_1 f_2 \cdots f_{n-2i}$. Let us define $b_1^{(i)}$ and $b_2^{(i)}$ by

$$b_1^{(i)} = (+,\cdots,+) \otimes (+,\cdots, \overset{n-2i}{\check{+}}, -,\cdots,-),$$
$$b_2^{(i)} = (+,\cdots, \overset{n-2i}{\check{+}}, -,-,+,\cdots,+) \otimes (+,\cdots, \overset{n-2i}{\check{+}}, +,+,-,\cdots,-).$$

The proofs of the following two lemmas are similar to that of Lemma 4.1.3 and 4.1.4.

Lemma 4.3.2. *The element* $P_i u_{n-2i} (0 \leq i \leq N_n)$ *is non-zero and is proportional to* $u_{n-2(i-1)}$.

Lemma 4.3.3. *Let* b *be an element of* $V^n \otimes V^n$ *which is a tensor product of global bases and has the weight* $\delta_{i0}\tilde{\Lambda}_n + \tilde{\Lambda}_{n-2i}$. *Set* $P_i b = \sum_{b'} F_b^{b'} b'$. *Then* $F_b^{b_1^{(i-1)}} \neq 0$ *if and only if* $b = b_1^{(i)}$ *or* $b_2^{(i)}$. *Moreover* $P_i b_1^{(i)} = q^{-1} y^{-1} b_1^{(i-1)}$ *and* $P_i b_2^{(i)} = x^{-1} b_1^{(i-1)}$.

Lemma 4.3.4. *If we write* $u_{n-2i} = b_1^{(i)} + \sum_b a_b b$, *then* $a_{b_2^{(i)}} = -q^{4i-3}$.

Proof. Let us set $b^-_{n-j} = (+,\cdots,+,\overset{n-j}{\breve{-}},+,\cdots,+,-)\otimes(+,\cdots,\overset{n-2i}{\check{+}},-,\cdots,-,\overset{n-j}{\check{+}},-,\cdots,-,+)$ and $b^+_{n-j} = (+,\cdots,+,\overset{n-2i+1}{\breve{-}},+,\cdots,+,\overset{n-j}{\breve{-}},+,\cdots,+) \otimes (+,\cdots,\overset{n-2i+1}{\check{+}},-,\cdots,-,\overset{n-j}{\check{+}},-,\cdots,-)$. By using

$$\begin{aligned} e_{n-j}(b^-_{n-j+1} - qb^-_{n-j}) &= 0 \quad \text{for} \quad 2 \le j \le 2i-1, \\ e_{n-j}(b^+_{n-j+1} - qb^+_{n-j}) &= 0 \quad \text{for} \quad 1 \le j \le 2i-2, \\ e_n(b^{(i)}_1 - qb^-_{n-1}) &= 0, \end{aligned}$$

it is easily verified, in a similar way as in the proof of Lemma 4.1.4, that u_{n-2i} must be of the form

$$\begin{aligned} u_{n-2i} =& b^{(i)}_1 + \sum_{j=1}^{2i-1}(-1)^j q^j b^-_{n-j} + \sum_{j=1}^{2i-2}(-1)^{j-1}q^{2i+j-1}b^+_{n-j} \\ &+ (\text{terms without the elements of the global base already appeared}). \end{aligned}$$

Since $b^{(i)}_2 = b_{n-2i+2,+}$, we have proved the lemma. □

By these lemmas we have in $V^1_x \otimes V^1_y$

$$P_i u_{n-2i} = q^{-1}x^{-1}y^{-1}(x - q^{4i-2}y)u_{n-2(i-1)}.$$

From this

$$\frac{\gamma_{n-2(i-1)}}{\gamma_{n-2i}} = \frac{y - q^{4i-2}x}{x - q^{4i-2}y} \quad \text{for} \quad 1 \le i \le N_n.$$

So we have proved

Proposition 4.3.5. *Let $z = xy^{-1}$. The R-matrix, up to a multiple of an element of $\mathbb{Q}(q)(z)$, is of the following form*

$$R(z) = \bigoplus_{j=0}^{N_n} \prod_{k=1}^{j}(z - q^{4k-2}) \prod_{i=j+1}^{N_n}(1 - q^{4i-2}z)P_{(\delta_{j0}\tilde{\Lambda}_n + \tilde{\Lambda}_{n-2j})}.$$

4.4. $(D^{(1)}_n, V^1\ \ )$ We use the same notations as in 4.3. Let us set $i_0 = 0$.

4.4.1. *Decomposition of tensor product*

Let $V(\Lambda_1)$ be the irreducible highest weight $U_q(\mathfrak{so}(2n))$-module with highest weight Λ_1 and $(L(\Lambda_1), B(\Lambda_1))$ its crystal base. By [KN] the elements of $B(\Lambda_1)$ are labeled in the following way.

$$B(\Lambda_1) = \{(i) | i \in \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{1}\}.$$

Then

$$V(\Lambda_1) \otimes V(\Lambda_1) \simeq V(2\Lambda_1) \oplus V(\Lambda_2) \oplus V(0). \tag{4.4.1}$$

4.4.2. *Construction of the Representation of $U'_q(\hat{\mathfrak{so}}(2n))$*

Define the actions of e_0 and f_0 on $V(\Lambda_1)$ by

$$\begin{array}{llll} f_0(\overline{1}) = (2), & f_0(\overline{2}) = (1), & f_0 b = 0 & \text{otherwise} \quad , \\ e_0(2) = (\overline{1}), & e_0(1) = (\overline{2}), & e_0 b = 0 & \text{otherwise} \quad . \end{array}$$

It is easily verified that $V(\Lambda_1)$ is a well-defined $U_q'(\hat{so}(2n))$-module with the actions of e_0, f_0 given above and $q^{h_0} = q^{-h_1-2(h_2+\cdots+h_{n-2})-h_{n-1}-h_n}$. We denote this $U_q'(\hat{so}(2n))$-module by V^1.

4.4.3. *Construction of a Polarization of* V^1 Let (,) be the polarization of the $U_q(so(2n))$-module $V(\Lambda_1)$. Note that for any $b \in B(\Lambda_1)$ and i $(1 \leq i \leq n)$, $\varphi_i(b) + \varepsilon_i(b) \leq 1$. Hence, as in 4.1.3, (,) is a polarization of $U_q'(\hat{so}(2n))$-module V^1.

4.4.4. *Calculation of R-matrix* By the decomposition (4.4.1), $\mathrm{R}(x/y)$ can be written as $\mathrm{R}(x/y) = \gamma_{2\Lambda_1} P_{2\Lambda_1} \oplus \gamma_{\Lambda_2} P_{\Lambda_2} \oplus \gamma_0 P_0$, where $P_{2\Lambda_1}$ etc. is, as in the previous sections, the projection to the corresponding $U_q(so(2n))$-irreducible component. Let u_0, u_{Λ_2} and $u_{2\Lambda_1}$ be the highest weight vectors in the $U_q(so(2n))$-module $V(\Lambda_1) \otimes V(\Lambda_1)$ with the corresponding highest weights. Direct calculations show the following lemmas.

Lemma 4.4.1. *The highest weight vectors* u_0, u_{Λ_2} *and* $u_{2\Lambda_1}$ *are, up to constants,*
(1) $u_0 = \sum_{i=1}^{n} (-1)^{i-1} q^{i-1} (i) \otimes (\overline{i}) + \sum_{i=0}^{n-1} (-1)^{n+i-1} q^{n+i-1} (\overline{n-i}) \otimes (n-i)$,
(2) $u_{\Lambda_2} = (1) \otimes (2) - q(2) \otimes (1)$,
(3) $u_{2\Lambda_1} = (1) \otimes (1)$.

Lemma 4.4.2. *With the expressions of* u_0, u_{Λ_2} *and* $u_{2\Lambda_1}$ *in Lemma 4.4.1 we have in* $V_x^1 \otimes V_y^1$
(1) $f_0 u_0 = q^{-1} x^{-1} y^{-1} (x - q^{2n-2} y) u_{\Lambda_2}$,
(2) $f_0 f_2 f_3 \cdots f_{n-2} f_n f_{n-1} \cdots f_2 u_{\Lambda_2} = q^{-1} x^{-1} y^{-1} (x - q^2 y) u_{2\Lambda_1}$.

By lemma 4.4.2

$$\frac{\gamma_0}{\gamma_{\Lambda_2}} = \frac{x - q^{2n-2} y}{y - q^{2n-2} x} \quad \text{and} \quad \frac{\gamma_{\Lambda_2}}{\gamma_{2\Lambda_1}} = \frac{x - q^2 y}{y - q^2 x}.$$

So we have proved

Proposition 4.4.3. *Let* $z = xy^{-1}$. *The R-matrix, up to a multiple of an element of* $\mathbf{Q}(q)(z)$, *is of the following form*

$$R(z) = (1 - q^2 z)(1 - q^{2n-2} z) P_{2\Lambda_1} \oplus (z - q^2)(1 - q^{2n-2} z) P_{\Lambda_2} \oplus (z - q^{2n-2})(z - q^2) P_0.$$

4.5. $(B_n^{(1)}, V^1$) Let $\mathfrak{g} = \hat{so}(2n+1)$ be the affine Lie algebra of type $B_n^{(1)}$. We assume that $(\langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is the Cartan matrix of type B_n. We set $i_0 = 0$. Let $U_q(so(2n+1))$be the subalgebra of $U_q(\hat{so}(2n+1))$associated with $\{h_i, \alpha_j | 1 \leq i, j \leq n\}$. In this case $q_n = q$ and $q_i = q^2 (i \neq n)$.

4.5.1. *Decomposition of tensor product*

Let $V(\Lambda_1)$ be the irreducible highest weight $U_q(so(2n+1))$-module with highest weight Λ_1 and $(L(\Lambda_1), B(\Lambda_1))$ its crystal base. By [KN] the elements of $B(\Lambda_1)$ are labeled in the following way.

$$B(\Lambda_1) = \{(i) | i \in \{1, 2, \cdots, n, 0, \overline{n}, \cdots, \overline{1}\}.$$

Then

$$V(\Lambda_1) \otimes V(\Lambda_1) \simeq V(2\Lambda_1) \oplus V(\Lambda_2) \oplus V(0). \tag{4.5.1}$$

The highest weight elements for the corresponding crystals are given by

$$(1) \otimes (1) \quad \text{for} \quad B(2\Lambda_1), \quad (1) \otimes (2) \quad \text{for} \quad B(\Lambda_2) \quad \text{and} \quad (1) \otimes (\overline{1}) \quad \text{for} \quad B(0).$$

4.5.2. *Construction of the Representation of* $U_q'(\hat{so}(2n+1))$
Define the actions of e_0 and f_0 on $V(\Lambda_1)$ by

$$\begin{aligned} &f_0(\overline{2}) = (1), \quad f_0(\overline{1}) = (2), \quad f_0 b = 0 \quad \text{otherwise}, \\ &e_0(1) = (\overline{2}), \quad e_0(2) = (\overline{1}), \quad e_0 b = 0 \quad \text{otherwise}. \end{aligned}$$

It is easily verified that $V(\Lambda_1)$ is a well-defined $U_q'(\hat{so}(2n+1))$-module with the actions of e_0, f_0 given above and $q^{h_0} = q^{-h_1 - 2(h_2 + \cdots + h_{n-1}) - h_n}$. We denote this $U_q'(\hat{so}(2n+1))$-module by V^1.

4.5.3. *Construction of a Polarization of* V^1 Let (,) be the polarization of the $U_q(so(2n+1))$-module $V(\Lambda_1)$. By using Lemma 4.1.2 we have

$$((i), (i)) = q_n [2]_n ((0), (0)) \quad \text{for} \quad i \in \{1, \cdots, n, \overline{n}, \cdots, \overline{1}\}.$$

It follows from this that (,) is a polarization of $U_q'(\hat{so}(2n+1))$-module V^1.

4.5.4. *Calculation of R-matrix* By the decomposition (4.5.1), $\mathrm{R}(x/y)$ can be written as $\mathrm{R}(x/y) = \gamma_{2\Lambda_1} P_{2\Lambda_1} \oplus \gamma_{\Lambda_2} P_{\Lambda_2} \oplus \gamma_0 P_0$, where $P_{2\Lambda_1}$ etc. are, as in the previous sections, the projections to the corresponding $U_q(so(2n+1))$-irreducible components. Let u_0, u_{Λ_2} and $u_{2\Lambda_1}$ be the highest weight vectors in the $U_q(so(2n+1))$-module $V(\Lambda_1) \otimes V(\Lambda_1)$ with the corresponding highest weights. Direct calculations show the following lemmas.

Lemma 4.5.1. *The highest weight vectors* u_0, u_{Λ_2} *and* $u_{2\Lambda_1}$ *are, up to constants,*

(1) $u_0 = \sum_{i=1}^{n} (-1)^{i-1} q^{2(i-1)} (i) \otimes (\overline{i}) + (-1)^n [2]_0^{-1} q^{2(n-1)} (0) \otimes (0)$
$\quad + \sum_{i=0}^{n-1} (-1)^{n-i-1} q^{2(n+i)} (\overline{n-i}) \otimes (n-i)$,
(2) $u_{\Lambda_2} = (1) \otimes (2) - q^2 (2) \otimes (1)$,
(3) $u_{2\Lambda_1} = (1) \otimes (1)$.

Lemma 4.5.2. *Consider the highest weight vectors* u_0, u_{Λ_2} *and* $u_{2\Lambda_1}$ *in Lemma 4.5.1. Then we have in* $V_x^1 \otimes V_y^1$

(1) $f_0 u_0 = q^{-2} x^{-1} y^{-1} (x - q^{4n-2} y) u_{\Lambda_2}$,
(2) $f_0 f_2 f_3 \cdots f_{n-1} f_n^{(2)} f_{n-1} \cdots f_2 u_{\Lambda_2} = q^{-2} x^{-1} y^{-1} (x - q^4 y) u_{2\Lambda_1}$,

By lemma 4.5.2

$$\frac{\gamma_{\Lambda_2}}{\gamma_0} = \frac{y - q^{4n-2} x}{x - q^{4n-2} y} \quad \text{and} \quad \frac{\gamma_{2\Lambda_1}}{\gamma_{\Lambda_2}} = \frac{y - q^4 x}{x - q^4 y}.$$

So we have proved

Proposition 4.5.3. *Let $z = xy^{-1}$. The R-matrix, up to a multiple of an element of $\mathbf{Q}(q)(z)$,is of the following form*

$$R(z) = (1-q^4z)(1-q^{4n-2}z)P_{2\Lambda_1} \oplus (1-q^{4n-2}z)(z-q^4)P_{\Lambda_2} \oplus (z-q^4)(z-q^{4n-2})P_0.$$

4.6. $(A^{(2)}_{2n-1}, V^1\quad)$ Let $\mathfrak{g}$ be the affine Lie algebra of type $A^{(2)}_{2n-1}$. We assume that $(\langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is the Cartan matrix of type C_n. We set $i_0 = 0$. Let $U_q(\mathfrak{sp}(n))$be the subalgebra of $U_q(\mathfrak{g})$associated with $\{h_i, \alpha_j | 1 \leq i, j \leq n\}$. In this case $q_n = q^2$ and $q_i = q (i \neq n)$.

4.6.1. *Decomposition of tensor product*
Let $V(\Lambda_1)$ be the irreducible highest weight $U_q(\mathfrak{sp}(n))$-module with highest weight Λ_1 and $(L(\Lambda_1), B(\Lambda_1))$ its crystal base. By [KN] the elements of $B(\Lambda_1)$ are labeled in the following way.

$$B(\Lambda_1) = \{(i) | i \in \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{1}\}.$$

Then

$$V(\Lambda_1) \otimes V(\Lambda_1) \simeq V(2\Lambda_1) \oplus V(\Lambda_2) \oplus V(0). \tag{4.6.1}$$

The highest weight elements for the corresponding crystals are given by

$$(1) \otimes (1) \quad \text{for} \quad B(2\Lambda_1), \quad (1) \otimes (2) \quad \text{for} \quad B(\Lambda_2) \quad \text{and} \quad (1) \otimes (\overline{1}) \quad \text{for} \quad B(0).$$

4.6.2. *Construction of the Representation of $U_q'(\mathfrak{g})$*
Define the actions of e_0 and f_0 on $V(\Lambda_1)$ by

$$\begin{aligned} &f_0(\overline{2}) = (1), \quad f_0(\overline{1}) = (2), \quad f_0 b = 0 \quad \text{otherwise}, \\ &e_0(1) = (\overline{2}), \quad e_0(2) = (\overline{1}), \quad e_0 b = 0 \quad \text{otherwise}. \end{aligned}$$

It is easily verified that $V(\Lambda_1)$ is a well-defined $U_q'(\mathfrak{g})$-module with the actions of e_0, f_0 given above and $q^{h_0} = q^{-h_1 - 2(h_2 + \cdots + h_n)}$. We denote this $U_q'(\mathfrak{g})$-module by V^1.

4.6.3. *Construction of a Polarization of V^1* Let (,) be the polarization of the $U_q(\mathfrak{g}_0)$-module $V(\Lambda_1)$. Note that for any $b \in B(\Lambda_1)$ and i $(1 \leq i \leq n)$, $\varphi_i(b) + \varepsilon_i(b) \leq 1$. Hence, as in 4.1.3, (,) is a polarization of $U_q'(\mathfrak{g})$-module V^1.

4.6.4. *Calculation of R-matrix* By the decomposition (4.6.1), $\mathrm{R}(x, y)$ can be written as $\mathrm{R}(x, y) = \gamma_{2\Lambda_1} P_{2\Lambda_1} \oplus \gamma_{\Lambda_2} P_{\Lambda_2} \oplus \gamma_0 P_0$, where $P_{2\Lambda_1}$ etc. are, as in the previous sections, the projections to the corresponding $U_q(\mathfrak{g}_0)$-irreducible components. Let u_0, u_{Λ_2} and $u_{2\Lambda_1}$ be the highest weight vectors in the $U_q(\mathfrak{g}_0)$-module $V(\Lambda_1) \otimes V(\Lambda_1)$ with the corresponding highest weights. Direct calculations show the following lemmas.

Lemma 4.6.1. *The highest weight vectors u_0, u_{Λ_2} and $u_{2\Lambda_1}$ are, up to constants,*
(1) $u_0 = \sum_{i=1}^{n} (-1)^{i-1} q^{i-1} (i) \otimes (\overline{i}) + \sum_{i=0}^{n-1} (-1)^{n+i} q^{n+i+1} (\overline{n-i}) \otimes (n-i)$,
(2) $u_{\Lambda_2} = (1) \otimes (2) - q(2) \otimes (1)$,
(3) $u_{2\Lambda_1} = (1) \otimes (1)$.

Lemma 4.6.2. *With the expressions of* u_0, u_{Λ_2} *and* $u_{2\Lambda_1}$ *in Lemma 4.6.1, we have in* $V_x^1 \otimes V_y^1$

(1) $f_0 u_0 = q^{-1}x^{-1}y^{-1}(x+q^{2n}y)u_{\Lambda_2}$,
(2) $f_0 f_2 f_3 \cdots f_{n-1} f_n f_{n-1} \cdots f_2 u_{\Lambda_2} = q^{-1}x^{-1}y^{-1}(x-q^2y)u_{2\Lambda_1}$.

By Lemma 4.6.2

$$\frac{\gamma_{\Lambda_2}}{\gamma_0} = \frac{y+q^{2n}x}{x+q^{2n}y} \quad \text{and} \quad \frac{\gamma_{2\Lambda_1}}{\gamma_{\Lambda_2}} = \frac{y-q^2x}{x-q^2y}.$$

So we have proved

Proposition 4.6.3. *Let* $z = xy^{-1}$. *The R-matrix, up to a multiple of an element of* $\mathbf{Q}(q)(z)$, *is of the following form*

$$R(z) = (1-q^2z)(1+q^{2n}z)P_{2\Lambda_1} \oplus (1+q^{2n}z)(z-q^2)P_{\Lambda_2} \oplus (z-q^2)(z+q^{2n})P_0.$$

4.7. $(A_{2n}^{(2)}, V^1\ )$ Let $\mathfrak{g}$ be the affine Lie algebra of type $A_{2n}^{(2)}$. We assume that $(\langle h_i, \alpha_j\rangle)_{1\le i,j\le n}$ and $(\langle h_i, \alpha_j\rangle)_{0\le i,j\le n-1}$ are the Cartan matrices of type C_n and B_n respectively. We set $i_0 = n$. Let $U_q(\mathfrak{sp}(n))$and $U_q(\mathfrak{so}(2n+1))$be the subalgebras of $U_q(\mathfrak{g})$associated with $\{h_i, \alpha_j | 1 \le i,j \le n\}$ and $\{h_i, \alpha_j | 0 \le i,j \le n-1\}$ respectively. We define a bijective map $\iota : \{0,1,\cdots,n-1\} \to \{1,\cdots,n\}$ by $\iota(p) = n-p$. In this case $q_0 = q$, $q_i = q^2 (i \ne 0, n)$ and $q_n = q^4$.

4.7.1. *Construction of the Representation of* $U_q'(\mathfrak{g})$
Let $V(\Lambda_1)$ be the irreducible highest weight $U_q(\mathfrak{so}(2n+1))$-module with highest weight Λ_1 and $(L(\Lambda_1), B(\Lambda_1))$ its crystal base. The parametrization of elements of $B(\Lambda_1)$ is already given in 4.5. Define the actions of e_n and f_n on $\iota^* V(\Lambda_1)$ by

$$f_n(\bar{1}) = (1), \quad f_n b = 0 \quad \text{otherwise},$$
$$e_n(1) = (\bar{1}), \quad e_n b = 0 \quad \text{otherwise}.$$

It is easily verified that $\iota^* V(\Lambda_1)$ is a well-defined $U_q'(\mathfrak{g})$-module by the actions of e_n, f_n given above and $q^{h_n} = q^{-(h_0+\cdots+h_{n-1})}$. We denote this $U_q'(\mathfrak{g})$-module by V^1.

4.7.2. *Construction of a Polarization of* V^1 Let (,) be the polarization of the $U_q(\mathfrak{so}(2n+1))$-module $V(\Lambda_{\iota(1)})$. As in 4.5.3, (,) is a polarization of $U_q'(\mathfrak{g})$-module V^1.

4.7.3. *Calculation of R-matrix* By the decomposition (4.5.1), $\mathrm{R}(x/y)$ can be written as $\mathrm{R}(x/y) = \gamma_{2\Lambda_{n-1}}\iota^* P_{2\Lambda_1} \oplus \gamma_{\Lambda_{n-2}}\iota^* P_{\Lambda_2} \oplus \gamma_0 \iota^* P_0$, where $P_{2\Lambda_1}$ etc. are, as in the previous sections, the projections to the corresponding $U_q(\mathfrak{so}(2n+1))$-irreducible components. Let u_0, $u_{\Lambda_{n-2}}$ and $u_{2\Lambda_{n-1}}$ be the highest weight vectors in the $U_q(\mathfrak{so}(2n+1))$-module $\iota^* V(\Lambda_1) \otimes \iota^* V(\Lambda_1)$ with the corresponding highest weights. Direct calculations show the following lemma.

Lemma 4.7.2. *With the expressions of u_0, $u_{\Lambda_{n-2}}$ and $u_{2\Lambda_{n-1}}$ in Lemma 4.5.1, we have in $V_x^1 \otimes V_y^1$*

(1) $f_n u_0 = q^{-4}x^{-1}y^{-1}(x + q^{4n+2}y)u_{2\Lambda_{n-1}}$,

(2) $f_n f_{n-1} \cdots f_1 f_0^{(2)} f_1 f_2 \cdots f_{n-2} u_{\Lambda_{n-2}} = q^{-2}x^{-1}y^{-1}(x - q^4 y)u_{2\Lambda_{n-1}}$.

By Lemma 4.7.2

$$\frac{\gamma_{2\Lambda_{n-1}}}{\gamma_0} = \frac{y + q^{4n+2}x}{x + q^{4n+2}y} \quad \text{and} \quad \frac{\gamma_{2\Lambda_{n-1}}}{\gamma_{\Lambda_{n-2}}} = \frac{y - q^4 x}{x - q^4 y}.$$

So we have proved

Proposition 4.7.3. *Let $z = xy^{-1}$. The R-matrix, up to a multiple of an element of $\mathbf{Q}(q)(z)$, of the following form*

$$R(z) = (1-q^4z)(1+q^{4n+2}z)\iota^* P_{2\Lambda_1} \oplus (1+q^{4n+2}z)(z-q^4)\iota^* P_{\Lambda_2} \oplus (1-q^4z)(z+q^{4n+2})P_0.$$

4.8. $(D_{n+1}^{(2)}, V^1 \quad)$ Let $\mathfrak{g}$ be the affine Lie algebra of type $D_{n+1}^{(2)}$. We assume that $(\langle h_i, \alpha_j \rangle)_{1 \le i,j \le n}$ is the Cartan matrix of type B_n. Let $U_q(\mathfrak{so}(2n+1))$be the subalgebra of $U_q(\mathfrak{g})$associated with $\{h_i, \alpha_j | 1 \le i, j \le n\}$. We set $i_0 = 0$. In this case $q_0 = q_n = q$, $q_i = q^2 (i \ne 0, n)$.

4.8.1. *Construction of the Representation of $U_q'(\mathfrak{g})$*
Let $V(\Lambda_1) \oplus V(0)$ be the direct sum of the irreducible highest weight $U_q(\mathfrak{so}(2n+1))$-modules with highest weights Λ_1 and 0 respectively and $(L(\Lambda_1) \oplus L(0), B(\Lambda_1) \oplus B(0))$ its crystal base. Since the dimension of any non-zero weight space of $V(\Lambda_1)$ is one, we denote, as usual, by b the lower global base corresponding to $b \in B(\Lambda_1)$. Let us denote by $(\cdot)$ the element of $B(0)$. The parametrization of elements of $B(\Lambda_1)$ is also given by [KN] as

$$B(\Lambda_1) = \{(i) | i \in \{1, 2, \cdots, n, 0, \overline{n}, \cdots, \overline{1}\}.$$

Define the actions of e_0 and f_0 on $V(\Lambda_1) \oplus V(0)$ by

$$\begin{aligned} &f_0(\overline{1}) = (\cdot), \quad f_0(\cdot) = [2]_0(1), \quad f_0 b = 0 \quad \text{otherwise}, \\ &e_0(\cdot) = [2]_0(\overline{1}), \quad e_0(1) = (\cdot), \quad e_0 b = 0 \quad \text{otherwise}. \end{aligned}$$

It is easily verified that $V(\Lambda_1) \oplus V(0)$ is a well-defined $U_q'(\mathfrak{g})$-module with the actions of e_0, f_0 given above and $q^{h_0} = q^{-2(h_1 + \cdots + h_{n-1}) - h_n}$. We denote this $U_q'(\mathfrak{g})$-module by V^1.

4.8.2. *Construction of a Polarization of V^1* Let $(\quad,\quad)_1$ be the polarization of the $U_q(\mathfrak{so}(2n+1))$-module $V(\Lambda_1)$. We shall define a symmetric bilinear form $(\ ,\ )$ on V^1 by

$$\begin{aligned} ((\cdot), u) = (u, (\cdot)) = 0 \quad &\text{for} \quad u \in V(\Lambda_1), \\ ((\cdot), (\cdot)) = q_0[2]_0((\cdot), (\cdot))_1&, \\ (u, v) = (u, v)_1 \quad &\text{for} \quad u, v \in V(\Lambda_1). \end{aligned}$$

Since $(\ ,\ )_1$ is positive definite, $(\ ,\ )$ is positive definite. As already proved in 4.5.3, $((\overline{1}),(\overline{1})) = ((1),(1))$. It follows that $(\ ,\ )$ is a polarization of the $U_q'(\mathfrak{g})$-module V^1.

4.8.3. *Decomposition of Tensor Product*
As a $U_q(\mathfrak{so}(2n+1))$-module we have the splitting

$$\begin{aligned}&(V(\Lambda_1)\oplus V(0))\otimes(V(\Lambda_1)\oplus V(0))\\ &\simeq V(2\Lambda_1)\oplus V(\Lambda_2)\oplus V(\Lambda_1)^{\oplus 2}\oplus V(0)^{\oplus 2}.\end{aligned}\tag{4.8.1}$$

Lemma 4.8.1. *The following $u_{2\Lambda_1}$, u_{Λ_2}, $u^1_{\Lambda_1}$, $u^2_{\Lambda_1}$, u^1_0 and u^2_0 are the highest weight vectors with the weights $2\Lambda_1$, Λ_2, Λ_1, Λ_1, 0 and 0 respectively.*
(1) $u_{2\Lambda_1} = (1)\otimes(1)$.
(2) $u_{\Lambda_2} = (1)\otimes(2) - q^2(2)\otimes(1)$.
(3) $u^1_{\Lambda_1} = (1)\otimes(\cdot)$.
(4) $u^2_{\Lambda_1} = (\cdot)\otimes(1)$.
(5) $u^1_0 = (\cdot)\otimes(\cdot)$.
(6) $u^2_0 = \sum_{i=1}^{n}(-1)^{i-1}q^{2(i-1)}(i)\otimes(\overline{i}) + (-1)^n q^{2(n-1)}[2]_n^{-1}(0)\otimes(0)$
$+\sum_{i=0}^{n-1}(-1)^{n+i+1}q^{2(n+i)}(\overline{n-i})\otimes(n-i)$.

4.8.4. *Calculation of R-matrix* We express the the R-matrix $R = \mathrm{R}(x/y)$ as

$$R(u_{2\Lambda_1}) = a^{2\Lambda_1}u_{2\Lambda_1},\quad R(u_{\Lambda_2}) = a^{\Lambda_2}u_{\Lambda_2},\quad R(u^i_{\Lambda_1}) = \sum_{j=1}^{2} a^{\Lambda_1}_{ij}u^j_{\Lambda_1},\quad R(u^i_0) = \sum_{j=1}^{2} a^0_{ij}u^j_0.$$

Lemma 4.8.2. *Consider the highest weight vectors defined in Lemma 4.8.1. Then we have in $V^1_x\otimes V^1_y$*
(1) $e_0u_{2\Lambda_1} = yu^1_{\Lambda_1} + q^2xu^2_{\Lambda_1}$,
(2) $f_0f_1\cdots f_{n-1}f_n^{(2)}f_{n-1}f_{n-2}\cdots f_1u_{2\Lambda_1} = x^{-1}y^{-1}(q^2xu^1_{\Lambda_1} + yu^2_{\Lambda_1})$,
(3) $f_0f_1\cdots f_{n-1}f_n^{(2)}f_{n-1}f_{n-2}\cdots f_2u_{\Lambda_2} = x^{-1}y^{-1}(xu^1_{\Lambda_1} - q^2yu^2_{\Lambda_1})$,
(4) $f_0u^1_0 = [2]_0x^{-1}y^{-1}(yu^1_{\Lambda_1} + xu^2_{\Lambda_1})$,
(5) $f_0u^2_0 = q^{-2}x^{-1}y^{-1}(xu^1_{\Lambda_1} + q^{4n}yu^2_{\Lambda_1})$.

From this we obtain

$$\begin{gathered}\begin{pmatrix} y & q^2x \\ q^2x & y\end{pmatrix}(a^{\Lambda_1}_{ij}) = a^{2\Lambda_1}\begin{pmatrix} x & q^2y \\ q^2y & x\end{pmatrix},\\ (x,-q^2y)(a^{\Lambda_1}_{ij}) = a^{\Lambda_2}(y,-q^2x),\\ \begin{pmatrix} q(1+q^2)y & q(1+q^2)x \\ x & q^{4n}y\end{pmatrix}(a^{\Lambda_1}_{ij}) = (a^0_{ij})\begin{pmatrix} q(1+q^2)x & q(1+q^2)y \\ y & q^{4n}x\end{pmatrix}.\end{gathered}$$

Let $P_{2\Lambda_1}$, P_{Λ_2}, $P_{u^i_{\Lambda_1}}(i=1,2)$ and $P_{u^j_0}(i=1,2)$ be the projections from $V^1\otimes V^1$ to $V(2\Lambda_1)$, $V(\Lambda_2)$, $U_q(\mathfrak{so}(2n+1))u^i_{\Lambda_1}(i=1,2)$ and $U_q(\mathfrak{so}(2n+1))u^i_0(i=1,2)$ respectively, where $u_{2\Lambda_1}$ etc. are the highest weight vectors in Lemma 4.8.1. Then we have

Proposition 4.8.3. *Let $z = xy^{-1}$. The R-matrix, up to a multiple of an element of $\mathbf{Q}(q)(z)$, is of the following form*

$$\begin{aligned}R(z) =&(1-q^4z^2)(1-q^{4n}z^2)P_{2\Lambda_1} \oplus (1-q^{4n}z^2)(z^2-q^4)P_{\Lambda_2}\\ &\oplus \oplus_{i=1}^{2} \sum_{j=1}^{2} a_{ij}^{\Lambda_1} P_{z^j_{\Lambda_1}} \oplus \oplus_{j=1}^{2} \sum_{k=1}^{2} a_{jk}^{0} P_{z_0^k}.\end{aligned}$$

Here $(a_{ij}^{\Lambda_1})$ and (a_{kl}^{0}) are given by

$$(a_{ij}^{\Lambda_1}) = (1-q^{4n}z^2)\begin{pmatrix}(1-q^4)z & q^2(1-z^2)\\ q^2(1-z^2) & (1-q^4)z\end{pmatrix}$$

$$\begin{aligned}a_{11}^0 &= q^2 + (1-q^2-q^4-q^{4n}-q^{4n+2}+q^{4n+4})z^2 + q^{4n+2}z^4,\\ a_{22}^0 &= q^{4n+2} + (1-q^2-q^4-q^{4n}-q^{4n+2}+q^{4n+4})z^2 + q^2z^4,\\ a_{12}^0 &= q(1+q^2)(1-q^4)z(1-z^2)\\ a_{21}^0 &= (1+q^2)^{-1}q(1+q^{4n-2})(1-q^{4n+2})z(1-z^2).\end{aligned}$$

The eigenvalues of $(a_{ij}^{\Lambda_1})$ are $(1-q^{4n}z^2)(z+q^2)(1-q^2z)$ and $(1-q^{4n}z^2)(z-q^2)(1+q^2z)$. Furthermore $\det(a_{ij}^0) = (z^2-q^{4n})(1-q^{4n}z^2)(z^2-q^4)(1-q^4z^2)$. In particular the eigenvalues of (a_{ij}^0) have no common zeros.

4.9. $(D_{n+1}^{(2)}, V^n\quad)$ We use the same notations as in 4.8. We set $i_0 = 0$.

4.9.1. *Decomposition of tensor product*

Let $V(\Lambda_n)$ be the irreducible highest weight $U_q(\mathfrak{so}(2n+1))$-module with highest weight Λ_n and $(L(\Lambda_n), B(\Lambda_n))$ its crystal base. Define $\tilde{\Lambda}_i (i \in \mathbf{Z})$ by

$$\tilde{\Lambda}_i = \begin{cases}\Lambda_i & \text{for } 1 \le i \le n,\\ 0 & \text{otherwise.}\end{cases}$$

By [KN] the elements of $B(\Lambda_n)$ are labeled in the following way.

$$B(\Lambda_n) = \{(m_i)_{i=1}^n | m_i = + \quad \text{or} \quad -\}.$$

Then

$$V(\Lambda_n) \otimes V(\Lambda_n) \simeq \oplus_{i=0}^{n} V(\delta_{i0}\tilde{\Lambda}_n + \tilde{\Lambda}_{n-i}). \tag{4.9.1}$$

The highest weight elements for the corresponding crystals are given by

$$(+,\cdots,+)\otimes(+,\cdots,\overset{i}{\check{+}},-,\cdots,-) \quad \text{for} \quad B(\delta_{i0}\tilde{\Lambda}_n + \tilde{\Lambda}_{n-i}).$$

4.9.2. *Construction of the Representation of $U_q'(\mathfrak{g})$*

Define the actions of e_0 and f_0 on $V(\Lambda_n)$ by

$$f_0 b = \begin{cases} (+, i_1, \cdots, i_{n-1}) & \text{if } b = (-, i_1, \cdots, i_{n-1}) \\ 0 & \text{otherwise} \end{cases},$$
$$e_0 b = \begin{cases} (-, j_1, \cdots, j_{n-1}) & \text{if } b = (+, j_1, \cdots, j_{n-1}) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that $V(\Lambda_n)$ is a well-defined $U_q'(\mathfrak{g})$-module with the actions of e_0, f_0 given above and $q^{h_0} = q^{-2(h_1+\cdots+h_{n-1})-h_n}$. We denote this $U_q'(\mathfrak{g})$-module by V^n.

4.9.3. *Construction of a Polarization of V^n* Let $(\ ,\ )$ be the polarization of the $U_q(\mathfrak{so}(2n+1))$-module $V(\Lambda_n)$. As in 4.3.3, $(\ ,\ )$ is a polarization of $U_q'(\mathfrak{g})$-module V^n.

4.9.4. *Calculation of R-matrix* By the decomposition (4.9.1), $\mathrm{R}(x/y)$ can be written as $\mathrm{R}(x/y) = \oplus_{i=1}^{n} \gamma_i P_{\delta_{in}\tilde{\Lambda}_n + \tilde{\Lambda}_i}$, where $P_{\delta_{in}\tilde{\Lambda}_n + \tilde{\Lambda}_i}$ is the projection $P_{\delta_{in}\tilde{\Lambda}_n + \tilde{\Lambda}_i} : V(\Lambda_n) \otimes V(\Lambda_n) \to V(\delta_{in}\tilde{\Lambda}_n + \tilde{\Lambda}_i)$. Let $u_i (0 \le i \le n)$ be the highest weight vector in the $U_q(\mathfrak{so}(2n+1))$-module $V(\Lambda_n) \otimes V(\Lambda_n)$ with the weight $\delta_{in}\tilde{\Lambda}_n + \tilde{\Lambda}_i$. We set $P_i = f_0 f_1 f_2 \cdots f_i$. Let us define $b_1^{(i)}$ and $b_2^{(i)}$ by

$$b_1^{(i)} = (+, \cdots, +) \otimes (+, \cdots, \overset{\overset{i}{\smile}}{+}, -, \cdots, -),$$
$$b_2^{(i)} = (+, \cdots, +, \overset{\overset{i+1}{\smile}}{-}, +, \cdots, +) \otimes (+, \cdots, +, \overset{\overset{i+1}{\smile}}{+}, -, \cdots, -).$$

The proofs of the following two lemmas are similar to those of Lemma 4.1.3 and 4.1.4.

Lemma 4.9.2. *The element $P_i u_i$ $(0 \le i \le n-1)$ is proportional to u_{i+1}.*

Lemma 4.9.3. *Let b be an element of $V^n \otimes V^n$ which is a tensor product of global bases and has the weight $\delta_{in}\tilde{\Lambda}_n + \tilde{\Lambda}_i$. Set $P_i b = \sum_{b'} F_b^{b'} b'$. Then $F_b^{b_1^{(i+1)}} \neq 0$ if and only if $b = b_1^{(i)}$ or $b_2^{(i)}$. Moreover $P_i b_1^{(i)} = q^{-1} y^{-1} b_1^{(i+1)}$ and $P_i b_2^{(i)} = x^{-1} b_1^{(i+1)}$.*

Lemma 4.9.4. *If we write $u_i = b_1^{(i)} + \sum_b a_b b$, then $a_{b_2^{(i)}} = (-1)^{n-i} q^{2(n-i)+3}$.*

Proof. Let $b_k = (+, \cdots, +, \overset{\overset{k}{\smile}}{-}, +, \cdots, +) \otimes (+, \cdots, \overset{\overset{i}{\smile}}{+}, -, \cdots, -, \overset{\overset{k}{\smile}}{+}, -, \cdots, -)$ for $i + 1 \le k \le n$ and $b_1^{(i)} = b_{n+1} = (+, \cdots, +) \otimes (+, \cdots, \overset{\overset{i}{\smile}}{+}, -, \cdots, -)$. By using

$$e_{k-1}(b_k - q^2 b_{k-1}) = 0 \quad \text{for} \quad i+2 \le k \le n,$$
$$e_n(b_{n+1} - q b_n) = 0,$$
$$e_n(b_1^{(i)} - q b_{n-1,-}) = 0,$$

it is easily verified that u_i must be of the form

$$u_i = b_1^{(i)} + \sum_{j=0}^{i-1} (-1)^{j+1} q^{2j+1} b_{n-j}$$
$$+ (\text{terms without the elements of the global base already appeared}).$$

□

By these lemmas we have in $V_x^1 \otimes V_y^1$

$$P_i u_i = q^{-1} x^{-1} y^{-1} (x + (-1)^{n-i} q^{2(n-i+2)} y) u_{i+1},$$

where u_i $(0 \le i \le n)$ are normalized as in Lemma 4.9.4. From this

$$\frac{\gamma_{i+1}}{\gamma_i} = \frac{y + (-1)^{n-i} q^{2(n-i+2)} x}{x + (-1)^{n-i} q^{2(n-i+2)} y} \quad \text{for} \quad 0 \le i \le n-1.$$

So we have proved

Proposition 4.9.5. *Let $z = xy^{-1}$. The R-matrix, up to a multiple of an element of $\mathbf{Q}(q)(z)$, is of the following form*

$$R(z) = \bigoplus_{r=0}^{n} \prod_{i=0}^{r-1} (1 + (-1)^{n-i} q^{2(n-i+2)} z) \prod_{j=r}^{n-1} (z + (-1)^{n-j} q^{2(n-j+2)}) P_{\delta_{rn} \tilde{\Lambda}_n + \tilde{\Lambda}_r}.$$

5. Applications of Fusion Construction

5.1. $(A_n^{(1)}, V(l(\Lambda_k - \Lambda_0)))$ Let us take $A_n^{(1)}$ as $\mathfrak{g}$ and let I and i_0 be as in 4.1. Let $V = V^k$ $(1 \le k \le n)$ be the $U_q(\mathfrak{g})$-module with the polarization constructed in 4.1.1. Then as an $U_q(\mathfrak{g}_{I \setminus \{i_0\}})$-module, $V = V^k$ is isomorphic to $V(\Lambda_k - \Lambda_0)$. Now, we shall employ the results and notations in 3. Taking $\lambda_0 = \Lambda_k - \Lambda_0$ the condition (3.1.1) is staisfied. The existence of V_{K_Z} is obvious. We have, by using the explicit form of the R matrix in Proposition 4.1.6

$$\varphi(z) = \prod_{j=1}^{min(n+1-k,k)} (1 - q^{2j} z).$$

Here $\varphi(u)$ is the one given in (3.1.6). Therefore if we set

$$r = 1 \tag{5.1.1}$$

the condition (3.3.4) is satisfied. We have

$$R(q^{2r}) = \varphi(q^{2r}) P_{2\lambda_0}.$$

Here $P_{2\lambda_0}$ is the projector to the $U_q(\mathfrak{g}_{I \setminus \{i_0\}})$-module $V(2\lambda_0)$.Hence we obtain

$$\mathrm{Im} R(q^{2r}) = V(2\lambda_0). \tag{5.1.2}$$

Therefore, we obtain by (3.3.9)

$$V_l \subset \bigcap_{j=0}^{l-2} V(\lambda_0)^{\otimes j} \otimes V(2\lambda_0) \otimes V(\lambda_0)^{\otimes l-2-j}. \tag{5.1.3}$$

Thus, applying [KN], we obtain

Lemma 5.1.1. *Let $I_0 = I\backslash\{0\}$. Then as an $U_q(\mathfrak{g}_{I_0})$-module, V_l is isomorphic to the irreducible highest weight module with highest weight $l(\Lambda_k - \Lambda_0)$*

By applying Proposition 3.4.4, we obtain

Proposition 5.1.2. *For $1 \leq l$ and $1 \leq k \leq n$, there exists a polarized $U_q'(\mathfrak{g})$-module V_l^k satisfying:*
(i) V_l^k has a crystal pseudo-base.
(ii) For any $j \in I$, V_l^k is isomorphic as a $U_q(\mathfrak{g}_{I\backslash\{j\}})$-module to the irreducible module with highest weight $l(\Lambda_{j+k} - \Lambda_j)$.

Proof. (i) is an immediate consequence of Proposition 3.4.4, and (ii) is already proved for $j = 0$. Since the Weyl group contains the cyclic permutation, $I \ni i \mapsto i+j \in I$, $wt(V_l^k) \ni l(\Lambda_{j+k}-\Lambda_j)$ and $wt(V_l^k) \subset l(\Lambda_{j+k}-\Lambda_j)+\sum_{i\neq j} \mathbf{Z}_{\leq 0}cl(\alpha_i)$. Hence V_l^k contains the $U_q(\mathfrak{g}_{I\backslash\{j\}})$-module $V(l(\Lambda_{j+k} - \Lambda_j))$. Comparing the dimensions, we obtain $V_l^k = V(l(\Lambda_{j+k} - \Lambda_j))$. □

5.2 $(C_n^{(1)}, V(l(\Lambda_n - \Lambda_0)))$ Let $\mathfrak{g}$ be of type $C_n^{(1)}$ and I, i_0 as in 4.2. In 4.2 we constructed $U_q'(\mathfrak{g})$-module V with polarization. It is obvious that V has $V_{K_{\mathbf{Z}}}$. As a $U_q(\mathfrak{g}_{I\backslash\{i_0\}})$-module, V is isomorphic to $V(\Lambda_n - \Lambda_0)$. Hence taking $\lambda_0 = \Lambda_n - \Lambda_0$, the condition (3.1.1) is satisfied. By the explicit form of the R-matrix given in Proposition 4.2.6,

$$\varphi(z) = \prod_{j=1}^{n}(1 - q^{2(j+1)}z).$$

Therefore, if we set

$$r = 2, \tag{5.2.1}$$

then the condition (3.3.4) is satisfied. We have

$$R(q^{2r}) = \varphi(q^{2r})P_{2\lambda_0}.$$

Here $P_{2\lambda_0}$ is the projection to the $U_q(\mathfrak{g}_{I\backslash\{i_0\}})$-module $V(2\lambda_0)$. Thus, by [KN] and Proposition 3.4.4, we have the following.

Proposition 5.2.1.
(i) V_l has a crystal pseudo-base.
(ii) V_l is isomorphic to $V(l(\Lambda_n - \Lambda_0))$ as an $U_q(\mathfrak{g}_{I\backslash\{0\}})$-module and is isomorphic to $V(l(\Lambda_0 - \Lambda_n))$ as an $U_q(\mathfrak{g}_{I\backslash\{n\}})$-module.

The last statements follows from the fact that the Weyl group of C_n contains -1.

5.3 $(D_n^{(1)}, V(l(\Lambda_n - \Lambda_0)))$ Let $\mathfrak{g}$ be of type $D_n^{(1)}$ and take I, i_0 as in 4.3. In 4.3, we constructed the polarized $U_q'(\mathfrak{g})$-module V. As a $U_q(\mathfrak{g}_{I\backslash\{i_0\}})$-module V is isomorphic to $V(\Lambda_n - \Lambda_0)$. It is obvious that $V_{K_{\mathbf{Z}}}$ as in 6.1 exists. Hence taking $\lambda_0 = \Lambda_n - \Lambda_0$, the condition (3.1.1) is satisfied. We have, by Proposition 4.3.5,

$$\varphi(z) = \prod_{j=1}^{N_n}(1 - q^{4j-2}z),$$

where N_n is the largest integer which does not exceed $n/2$. Therefore, if we set

$$r = 1, \tag{5.3.1}$$

then the condition (3.3.4) is satisfied. We have

$$R(q^{2r}) = \varphi(q^{2r})P_{2\lambda_0}.$$

Hence we have by (3.3.9), [KN] and Proposition 3.4.4,

Proposition 5.3.1.
(i) V_l has a crystal pseudo-base.
(ii) V_l is an irreducible $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module with highest weight $l(\Lambda_n - \Lambda_0)$ and an irreducible module with highest weight $l(\Lambda_0 - \Lambda_n)$ or $l(\Lambda_1 - \Lambda_n)$ as an $U_q(\mathfrak{g}_{I\setminus\{n\}})$-module according that n is even or odd.

(ii) follows from the fact that $\Lambda_0 - \Lambda_n$ or $\Lambda_1 - \Lambda_n$ is in the Weyl group orbit of $\Lambda_n - \Lambda_0$ according to the parity of n.

5.4 $(D_n^{(1)}, V(l(\Lambda_1 - \Lambda_0))))$ Let $\mathfrak{g}$ be of type $D_n^{(1)}$ and take I as in 4.4 and $i_0 = 0$. Let V be the polarized $U_q'(\mathfrak{g})$-module constructed in 4.4. Then V is isomorphic to $V(\Lambda_1 - \Lambda_0)$ as an $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module. Hence taking $\Lambda_1 - \Lambda_0$ as λ_0, the condition (3.1.1) is satisfied. By Proposition 4.4.3, we have

$$\varphi(z) = (1 - q^2 z)(1 - q^{2n-2} z).$$

We set

$$r = 1. \tag{5.4.1}$$

Then the condition (3.3.4) is satisfied and

$$R(q^{2r}) = \varphi(q^{2r})P_{2\lambda_0}. \tag{5.4.2}$$

Thus we obtain, by (3.3.9), [KN] and Proposition 3.4.4

Proposition 5.4.1.
(i) V_l is an irreducible module with highest weight $l(\Lambda_1 - \Lambda_0)$ as a $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module.
(ii) V_l has a crystal pseudo-base.

5.5 $(B_n^{(1)}, V(l(\Lambda_1 - \Lambda_n)))$ Let $\mathfrak{g}$ be of type $B_n^{(1)}$ and $I, i_0 = 0$ as in 4.5. We constructed in 4.5, the polarized $U_q'(\mathfrak{g})$-module V. It is an irreducible module with highest weight $\Lambda_1 - \Lambda_0$ as an $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module. Hence taking $\Lambda_1 - \Lambda_0$ as λ_0, the condition (3.1.1) is satisfied. We have, by Proposition 4.5.3,

$$\varphi(z) = (1 - q^4 z)(1 - q^{4n-2} z).$$

Set

$$r = 2. \tag{5.5.1}$$

Then the condition (3.3.4) is satisfied and

$$R(q^{2r}) = \varphi(q^{2r})P_{2\lambda_0}. \tag{5.5.2}$$

Hence by (3.3.9), [KN] and Proposition 3.4.4, we obtain

Proposition 5.5.1.

(i) V_l is an irreducible module with highest weight $l(\Lambda_1 - \Lambda_0)$ as an $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module.

(ii) V_l has a crystal pseudo base.

5.6 $(A^{(2)}_{2n-1}, V(l(\Lambda_1 - \Lambda_0)))$ Let $\mathfrak{g}$ be of type $A^{(2)}_{2n-1}$ and $I, i_0 = 0$ as in 4.6, where we constructed the polarized $U'_q(\mathfrak{g})$-module V. It is an irreducible module with highest weight $\Lambda_1 - \Lambda_0$ as an $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module. Hence taking $\lambda_0 = \Lambda_1 - \Lambda_0$, the condition (3.1.1) is satisfied. We have by Proposition 4.6.3

$$\varphi(z) = (1 - q^2 z)(1 - q^{2n} z).$$

We set

$$r = 1. \tag{5.6.1}$$

Then the condition (3.3.4) is satisfied and

$$R(q^{2r}) = \varphi(q^{2r}) P_{2\lambda_0}.$$

Hence by (3.3.9), [KN] and Proposition 3.4.4, we have

Proposition 5.6.1.

(i) V_l is an irreducible module with highest weight $l(\Lambda_1 - \Lambda_0)$ as an $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module.

(ii) V_l has a crystal pseudo-base.

5.7 $(A^{(2)}_{2n}, \oplus_{0\le k\le l/2} V((l-2k)(\Lambda_{n-1} - \Lambda_n)))$ Let $\mathfrak{g}$ be of type $A^{(2)}_{2n}$ and $I, i_0 = n$ as in 4.7, where we constructed the polarized $U'_q(\mathfrak{g})$-module V. It is an irreducible module with highest weight $\Lambda_{n-1} - \Lambda_n$ as $U_q(\mathfrak{g}_{I\setminus\{n\}})$-module. Hence taking $\Lambda_{n-1} - \Lambda_n$ as λ_0 the condition (3.1.1) is satisfied. Note that

$$\alpha_n = 2(\Lambda_n - \Lambda_{n-1}) \tag{5.7.1}$$

We have by, Proposition 4.7.1,

$$\varphi(z) = (1 - q^4 z)(1 + q^{4n+2} z). \tag{5.7.2}$$

We set

$$r = 2. \tag{5.7.3}$$

Then the condition (3.3.4) is satisfied and

$$R(q^{2r}) = (1 - q^8)\big((1 + q^{4n+6}) P_{2\lambda_0} + (q^4 + q^{4n+2}) P_0\big), \tag{5.7.4}$$

where P_λ is the $U_q(\mathfrak{g}_{I\setminus\{n\}})$-linear projector to $V(\lambda)$. We set as in §3

$$V_l = \mathrm{Im} R_l. \tag{5.7.5}$$

In order to apply Proposition 3.4.5, we shall prove

$$(5.7.6) \qquad \dim(V_l)_\lambda \leq \sum_{0\leq k\leq l/2} \dim V((l-2k)(\Lambda_{n-1}-\Lambda_n))_\lambda$$

for any λ. Let W be the kernel of $R(q^{2r})$. Set $U = V^{\otimes l}/\sum_{i=0}^{l-2} V^{\otimes i}\otimes W\otimes V^{\otimes(l-i-2)}$. Then by (3.3.10) V_l is a quotient of U. Hence

$$\dim(V_l)_\lambda \leq \dim U_\lambda.$$

Set $S(q) = (1+q^{4n+6})P_{2\lambda_0}+(q^4+q^{4n+2})P_0$. Then $R(q^2) = (1-q^8)S$. Then at $q=1$, U is isomorphic to $S^l(V)$. Thus $\dim U_\lambda$ is equal or less than the weight multiplicity of $U_q(\mathfrak{g}_{I\setminus\{n\}})$-module $S^l(V)$. Setting $\epsilon_i = \Lambda_i - \Lambda_{i+1}$, $\mathrm{ch}V = 1+\sum_{0\leq i\leq n-1}(e^{\epsilon_i}+e^{-\epsilon_i})$. Thus we obtain

$$\sum_l \mathrm{ch}(S^l V)t^l = \frac{1}{(1-t)\prod_{0\leq i\leq n-1}(1-e^{\epsilon_i}t)(1-e^{-\epsilon_i}t)}$$

This implies

$$(5.7.7) \qquad \sum_l \mathrm{ch}(V_l)t^l \leq \frac{1}{(1-t)\prod_{0\leq i\leq n-1}(1-e^{\epsilon_i}t)(1-e^{-\epsilon_i}t)}$$

This means that the coefficient of $e^\lambda t^k$ ($\lambda\in P, 0\leq k$) of the left hand side is less than or equal to that of the right hand side.

Lemma 5.7.1.

$$\frac{1}{(1-t)\prod_{0\leq i\leq n-1}(1-e^{\epsilon_i}t)(1-e^{-\epsilon_i}t)} = \sum_{0\leq 2k\leq l} ch(V((l-2k)(\Lambda_{n-1}-\Lambda_n)))t^l.$$

Proof. Let us first calculate the character of $V(j(\Lambda_{n-1}-\Lambda_n))$. By [KN], crystal bases of $V(j(\Lambda_n-\Lambda_0))$ consists of $a_1\otimes\cdots\otimes a_j$ where $1\leq a_1\leq\cdots\leq a_j\leq\overline{1}$. Here $a_1,\cdots,a_j$ are elements of $\{1,2,\cdots,n,0,\overline{n},\cdots,\overline{1}\}$ with the ordering $1<2<\cdots<n<0<\overline{n}<\cdots<\overline{1}$. Moreover 0 does not appear more than once in $a_1,\cdots,a_j$. Note that i has weight ϵ_i, $\overline{i}$ has weight $-\epsilon_i$ and 0 has weight 0. Thus we have

$$(5.7.8) \qquad \sum \mathrm{ch}(V(j(\Lambda_{n-1}-\Lambda_n)))t^j = \frac{1+t}{\prod(1-e^{\epsilon_i}t)(1-e^{-\epsilon_i}t)}.$$

Thus we have the desired result. □

As a corollary of this lemma, we obtain (5.7.6). Thus we can apply Proposition 4.4.5 and we obtain

Proposition 5.7.2.
(i) V_l has a crystal pseudo-base.
(ii) V_l is isomorphic to

$$\oplus_{0\leq k\leq l/2} V((l-2k)(\Lambda_{n-1}-\Lambda_n))$$

as a $U_q(\mathfrak{g}_{I\setminus\{n\}})$-module.

5.8 $(D^{(2)}_{n+1}, \oplus_{j=0}^{l} V(j(\Lambda_1 - 2\Lambda_0)))$ Let $\mathfrak{g}$ be of type $D^{(2)}_{n+1}$ and I, i_0 as in 4.8, where we constructed the polarized $U_q'(\mathfrak{g})$-module V. This is isomorphic to $V(\Lambda_1 - 2\Lambda_0) \oplus V(0)$ as a $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module. This admits a $V_{K_\mathbb{Z}}$ and taking $\lambda_0 = \Lambda_1 - 2\Lambda_0$, the condition (3.1.1) is satisfied. By Proposition 4.8.3, we have

$$\varphi(z) = (1 - q^4 z^2)(1 - q^{4n} z^2). \tag{5.8.1}$$

Let us take

$$r = 1. \tag{5.8.2}$$

Then the condition (3.3.4) is satisfied. By the calculation, $N = \mathrm{Ker}(q^{2r})$ contains u_{Λ_2}, $u^1_{\Lambda_1} - u^2_{\Lambda_1}$ and $q(1+q^2)(1+q)u^1_0 - (1-q^{4n+2})(1-q)^{-1}u^2_0$. Hence, by Lemma 4.8.1, at $q = 1$, N contains $(\cdot) \wedge V(\Lambda_1)$, $\wedge^2 V(\Lambda_1)$ and $(\cdot) \otimes (\cdot) + u$, where u is an element of $V(\Lambda_1) \otimes V(\Lambda_1)$. Hence

$$V^{\otimes l} / (\sum V^{\otimes j} \otimes N \otimes V^{\otimes (l-2-j)})$$

is generated by $S^l(V(\Lambda_1))$ and $(\cdot) \otimes S^{l-1}(V(\Lambda_1))$. Hence we obtain

$$\begin{aligned} \sum \mathrm{ch}(V_l) t^l &\le \sum \mathrm{ch} S^l(V(\Lambda_1)) t^l + \sum \mathrm{ch} S^l(V(\Lambda_1)) t^{l+1} \\ &= (1+t) \sum \mathrm{ch} S^l(V(\Lambda_1)) t^l \\ &= \frac{1+t}{(1-t) \prod_{j=1}^{l} (1 - e^{\epsilon_j} t)(1 - e^{-\epsilon_j} t)}. \end{aligned}$$

On the other hand by (5.7.6)

$$\sum_{0 \le j \le l} \mathrm{ch} V(j(\Lambda_1 - 2\Lambda_0)) t^l = \frac{1+t}{(1-t) \prod (1 - e^{\epsilon_j} t)(1 - e^{-\epsilon_j} t)}.$$

Thus we obtain

$$\dim(V_l)_\lambda \le \sum_{j=0}^{l} \dim V(j(\Lambda_1 - 2\Lambda_0))_\lambda$$

for any λ. Thus we can apply Proposition 3.4.5 and we obtain the following results.

Proposition 5.8.1.
(i) V_l has a crystal pseudo-base.
(ii) V_l is isomorphic to $\oplus_{j=0}^{l} V(j(\Lambda_1 - 2\Lambda_0))$ as a $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module.

5.9 $(D^{(2)}_{n+1}, V(l(\Lambda_n - \Lambda_0)))$ Let $\mathfrak{g}$ be of type $D^{(2)}_{n+1}$ and $I, i_0 = 0$ as in 4.9, where we constructed the polarized $U_q'(\mathfrak{g})$-module V and calculated its R-matrix. This V

is isomorphic to $V(\Lambda_n - \Lambda_0)$ as a $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module. Hence setting $\lambda_0 = \Lambda_n - \Lambda_0$, the condition (3.1.1) is satisfied. We have, by Proposition 4.9.5,

$$\varphi(z) = \prod_{i=1}^{n}(1 + (-1)^i q^{2(i+2)} z). \tag{5.9.1}$$

Hence setting

$$r = 3 \tag{5.9.2}$$

the condition (3.3.4) is satisfied and

$$R(q^{2r}) = \varphi(q^{2r}) P_{2\lambda_0}.$$

Thus by (3.3.9), [KN] and Proposition 3.4.4, we have

Proposition 5.9.1. *For $l \geq 1$, there exists a $U_q'(\mathfrak{g})$-module V_l such that*

(i) V_l has a crystal pseudo-base,
(ii) V_l is isomorphic to $V(l(\Lambda_n - \Lambda_0))$ as a $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module.

6. Perfectness of the graphs.

In this section we prove the perfectness of the crystals introduced in 1, whose representations have been constructed in 5.

Let $U_q(\mathfrak{g})$ and $U_q(\mathfrak{g}')$ be quantum universal enveloping algebras with I and I' as index sets of simple roots and with A and A' as generalized Cartan matrices. Suppose that a map $\iota : I \longrightarrow I'$ satisfies $(A)_{jj'} = (A')_{\iota(j)\iota(j')}$. If B' is a crystal for $U_q(\mathfrak{g}')$, we make a crystal B for $U_q(\mathfrak{g})$ by putting $B = B'$ as a set and drawing a j arrow from b to b' if and only if there is an $\iota(j)$ arrow from b to b'. We denote the identity map from B' to B also by ι^*.

6.1. *$\mathfrak{sl}(2)$-crystals* Here we give the rule of arrow in tensor products of $\mathfrak{sl}(2)$-crystals. We denote by $I = \{1\}$ the index set of the simple root for $\mathfrak{sl}(2)$. For $j \in \mathbf{Z}/2$, let

$$B(j) = \{u_m(j) \mid -j \leq m \leq j, m \in j + \mathbf{Z}\} \tag{6.1.1}$$

be the crystal of $(2j+1)$ dimensional irreducible $\mathfrak{sl}(2)$-module. The arrows in $B(j)$ are

$$u_m(j) \xrightarrow{1} u_{m-1}(j) \quad (-j < m \leq j).$$

We abbreviate $u_0(0)$, $u_{1/2}(1/2)$, $u_{-1/2}(1/2)$ by $0, +, -$, respectively. The following proposition is immediate and is useful to describe the arrows in tensor products of $\mathfrak{sl}(2)$-crystals.

Proposition 6.1.1. *Let B_1, B_2 be crystals for $U_q(\mathfrak{sl}(2))$. The following are morphisms of crystals.*

$$\begin{array}{ccc} B_1 \otimes B_2 & \longrightarrow & B_1 \otimes B(0) \otimes B_2, \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ b_1 \otimes b_2 & \mapsto & b_1 \otimes 0 \otimes b_2 \end{array} \tag{6.1.2}$$

$$\begin{array}{ccc} B_1 \otimes B_2 & \longrightarrow & B_1 \otimes B(1/2) \otimes B(1/2) \otimes B_2, \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ b_1 \otimes b_2 & \mapsto & b_1 \otimes + \otimes - \otimes b_2 \end{array} \tag{6.1.3}$$

$$\begin{array}{ccc} B(j) & \longrightarrow & B(1/2)^{\otimes(2j)}. \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ u_m(j) & \mapsto & \underbrace{- \otimes \cdots \otimes -}_{j-m} \otimes \underbrace{+ \otimes \cdots \otimes +}_{j+m}. \end{array} \tag{6.1.4}$$

For example, the following is a string of 1-arrows in $B(1/2)^{\otimes 4} \otimes B(0) \otimes B(1/2)^{\otimes 2}$.

$$\begin{tikzcd}
+\otimes+\otimes+\otimes-\otimes 0\otimes-\otimes+ \arrow[d, "1"] \\
-\otimes+\otimes+\otimes-\otimes 0\otimes-\otimes+ \arrow[d, "1"] \\
-\otimes+\otimes+\otimes-\otimes 0\otimes-\otimes-.
\end{tikzcd}$$

The following proposition follows from [KN].

Proposition 6.1.2. *Take*

$$b = (+)^{\otimes x_1} \otimes (-)^{\otimes y_1} \otimes \cdots \otimes (+)^{\otimes x_k} \otimes (-)^{\otimes y_k} \tag{6.1.5}$$

in $B(1/2)^{\otimes M}$ *where* $M = \sum_{j=1}^{k}(x_k + y_k)$. *Then*

$$\mathrm{wt}\, b = \sum_{j=1}^{k}(x_k - y_k)\Lambda_1 \tag{6.1.6}$$

$$\varepsilon_1(b) = \sum_{j=1}^{k}(p_j)_+, \tag{6.1.7}$$

$$\varphi_1(b) = (-p_k)_+ \tag{6.1.8}$$

where the p_j *are defined inductively by*

$$p_0 = 0, \quad p_j = y_j - x_j - (-p_{j-1})_+ \tag{6.1.9}$$

and

$$\begin{aligned} x_+ &= x \quad \text{if} \quad x \geq 0 \\ &= 0 \quad \text{if} \quad x < 0. \end{aligned}$$

We omit the proof.

6.2. *$\mathfrak{sl}(n+1)$-crystals*

We use the description of crystals for $U_q\big(\mathfrak{sl}(n+1)\big)$ given by [KN]. We recollect some of their results here.

Let $\Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_l}$ $(n \geq k_1 \geq \cdots \geq k_l > 0)$ be a dominant integral weight for $\mathfrak{sl}(n+1)$. We consider $B(\Lambda)$, the crystal for $U_q\big(\mathfrak{sl}(n+1)\big)$ associated with the irreducible representation with highest weight Λ. The crystal $B(\Lambda)$ is given as follows. Let Y be the Young diagram with the columns of length $k_1, \ldots, k_l$. We identify Y with the set of pairs of integers (j, j') such that $1 \leq j \leq k_{j'}$, $1 \leq j' \leq l$. Then $B(\Lambda)$ consists of the maps

$$\begin{array}{cccc} b: & Y & \longrightarrow & \{1, \ldots, n+1\}, \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (j, j') & \longmapsto & b_{jj'} \end{array} \tag{6.2.1}$$

satisfying $b_{jj'} \leq b_{j\,j'+1}$ and $b_{jj'} < b_{j+1\,j'}$. We call this b a standard tableau as usual.

Fix i, and let $\phi_i : \{1\} \longrightarrow J$ be the map given by $\phi_i(1) = i$, and let $B_i = \phi_i^*(B(\Lambda))$. We are going to give the arrows in B_i. Set $\widetilde{B} = \oplus_d B_d$ where the sum ranges over the maps $d : \{1, \dots, M\} \longrightarrow \{0, 1/2\}$ and $B_d = B(d(1)) \otimes \cdots \otimes B(d(M))$. Define $\tau : B_i \longrightarrow \widetilde{B}$ as follows. For a given $b \in B(\Lambda)$, we shall define $d = (d(1), \dots, d(M))$ and $\tau(b) = s_1 \otimes \cdots \otimes s_M \in B_d$. Suppose that $(j, j') \in Y$ is the m-th element in the sequence

$$\begin{aligned}(6.2.2)\qquad &(1,l),(2,l),\dots,(k_l,l),\\ &(1,l-1),(2,l-1),\dots,(k_{l-1},l-1),\\ &\dots\\ &(1,1),(2,1),\dots,(k_1,1).\end{aligned}$$

Then we define

$$(6.2.3)\qquad \begin{aligned} d(m) &= 1/2 \quad \text{if } b_{jj'} = i \text{ or } i+1\\ &= 0 \quad\ \ \text{otherwise},\\ s(m) &= + \quad \text{if } b_{jj'} = i\\ &= - \quad \text{if } b_{jj'} = i+1\\ &= 0 \quad \text{otherwise}.\end{aligned}$$

The map τ is a morphism of crystals, and the i-arrows in $B(\Lambda)$ can be read from the 1-arrows in $\widetilde{B}$ by using Proposition 6.1.2.

It is convenient to use a different labeling of $B(l\Lambda_k)$. Let Λ be a corresponding weight of a Young diagram $Y = (l_1, \cdots, l_n)$.

Consider the set of tables of non negative integers $(x_{ji})_{1\le j\le n, 1\le i\le n+1}$ such that

$$(6.2.4)\qquad \sum_i x_{ji} = l_j \quad (1 \le j \le n),$$

$$(6.2.5)\qquad \sum_{i'\le i} x_{ji'} \ge \sum_{i'\le i} x_{j+1\, i'+1} \quad (1 \le j \le n, 1 \le i \le n).$$

We denote this set by $X(\Lambda)$. Define the map

$$(6.2.6)\qquad \begin{array}{cccc} x: & B(\Lambda) & \longrightarrow & X(\Lambda)\\ & \Cup & & \Cup\\ & b & \mapsto & x(b) = (x_{ji}) \end{array}$$

by

$$(6.2.7)\qquad x_{ji} = \sharp\{j' \mid b_{jj'} = i\}$$

The following is immediate.

Proposition 6.2.1. *The map x given by (6.2.6-7) is bijective. The weight of b is given by*

$$\operatorname{wt} b = \sum_{i=1}^{n+1} \sum_{j=1}^{k} x_{ji}(\Lambda_i - \Lambda_{i-1}).$$

Here we set $\Lambda_0 = \Lambda_{n+1} = 0$.

For $b, b' \in B(\Lambda)$ we write $b \xRightarrow{i\,r} b'$ if and only if $b' = \tilde{f}_i^r b$ and $\tilde{f}_i b' = 0$. We also write $b \xLeftarrow{i\,r} b'$ if and only if $b' = \tilde{e}_i^r b$ and $\tilde{e}_i b' = 0$. Note that $b \xRightarrow{i\,r} b'$ and $b' \xLeftarrow{i\,r} b$ are not equivalent. If $b \xRightarrow{i\,r} b'$, then $r = \varphi_i(b)$. If $b' \xLeftarrow{i\,r} b$, then $r = \varepsilon_i(b')$.

Now set $B = B(\Lambda)$ where Λ is a dominant integral weight of $\mathfrak{sl}(n+1)$. Consider the sequence

$$\begin{array}{l} 1, 2, \dots, n-1, n, \\ 1, 2, \dots, n-1, \\ \dots \\ 1, 2, \\ 1 \end{array}$$

For $b \in B$ we define $r_{ji} \in \mathbf{Z}_{\geq 0}$ and $b(j,i) \in B$ $(1 \leq i \leq n+1-j)$ in such a way that

$$\begin{array}{llllllll} b & \xRightarrow{1\,r_{11}} & b(1,1) & \xRightarrow{2\,r_{12}} & \cdots & \xRightarrow{n-1\,r_{1\,n-1}} & b(1,n-1) & \xRightarrow{n\,r_{1n}} \quad b(1,n) \\ & \xRightarrow{1\,r_{21}} & b(2,1) & \xRightarrow{2\,r_{22}} & \cdots & \xRightarrow{n-1\,r_{2\,n-1}} & b(2,n-1) & \\ & \dots & & & & & & \\ & \xRightarrow{1\,r_{n-1\,3}} & b(n-1,1) & \xRightarrow{2\,r_{n-1\,2}} & b(n-1,2) & & & \\ & \xRightarrow{1\,r_{n,1}} & b(n,1). & & & & & \end{array}$$

Then we have

Proposition 6.2.2. *The last element $b(n,1)$ is the lowest weight element in B, and*

$$r_{ji} \leq r_{j-1\,i+1}. \tag{6.2.8}$$

If we replace the arrow $\Leftarrow$ to $\Rightarrow$, then the last element $b(n,1)$ is the highest weight element in B, and we have (6.2.8).

Proof. Because of the symmetry $e_i \leftrightarrow f_i$, $e^h \leftrightarrow e^{-h}$ of $U_q(\mathfrak{sl}(n+1))$, it is enough to prove the case for $\Leftarrow$. We call the subset of Y given by $\{(r,i) \in Y \mid b_{ri} = s\}$ the (r,s)-block of b. The value of $b(j,i) : Y \longrightarrow \{1, \dots, n+1\}$ is constant on each (r,s)-block. More precisely, for $b' \xLeftarrow{i\,r_{ji}} b''$, we have

$$\begin{aligned} b''\big|_{(r,s)-\text{block}} &= b'\big|_{(r,s)-\text{block}} - 1 \quad && \text{if } j+i = s, 1 \leq j \leq s-r \\ &= b'\big|_{(r,s)-\text{block}} && \text{otherwise.} \end{aligned}$$

Namely, the value of the (r,s)-block decreases one by one in the process of $\xRightarrow{i\,r_{ji}}$ at $(j,i) = (1, s-1), \dots, (s-r, r)$. Therefore, we have $b(n,1)|_{(r,s)-\text{block}} = r$ and $r_{ji} = \sum_{1 \leq j' \leq i} x_{j'\,j+i}$. The assertion immediately follows from this. □

In the following subsections, we shall give the proofs of the results in 1. Note that in those proofs we may only estimate $\langle c, \varepsilon(b) \rangle$ but $\langle c, \varphi(b) \rangle$ by considering duals of crystals (see [KMN^2],§5).

6.3. $(A_n^{(1)}, B(l\Lambda_k))$ $(n \geq 2, 1 \leq k \leq n)$

We shall use the notations in 1.2.

Proposition 6.3.1. *Let $J' = \{1, \dots, n-1\}$ be the index set of the simple roots for $U_q(\mathfrak{sl}(n))$, and let $\iota' : J' \longrightarrow J$ be $\iota'(j) = j$. Set $B' = \iota'^*(B(l\Lambda_k))$. The crystal B' splits into $l+1$ connected components,*

$$B' = \oplus_{m=0}^{l} B\big(m\Lambda_k + (l-m)\Lambda_{k-1}\big). \tag{6.3.1}$$

Here we mean $\Lambda_n = \Lambda_0 = 0$. If B_1 and B_2 are both isomorphic to B', then the isomorphism $B_1 \longrightarrow B_2$ is unique.

Proof. An element $b \in B(l\Lambda_k)$ corresponds to a highest weight element in B' if and only if

$$\begin{aligned} b_{jj'} &= j \quad (1 \le j \le k-1, 1 \le j' \le l), \\ b_{kj'} &= k \quad (1 \le j' \le m) \\ &= n+1 \quad (m+1 \le j' \le l), \end{aligned}$$

for some $0 \le m \le l$. The weight of $\iota'^*(b)$ is $m\Lambda_k + (l-m)\Lambda_{k-1}$. Hence we have (6.3.1). The uniqueness of the isomorphism $B_1 \longrightarrow B_2$ follows from the fact that the highest weights of the connected components of B' are all different. □

Set

$$k' = n + 1 - k. \tag{6.3.2}$$

Proof of Proposition 1.2.1.

The existence of $B^{k,l}$ is proved in Section 5. Let us prove the uniqueness. Suppose that B_1 and B_2 are crystals of $U_q(\widehat{\mathfrak{sl}}(n+1))$ such that $\iota^{(i)*}(B_1) = \iota^{(i)*}(B_2) = B(l\Lambda_k)$ for $i = 0, n$. Let $\tau^{(i)} : \iota^{(i)*}(B_1) \longrightarrow \iota^{(i)*}(B_2)$ $(i = 0, n)$ be the isomorphisms of $\mathfrak{sl}(n+1)$-crystals. By Proposition 6.3.1 we have $\tau^{(0)} = \tau^{(n)}$ as a morphism of $\mathfrak{sl}(n+1)$-crystal and these maps can be extended to the isomorphism $\tau : B_1 \longrightarrow B_2$ such that $\tau|_{\iota^{(i)*}(B_1)} = \tau^{(i)}$. □

The proof of Theorem 1.2.2 is divided into several parts. Without loss of generality we can assume that $k \le k'$. For $b \in B^{k,l}$ we set $(x_{ji}) = x(\iota^{(0)*}(b))$. For convenience we set $x_{00} = l$, $x_{01} = \cdots = x_{0\,n+1} = 0$ and $x_{k+1\,0} = \cdots = x_{k+1\,n+1} = 0$. The following is immediate from Proposition 6.1.2.

Proposition 6.3.2.

$$\varepsilon_i(b) = \sum_{j=1}^{k} (p_{ji})_+ \quad \varphi_i(b) = (-p_{k+1\,i})_+$$

where p_{ji} $(0 \le j \le k+1, 1 \le i \le n)$ are defined inductively by

$$p_{0i} = 0, \quad p_{ji} = x_{j\,i+1} - x_{j-1\,i} - (-p_{j-1\,i})_+.$$

Proposition 6.3.3. *For $b \in B$ we have*

$$\sum_{i=1}^{n} \varepsilon_i(b) \ge l - x_{kk}.$$

The equality holds if and only if $p_{ii} \geq 0$ $(2 \leq i \leq k)$ *and* $p_{ki} \geq 0$ $(k+1 \leq i \leq n-1)$. $\blacksquare$

Proof. Note that $p_{ji} \leq 0$ if $j > i$, and also that $p_{kn} \geq 0$. We have $\varepsilon_1(b) = x_{12}$. For $2 \leq i \leq k$, by use of $x_+ - (-x)_+ = x$ we have

$$\begin{aligned}
\varepsilon_i(b) = \sum_{j=1}^{i} (p_{ji})_+ &\geq \sum_{j=1}^{i-1} (p_{ji})_+ + p_{ii} \\
&= \sum_{j=1}^{i-1} (p_{ji})_+ + x_{i\,i+1} - x_{i-1\,i} - (-p_{i-1\,i})_+ \\
&= \sum_{j=1}^{i-2} (p_{ji})_+ + p_{i-1\,i} + x_{i\,i+1} - x_{i-1\,i} \\
&= \sum_{j=1}^{i} (x_{j\,i+1} - x_{j-1\,i}).
\end{aligned}$$

For $k+1 \leq i \leq n$, we have

$$\varepsilon_i(b) = \sum_{j=1}^{k} (p_{ji})_+ \geq \sum_{j=1}^{k-1} (p_{ji})_+ + p_{ki} = \sum_{j=1}^{k} (x_{j\,i+1} - x_{j-1\,i}).$$

Therefore, we have

$$\sum_{i=1}^{n} \varepsilon_i(b) \geq x_{k\,k+1} + \cdots + x_{k\,n+1} = l - x_{kk}. \qquad \square$$

We set $(x'_{ji}) = x(\bar{\iota}^{(k)*}(b))$. Define

$$l_j = \sum_{i=j}^{k} x_{ji} \quad (0 \leq j \leq k) \qquad l'_j = \sum_{i=j}^{k} x'_{ji} \quad (0 \leq j \leq k'). \tag{6.3.3}$$

These integers are the same for b and b' if $b \xrightarrow{i} b'$ for some $i \neq 0, k$. Note that $l_0 = l'_0 = l$. We are going to show

Proposition 6.3.4.

$$\begin{aligned}
l'_i &= l - l_{k+1-i} \quad \textit{if} \quad 1 \leq i \leq k+1 \\
&= 0 \quad \textit{if} \quad k+1 \leq i \leq k'.
\end{aligned}$$

Proof. Define $\overline{b} \in B^{k,l}$ by $(\bar{x}_{ji}) = x(\overline{b})$ where

$\bar{x}_{11} = l_1, \bar{x}_{12} = \cdots = \bar{x}_{1k} = 0, \bar{x}_{1\,k+1} = l - l_1,$
$\bar{x}_{22} = l_2, \bar{x}_{23} = \cdots = \bar{x}_{2k} = 0, \bar{x}_{2\,k+1} = l_1 - l_2, \bar{x}_{2\,k+2} = l - l_1,$
$\cdots$
$\bar{x}_{kk} = l_k, \bar{x}_{k\,k+1} = l_{k-1} - l_k, \ldots, \bar{x}_{k\,2k} = l - l_1.$

Then $\iota^{(0,k)*}(\overline{b})$ is the highest weight element in the connected component of the crystal $\iota^{(0,k)*}(B^{k,l})$ that contains $\iota^{(0,k)*}(b)$. By Proposition 6.2.1 the weight of $\overline{b}$ is given by

$$\Lambda = \sum_{j=1}^{k-1}(l_j - l_{j+1})(\Lambda_j + \Lambda_{2k-j}) + (2l_k - l)\Lambda_k + (l - l_1)\Lambda_{2k} - l_1\Lambda_0.$$

Therefore the weight of $\bar{\iota}^{(k)*}(\overline{b})$ is

$$\bar{\Lambda} = \sum_{j=1}^{k-1}(l_j - l_{j+1})(\Lambda_{k-j} + \Lambda_{n+1-k+j}) + (l - l_1)\Lambda_{k'} - l_1\Lambda_k.$$

On the other hand, in terms of l'_j, $\bar{\Lambda}$ reads as

$$\bar{\Lambda} = \sum_{i=1}^{k-1}(l'_i - l'_{i+1})(\Lambda_i - \Lambda_{n+1-i}) - (l - l'_k)\Lambda_k + l'_k\Lambda_{k'}.$$

Comparing these two expressions of $\bar{\Lambda}$, we get the assertion. □

Proposition 6.3.5. *For $b \in B^{k,l}$ we have*

$$\varepsilon_0(b) \geq x_{kk}.$$

The equality holds if and only if $p'_j \leq 0$ for $2 \leq j \leq k$ where p'_j are defined inductively by

$$p'_0 = 0, p'_j = x'_{j\,k+1} - x'_{j-1\,k} - (-p'_{j-1})_+.$$

Proof. From Proposition 6.3.4 we have $l'_1 = l - l_k = l - x_{kk}$. Therefore, we have

$$\varepsilon_0(b) = \varepsilon_k(\bar{\iota}^{(k)*}(b)) = \sum_{j=1}^{k}(p'_j)_+ \geq x'_{1\,k+1} = l - l'_1 = x_{kk}. \qquad \square$$

From Proposition 6.3.5 and 6.3.7 we have

Corollary 6.3.6. *For $b \in B^{k,l}$ we have $\sum_{i=0}^{n}\varepsilon_i(b) \geq l$.*

Since $c = \sum_{i=0}^{n} h_i$, b is minimal if and only if $\sum_{i=0}^{n}\varepsilon_i(b) = l$.

Proposition 6.3.7.

$$\begin{aligned}
x'_{j\,k} &= l_{k-j} - l_{k-j+1} - r_{1j} + r_{1\,j+1} \quad \text{if} \quad 1 \leq j \leq k-1\\
&= l - l_1 \quad \text{if} \quad j = k\\
&= 0 \quad \text{if} \quad k+1 \leq j \leq k',\\
x'_{j\,k+1} &= l_k \quad \text{if} \quad j = 1\\
&= r_{k+2-j\,n+2-j} - r_{k+1-j\,n+1-j} \quad \text{if} \quad 2 \leq j \leq k\\
&= r_{1k'} \quad \text{if} \quad j = k+1\\
&= 0 \quad \text{if} \quad k+2 \leq j \leq k'.
\end{aligned}$$

Proof. We connect (x'_{ji}) and (x_{ji}) in the following scheme.

(6.3.5)
$$\begin{tikzcd}
\iota^{(0,k)*}(b) \arrow[r, "\tilde{f}_i{}'s"] & \iota^{(0,k)*}(\underline{b}) : \text{ the lowest weight element in } \iota^{(0,k)*}(B^{k,l}) \arrow[d, "\text{Proposition 6.3.4}"] \\
\overline{\iota}^{(0,k)*}(b) & \overline{\iota}^{(0,k)*}(\underline{b}) : \text{ the lowest weight element in } \overline{\iota}^{(0,k)*}(B^{k,l}). \arrow[l, "\tilde{e}_i{}'s"']
\end{tikzcd}$$

We use l_j and l'_j of (6.3.3). We set formally $l_{k+1} = 0$. Let $\underline{b} \in B^{k,l}$ be given by $(y_{ji}) = x\big(\iota^{(0)*}(\underline{b})\big)$ where

$$\begin{aligned} y_{ji} &= l_{k+j-i} - l_{k+j-i+1} \quad \text{if} \quad j \le i \le k \\ &= 0 \quad \text{if} \quad k+1 \le i \le k'+j-1 \\ &= l - l_j \quad \text{if} \quad i = k'+j. \end{aligned}$$

Then $\iota^{(0,k)*}(\underline{b})$ is the lowest weight element in the connected component of $\iota^{(0,k)*}(B^{k,l})$ that contains $\iota^{(0,k)*}(b)$. Set $(z_{ji}) = x\big(\overline{\iota}^{(k)*}(\underline{b})\big)$. Note that $\overline{\iota}^{(k)*}(B^{k,l}) = B(l\Lambda_{k'})$. We have

$$\begin{aligned} z_{ji} &= l'_{k+j-i} - l'_{k+j-i+1} \quad \text{if} \quad j \le i \le k = l_{i-j} - l_{i-j+1} \\ &= 0 \quad \text{if} \quad k+1 \le i \le k+j-1 \\ &= l - l'_j = l_{k+1-j} \quad \text{if} \quad i = k+j. \end{aligned} \tag{6.3.6}$$

By proposition 6.2.2, $\underline{b}$ is obtained from b by the process

$$\begin{aligned}
b\ & \overset{k-1\; r_{k-1\,k-1}}{\Longrightarrow} b(k-1,k-1) \overset{k-2\; r_{k-2\,k-2}}{\Longrightarrow} \cdots \overset{2\; r_{22}}{\Longrightarrow} b(2,2) \overset{1\; r_{11}}{\Longrightarrow} b(1,1) \\
& \overset{k-1\; r_{k-2\,k-1}}{\Longrightarrow} b(k-2,k-1) \overset{k-2\; r_{k-3\,k-2}}{\Longrightarrow} \cdots \overset{2\; r_{12}}{\Longrightarrow} b(1,2) \\
& \cdots \\
& \overset{k-1\; r_{1\,k-1}}{\Longrightarrow} b(1,k-1) \\
& \overset{k+1\; r_{k\,k+1}}{\Longrightarrow} b(k,k+1) \overset{k+2\; r_{k\,k+2}}{\Longrightarrow} \cdots \overset{n-1\; r_{k\,n-1}}{\Longrightarrow} b(k,n-1) \overset{n\; r_{k\,n}}{\Longrightarrow} b(k,n) \\
& \overset{k+1\; r_{k-1\,k+1}}{\Longrightarrow} b(k-1,k+1) \overset{k+2\; r_{k-1\,k+2}}{\Longrightarrow} \cdots \overset{n-1\; r_{k-1\,n-1}}{\Longrightarrow} b(k-1,n-1) \\
& \cdots \\
& \overset{k+1\; r_{1\,k+1}}{\Longrightarrow} b(1,k+1) \overset{k+2\; r_{1\,k+2}}{\Longrightarrow} \cdots \overset{k'\; r_{1k'}}{\Longrightarrow} b(1,k') = \underline{b}.
\end{aligned} \tag{6.3.7}$$

If $k = k'$, the last line is $\overset{k+1\; r_{2\,k+1}}{\Longrightarrow} b(2,k+1) = \underline{b}$. For convenience we put $r_{1k} = 0$.
With these r_{ji} we get

$$\begin{aligned}
\overline{\iota}^{(k)*}(b) = & \tilde{e}_1^{r_{k-1\,k-1}} \tilde{e}_2^{r_{k-2\,k-2}} \cdots \tilde{e}_{k-2}^{r_{2\,2}} \tilde{e}_{k-1}^{r_{1\,1}} \\
& \tilde{e}_1^{r_{k-2\,k-1}} \tilde{e}_2^{r_{k-3\,k-2}} \cdots \tilde{e}_{k-2}^{r_{1\,2}} \\
& \cdots \\
& \tilde{e}_1^{r_{1\,k-1}} \\
& \tilde{e}_n^{r_{k\,k+1}} \tilde{e}_{n-1}^{r_{k\,k+2}} \cdots \tilde{e}_{k+2}^{r_{k\,n-1}} \tilde{e}_{k+1}^{r_{k\,n}} \\
& \tilde{e}_n^{r_{k-1\,k+1}} \tilde{e}_{n-1}^{r_{k-1\,k+2}} \cdots \tilde{e}_{k+2}^{r_{k-1\,n-1}} \\
& \cdots \\
& \tilde{e}_n^{r_{1\,k+1}} \tilde{e}_{n-1}^{r_{1\,k+2}} \cdots \tilde{e}_{2k}^{r_{1\,k'}}\ \overline{\iota}^{(k)*}(\underline{b})
\end{aligned} \tag{6.3.8}$$

If $k = k'$, the last line is $\tilde{e}_n^{r_{2\,k+1}}\bar{\iota}^{(k)*}(\underline{b})$. Note that in the first half of (6.3.8) the $\tilde{e}_i$'s are with $i \in \{1, \dots, k-1\}$, and in the latter half $\tilde{e}_i$'s are with $i \in \{k+1, \dots, n\}$. Hence they commute each other. Furthermore, $\tilde{e}_{k-1}$ in the first row commutes with all the $\tilde{e}_i$'s in the second and the later rows except for the $\tilde{e}_{k-2}$ in the second row, and this $\tilde{e}_{k-2}$ commutes with all the $\tilde{e}_i$'s in the third and the later rows except for the $\tilde{e}_{k-3}$ in the third row, and so on. Therefore, we can write

$$\bar{\iota}^{(k)*}(b) = (\text{a sequence of } \tilde{e}_1, \dots, \tilde{e}_{k-2}, \tilde{e}_{k+2}, \dots, \tilde{e}_n)b'' \tag{6.3.9}$$

$$\begin{aligned} b'' &= \tilde{e}_{k-1}^{r_{11}}\tilde{e}_{k-2}^{r_{12}} \cdots \tilde{e}_1^{r_{1\,k-1}}\tilde{e}_{k+1}^{r_{k\,n}}\tilde{e}_{k+2}^{r_{k-1\,n-1}} \cdots \tilde{e}_{2k}^{r_{1\,k'}}\ \bar{\iota}^{(k)*}(\underline{b}) \quad \text{if} \quad k < k' \\ &= \tilde{e}_{k-1}^{r_{11}}\tilde{e}_{k-2}^{r_{12}} \cdots \tilde{e}_1^{r_{1\,k-1}}\tilde{e}_{k+1}^{r_{kn}}\tilde{e}_{k+2}^{r_{k-1\,n-1}} \cdots \tilde{e}_n^{r_{2\,k+1}}\bar{\iota}^{(k)*}(\underline{b}) \quad \text{if} \quad k = k'. \end{aligned} \tag{6.3.10}$$

In (6.3.9) neither $\tilde{e}_{k-1}$ nor $\tilde{e}_{k+1}$ appears. Hence, if we set $(x''_{ji}) = x(b'')$, then we have $x'_{jk} = x''_{jk}$ and $x'_{j\,k+1} = x''_{j\,k+1}$. Proposition 6.2.2 implies that $r_{1j} \geq r_{1\,j+1}$ and $r_{k+1-j\,n+1-j} \geq r_{k-j\,n-j}$. Therefore, from (6.3.6) and (6.3.10) we get x'_{jk} and $x'_{j\,k+1}$ as above. □

Proof of Theorem 1.2.3.
Let $l_{j\,i} = \sum_{j \leq i' \leq i} x_{j\,i'}$. Consider the following statements for $1 \leq j \leq k-1$.

$(A)_j$ $\quad p_{i+1-j\,i} \geq 0 \quad (j+1 \leq i \leq k),$
$\qquad p_{k+1-j\,i} \geq 0 \quad (k+1 \leq i \leq n-j).$

$(B)_j$ $\quad x_{i+1-j\,i+1} = x_{i-j\,i} \quad (j+1 \leq i \leq k),$
$\qquad x_{k+1-j\,i+1} = x_{k-j\,i} \quad (k+1 \leq i \leq n-j).$

$(C)_j$ $\quad p'_{j+1} \geq 0.$

$(D)_j$ $\quad p'_{j+1} \leq 0.$

$(E)_j$ $\quad r_{i+1-j\,i} = l_{i+1-j,i} - l_{k+1-j} \quad (j \leq i \leq k-1),$
$\qquad r_{k+1-j\,i} = l_{k+1-j,i} - l_{k+1-j} \quad (k+1 \leq i \leq n+1-j).$

$(F)_j$ $\quad r_{1j} = l_{1,j} - l_{k+1-j},$
$\qquad r_{k+1-j\,n+1-j} = l_{k+1-j,n+1-j} - l_{k+1-j}.$

From Proposition 6.3.3 and 6.3.4, the equality $\sum_{i=0}^n \varepsilon_i(b) = l$ holds if and only if the following $(A)_1$ and $(D)_j$ $(1 \leq j \leq k-1)$ hold. We will show the equivalence of the following.

(i) $\quad (A)_1$ and $(D)_j$ $\quad (1 \leq j \leq k-1)$.
(ii) $\quad (B)_j$ $\quad (1 \leq j \leq k-1)$.

Note that (ii) implies Theorem 1.2.3.

First we show that $(A)_j (1 \leq j \leq j_0)$ implies $(E)_{j_0}$. Let us trace (6.3.7), the process of the standard tableau $\bar{b}$ changing to $\underline{b}$, more closely under the condition $(A)_j (1 \leq j \leq j_0)$. Note that at each r_{ji} step $\overset{i\,r_{ji}}{\Longrightarrow}$ the standard tableau changes some nodes from i to $i+1$. We claim the following.

(6.3.11) Assume $(A)_j$ $(1 \leq j \leq j_0)$. Then, for $1 \leq j \leq j_0$, the changes for $\overset{i\,r_{i+1-j,i}}{\Longrightarrow}$ $(j \leq i \leq k-1)$ take place on the $(i+1-j)$-th row, and the changes for $\overset{i\,r_{k+1-j,i}}{\Longrightarrow}$ $(k+1 \leq i \leq n-j)$ take place on the $(k+1-j)$-th row. Furthermore, $(E)_j$ and $(F)_{j+1}$ are valid.

Let us prove this by induction on j_0. Since $\tilde{f}_i$ in the process from b to $b(1,k-1)$ and those from $b(1,k-1)$ to $\underline{b}$ commute, it is enough to prove the two cases separately. Since the proofs for them are similar we give it for the first half.

First consider the case $j_0 = 1$. Since $p_{k-1\,k-1} \geq 0$, we have $p_{k\,k-1} = x_{kk} - x_{k-1\,k-1} \leq 0$. Therefore, we have $\varphi_{k-1}(b) = (-p_{k+1\,k-1})_+ = (-p_{k\,k-1})_+ = x_{k-1\,k-1} - x_{kk} = l_{k-1\,k-1} - l_k$. This implies that the changes for $\overset{k-1\,r_{k-1\,k-1}}{\Longrightarrow}$ are on the $(k-1)$-th row. Hence, by the definition of $p_{j\,i}$, $p_{k-2\,k-2}$ of $b(k-1,k-1)$ is equal to that b. Therefore, noting that $l_{k-1,k-1}$ for $b(k-1,k-1)$ has changed to l_k, we have $\varphi_{k-2}\big(b(k-1,k-1)\big) = (-p_{k+1\,k-2})_+ = (-p_{k-1\,k-2})_+ = l_{k-2\,k-2} - l_k$.

By continuing this we can prove that changes for $\overset{i\,r_{ii}}{\Longrightarrow}$ $(1 \leq i \leq k-1)$ take place only on the i-th row. Therefore follows the first half of $(E)_1$. Furthermore, it turns out that for $b(1,1)$ and after, the change from 2 to 3 is possible only on the first row. This proves $(F)_2$.

Now let us proceed the induction step to $j_0 = 2$. Define $b(2)$ by

$$b(1,1)\overset{k+1\,r_{k\,k+1}}{\Longrightarrow}\cdots\overset{n\,r_{kn}}{\Longrightarrow}b(2).$$

Compare $b(2)$ with b. Among l_{ji} those which has changed their values are l_{ii} $(1 \leq i \leq k-1)$ and l_{ki} $(k+1 \leq i \leq n)$. For $b(2)$ we have $l_{ii} = l_k$ $(1 \leq i \leq k-1)$ and $l_{ki} = l_k$ $(k+1 \leq i \leq n)$. Moreover, p_{ji} appearing in $(A)_j$ $(2 \leq j \leq k-1)$ have not changed. Therefore, the same structure of the proof remains for the next step. By repeating this we can prove the assertion.

Now we go back to the equivalence of (i) and (ii).

Let us prove (ii) $\Rightarrow$ (i). From $(B)_j$ $(1 \leq j \leq k-1)$ it is easy to see $(A)_j$ $(1 \leq j \leq k-1)$. Therefore, we have $(E)_j$. Note that the convention $r_{1k} = 0$ is consistent with $(E)_k$ with $i = k$. Substituting all these to (6.3.11) we have

$$\begin{aligned} x'_{j\,k} &= l_{1\,j+1} - l_{1j} = x_{1\,j+1}, \\ x'_{j+1\,k+1} &= l_{k+1-1\,n+1-j} - l_{k+1-j} - l_{k-j\,n-j} + l_{k-j}, \end{aligned}$$

for $1 \leq j \leq k-1$. Note that from $(B)_j$ we have $l_{k-j} - l_{k+1-j} = x_{k-j\,k} + x_{k-j\,k-j} - x_{k+1-j\,k+1-j}$ and $l_{k-j\,k-j} - l_{k+1-j\,n+1-j} = x_{k-j\,k-j} - x_{k+1-j\,k+1-j}$. Therefore, we have $x'_{j+1\,k+1} = x_{k-j\,k} = x_{1\,j+1}$. Hence we obtain $(D)_j$ for $1 \leq j \leq k-1$.

Let us prove (i) $\Rightarrow$ (ii). We assume (i).Note that $(C)_0$ is valid. Assume $(A)_{j'}$ for $1 \leq j' \leq j$ and $(C)_{j-1}$. From $(A)_j$ we have

$$0 \leq \sum_{i=2}^{k+1-j} p_{i\,i+j-1} \leq x_{k+1-j\,k+1} - x_{1\,j+1}, \tag{6.3.12}$$

$$0 \leq \sum_{i=k+1}^{n-j} p_{k+1-j\,i} \leq \sum_{i=k+1}^{n-j} \big(x_{k+1-j\,i+1} - x_{k-j\,i}\big). \tag{6.3.13}$$

Therefore, we have

$$x_{k+1-j\,k+1} - x_{1\,j-1} + \sum_{i=k+1}^{n-j} \big(x_{k+1-j\,i+1} - x_{k-j\,i}\big) \geq 0. \tag{6.3.14}$$

From $(C)_{j-1}$ and $(D)_j$ we have $x'_{j\,k} - x'_{j+1\,k+1} \geq 0$. Therefore, by using Proposition 6.3.7, we have

$$\begin{aligned} 0 &\leq x'_{j\,k} - x'_{j+1\,k+1} \\ &= l_{k-j} - l_{k+1-j} - r_{1j} + r_{1\,j+1} - r_{k+1-j\,n+1-j} + r_{k-j\,n-j}. \end{aligned}$$

From (6.3.11) we have $(F)_j$ and $(F)_{j+1}$. Therefore, we have

$$x_{k+1-j\,k+1} - x_{1\,j-1} + \sum_{i=k+1}^{n-j} \left(x_{k+1-j\,i+1} - x_{k-j\,i}\right) \leq 0. \tag{6.3.15}$$

Comparing (6.3.12-15) we have $(B)_j$ and $p'_{j+1} = 0$ (in particular, $(C)_j$.) From $(B)_j$ and $(A)_j$ follows $(A)_{j+1}$. Thus, by induction we have proved (ii). □

Suppose that $b \in B^{k,l}$ is minimal and set $(x_{ji}) = x(b)$. Set $a_i = x_{ii}$ $(1 \leq i \leq k)$ and $b_i = x_{1i}$ $(2 \leq i \leq k')$. Let $\Lambda = \sum_{i=0}^{n} \lambda_i \Lambda_i$ be a dominant integral weight of level l, i.e., $\lambda_i \in \mathbf{Z}_{\geq 0}$, $\sum_{i=0}^{n} \lambda_i = l$. We claim that there exists a unique b such that $\varepsilon_i(b) = \lambda_i$.

Proposition 6.3.8. *With the notations as above, the equalities* $\varepsilon_i(b) = \lambda_i$ $(0 \leq i \leq n)$ *hold if and only if*

$$\begin{aligned} \lambda_i &= a_k \quad i = 0 \\ &= b_{i+1} \quad 1 \leq i \leq k' - 1 \\ &= a_{i+1-k'} - a_{i+2-k'} \quad k' \leq i \leq n. \end{aligned}$$

Proof. The assertion follows from Proposition 6.3.2 and Proposition 1.2.3. □

Proposition 6.3.9. *Let* Λ *and* b *be given as above. Set* $\Lambda' = \Lambda + \mathrm{wt}\, b$. *Then we have*

$$\begin{aligned} \lambda'_i &= \lambda_{i+k'-1} \quad (0 \leq i \leq k-1) \\ &= \lambda_{i-k} \quad (k \leq i \leq n), \end{aligned}$$

where $\Lambda' = \sum_{i=0}^{n} \lambda'_i \Lambda_i$.

Proof. Set $m_i = \sharp\{(j,j') \mid m_{jj'} = i\}$ where $(m_{jj'}) = m(b)$. We have $\lambda'_i = \lambda_i + m_i - m_{i+1}$. If $k \leq k' - 1$, then we have

$$\begin{aligned} m_i - m_{i+1} &= b_{k'} - a_k \quad (i = 0) \\ &= a_i - a_{i+1} - b_{i+1} \quad (1 \leq i \leq k-1) \\ &= a_k - b_{i+1} \quad (i = k) \\ &= b_{i+1-k} - b_{i+1} \quad (k+1 \leq i \leq k'-1) \\ &= b_{i+1-k} + a_{i+2-k'} - a_{i+1-k'} \quad (k' \leq i \leq n). \end{aligned}$$

If $k \geq k'$, then we have

$$\begin{aligned} m_i - m_{i+1} &= b_{k'} - a_k \quad (i = 0) \\ &= a_i - a_{i+1} - b_{i+1} \quad (1 \leq i \leq k'-1) \\ &= a_i - a_{i+1} + a_{i+2-k'} - a_{i+1-k'} \quad (k' \leq i \leq k-1) \\ &= a_k + a_{i+2-k'} - a_{i+1-k'} \quad (i = k) \\ &= b_{i+1-k} + a_{i+2-k'} - a_{i+1-k'} (k+1 \leq i \leq n). \end{aligned}$$

The assertion then follows from Proposition 6.3.8. □

Proposition 6.3.10. *$B^{k,l} \otimes B^{k,l}$ is connected.*

Proof. Let $u_0 \in B^{k,l}$ be the highest weight element of $\iota^{(0)*}(B^{k,l})$ and $v_0 \in B^{k,l}$ be the lowest weight element of $\iota^{(k)*}(B^{k,l})$. For any $b_1 \otimes b_2 \in B^{k,l} \otimes B^{k,l}$, we shall show that $b_1 \otimes b_2$ is connected to $u_0 \otimes v_0$. By the description of the actions of $\tilde{f}_i$ in [KN], we can easily obtain that $\tilde{f}_i u_0 = 0$ for $i = 1, \cdots, k-1, k+1, \cdots, n$. Assuming that $\tilde{f}_0 u_0 \neq 0$, we have $wt\iota^{(0)*}(\tilde{f}_0 u_0) = l\Lambda_k + \alpha_1 + \cdots + \alpha_n$ by $\alpha_0 = -\sum_{i=1}^{n} \alpha_i$ in the sense of classical weight. This is a contradiction to the fact that u_0 is the highest element of $\iota^{(0)*}(B^{k,l}) \cong B(l\Lambda_k)$. Then $\tilde{f}_0 u_0 = 0$. Therefore, we have $\varphi_j(u_0) = 0$ for $j \neq k$. Let $u_0 \otimes b$ be the highest weight element of the connected component of $\iota^{(0)*}(B^{k,l} \otimes B^{k,l})$ that contains $b_1 \otimes b_2$, hence $b_1 \otimes b_2$ is connected to $u_0 \otimes b$ by $\tilde{f}_1, \cdots, \tilde{f}_n$. b can be written in the following form: $b = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} v_0\, (i_j \neq k)$. By $\varphi_j(u_0) = 0$ $(j \neq k)$, if $j \neq k$, then $\tilde{f}_j(u_0 \otimes b) = u_0 \otimes \tilde{f}_j b$. Hence, $\tilde{f}_{i_l} \cdots \tilde{f}_{i_1}(u_0 \otimes b) = u_0 \otimes v_0$. □

Now, we have completed the proof of the Theorem 1.2.2 by Theorem 1.2.3, Proposition 6.3.10, Corollary 6.3.6, and Remark 1.2.4.

The following will not be used in the rest of the paper, and the proof is omitted. Let $c = (c_0, c_1 \ldots, c_k)$ and $c^* = (c_2^*, \ldots, c_{k'-1}^*)$ be a partition of $\{1, \ldots, n+1\}$ such that $1 = c_0 < c_1 < \cdots < c_k = n+1$ and $c_2^* < \cdots < c_{k'-1}^*$. For c, c' we denote $c \leq c'$ if and only if $c_j \leq c'_j (\forall j)$. Define

$$\Delta(c,b) = \sum_{j=2}^{k'-1} x_{c_j^*+1-j\, c_j^*},$$

$$B(c) = \{b \in B \mid \Delta(c,b) \leq \Delta(c',b) \quad \text{if} \quad c' \leq c, \Delta(c,b) < \Delta(c',b) \quad \text{otherwise}\ \},$$

$$\begin{aligned} F(c)_{ji} &= 1 \quad \text{if} \quad i = c_{j-1} \\ &= -1 \quad \text{if} \quad i = c_j \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Proposition 6.3.11. *With the notations as above, we have the following.*

(1) $B = \sqcup_c B(c)$ *is a disjoint union.*

(2) *If* $b \in B(c)$, *then* $\tilde{f}_0 b = 0$ *if and only if* $c = (1, k, k+1, \ldots, n+1)$ *and* $x'_{1k'} = 0$.

(3) *Suppose that* $\tilde{f}_0 b \neq 0$. *Then we have* $y_{ji} = x_{ji} + F(c)_{ji}$ *where* $(y_{ji}) = x(\tilde{f}_0 b)$.

6.4. $(C_n^{(1)}, B(l\Lambda_n))$ $(n \geq 2)$

We shall use the notations in 1.3.

Lemma 6.4.1. *We have*

$$\begin{aligned} \omega_{j\,i} &= x_{j\,i+1} + x_{j^* \bar{i}} - x_{j-1\,i} - x_{j^*-1\,\overline{i+1}} \quad \textit{if} \quad 1 \leq j \leq i < n \\ &= x_{j\,\bar{n}} - x_{j-1\,n} \quad \textit{if} \quad 1 \leq j \leq i = n. \end{aligned}$$

The proof is immediate. From this follows

Proposition 6.4.2. *For $b \in B$ we have*

$$\varepsilon_i(b) = \sum_{j=1}^{i} (p_{ji})_+$$

where $p_{ji} \quad (0 \le j \le i \le n)$ are defined inductively by

$$p_{0i} = 0,$$
$$p_{ji} = \omega_{j\,i} - \left(-p_{j-1\,i}\right)_+ \quad (1 \le j \le i \le n).$$

Then, we have

Proposition 6.4.3. *For $b \in B$ we have*

$$\sum_{i=1}^{n} \varepsilon_i(b) \ge l - l_n.$$

The equality holds if and only if $p_{ii} \ge 0 \quad (2 \le i \le n)$.

Proof. Recall the proof of Proposition 6.3.3. Similarly, we have $\varepsilon_i(b) \ge \sum_{j=1}^{i} \omega_{j\,i}$, where the equality holds if and only if $p_{ii} \ge 0 \quad (2 \le i \le n)$. The right-hand side then reduces to $l - l_n$. □

Proof of Proposition 1.3.1. The existence is given in Section 5. Let us prove the uniqueness. Let $J' = \{1, \dots, n-1\}$ be the index set of the simple roots for $\mathfrak{sl}(n)$, and let $\iota^{(0,n)} : J' \longrightarrow I$ be $\iota^{(0,n)}(j) = j$. Set $B' = \iota^{(0,n)*}(B^{n,l})$.

Proposition 6.4.4. *As a crystal of $U_q\left(\widehat{\mathfrak{sl}}\,(n)\right)$, B' splits as*

$$B' = \oplus_{0 \le l_n \le \cdots \le l_1 \le l} B\left(\Lambda_{l_1, \dots, l_n}\right)$$

where

$$\Lambda_{l_1,\dots,l_n} = (l - 2l_0)\Lambda_0 + \sum_{i=1}^{n-1} 2(l_i - l_{i+1})\Lambda_i + (2l_n - l)\Lambda_n.$$

If B_1 and B_2 are both isomorphic to B', then the isomorphism $B_1 \longrightarrow B_2$ is unique.

We omit the proof of Proposition 6.4.4 which is similar to that of Proposition 6.3.1. From this follows the uniqueness of $B^{n,l}$ similarly to the proof of Poposition 1.2.1.(see 6.3.) □

Proof of Theorem 1.3.2.

Set $\left(x'_{jk}\right) = x\left(\bar{\iota}^*(b)\right)$. Similarly, we define y'_{jk}, z'_{jj^*}, and so on.

Proposition 6.4.5. *For $b \in B^{n,l}$ we have*

$$\varepsilon_0(b) \ge l_n.$$

The equality holds if and only if $p'_j \le 0 \quad (2 \le j \le n)$ where p'_j are defined inductively by

$$p'_0 = 0, \quad p'_j = z'_{j-1\,j} - z'_{j\,j-1} - \left(-p'_{j-1}\right)_+.$$

In particular, we have $p_1' = l - l_1' \geq 0$.

Proof. By an argument similar to that was used in the proof of Proposition 6.3.5, we have $l_i' = l - l_{n+1-i}$. Therefore, we have

$$\varepsilon_0(b) = \varepsilon_n\left(\bar{\iota}^*(b)\right) = \sum_{j=1}^{n} \left(p_j'\right)_+ \geq l - l_1' = l_n. \qquad \square$$

Corollary 6.4.6. *For* $b \in B^{n,l}$, *we obtain* $\langle c, \varepsilon(b)\rangle = \sum_{i=0}^{n} \varepsilon_i(b) \geq l$.

Proof of Theorem 1.3.3.

Let b be an element in $B^{n,l}$ such that $\langle c, \varepsilon(b)\rangle = l$ and $\underline{b}$ be the unique element in $B^{n,l}$ such that $\iota^{(0,n)*}(\underline{b})$ is the lowest element in the connected component of B' that contains $\iota^{(0,n)*}(b)$. This is obtained by

$$\begin{aligned}
\iota^{(0)*}(b) \;&\xRightarrow{n-1\; r_{1\,n-1}} b(1,n-1) \xRightarrow{n-2\; r_{2\,n-2}} \cdots \xRightarrow{2\; r_{1\,2}} b(1,2) \xRightarrow{1\; r_{1\,1}} b(1,1) \\
&\xRightarrow{n-1\; r_{2\,n-1}} b(2,n-1) \xRightarrow{n-2\; r_{2\,n-2}} \cdots \xRightarrow{2\; r_{2\,2}} b(2,2) \\
&\cdots \\
&\xRightarrow{n-1\; r_{n-1\,n-1}} b(n-1,n-1) = \iota^{(0)*}(\underline{b})
\end{aligned}$$

Here r_{ji} are determined in such a way that for each pair $b_1 i^r \to b_2$ we have $\tilde{f}_i b_2 = 0$. By Proposition 6.2.2 we have $r_j < r_{j+1} \quad (1 \leq j \leq n-2)$ where $r_j = r_{n-j\,n-j}$. With these r_j set $b'' = \tilde{e}_{n-1}^{r_{n-1}} \cdots \tilde{e}_1^{r_1} \underline{b}$. Then, we have $x_{jn}' = x_{jn}''$ and $x_{j\,n+1}' = x_{j\,n+1}''$ where $\left(x_{ji}''\right) = x\left(\iota^*(b'')\right)$. In this way we have

$$z_{j\,j+1}' = 2l_{j+1}' + r_{n-j} - r_{n-j-1} \quad (1 \leq j \leq n-1).$$

Here we set $r_0 = 0$.

From Proposition 6.4.3 and 6.4.5, the equality $\sum_{i=o}^{n} \varepsilon_i(b) = l$ holds if and only if the following $(A)_1$ and $(D)_j$ $(1 \leq j \leq n-1)$. Consider the following statements.

$$\begin{array}{ll}
(A)_j & p_{i+1-j\,i} \geq 0 \quad (j+1 \leq i \leq n), \\
(B)_j & \omega_{i+1-j\,i} = 0 \quad (j+1 \leq i \leq n), \\
(C)_j & p_{j+1}' = 0, \\
(D)_j & p_{j+1}' \geq 0, \\
(E)_j & r_{n-j-1} = z_{1\,n-j} - 2l_{n-j}.
\end{array}$$

We will show the equivalence of

(i) $(B)_j$ $(1 \leq j \leq n-1)$.

(ii) $(A)_1$ and $(D)_j$ $(1 \leq j \leq n-1)$.

Note that (i) implies Theorem 1.3.3. Let us derive (i) from (ii). Assume that $(A)_1, \ldots, (A)_j$ and $(C)_{j-1}$ are valid. From $(A)_j$ we have $\omega_{i+1-j\,i} \geq 0$. Therefore, we have

$$\sum_{i=j+1}^{n} \omega_{i+1-j\,i} = z_{1\,n+1-j} - z_{1\,n-j} + l_{n-j} - l_{n+1-j} \geq 0.$$

Note that $(E)_0 : r_{n-1} = z_{1n} - 2l_n$ holds in any case, and noting that the changes for $r_{i,i}$ take place on the first row, we have that $(A)_1, \cdots, (A)_j$ imply $(E)_1, \cdots, (E)_j$. From $(C)_{j-1}$ and $(D)_j$ we get

$$(6.4.1) \qquad p'_{j+1} = z'_{j\,j+1} - z'_{j+1\,j} = l'_{j+1} - l'_j + r_{n-j} - r_{n-j-1} \geq 0.$$

Then, by using $(E)_{j-1}$, $(E)_j$ and $l'_j + l_{n+1-j} = l$, we get

$$(6.4.2) \qquad p'_{j+1} = z_{1\,n+1-j} - z_{1\,n-j} + l_{n-j} - l_{n+1-j} \geq 0.$$

From (6.4.1) and (6.4.2) we have $(B)_j$ and $(C)_j$. From $(B)_j$ and $(A)_j$, we have $(A)_{j+1}$. Thus we proved (i) by assuming (ii).

Next, we show that (i) implies (ii). Note that (i) means $\omega_{ji} = 0$ $(1 \leq j \leq i \leq n)$. Note also that $p_{1i} \geq 0$ $(1 \leq i \leq n)$. From these we can inductively show $p_{ji} = 0$ $(2 \leq j \leq i \leq n)$. In particular, we have $(A)_j$ $(1 \leq j \leq n-1)$. As we noted before, $(A)_1, \cdots, (A)_j$ imply $(E)_1, \cdots, (E)_j$. Therefore, we have $p'_{j+1} = z_{1\,n+1-j} - z_{1\,n-j} + l_{n-j} - l_{n+1-j}$. Since $\omega_{ji} = 0$ $(2 \leq j \leq i \leq n)$, $p'_{j+1} = 0$. Hence, we obtain $(C)_j$ and then $(D)_j$. □

Proposition 6.4.7. *$B^{n,l} \otimes B^{n,l}$ is connected.*

The proof is similar to that of Proposition 6.3.10.

Now, we completed the proof of Theorem 1.3.2 by Theorem 1.3.3, Proposition 6.4.7, Corollary 6.4.6 and Remark 1.3.4. □

6.5. $(D^{(2)}_{n+1}, B(l\Lambda_n))$ $(n \geq 2)$

We shall use the notations in 1.4. We describe the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $(m_{j,j'})$ as follows. First we read each column of $(m_{j,j'})$ from the right to the left and obtain the sequence $u_1 \cdots u_n$ where u_k is the k-th column from the right.

(1) The case $i \neq n$. If there are i and $\overline{i+1}$ in u_k then we identify it with $+$, if there are $i+1$ and $\bar{i}$ in u_k, then we identify it with $-$ and otherwise, we identify with 0. Then we obtain the actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(1 \leq i < n)$.

(2) The case $i = n$. If there are n in u_k then we identify it with $+$, if there are $\bar{n}$ in u_k, then we identify it with $-$ and otherwise, we identify with 0. Then we obtain the actions of $\tilde{e}_i$ and $\tilde{f}_i$.

We also use

$$l_j = y_{j\,n} \quad (1 \leq j \leq n),$$
$$\omega_{j\,i} = \begin{cases} x_{j\,i+1} - x_{j-1\,i} & (1 \leq j \leq i < n), \\ x_{j\,\bar{n}} - x_{j-1\,n} & (1 \leq j \leq i = n). \end{cases}$$

Note that $\omega_{1\,i} = x_{1\,i+1} - x_{0\,i} = x_{1\,i+1} \geq 0$.

Proposition 6.5.1. *For $b \in B$ we have*

$$\varepsilon_i(b) = \sum_{j=1}^{i} (p_{ji})_+$$

where p_{ji} $(0 \leq j \leq i \leq n)$ are defined inductively by

$$p_{0i} = 0,$$
$$p_{ji} = \omega_{j\,i} - \left(-p_{j-1\,i}\right)_+ \quad (1 \leq j \leq i \leq n).$$

Then, we have

Proposition 6.5.2. *For $b \in B$ we have*

$$2\sum_{i=1}^{n-1} \varepsilon_i(b) + \varepsilon_n(b) \geq l - l_n.$$

The equality holds if and only if $p_{ii} \geq 0 \quad (2 \leq i \leq n)$.

Proof. Recall the proof of Proposition 6.3.3. Similarly, we have $\varepsilon_i(b) \geq \sum_{j=1}^{i} \omega_{j\,i}$, where the equality holds if and only if $p_{ii} \geq 0 \quad (2 \leq i \leq n)$. The right-hand side then reduces to $l - l_n$. □

Proof of Proposition 1.4.1. The existence is given in Section 5. Let us prove the uniqueness. Let $J' = \{1, \ldots, n-1\}$ be the index set of the simple roots for $\mathfrak{sl}(n)$, and let $\iota^{(0,n)} : J' \longrightarrow I$ be $\iota^{(0,n)}(j) = j$. Set $B' = \iota^{(0,n)*}(B^{n,l})$.

Proposition 6.5.3. *As a crystal of $U_q(\mathfrak{sl}(n))$, B' splits as*

$$B' = \oplus_{0 \leq l_n \leq \cdots \leq l_1 \leq l_0 = l} B\left(\Lambda_{l_1,\ldots,l_n}\right)$$

where

$$\Lambda_{l_1,\ldots,l_n} = \sum_{i=0}^{n-1} (l_i - l_{i+1})\Lambda_i.$$

If B_1 and B_2 are both isomorphic to B', then the isomorphism $B_1 \longrightarrow B_2$ is unique.

We omit the proof of Proposition 6.5.3 which is similar to that of Proposition 6.3.1. From this follows the uniqueness of $B^{n,l}$ similarly to the proof of Proposition 1.2.1.(see 6.3.) □

Now, we are going to prove Theorem 1.4.2. Set $\left(x'_{j\,k}\right) = x\left(\bar{\iota}^*(b)\right)$. Similarly, we define $x'_{j\,k}, y'_{j\,k}$, and so on.

Proposition 6.5.4. *For $b \in B^{n,l}$ we have*

$$\varepsilon_0(b) \geq l_n.$$

The equality holds if and only if $p'_j \leq 0 \quad (2 \leq j \leq n)$ where p'_j are defined inductively by

$$p'_0 = 0, \quad p'_j = x'_{j,\bar{n}} - x'_{j-1,n} - (-p'_{j-1})_+.$$

In particular, we have $p'_1 = l - l'_1 \geq 0$.

Proof. By an argument similar to that was used in the proof of Proposition 6.3.5, we have $l'_i = l - l_{n+1-i}$. Therefore, we have

$$\varepsilon_0(b) = \varepsilon_n\left(\bar{\iota}^*(b)\right) = \sum_{j=1}^{n} \left(p'_j\right)_+ \geq l - l'_1 = l_n.$$

□

Corollary 6.5.5. *For $b \in B^{n,l}$, we have*

$$\langle c, \varepsilon(b)\rangle = \varepsilon_0(b) + 2\sum_{i=1}^{n-1} \varepsilon_i(b) + \varepsilon_n(b) \geq l.$$

Proof of Theorem 1.4.3.

Let b be an element in $B^{n,l}$ such that $\langle c, \varepsilon(b)\rangle = l$ and $\underline{b}$ be the unique element in $B^{n,l}$ such that $\iota^{(0,n)*}(\underline{b})$ is the lowest element in the connected component of B' that contains $\iota^{(0,n)*}(b)$. This is obtained by

$$\begin{aligned}
\iota^{(0)*}(b) \ & {}^{n-1}\overset{r_{1\,n-1}}{\Longrightarrow}{}^{n-1} b(1,n-1) {}^{n-2}\overset{r_{2\,n-2}}{\Longrightarrow}{}^{n-2} \ \ldots \overset{2\,r_{1,2}}{\Longrightarrow} b(1,2) \overset{1\,r_{1,1}}{\Longrightarrow} b(1,1) \\
& {}^{n-1}\overset{r_{2\,n-1}}{\Longrightarrow}{}^{n-1} b(2,n-1) {}^{n-2}\overset{r_{2\,n-2}}{\Longrightarrow}{}^{n-2} \ \ldots \overset{2\,r_{2,2}}{\Longrightarrow} b(2,2) \\
& \ldots \\
& {}^{n-1}\overset{r_{n-1\,n-1}}{\Longrightarrow}{}^{n-1} b(n-1,n-1) = \iota^{(0)*}(\underline{b})
\end{aligned}$$

Here r_{ji} are determined in such a way that for each pair $b_1(i)^r \to b_2$ we have $\tilde{f}_i b_2 = 0$. By Proposition 6.2.2 we have $r_j < r_{j+1} \quad (1 \leq j \leq n-2)$ where$r_j = r_{n-j\,n-j}$. With these r_j set $b'' = \tilde{e}_{n-1}^{r_{n-1}} \cdots \tilde{e}_1^{r_1}\underline{b}$. Then, we have $x'_{jn} = x''_{jn}$ and $x'_{j\,n+1} = x''_{j\,n+1}$ where $\left(x''_{ji}\right) = x\left(\iota^*(b'')\right)$. In this way we have

$$\text{(6.5.1)}\quad x'_{j+1,\overline{n}} - x'_{j,n} = 2(r_{n-j} - r_{n-j-1}) - (y'_{j,n} - y'_{j+1,n}) \quad (1 \leq j \leq n-1)$$

Here we set $r_0 = 0$.

From Proposition 6.5.2 and 6.5.4, the equality $\varepsilon_0(b) + 2\sum_{i=1}^{n-1}\varepsilon_i(b) + \varepsilon_n(b) = l$ holds if and only if the following $(A)_1$ and $(D)_j$ $(1 \leq j \leq n-1)$. Consider the following statements.

$$\begin{aligned}
&(A)_j \quad p_{i+1-j\,i} \geq 0 \quad (j+1 \leq i \leq n), \\
&(B)_j \quad \omega_{i+1-j\,i} = 0 \quad (j+1 \leq i \leq n), \\
&(C)_j \quad p'_{j+1} = 0, \\
&(D)_j \quad p'_{j+1} \leq 0, \\
&(E)_j \quad r_{n-j-1} = y_{1,j+1} - y_{n-j,n} \quad (0 \leq j \leq n-1).
\end{aligned}$$

We will show the equivalence of the following (i) and (ii)
(i) $(B)_j \quad (1 \leq j \leq n-1)$,
(ii) $(A)_1$ and $(D)_j$ $(1 \leq j \leq n-1)$.

Let us derive (i) from (ii). Assume that $(A)_1, \ldots, (A)_j$ and $(C)_{j-1}$ are valid. From $(A)_j$ we have $\omega_{i+1-j\,i} \geq 0$. Therefore, we have

$$\text{(6.5.2)}\qquad x_{n-j\,n} \geq x_{n-j-1\,n-1} \geq x_{n-j-2\,n-2} \geq \cdots \geq x_{2\,j+2} \geq x_{1\,j+1},$$

$$\text{(6.5.3)}\qquad x_{n-j+1\,\overline{n}} \geq x_{n-j\,n} \text{and then } 2y_{n-j\,n-1} \geq y_{n-j+1\,n} + y_{n-j\,n}.$$

Note that $(E)_0 : r_{n-1} = y_{1\,1} - y_{n\,n}$ holds in any case, and noting that the changes for $r_{i,i}$ take place on the first row, we have that $(A)_1, \cdots, (A)_j$ imply $(E)_1, \cdots, (E)_j$. From $(C)_{j-1}$ and $(D)_j$ we get

$$\text{(6.5.4)}\qquad p'_{j+1} = x'_{j+1\,\overline{n}} - x'_{j\,n} \leq 0$$

Then, by using $(E)_{j-1}$, $(E)_j$, (6.5.1), (6.5.4) and $y'_{j\,n}+y_{n+1-j\,n}=l$, we get

$$p'_{j+1} = 2(y_{1\,j}-y_{n-j+1\,n}-y_{1\,j+1}+y_{n-j\,n})-(y_{n-j\,n}-y_{n+1-j\,n}) \\ = 2(y_{1\,j}-y_{1\,j+1})+(y_{n-j\,n}-y_{n-j+1\,n}) \leq 0. \tag{6.5.5}$$

From (6.5.3) and (6.5.5) we obtain $y_{1\,j+1}-y_{1\,j} \geq y_{n-j\,n}-y_{n-j\,n-1}$ and then

$$x_{1\,j+1} \geq x_{n-j\,n}. \tag{6.5.6}$$

From (6.5.2) and (6.5.6),

$$x_{1\,j+1} = x_{2\,j+2} = \cdots = x_{n-j\,n}. \tag{6.5.7}$$

Then we have $(B)_j$ and $(C)_j$. From $(B)_j$ and $(A)_j$, we have $(A)_{j+1}$. Thus we proved (i) by assuming (ii).

Next, we show that (i) implies (ii). Assume $(C)_{j-1}$ Note that (i) means $\omega_{ji}=0$ $(1\leq j\leq i\leq n)$ and $p_{1i}\geq 0$ $(1\leq i\leq n)$. From these we can inductively show $p_{ji}=0$ $(2\leq j\leq i\leq n)$. In particular, we have $(A)_j$ $(1\leq j\leq n-1)$. As we noted before, $(A)_1,\cdots,(A)_j$ imply $(E)_1,\cdots,(E)_j$. Therefore, we have

$$p'_{j+1} = 2(y_{1\,j}-y_{1\,j+1})+(y_{n-j\,n}-y_{n-j+1\,n}). \tag{6.5.8}$$

From $(B)_j$, we have $x_{n-j\,n}=x_{1\,j+1}$. Hence $y_{n-j\,n}-y_{n-j\,n-1}=y_{1\,j+1}-y_{1\,j}$. From this and (6.5.8), we obtain

$$\begin{aligned} p'_{j+1} &= -2(y_{n-j\,n}-y_{n-j\,n-1})+(y_{n-j\,n}-y_{n-j+1\,n}) \\ &= (y_{n-j\,n-1}-y_{n-j\,n})+(y_{n-j\,n-1}-y_{n-j+1\,n}) \\ &= -x_{n-j\,n}+x_{n-j+1\,\overline{n}} = \omega_{n-j+1\,n} = 0. \end{aligned}$$

Hence, we obtain $(C)_j$ and then $(D)_j$ □

Proposition 6.5.6. *$B^{n,l}\otimes B^{n,l}$ is connected.*

The proof is similar to that of Proposition 6.3.10.

Now, we have completed the proof of Theorem 1.4.2 by Theorem 1.4.3, Proposition 6.5.6, Corollary 6.5.5 and Remark 1.4.4. □

6.6. *($D_n^{(1)}$,$B(l\Lambda_{n-1})$ and $B(l\Lambda_n)$)* $(n\geq 4)$

We shall use the notations in 1.5. We describe the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $(m_{j\,j'})$ as follows. First, we read each column of $(m_{j\,j'})$ from the right to the left and then obtain a sequence $u_1u_2\cdots u_l$ where u_k is the k-th column from the right side. Then the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on u_k are as follows.

(1) In the case $i\neq n$.

(i) If u_k contains i and $\overline{i+1}$, then $\tilde{f}_iu_k$ is obtained by replacing i and $\overline{i+1}$ with $i+1$ and $\overline{i}$ and otherwise, $\tilde{f}_iu_k=0$.

(ii) If u_k contains $i+1$ and $\overline{i}$, then $\tilde{e}_iu_k$ is obtained by replacing $i+1$ and $\overline{i}$ with i and $\overline{i+1}$ and otherwise, $\tilde{e}_iu_k=0$.

(2) In the case $i = n$.

(i) If u_k contains $n-1$ and n, then $\tilde{f}_n u_k$ is obtained by replacing $n-1$ and n with $\overline{n}$ and $\overline{n-1}$ and otherwise, $\tilde{f}_n u_k = 0$.

(ii) If u_k contains $\overline{n}$ and $\overline{n-1}$, then $\tilde{e}_n u_k$ is obtained by replacing $\overline{n}$ and $\overline{n-1}$ with $n-1$ and n and otherwise, $\tilde{e}_n u_k = 0$.

From these descriptions, we obtain the actions of $\tilde{e}_i$ and $\tilde{f}_i$ by the following way.

(a) In the case $i \neq n$. If u_k contains i and $\overline{i+1}$, then u_k is identified with $+$, if u_k contains $i+1$ and $\overline{i}$, then u_k is identified with $-$ and otherwise, u_k is identified with 0.

(b) In the case $i = n$. If u_k contains $n-1$ and n, then u_k is identified with $+$, if u_k contains $\overline{n}$ and $\overline{n-1}$, then u_k is identified with $-$ and otherwise, u_k is idnetifed with 0.

For $b \in B$ (resp. B') we also use the following notations;

$$\omega_{j\,i} = x_{j\,i+1} - x_{j-1\,i} \quad (1 \le j \le i \le n-2).$$

If n is even, for $j = 2, \cdots, n$

$$\omega_{j\,n-1} = \begin{cases} x_{j\,\overline{n-1}} - x_{j-3\,n-1} & \text{if } j \text{ is odd (resp. even),} \\ 0 & \text{if } j \text{ is even (resp. odd),} \end{cases}$$

$$\omega_{j\,n} = \begin{cases} x_{j\,\overline{n-1}} - x_{j-3\,n-1} & \text{if } j \text{ is even (resp. odd),} \\ 0 & \text{if } j \text{ is odd (resp. even).} \end{cases}$$

If n is odd, for $j = 2, \cdots, n$

$$\omega_{j\,n-1} = \begin{cases} x_{j\,\overline{n-1}} - x_{j-3\,n-1} & \text{if } j \text{ is even (resp. odd),} \\ 0 & \text{if } j \text{ is odd (resp. even),} \end{cases}$$

$$\omega_{j\,n} = \begin{cases} x_{j\,\overline{n-1}} - x_{j-3\,n-1} & \text{if } j \text{ is odd (resp. even),} \\ 0 & \text{if } j \text{ is even (resp. odd).} \end{cases}$$

Proposition 6.6.1. *For $b \in B$ we have*

$$\varepsilon_i(b) = \sum_{j=1}^{i} (p_{j\,i})_+$$

where $p_{j\,i}$ are defined by inductively as follows;

for $0 \le j \le i \le n-2$,

$$p_{0\,i} = 0,$$
$$p_{j\,i} = \omega_{j\,i} - (-p_{j-1\,i})_+ \quad (1 \le j \le i \le n-2),$$

for $0 \le j \le i = n-1$ or n,

$$p_{0\,i} = p_{1\,i} = 0,$$
$$p_{j\,i} = \omega_{j\,i} - (-p_{j-2\,i})_+ \quad (1 \le j \le i = n-1, n).$$

Lemma 6.6.2. *For $b \in B$ we have*

$$\varepsilon_1(b)+2\varepsilon_2(b)+\cdots+2\varepsilon_{n-2}(b) \geq \sum_{k=1}^{n-3}(|p_{k\,k}|+2y_{k\,n-1}-2y_{k\,n-2})+y_{n-2\,n-1}-y_{n-2\,n-2}.$$

The equality holds if and only if $p_{i-1\,i} \geq 0$ $(2 \leq i \leq n-2)$.

Proof. From the identities $x_+-(-x)_+=x$ and $|x|+x=(x)_+$, for $2 \leq i \leq n-2$

$$(p_{1\,i})_+ +\cdots+(p_{i-1\,i})_+ +2(p_{i\,i})_+ =\omega_{1\,i}+\cdots+\omega_{i\,i}+|p_{i\,i}|, \tag{6.6.1}$$

$$(p_{1\,i})_+ +\cdots+(p_{i-1\,i})_+ \geq \omega_{1\,i}+\cdots+\omega_{i\,i}. \tag{6.6.2}$$

The equality of (6.6.2) holds if and only if $p_{i-1\,i} \geq 0$. Hence, by the definitions of $\omega_{j\,i}$, (6.6.1) and (6.6.2) we obtain the desired result. □

Lemma 6.6.3. *For $b \in B$ we have*

$$\varepsilon_{n-1}(b)+\varepsilon_n(b) \geq l+y_{n-2\,n-2}-y_{n-2\,n-1}-y_{n-1\,n-1}+2\sum_{k=1}^{n-3}(y_{k\,n-2}-y_{k\,n-1}). \tag{6.6.3}$$

The equality of (6.6.3) holds if and only if $p_{n-1\,n-1} \geq 0$ and $p_{n\,n} \geq 0$.

Proof. From the identity $x_+-(-x)_+=x$, we obtain

(i) if n is even,

$$\begin{aligned} &\varepsilon_{n-1}(b) \geq \omega_{1\,n-1}+\omega_{3\,n-1}+\cdots+\omega_{n-1\,n-1}, \\ &\varepsilon_n(b) \geq \omega_{2\,n}+\omega_{4\,n}+\cdots+\omega_{n\,n}, \end{aligned} \tag{6.6.4}$$

(ii) if n is odd,

$$\begin{aligned} &\varepsilon_{n-1}(b) \geq \omega_{2\,n-1}+\omega_{4\,n-1}+\cdots+\omega_{n-1\,n-1}, \\ &\varepsilon_n(b) \geq \omega_{1\,n}+\omega_{3\,n}+\cdots+\omega_{n\,n}. \end{aligned} \tag{6.6.5}$$

The equalities of (6.6.4) and (6.6.5) hold if and only if $p_{n-1\,n-1} \geq 0$ and $p_{n\,n} \geq 0$. From (6.6.4), (6.6.5) and the definitions of $\omega_{k\,n-1}$ and $\omega_{k\,n}$ we obtain the desired result. □

By Lemma 6.6.2 and Lemma 6.6.3, we obtain the following.

Proposition 6.6.4. *For $b \in B$ we have*

$$\varepsilon_1(b)+2(\varepsilon_2(b)+\cdots\varepsilon_{n-2}(b))+\varepsilon_{n-1}(b)+\varepsilon_n(b) \geq l-y_{n-1\,n-1}+\sum_{k=2}^{n-1}|p_{k\,k}|. \tag{6.6.6}$$

The equality of (6.6.6) holds if and only if $p_{k-1\,k} \geq 0$ $(2 \leq k \leq n-2)$, $p_{n-1\,n-1} \geq 0$ and $p_{n\,n} \geq 0$.

Proof of Proposition 1.5.1. The existence is given in Section 5. Let us prove the uniqueness. Let $J'=\{1,2,\cdots,n-1\}$ be the index set of the simple roots for $\mathfrak{sl}(n)$ and let $\iota^{(0,n)}: J' \longrightarrow I$ be $\iota^{(0,n)}(j)=j$. Set $\tilde{B}=\iota^{(0,n)*}(B^{n,l})$.

proposition 6.6.5. *As a crystal of $U_q(\mathfrak{sl}(n))$, $\tilde{B}$ splits as follows.*
(i) If n is even,

$$\tilde{B} = \bigoplus_{0 \le l_n \le l_{n-2} \le \cdots \le l_2 \le l_0 = l} B(\Lambda_{l_2, l_4, \cdots, l_n}),$$

where $l_k = y_{k\,n}$ and $\Lambda_{l_2,\cdots,l_n} = \sum_{k=0,2,4,\cdots,n}(l_k - l_{k+2})\Lambda_k$.
(ii) If n is odd,

$$\tilde{B} = \bigoplus_{0 \le l_n \le l_{n-2} \le \cdots \le l_3 \le l_1 = l} B(\Lambda_{l_1, l_3, \cdots, l_n}),$$

where $l_k = y_{k\,n}$ and $\Lambda_{l_1,l_3,\cdots,l_n} = \sum_{k=1,3,\cdots,n}(l_k - l_{k+2})\Lambda_k$.
If B_1 and B_2 are both isomorphic to $\tilde{B}$, then the isomorphism $B_1 \longrightarrow B_2$ is unique.

We omit the proof of Proposition 6.6.5 which is similar to that of Proposition 6.3.1. From this follows the uniqueness of $B^{n,l}$ similarly to the proof of Proposition 1.2.1.(see 6.3.). □

Now, we are going to prove Theorem 1.5.2.
Set $(x'_{j\,k}) = x(\bar{\iota}^*(b))$. Similarly, we define $y'_{j\,k}$, $\omega'_{j\,k}$ and so on.

Proposition 6.6.6. *For $b \in B^{n,l}$ we have*

$$\varepsilon_0(b) \ge y_{n\,n}, \tag{6.6.7}$$

where $p'_{j\,n}$ is defined by the same formulas as $p_{j\,n}$ for $(x'_{j\,k})$. The equality of (6.6.7) holds if and only if $p'_{j\,n} \le 0$ for $j = 3, \cdots, n$.

Proof. By a similar argument to that was used in the proof of Proposition 6.3.5, we have

$$y'_{j\,n} = l - y_{n-j+2\,n} \quad (j \text{ is even}).$$

Therefore,

$$\varepsilon_0(b) = \varepsilon_n(\bar{\iota}^*(b)) = \sum_{j=1}^{n}(p'_{j\,n})_+ \ge (p'_{2\,n})_+ \ge \omega_{2\,n}{}' = x'_{2\,\overline{n-1}} = l - y'_{2\,n} = y_{n\,n}. \tag{6.6.9}$$

Note that if n is odd, $\bar{\iota}^*(B^{n,l}) = B(l\Lambda_{n-1}) = \tilde{B}$. The equality of (6.6.9) holds if and only if $p'_{j\,n} \le 0$ for $j = 3, \cdots, n$. □

Corollary 6.6.7. *For $b \in B^{n,l}$ (resp. $B^{n-1,l}$), we have*

$$\langle c, \varepsilon(b)\rangle = \varepsilon_0(b) + \varepsilon_1(b) + 2\sum_{k=2}^{n-2}\varepsilon_k(b) + \varepsilon_{n-1}(b) + \varepsilon_n(b) \ge l.$$

Proof of Theorem 1.5.3.
Let $\bar{b}$ be the unique element in $B^{n,l}$ such that $\iota^{(0,n)*}(\bar{b})$ is the lowest element in the connected component of B' that contains $\iota^{(0,n)*}(b)$. This is obtained by

$$\begin{aligned}
\iota^{(0)*}(b) &\xRightarrow{n-1\; r_{1\,n-1}} b(1, n-1) \xRightarrow{n-2\; r_{2\,n-2}} \cdots \xRightarrow{2\; r_{1,2}} b(1,2) \xRightarrow{1\; r_{1,1}} b(1,1) \\
&\xRightarrow{n-1\; r_{2\,n-1}} b(2, n-1) \xRightarrow{n-2\; r_{2\,n-2}} \cdots \xRightarrow{2\; r_{2,2}} b(2,2) \\
&\cdots \\
&\xRightarrow{n-1\; r_{n-1\,n-1}} b(n-1, n-1) = \iota^{(0)*}(\underline{b})
\end{aligned}$$

Here $r_{j,i}$ are determined in such a way that for each pair $b_1(i)^r \rightarrow_2$ we have $\tilde{f}_i b_2 = 0$. By Proposition 6.2.2, we have $r_j \leq r_{j+1}$ $(1 \leq j \leq n-2)$ and $\overline{r}_j \leq \overline{r}_{j+1}$ $(1 \leq j \leq n-3)$ where $r_j = r_{n-j\,n-j}$, $\overline{r}_j = r_{n-j-1\,n-j}$. With these r_j and $\overline{r}_j$ set

$$b'' = \tilde{e}_{n-2}^{\overline{r}_{n-2}} \tilde{e}_{n-3}^{\overline{r}_{n-3}} \cdots \tilde{e}_1^{\overline{r}_1} \tilde{e}_{n-1}^{r_{n-1}} \tilde{e}_{n-2}^{r_{n-2}} \cdots \tilde{e}_1^{r_1} \underline{b}.$$

We set $(x''_{j\,k}) = x(b'')$. We have $x'_{j\,n-1} = x''_{j\,n-1}$, $x'_{j\,n} = x''_{j\,n}$, $x'_{j\,\overline{n-1}} = x''_{j\,\overline{n-1}}$

From Proposition 6.6.4 and Proposition 6.6.6, the equality of (6.6.10) holds if $p_{k-1\,k} \geq 0$ $\;(2 \leq k \leq n-2)$, $p_{n-1\,n-1} \geq 0$, $p_{(n,n)} \geq 0$, $p_{k\,k} = 0$ $\;(2 \leq k \leq n-2)$ and $p'_{j\,n} \leq 0$ $\;(4 \leq j \leq n$ and j is even.).

Now, for $1 \leq j \leq [\frac{n-2}{2}]$ we consider the following statements where [ ] is the Gauss's symbol.

$(A)_j$ $\quad p_{i-2j+1\;i} \geq 0\,, p_{i-2j+2\;i} = 0 \quad (2j \leq i \leq n-2)\,,$
$\qquad\; p_{n-2j+2\;n} \geq 0$ and $p_{n-2j+1\;n-1} \geq 0$.

$(B)_j$ $\quad \omega_{i-2j+1\;i} = 0 \quad (2j \leq i \leq n)$ and $\omega_{i-2j+2\;i} = 0 \quad (2j-1 \leq i \leq n)$.

$(C)_j$ $\quad p'_{2j+2\;n} = 0$.

$(D)_j$ $\quad p'_{2j+2\;n} \leq 0$.

$(E)_j$ $\quad r_{n-2j-2} = r_{n-2j-1} = y_{1\;2j+1} - y_{n-2j\;n}$,
$\qquad \overline{r}_{n-2j-2} = (y_{2\;2j+2} - y_{n-2j\;n}) + (y_{1\;2j+2} - y_{1\;2j+1})$,
$\qquad \overline{r}_{n-2j-3} = y_{2\;2j+2} - y_{n-2j\;n}$.

Here, the equality of (6.6.10) holds if and only if $(A)_1$ and $(D)_j$ $\quad (1 \leq j \leq [\frac{n-2}{2}])$.

Let us show the equivalence of
(i) $(B)_j$ $\quad (1 \leq j \leq [\frac{n-2}{2}])$.
(ii) $(A)_1$ and $(D)_j$ $\quad (1 \leq j \leq [\frac{n-2}{2}])$.

Note that $(B)_j$ implies THeorem 1.5.3. Let us derive (i) from (ii). Assume that $(A)_1, \cdots, (A)_j$ and $(C)_{j-1}$ are valid. From $(A)_j$, we obtain

$$\omega_{i-2j+1\,i} \geq 0 \quad (2j \leq i \leq n-2), \tag{6.6.11}$$

$$\omega_{i-2j+2\,i} \geq 0 \quad (2j \leq i \leq n-2), \tag{6.6.12}$$

$$\omega_{n-2j+2\,n} \geq 0, \tag{6.6.13}$$

$$\omega_{n-2j+1\,n-1} \geq 0. \tag{6.6.14}$$

Also, from $p_{i-2j+1\,i} \geq 0$ we obtain

$$\omega_{i-2j+1\,i} \geq (-p_{i-2j\,i})_+ \geq -p_{i-2j\,i} = -\omega_{i-2j\,i} + (-p_{i-2j-1\,i})_+. \tag{6.6.15}$$

This implies

$$\omega_{i-2j+1\,i} + \omega_{i-2j\,i} \geq 0 \quad (2j+1 \leq i \leq n-2). \tag{6.6.16}$$

From (6.6.11) and (6.6.13), we get

$$x_{n-2j+2\,\overline{n-1}} \geq x_{n-2j-1\,n-1} \geq x_{n-2j-2\,n-2} \geq \cdots x_{2\,2j+2} \geq x_{1\,2j+1}. \tag{6.6.17}$$

From (6.6.14),

$$x_{n-2j+1\,\overline{n-1}} \geq x_{n-2j-2\,n-1}. \tag{6.6.18}$$

From (6.6.12),

$$x_{n-2j\,n-1} \geq x_{n-2j-1\,n-2} \geq \cdots \geq x_{2\,2j+1} \geq x_{1\,2j}. \tag{6.6.19}$$

From (6.6.16),

$$x_{n-2j-1\,n-1} + x_{n-2j-2n-1} \geq x_{n-2j-2\,n-2} + x_{n-2j-3\,n-2} \geq \cdots \geq x_{2\,2j+2} + x_{1\,2j+2}. \tag{6.6.20}$$

From (6.6.17)–(6.6.20) we obtain the following,

$$\begin{aligned}
&(6.6.17)' \quad && y_{n-2j\,n-2} - y_{n-2j+2\,n} \geq y_{1\,2j+1} - y_{1\,2j},\\
&(6.6.18)' \quad && y_{n-2j-1\,n-2} - y_{n-2j\,n-1} \geq y_{n-2j-2\,n-1} - y_{n-2j-2\,n-2},\\
&(6.6.19)' \quad && y_{n-2j\,n-1} - y_{n-2j\,n-2} \geq y_{2\,2j+1} - y_{2\,2j},\\
&(6.6.20)' \quad && (y_{n-2j-1\,n-1} - y_{n-2j-1\,n-2}) + (y_{n-2j-2\,n-1} - y_{n-2j-2\,n-2})\\
& && \qquad \geq (y_{2\,2j+2} - y_{2\,2j+1}) + (y_{1\,2j+2} - y_{1\,2j+1}).
\end{aligned}$$

From (6.6.17)'–(6.6.20)', we obtain

$$y_{n-2j-1\,n-1} - y_{n-2j+2\,n} \geq (y_{2\,2j+2} - y_{2\,2j}) + (y_{1\,2j+2} - y_{1\,2j}). \tag{6.6.21}$$

Note that $(E)_0 : r_{n-2} = y_{1\,1} - y_{n\,n}, \overline{r}_{n-2} = (y_{2\,2} - y_{n-2\,n-2}) + (y_{1\,2} - y_{1\,1})$ and $\overline{r}_{n-3} = y_{2\,2} - y_{n\,n}$ holds in any case and also that $(A)_1, \cdots, (A)_j$ imply $(E)_1, \cdots, (E)_j$.

From $(C)_{j-1} : p'_{2j\,n} = 0$ we obtain

$$p'_{2j+2n} = \omega'_{2j+2n} = x'_{2j+2\,\overline{n-1}} - x'_{2j-1\,n-1}.$$

From this and $(D)_j$,

$$x'_{2j+2\,\overline{n-1}} \leq x'_{2j-1\,n-1}. \tag{6.6.22}$$

Similarly to the previous cases we get

$$\begin{aligned}
x'_{2j-1\,n-1} &= (y_{n-2j\,n} - y_{n-2j+2\,n}) - (r_{n-2j+1} - r_{n-2j-1}) - (\overline{r}_{n-2j} - \overline{r}_{n-2j-1}),\\
x'_{2j+2\,\overline{n-1}} &= \overline{r}_{n-2j-1} - \overline{r}_{n-2j-2}.
\end{aligned}$$

Therefore, from (6.6.22) and these we obtain

$$\overline{r}_{n-2j-1} - \overline{r}_{n-2j-2} \leq (y_{n-2j\,n} - y_{n-2j+2\,n}) - (r_{n-2j+1} - r_{n-2j-1}) - (\overline{r}_{n-2j} - \overline{r}_{n-2j-1}). \tag{6.6.23}$$

From (6.6.23), $(E)_{j-1}$ and $(E)_j$

$$(y_{2\,2j+2} - y_{2\,2j}) + (y_{1\,2j+2} - y_{1\,2j}) \geq y_{n-2j\,n} - y_{n-2j+2\,n}. \tag{6.6.24}$$

From (6.6.21) and (6.6.24) we have the equalities (6.6.11–22) and (6.6.17)'–(6.6.20)'. Hence, we obtain $(B)_1, \cdots, (B)_j$. In particular, from the equalities (6.6.22) we get $\omega_{2j+2n} = 0$. From this and $(C)_{j-1}$, $p'_{2j+2\,n} = -(-p'_{2j\,n})_+ = 0$. Hence, we obtain $(C)_j$. From $\omega_{n-2j+2\,n} = 0$ in $(B)_j$, $p_{n-2j+2\,n} \geq 0$ in $(A)_j$ and $p_{n-2j+2\,n} = \omega_{n-2j+2\,n} - (-p_{n-2j\,n})_+$ we obtain $p_{n-2j\,n} \geq 0$. Similarly we obtain $p_{n-2j-1\,n} \geq 0$.

Therefore, we obtain $(A)_{j+1}$ from $(A)_j$ and $(B)_j$. Thus, we proved (i) by assuming (ii).

Next, we show that (i) implies (ii). Note that (i) means $\omega_{j\,i} = 0 \quad (1 \le j \le i \le n)$. Also note that $p_{1\,i} \ge 0 \quad (1 \le i \le n)$, $p_{2\,n-1} \ge 0$ and $p_{2\,n} \ge 0$. From these we can show $p_{j\,i} = 0 \quad (2 \le j \le i \le n)$ inductively. In particular, we have $(A)_j \quad (1 \le j \le [\frac{n-2}{2}])$. As noted before, $(A)_1, \cdots, (A)_j$ imply $(E)_1, \cdots, (E)_j$. Therefore, we get

$$\begin{aligned} p'_{2j+2\,n} \le \omega'_{2j+2,n} &= x'_{2j+2\,\overline{n-1}} - x'_{2j-1\,n-1} \\ &= (y_{n-2j+2\,n} - y_{n-2j\,n}) + (y_{2\,2j+2} - y_{2\,2j}) + (y_{1\,2j+2} - y_{1\,2j}). \end{aligned} \tag{6.6.25}$$

We have

$$y_{n-2j+2\,n} - y_{n-2j\,n} = -x_{n-2j+1\,\overline{n}} - x_{n-2j\,n}, \tag{6.6.26}$$

$$y_{2\,2j+2} - y_{2\,2j} = x_{2\,2j+2} + x_{2\,2j+1}, \tag{6.6.27}$$

$$y_{1\,2j+2} - y_{1\,2j} = x_{1\,2j+2} + x_{1\,2j+1}. \tag{6.6.28}$$

Note that if $i = n$ or $\overline{n}$, $x_{j\,i} = x_{j-1\,n-1} + x_{j+1\,\overline{n-1}}$. Hence, from this and (6.6.26),

$$y_{n-2j+2\,n} - y_{n-2j\,n} = -x_{n-2j\,n-1} - x_{n-2j+2\,\overline{n-1}} - x_{n-2j-1\,n-1} - x_{n-2j+1\,\overline{n-1}}. \tag{6.6.29}$$

From $(B)_j$,

$$x_{2\,2j+2} + x_{2\,2j+1} = x_{n-2j-1\,n-1} + x_{n-2j\,n-1}, \tag{6.6.30}$$

$$x_{1\,2j+2} + x_{1\,2j+1} = x_{n-2j-2\,n-1} + x_{n-2j-1\,n-1}. \tag{6.6.31}$$

Hence, from (6.6.25)–(6.6.31),

$$\begin{aligned} p'_{2j+2\,n} &\le -(x_{n-2j+2\,\overline{n-1}} - x_{n-2j-1\,n-1}) - (x_{n-2j+1\,\overline{n-1}} - x_{n-2j-2\,n-1}) \\ &= \omega_{n-2j+2\,n} - \omega_{n-2j+1\,\overline{n}} = 0. \end{aligned}$$

Thus, we obtain $(D)_j$. □

Proposition 6.6.8. *$B^{n,l} \otimes B^{n,l}$ is connected.*

The proof is similar to that of Proposition 6.3.10.

Now, we have completed the proof of Theorem 1.5.2 by Theorem 1.5.3, Corollary 6.6.7, Proposition 6.6.8 and Remark 1.5.4. □

6.7. $(A^{(2)}_{2n-1}, B(l\Lambda_1) \quad (n \ge 3))$

We shall use the notations in 1.6. For $i = 1, \cdots, n$, the rule of drawing i-arrow on $B(l\Lambda_1)$ is given in [KN]. From that rule, it is easy to see that

$$\begin{aligned} &\varepsilon_i(b) = \bar{x}_i(b) + (x_{i+1}(b) - \bar{x}_{i+1}(b))_+ \quad \text{for } i = 1, \cdots, n-1, \\ &\varphi_i(b) = x_i(b) + (\bar{x}_{i+1}(b) - x_{i+1}(b))_+ \quad \text{for } i = 1, \cdots, n-1, \\ &\varepsilon_n(b) = \bar{x}_n(b), \quad \varphi_n(b) = x_n(b). \end{aligned} \tag{6.7.1}$$

We define a bijection $\sigma : B(l\Lambda_1) \longrightarrow B(l\Lambda_1)$ by

$$\begin{aligned} x_1(\sigma(b)) &= \bar{x}_1(b), \quad \bar{x}_1(\sigma(b)) = x_1(b), \\ x_i(\sigma(b)) &= x_i(b), \quad \bar{x}_i(\sigma(b)) = \bar{x}_i(b) \quad \text{for } i = 2, \cdots, n. \end{aligned}$$

Then it is straightforward to see that $\sigma^2 = id_{B(l\Lambda_1)}$. Now we define the rule of 0-arrow by

$$\tilde{f}_0(b) = \sigma \tilde{f}_1 \sigma(b). \tag{6.7.2}$$

That is, $b \xrightarrow{0} b'$ if and only if $\sigma(b) \xrightarrow{1} \sigma(b')$. Note that

$$\begin{aligned} \varepsilon_0(b) &= x_1(b) + (x_2(b) - \bar{x}_2(b))_+, \\ \varphi_0(b) &= \bar{x}_1(b) + (\bar{x}_2(b) - x_2(b))_+. \end{aligned} \tag{6.7.3}$$

Now, let us denote by $B^{1,l}$ the crystal $B(l\Lambda_1)$ endowed with 0-arrows defined as above.

Proof of Proposition 1.6.1.

For $b, b' \in B^{1,l}$, by definition, $b \xrightarrow{0} b'$ if and only if $\sigma(b) \xrightarrow{1} \sigma(b')$. It immediately follows from the rule of arrows given in [KN] that $b \xrightarrow{i} b'$ if and only if $\sigma(b) \xrightarrow{i} \sigma(b')$ for $i = 2, \cdots, n$. Therefore we obtain an isomorphism $\iota_0^*(B^{1,l}) \cong \iota_1^*(B^{1,l}) \cong B(l\Lambda_1)$ of crystals for $U_q(C_n)$ induced by the map σ. In particular, $B^{1,l}$ is a crystal for $U_q(\mathfrak{g}_{I\setminus\{1\}})$ of type $U_q(C_n)$. It remains to show the commutativity of the 0-arrow and 1-arrow. For $b \in B^{1,l}$, by the definition of 0-arrow, we observe that if $x_2(b) \geq \bar{x}_2(b)$, $\bar{x}_1(b) \geq 1$, then

$$\begin{aligned} x_2(\tilde{f}_0(b)) &= x_2(b) + 1, \\ \bar{x}_1(\tilde{f}_0(b)) &= \bar{x}_1(b) - 1, \\ x_i(\tilde{f}_0(b)) &= x_i(b) \quad \text{for } i \neq 2, \\ \bar{x}_i(\tilde{f}_0(b)) &= \bar{x}_i(b) \quad \text{for } i \neq 1, \end{aligned}$$

if $x_2(b) \geq \bar{x}_2(b)$, $\bar{x}_1(b) = 0$, then $\tilde{f}_0(b) = 0$, and if $x_2(b) < \bar{x}_2(b)$, then

$$\begin{aligned} x_1(\tilde{f}_0(b)) &= x_1(b) + 1, \\ \bar{x}_2(\tilde{f}_0(b)) &= \bar{x}_2(b) - 1, \\ x_i(\tilde{f}_0(b)) &= x_i(b) \quad \text{for } i \neq 1, \\ \bar{x}_i(\tilde{f}_0(b)) &= \bar{x}_i(b) \quad \text{for } i \neq 2. \end{aligned}$$

On the other hand, by the rule of arrows given in [KN], we have if $x_2(b) \geq \bar{x}_2(b)$, $x_1(b) \geq 1$, then

$$\begin{aligned} x_1(\tilde{f}_1(b)) &= x_1(b) - 1, \\ x_2(\tilde{f}_1(b)) &= x_2(b) + 1, \\ x_i(\tilde{f}_0(1)) &= x_i(b) \quad \text{for } i \neq 1, 2, \\ \bar{x}_i(\tilde{f}_1(b)) &= \bar{x}_i(b) \quad \text{for all } i, \end{aligned}$$

if $x_2(b) \geq \bar{x}_2(b)$, $x_1(b) = 0$, then $\tilde{f}_1(b) = 0$, and if $x_2(b) < \bar{x}_2(b)$, then

$$\begin{aligned} \bar{x}_1(\tilde{f}_1(b)) &= \bar{x}_1(b) + 1, \\ \bar{x}_2(\tilde{f}_1(b)) &= \bar{x}_2(b) - 1, \\ x_i(\tilde{f}_i(b)) &= x_i(b) \quad \text{for all } i, \\ \bar{x}_i(\tilde{f}_1(b)) &= \bar{x}_i(b) \quad \text{for } i \neq 1, 2. \end{aligned}$$

Now it is straightforward to check that $\tilde{f}_0\tilde{f}_1 = \tilde{f}_1\tilde{f}_0$.

To prove the uniqueness of $B^{1,l}$, we need the following proposition.

Proposition 6.7.1. *Let $J' = \{2, \cdots, n\}$ be the index set for the simple roots for $U_q(C_{n-1})$, and define a map $\iota' : J' \to I$ by $\iota'(j) = j$ for $j \in J'$. Then $\iota'^*(B^{1,l})$ splits into a direct sum of mutually distinct crystals for $U_q(C_{n-1})$ with highest weight*

$$(\bar{t}_1 - t_1 - t_2)\Lambda_0 + (t_1 - \bar{t}_1 - t_2)\Lambda_1 + t_2\Lambda_2,$$

where t_1, t_2, $\bar{t}_1$ are nonnegative integers such that $t_1 + t_2 + \bar{t}_1 = l$.

Proof. Let b be an element of $B^{1,l}$ with $x_i(b) = t_i$, $\bar{x}_i(b) = \bar{t}_i$ for $i = 1, \cdots, n$. Then b is a highest weight element of $\iota'^*(B^{1,l})$ if and only if $\bar{t}_2 = 0$, $t_i = \bar{t}_i = 0$ for $i = 3, \cdots, n$. In this case, the weight of b is

$$(\bar{t}_1 - t_1 - t_2)\Lambda_0 + (t_1 - \bar{t}_1 - t_2)\Lambda_1 + t_2\Lambda_2.$$

It is clear that $t_1 + t_2 + \bar{t}_1 = l$, and it is easy to see that there is at most one highest element in $\iota'^*(B^{1,l})$ with a given highest weight. □

Now we prove the uniqueness of $B^{1,l}$.

Theorem 6.7.2. *Let B be a crystal for $U_q(A^{(2)}_{2n-1})$ such that $\iota_0^*(B) \cong B(l\Lambda_1)$ and $\iota_1^*(B) \cong B(l\Lambda_1)$ as crystals for $U_q(C_n)$. Then there exists a unique isomorphism $\psi : B^{1,l} \to B$ as crystals for $U_q(A^{(2)}_{2n-1})$.*

Proof. Let $\psi_0 : \iota_0^*(B^{1,l}) \to \iota_0^*(B)$ and $\psi_1 : \iota_1^*(B^{1,l}) \to \iota_1^*(B)$ be the isomorphisms of crystals for $U_q(C_n)$. Observe that $\iota'^*(B)$ splits into a direct sum of mutually distinct crystals with highest weight for $U_q(C_{n-1})$. Therefore ψ_0 and ψ_1 must coincide on highest weight elements of $U_q(C_{n-1})$, and hence for all elements of $B^{1,l}$. Thus we obtain a unique isomorphism $\psi : B^{1,l} \to B$ as crystals for $U_q(A^{(2)}_{2n-1})$. □

In Section 5, we proved that there exists a finite dimensional irreducible representation V of $U_q(A^{(2)}_{2n-1})$ with crystal base (L, B) such that $\iota_0^*(B) \cong B(l\Lambda_1)$ and $\iota_1^*(B) \cong B(l\Lambda_1)$ as crystals for $U_q(C_n)$. Hence by Theorem 6.7.2, there is a unique isomorphism $B^{1,l} \cong B$ as crystals for $U_q(A^{(2)}_{2n-1})$. Hence, we have completed the proof of Proposition 1.6.1. □

Proposition 6.7.3. *The crystal $B^{1,l} \otimes B^{1,l}$ is connected.*

Proof. Similar to Proposition 6.3.10. ☐

Proof of Theorem 1.6.2 and Theorem 1.6.3.
We first show that $\langle c, \varepsilon(b)\rangle \geq l$ for all $b \in B^{1,l}$. Since $c = h_0 + h_1 + 2h_2 + \cdots + 2h_{n-1} + 2h_n$, we have from (6.7.1) and (6.7.3)

$$\langle c, \varepsilon(b)\rangle = x_1(b) + \bar{x}_1(b) + 2\sum_{i=2}^{n}(\bar{x}_i(b) + (x_i(b) - \bar{x}_i(b))_+). \tag{6.7.4}$$

Set $S_0 = \{j \in J' \mid x_j(b) = \bar{x}_j(b)\}$, $S_1 = \{j \in J' \mid x_j(b) > \bar{x}_j(b)\}$, and $S_2 = \{j \in J' \mid x_j(b) < \bar{x}_j(b)\}$. Then (6.7.4) becomes

$$\begin{aligned}\langle c, \varepsilon(b)\rangle &= x_1(b) + \bar{x}_1(b) + 2\sum_{j\in S_0}\bar{x}_j(b) + 2\sum_{j\in S_1}x_j(b) \\ &\quad + 2\sum_{j\in S_2}\bar{x}_j(b) \\ &\geq x_1(b) + \bar{x}_1(b) + \sum_{j\in S_0}(x_j(b) + \bar{x}_j(b)) \\ &\quad + \sum_{j\in S_1}(x_j(b) + \bar{x}_j(b)) + \sum_{j\in S_2}(x_j(b) + \bar{x}_j(b)) \\ &= \sum_{i=1}^{n} x_i(b) + \sum_{i=1}^{n}\bar{x}_i(b) = l.\end{aligned} \tag{6.7.5}$$

Now let $\Lambda = \sum_{i=0}^{n} k_i\Lambda_i$ be a dominant integral weight of level l, i.e.,

$$\langle \Lambda, c\rangle = k_0 + k_1 + 2k_2 + \cdots + 2k_{n-1} + 2k_n = l. \tag{6.7.6}$$

We will show that there exists a unique element $b \in B^{1,l}$ such that $\varepsilon_i(b) = k_i$ for all $i = 0, \cdots, n$. For existence, we take $b \in B^{1,l}$ with

$$\begin{aligned}&x_1(b) = k_0, \quad \bar{x}_1(b) = k_1, \\ &x_i(b) = \bar{x}_i(b) = k_i \quad \text{for } i = 2, \cdots, n.\end{aligned}$$

Then it is easy to see that $\varepsilon_i(b) = k_i$ for $i = 0, 1, \cdots, n$.

For the uniqueness, let b' be an element of $B^{1,l}$ such that $\varepsilon_i(b') = k_i$ for all $i = 0, 1, \cdots, n$. Then $\langle c, \varepsilon(b')\rangle = l$. In (6.7.5), the equality holds if and only if $S_1 = S_2 = \phi$, i.e., $x_i(b') = \bar{x}_i(b')$ for all $i = 2, \cdots, n$. Hence we have

$$\begin{aligned}&\varepsilon_0(b') = x_1(b') = k_0, \\ &\varepsilon_1(b') = \bar{x}_1(b') = k_1, \\ &\varepsilon_i(b') = x_i(b') = \bar{x}_i(b') = k_i \quad \text{for } i = 2, \cdots, n,\end{aligned}$$

which completes the proof of Theorem 1.6.3 and then by the arguments above, Remark 1.6.4 and Proposition 6.7.3 we have completed the proof of Theorem 1.6.2.
☐

Proposition 6.7.4. *Let Λ and b be as in the proof of Theorem 1.6.2. Then we have*

$$\Lambda' = \Lambda + af(wt(b)) = k_1\Lambda_0 + k_0\Lambda_1 + \sum_{i=2}^{n} k_i\Lambda_i,$$

and the minimal vector b' for Λ' is given by

$$\begin{aligned} &x_1(b') = k_1, \quad \bar{x}_1(b') = k_0, \\ &x_i(b') = \bar{x}_i(b') = k_i \quad \text{for } i = 2, \cdots, n. \end{aligned}$$

Thus the ground-state path of weight Λ is the sequence $(b', b, b', b, b', b, \cdots)$.

6.8. $(B_n^{(1)}, B(l\Lambda_1)\,(n \geq 3))$

First note that the proof of Proposition 1.7.1 is similar to that of Proposition 1.6.1.

We shall use the notations in 1.7. For $i = 1, \cdots, n$, the rule of drawing i-arrow on $B(l\Lambda_1)$ is given in [KN]. From that rule, it is easy to see that

$$\begin{aligned} &\varepsilon_i(b) = \bar{x}_i(b) + (x_{i+1}(b) - \bar{x}_{i+1}(b))_+ \quad \text{for } i = 1, \cdots, n-1, \\ &\varphi_i(b) = x_i(b) + (\bar{x}_{i+1}(b) - x_{i+1}(b))_+ \quad \text{for } i = 1, \cdots, n-1, \\ &\varepsilon_n(b) = 2\bar{x}_n(b) + x_0(b), \quad \varphi_n(b) = 2x_n(b) + x_0(b). \end{aligned} \tag{6.8.1}$$

We define a bijection $\sigma : B(l\Lambda_1) \longrightarrow B(l\Lambda_1)$ by

$$\begin{aligned} &x_0(\sigma(b)) = x(b), \quad x_1(\sigma(b)) = \bar{x}_1(b), \quad \bar{x}_1(\sigma(b)) = x_1(b), \\ &x_i(\sigma(b)) = x_i(b), \quad \bar{x}_i(\sigma(b)) = \bar{x}_i(b) \quad \text{for } i = 2, \cdots, n. \end{aligned}$$

It is easy to see that $\sigma^2 = id_{B(l\Lambda_1)}$. Now we define the rule of 0-arrow by

$$\tilde{f}_0(b) = \sigma\tilde{f}_1\sigma(b). \tag{6.8.2}$$

That is, $b \xrightarrow{0} b'$ if and only if $\sigma(b) \xrightarrow{1} \sigma(b')$. Note that

$$\begin{aligned} &\varepsilon_0(b) = x_1(b) + (x_2(b) - \bar{x}_2(b))_+, \\ &\varphi_0(b) = \bar{x}_1(b) + (\bar{x}_2(b) - x_2(b))_+. \end{aligned} \tag{6.8.3}$$

Let us denote by $B^{1,l}$ the crystal $B(l\Lambda_1)$ endowed with 0-arrows.

Proof of Proposition 1.7.1.

To prove the uniqueness of $B^{1,l}$, we need the following result.

Proposition 6.8.1. *Let $J' = \{2, \cdots, n\}$ be the index set for the simple roots for $U_q(B_{n-1})$, and define a map $\iota' : J' \to I$ by $\iota'(j) = j$ for $j \in J'$. Then $\iota'^*(B^{1,l})$ splits into a direct sum of mutually distinct crystals for $U_q(B_{n-1})$ with highest weight*

$$(\bar{t}_1 - t_1 - t_2)\Lambda_0 + (t_1 - \bar{t}_1 - t_2)\Lambda_1 + t_2\Lambda_2,$$

where t_1, t_2, $\bar{t}_1$ are nonnegative integers such that $t_1 + t_2 + \bar{t}_1 = l$.

Proof. Similar to Proposition 6.7.1. □

Now we prove the uniqueness of $B^{1,l}$.

Theorem 6.8.2. *Let B be a crystal for $U_q(B_n^{(1)})$ such that $\iota_0^*(B) \cong B(l\Lambda_1)$ and $\iota_1^*(B) \cong B(l\Lambda_1)$ as crystals for $U_q(B_n)$. Then there exists a unique isomorphism $\psi : B^{1,l} \to B$ as crystals for $U_q(B_n^{(1)})$.*

Proof. Similar to Proposition 6.7.2 using Proposition 6.8.1 instead of Proposition 6.7.1. □

In Section 5, we proved that there exists a finite dimensional irreducible representation V of $U_q(B_n^{(1)})$ with crystal base (L, B) such that $\iota_0^*(B) \cong B(l\Lambda_1)$ and $\iota_1^*(B) \cong B(l\Lambda_1)$ as crystals for $U_q(B_n)$. Hence by Theorem 6.8.2, there is a unique isomorphism $B^{1,l} \cong B$ as crystals for $U_q(B_n^{(1)})$. Now, we have completed the proof of Proposition 1.7.1. □

Proposition 6.8.3. *The crystal $B^{1,l} \otimes B^{1,l}$ is connected.*

Proof. Similar to Proposition 6.3.10. □

Proof of Theorem 1.7.2 and Theorem 1.7.3.

We first show that $\langle c, \varepsilon(b)\rangle \geq l$ for all $b \in B^{1,l}$. Since $c = h_0 + h_1 + 2h_2 + \cdots + 2h_{n-1} + h_n$, we have from (6.8.1) and (6.8.3),

$$\langle c, \varepsilon(b)\rangle = x_1(b) + \bar{x}_1(b) + 2\sum_{i=2}^{n}(\bar{x}_i(b) + (x_i(b) - \bar{x}_i(b))_+) + x_0(b). \tag{6.8.4}$$

Set $S_0 = \{j \in J' |\ x_j(b) = \bar{x}_j(b)\}$, $S_1 = \{j \in J' |\ x_j(b) > \bar{x}_j(b)\}$, and $S_2 = \{j \in J' |\ x_j(b) < \bar{x}_j(b)\}$. Then by (6.8.4) we have

$$\begin{aligned}\langle c, \varepsilon(b)\rangle &= x_0(b) + x_1(b) + \bar{x}_1(b) + 2\sum_{j\in S_0}\bar{x}_j(b)\\ &\quad + 2\sum_{j\in S_1}x_j(b) + 2\sum_{j\in S_2}\bar{x}_j(b)\\ &\geq x_0(b) + x_1(b) + \bar{x}_1(b) + \sum_{j\in S_0}(x_j(b) + \bar{x}_j(b))\\ &\quad + \sum_{j\in S_1}(x_j(b) + \bar{x}_j(b)) + \sum_{j\in S_2}(x_j(b) + \bar{x}_j(b))\\ &= x_0(b) + \sum_{i=1}^{n}x_i(b) + \sum_{i=1}^{n}\bar{x}_i(b) = l.\end{aligned} \tag{6.8.5}$$

Now let $\Lambda = \sum_{i=0}^{n} k_i\Lambda_i$ be a dominant integral weight of level l, i.e.,

$$\langle \Lambda, c\rangle = k_0 + k_1 + 2k_2 + \cdots + 2k_{n-1} + k_n = l. \tag{6.8.6}$$

We will show that there exists a unique element $b \in B^{1,l}$ such that $\varepsilon_i(b) = k_i$ for all $i = 0, \cdots, n$. For existence, we take $b \in B^{1,l}$ with

$$\begin{aligned} x_0(b) &= 0 \quad \text{if } k_n \text{ is even,} \\ &= 1 \quad \text{if } k_n \text{ is odd,} \\ x_1(b) &= k_0, \quad \bar{x}_1(b) = k_1, \\ x_i(b) &= \bar{x}_i(b) = k_i \quad \text{for } i = 2, \cdots, n-1, \\ x_n(b) &= \bar{x}_n(b) = \frac{k_n - x_0(b)}{2}. \end{aligned}$$

The proof of the uniqueness is similar to the argument in the proof of Theorem 1.6.2. Hence, we have completed the proof of Theorem 1.7.3 and then that of Theorem 1.7.2 by the arguments above, Remark 1.7.4 and Proposition 6.8.3. □

Proposition 6.8.4. *Let Λ and b be as in the proof of Theorem 1.7.2. Then we have*

$$\Lambda' = \Lambda + af(wt(b)) = k_1\Lambda_0 + k_0\Lambda_1 + \sum_{i=2}^{n} k_i\Lambda_i,$$

and the minimal vector b' for Λ' is given by

$$\begin{aligned} x_1(b') &= k_1, \quad \bar{x}_1(b') = k_0, \\ x_i(b') &= \bar{x}_i(b') = k_i \quad \text{for } i = 2, \cdots, n. \end{aligned}$$

Thus the ground-state path of weight Λ is the sequence $(b', b, b', b, b', b, \cdots)$.

6.9. $(D_n^{(1)}, B(l\Lambda_1)\,(n \geq 4))$

First note that the proof of Proposition 1.8.1 is similar to that of Proposition 1.6.1.

We shall use the notations in 1.8. For $i = 1, \cdots, n$, the rule of drawing i-arrow on $B(l\Lambda_1)$ is given in [KN]. From that rule, it is easy to see that

$$\begin{aligned} \varepsilon_i(b) &= \bar{x}_i(b) + (x_{i+1}(b) - \bar{x}_{i+1}(b))_+ \quad \text{for } i = 1, \cdots, n-2, \\ \varphi_i(b) &= x_i(b) + (\bar{x}_{i+1}(b) - x_{i+1}(b))_+ \quad \text{for } i = 1, \cdots, n-2, \\ \varepsilon_{n-1}(b) &= \bar{x}_{n-1}(b) + x_n(b), \quad \varphi_{n-1}(b) = x_{n-1}(b) + \bar{x}_n(b), \\ \varepsilon_n(b) &= \bar{x}_{n-1}(b) + \bar{x}_n(b), \quad \varphi_n(b) = x_{n-1}(b) + x_n(b). \end{aligned} \tag{6.9.1}$$

We define a bijection $\sigma : B(l\Lambda_1) \to B(l\Lambda_1)$ by

$$\begin{aligned} x_1(\sigma(b)) &= \bar{x}_1(b), \quad \bar{x}_1(\sigma(b)) = x_1(b), \\ x_i(\sigma(b)) &= x_i(b), \quad \bar{x}_i(\sigma(b)) = \bar{x}_i(b) \quad \text{for } i = 2, \cdots, n. \end{aligned}$$

It is easy to see that $\sigma^2 = id_{B(l\Lambda_1)}$. Now we define the rule of 0-arrow by

$$\tilde{f}_0(b) = \sigma\tilde{f}_1\sigma(b). \tag{6.9.2}$$

That is, $b\xrightarrow{0}b'$ if and only if $\sigma(b)\xrightarrow{1}\sigma(b')$. Note that

$$\begin{aligned}\varepsilon_0(b) &= x_1(b) + (x_2(b) - \bar{x}_2(b))_+,\\ \varphi_0(b) &= \bar{x}_1(b) + (\bar{x}_2(b) - x_2(b))_+.\end{aligned} \tag{6.9.3}$$

Let us denote by $B^{1,l}$ the crystal $B(l\Lambda_1)$ endowed with 0-arrows.

Proof of Proposition 1.8.1.
For the uniqueness of $B^{1,l}$, we need the following result.

Proposition 6.9.1. *Let $J' = \{2, \cdots, n\}$ be the index set for the simple roots for $U_q(D_{n-1})$, and define a map $\iota' : J' \to I$ by $\iota'(j) = j$ for $j \in J'$. Then $\iota'^*(B^{1,l})$ splits into a direct sum of mutually distinct crystals for $U_q(D_{n-1})$ with highest weight*

$$(\bar{t}_1 - t_1 - t_2)\Lambda_0 + (t_1 - \bar{t}_1 - t_2)\Lambda_1 + t_2\Lambda_2,$$

where t_1, t_2, $\bar{t}_1$ are nonnegative integers such that $t_1 + t_2 + \bar{t}_1 = l$.

Proof. Similar to Proposition 6.7.1. □

Now we have the uniqueness of $B^{1,l}$.

Theorem 6.9.2. *Let B be a crystal for $U_q(D_n^{(1)})$ such that $\iota_0^*(B) \cong B(l\Lambda_1)$ and $\iota_1^*(B) \cong B(l\Lambda_1)$ as crystals for $U_q(D_n)$. Then there exists a unique isomorphism $\psi : B^{1,l} \to B$ as crystals for $U_q(D_n^{(1)})$.*

Proof. Similar to Proposition 6.7.2 using Proposition 6.9.1 instead of Proposition 6.7.1. □

In Section 5, we proved that there exists a finite dimensional irreducible representation V of $U_q(D_n^{(1)})$ with crystal base (L, B) such that $\iota_0^*(B) \cong B(l\Lambda_1)$ and $\iota_1^*(B) \cong B(l\Lambda_1)$ as crystals for $U_q(D_n)$. Hence by Theorem 6.9.2, there is a unique isomorphism $B^{1,l} \cong B$ as crystals for $U_q(D_n^{(1)})$. Now, we have completed the proof of Proposition 1.8.1. □

Proposition 6.9.3. *The crystal $B^{1,l} \otimes B^{1,l}$ is connected.*

Proof. Similar to Proposition 6.3.10. □

Proof of Theorem 1.8.2 and Theorem 1.8.3.
We first show that $\langle c, \varepsilon(b)\rangle \geq l$ for all $b \in B^{1,l}$. Since $c = h_0 + h_1 + 2h_2 + \cdots + 2h_{n-2} + h_{n-1} + h_n$, we have from (6.9.1) and (6.9.3),

$$\begin{aligned}\langle c, \varepsilon(b)\rangle =& x_1(b) + \bar{x}_1(b) + 2\sum_{i=2}^{n-1}(\bar{x}_i(b) + (x_i(b) - \bar{x}_i(b))_+)\\ &+ x_n(b) + \bar{x}_n(b).\end{aligned} \tag{6.9.4}$$

Set $S_0 = \{j = 2, \cdots, n-1 \mid x_j(b) = \bar{x}_j(b)\}$, $S_1 = \{j = 2, \cdots, n-1 \mid x_j(b) > \bar{x}_j(b)\}$, and $S_2 = \{j = 2, \cdots, n-1 \mid x_j(b) < \bar{x}_j(b)\}$. Then by (6.9.4) we have

$$\begin{aligned}\langle c, \varepsilon(b)\rangle &= x_1(b) + \bar{x}_1(b) + 2\sum_{j\in S_0} \bar{x}_j(b) + 2\sum_{j\in S_1} x_j(b)\\ &+ 2\sum_{j\in S_2} \bar{x}_j(b) + x_n(b) + \bar{x}_n(b)\\ &\geq x_1(b) + \bar{x}_1(b) + \sum_{j\in S_0}(x_j(b) + \bar{x}_j(b)) + \sum_{j\in S_1}(x_j(b) + \bar{x}_j(b))\\ &+ \sum_{j\in S_2}(x_j(b) + \bar{x}_j(b)) + x_n(b) + \bar{x}_n(b)\\ &= \sum_{i=1}^{n} x_i(b) + \sum_{i=1}^{n} \bar{x}_i(b) = l.\end{aligned} \tag{6.9.5}$$

Now let $\Lambda = \sum_{i=0}^{n} k_i\Lambda_i$ be a dominant integral weight of level l, i.e.,

$$\langle \Lambda, c\rangle = k_0 + k_1 + 2k_2 + \cdots + 2k_{n-2} + k_{n-1} + k_n = l. \tag{6.9.6}$$

We will show that there exists a unique element $b \in B^{1,l}$ such that $\varepsilon_i(b) = k_i$ for all $i = 0, \cdots, n$. For existence, we take $b \in B^{1,l}$ with

$$\begin{aligned} x_1(b) &= k_0, \quad \bar{x}_1(b) = k_1,\\ x_i(b) &= \bar{x}_i(b) = k_i \quad \text{for } i = 2, \cdots, n-2,\\ x_{n-1}(b) &= \bar{x}_{n-1}(b) = \min(k_{n-1}, k_n)\\ x_n(b) &= (k_{n-1} - k_n)_+,\\ \bar{x}_n(b) &= (k_n - k_{n-1})_+.\end{aligned}$$

The proof of the uniqueness is similar to the argument in the proof of Theorem 1.6.2. Hence, we have completed the proof of Theorem 1.8.3 and then that of Theorem 1.8.2 by the arguments above, Remark 1.8.4 and Proposition 6.9.3. □

Proposition 6.9.4. *Let Λ and b be as in the proof of Theorem 1.8.2. Then we have*

$$\Lambda' = \Lambda + af(wt(b)) = k_1\Lambda_0 + k_0\Lambda_1 + \sum_{i=2}^{n-2} k_i\Lambda_i + k_n\Lambda_{n-1} + k_{n-1}\Lambda_n,$$

and the minimal vector b' for Λ' is given by

$$\begin{aligned} x_1(b') &= k_1, \quad \bar{x}_1(b') = k_0,\\ x_i(b') &= \bar{x}_i(b') = k_i \quad \textit{for } i = 2, \cdots, n-2,\\ x_{n-1}(b) &= \bar{x}_{n-1}(b) = min(k_{n-1}, k_n)\\ x_n(b) &= (k_n - k_{n-1})_+,\\ \bar{x}_n(b) &= (k_{n-1} - k_n)_+.\end{aligned}$$

Thus the ground-state path of weight Λ *is the sequence* $(b', b, b', b, b', b, \cdots)$.

6.10. $(D^{(2)}_{n+1}, B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)\,(n \geq 2))$

We shall use the notations in 1.9. For $i = 1, \cdots, n$, the rule of drawing i-arrow on $\tilde{B}$ is given in [KN]. From that rule, for $b \in \tilde{B}$ and $i = 1, \cdots, n-1$, we observe that if $x_{i+1}(b) \geq \bar{x}_{i+1}(b)$, $x_i(b) \geq 1$, then

$$\begin{aligned} x_i(\tilde{f}_i(b)) &= x_i(b) - 1, \\ x_{i+1}(\tilde{f}_i(b)) &= x_{i+1}(b) + 1, \\ x_j(\tilde{f}_i(b)) &= x_j(b) \text{ for } j \neq i, i+1, \\ \bar{x}_j(\tilde{f}_i(b)) &= \bar{x}_j(b) \text{ for } j = 1, \cdots, n, \end{aligned} \tag{6.10.1}$$

if $x_{i+1}(b) \geq \bar{x}_{i+1}(b)$, $x_i(b) = 0$, then $\tilde{f}_i(b) = 0$, and if $x_{i+1}(b) < \bar{x}_{i+1}(b)$, then

$$\begin{aligned} \bar{x}_i(\tilde{f}_i(b)) &= \bar{x}_i(b) + 1, \\ \bar{x}_{i+1}(\tilde{f}_i(b)) &= \bar{x}_{i+1}(b) - 1, \\ \bar{x}_j(\tilde{f}_i(b)) &= \bar{x}_j(b) \text{ for } j \neq i, i+1, \\ x_j(\tilde{f}_i(b)) &= x_j(b) \text{ for } j = 0, 1, \cdots, n. \end{aligned} \tag{6.10.2}$$

We also note that if $x_0(b) = 1$, then

$$\begin{aligned} x_0(\tilde{f}_n(b)) &= 0, \\ \bar{x}_n(\tilde{f}_n(b)) &= \bar{x}_n(b) + 1, \\ \bar{x}_i(\tilde{f}_n(b)) &= \bar{x}_i(b) \text{ for } i = 1, \cdots n-1, \\ x_i(\tilde{f}_n(b)) &= x_i(b) \text{ for } i = 1, \cdots, n, \end{aligned} \tag{6.10.3}$$

if $x_0(b) = 0$, $x_n(b) \geq 1$, then

$$\begin{aligned} x_0(\tilde{f}_n(b)) &= 1, \\ x_n(\tilde{f}_n(b)) &= x_n(b) - 1, \\ x_i(\tilde{f}_n(b)) &= x_i(b) \text{ for } i = 1, \cdots, n-1, \\ \bar{x}_i(\tilde{f}_n(b)) &= \bar{x}_i(b) \text{ for } i = 1, \cdots, n, \end{aligned} \tag{6.10.4}$$

and if $x_0(b) = 0$, $x_n(b) = 0$, then $\tilde{f}_n(b) = 0$. It easily follows that for $i = 1, \cdots, n-1$,

$$\begin{aligned} \varepsilon_i(b) &= \bar{x}_i(b) + (x_{i+1}(b) - \bar{x}_{i+1}(b))_+, \\ \varphi_i(b) &= x_i(b) + (\bar{x}_{i+1}(b) - x_{i+1}(b))_+, \end{aligned} \tag{6.10.5}$$

and

$$\begin{aligned} \varepsilon_n(b) &= 2\bar{x}_n(b) + x_0(b), \\ \varphi_n(b) &= 2x_n(b) + x_0(b). \end{aligned} \tag{6.10.6}$$

We define a bijection $\sigma : \tilde{B} \to \tilde{B}$ as follows. Let $b = (b_k)_{k=1}^{j} \in B(j\Lambda_1)$. For $i = 1, \cdots, n-1$, we define

$$\begin{aligned} x_i(\sigma(b)) &= (\bar{x}_{n-i+1}(b) - x_{n-i+1}(b))_+ + \min(x_{n-i}(b), \bar{x}_{n-i}(b)), \\ \bar{x}_i(\sigma(b)) &= (x_{n-i+1}(b) - \bar{x}_{n-i+1}(b))_+ + \min(x_{n-i}(b), \bar{x}_{n-i}(b)). \end{aligned} \tag{6.10.7}$$

We also define

$$\begin{aligned} x_0(\sigma(b)) &= 0 \quad \text{if } l - s(b) \quad \text{is even,} \\ &= 1 \quad \text{if } l - s(b) \quad \text{is odd,} \end{aligned} \tag{6.10.8}$$

$$\begin{aligned} x_n(\sigma(b)) &= \left[\frac{l - s(b)}{2}\right] + (\bar{x}_1(b) - x_1(b))_+, \\ \bar{x}_n(\sigma(b)) &= \left[\frac{l - s(b)}{2}\right] + (x_1(b) - \bar{x}_1(b))_+, \end{aligned} \tag{6.10.9}$$

where $[x]$ denotes the greatest integer $\leq x$. Then it is straightforward to see that $\sigma^2 = id_{B^{1,l}}$. Note that $s(\sigma(b)) = l - t(b)$, where $t(b) = x_0(b) + 2\min(x_n(b), \bar{x}_n(b))$.

Now we define the rule of 0-arrow by

$$\tilde{f}_0(b) = \sigma \tilde{f}_n \sigma(b). \tag{6.10.10}$$

That is, $b \xrightarrow{0} b'$ if and only if $\sigma(b) \xrightarrow{n} \sigma(b')$. Observe that if $s(b) \leq l-1$ and $x_1(b) \geq \bar{x}_1(b)$, then

$$\begin{aligned} x_1(\tilde{f}_0(b)) &= x_1(b) + 1, \\ x_i(\tilde{f}_0(b)) &= x_i(b) \text{ for } i = 2, \cdots, n, \\ \bar{x}_i(\tilde{f}_0(b)) &= \bar{x}_i(b) \text{ for } i = 1, \cdots, n, \\ x_0(\tilde{f}_0(b)) &= x_0(b), \end{aligned} \tag{6.10.11}$$

if $s(b) = l$, $x_1(b) \geq \bar{x}_1(b)$, then $\tilde{f}_0(b) = 0$, and if $x_1(b) < \bar{x}_1(b)$, then

$$\begin{aligned} x_i(\tilde{f}_0(b)) &= x_i(b) \text{ for } i = 0, 1, \cdots, n, \\ \bar{x}_1(\tilde{f}_0(b)) &= \bar{x}_1(b) - 1, \\ \bar{x}_i(\tilde{f}_0(b)) &= \bar{x}_i(b) \text{ for } i = 2, \cdots, n. \end{aligned} \tag{6.10.12}$$

Thus we have

$$\begin{aligned} \varepsilon_0(b) &= l - s(b) + 2(x_1(b) - \bar{x}_1(b))_+, \\ \varphi_0(b) &= l - s(b) + 2(\bar{x}_1(b) - x_1(b))_+. \end{aligned} \tag{6.10.13}$$

Let us denote by $B^{1,l}$ the crystal $\tilde{B}$ endowed with 0-arrows defined as above.

Proof of Proposition 1.9.1.

By (6.10.1)-(6.10.4) and (6.10.7)-(6.10.9), it is straightforward to check that $b \xrightarrow{i} b'$ if and only if $\sigma(b) \xrightarrow{n-i} \sigma(b')$ for $i = 1, \cdots, n-1$. It is immediate from the definition that $b \xrightarrow{0} b'$ if and only if $\sigma(b) \xrightarrow{n} \sigma(b')$. Therefore we obtain an isomorphism of crystals for $U_q(B_n)$ induced by the map σ :

$$\iota_0^*(B^{1,l}) \cong \iota_n^*(B^{1,l}) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1).$$

In particular, $B^{1,l}$ is a crystal for $U_q(\mathfrak{g}_{I\setminus\{n\}})$ of type $U_q(B_n)$. By (6.10.3), (6.10.4), and (6.10.11)-(6.10.13), it is easy to check that $\tilde{f}_0\tilde{f}_n = \tilde{f}_n\tilde{f}_0$.

Proposition 6.10.1. *Let $J' = \{1,\cdots,n-1\}$ be the index set for the simple roots for $U_q(\mathfrak{sl}(n))$, and define a map $\iota' : J' \to I$ by $\iota'(j) = j$ for $j \in J'$. Then $\iota'^*(B^{1,l})$ splits into a direct sum of crystals for $U_q(\mathfrak{sl}(n))$ with highest weight*

$$-2t_1\Lambda_0 + t_1\Lambda_1 + (\bar{t}_n - t_n)\Lambda_{n-1} + 2(t_n - \bar{t}_n)\Lambda_n,$$

where t_1, t_n, $\bar{t}_n$ are nonnegative integers such that $t_n \le \bar{t}_n$, $t_1 + t_n + \bar{t}_n \le l$.

Proof. Let b be an element of $B^{1,l}$ with $x_i(b) = t_i$, $\bar{x}_i(b) = \bar{t}_i$, $(i = 1,\cdots,n)$ and $x_0(b) = t_0$. Then b is a highest weight element of $\iota'^*(B^{1,l})$ if and only if $\bar{t}_1 = 0$, $t_n \le \bar{t}_n$, and $t_i = \bar{t}_i = 0$ for $i = 1,\cdots,n-1$. In this case, the weight of b is

$$-2t_1\Lambda_0 + t_1\Lambda_1 + (\bar{t}_n - t_n)\Lambda_{n-1} + 2(t_n - \bar{t}_n)\Lambda_n.$$

It is clear that $t_1 + t_n + \bar{t}_n \le l$. □

Now we prove the uniqueness of $B^{1,l}$.

Theorem 6.10.2. *Let B be a crystal for $U_q(D_{n+1}^{(2)})$ such that*

$$\iota_0^*(B) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

and

$$\iota_n^*(B) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

as crystals for $U_q(B_n)$. Then there exists a unique isomorphism $\psi : B^{1,l} \to B$ as crystals for $U_q(D_{n+1}^{(2)})$.

Let $\psi_0 : \iota_0^*(B^{1,l}) \to \iota_0^*(B)$ and $\psi_n : \iota_n^*(B^{1,l}) \to \iota_n^*(B)$ be the isomorphisms of crystals for $U_q(B_n)$. We wish to show that ψ_0 and ψ_n coincide on every element of $B^{1,l}$. We first prove the following lemma.

Lemma 6.10.3. *Let $b = (b_k)_{k=1}^{j}$ be an element of $B^{1,l}$ such that $b_k = 1$ for all $k = 1,\cdots,j$. Then $\psi_n(b) = \psi_0(b)$.*

Proof. We use a downward induction on j. If $j = l$, it is obvious. Suppose $j < l$. Since b is a highest weight element of $\iota'^*(B^{1,l})$ with highest weight $-2j\Lambda_0 + j\Lambda_1$, $\psi_n(b)$ is also a highest weight element of $\iota'^*(B)$ with the same highest weight. Therefore $\psi_n(b) = \psi_0(b')$ for some highest weight element b' of $\iota'^*(B^{1,l})$ with highest weight $-2j\Lambda_0 + j\Lambda_1$. Thus by Proposition 6.10.1, b' has the form $x_1(b') = j$, $x_i(b') = \bar{x}_i(b') = 0$, for $i = 1,\cdots,n-1$, $x_n(b') = \bar{x}_n(b') = t$, and $x_0(b') = t_0$, where t is a nonnegative integer, $t_0 = 0$ or 1, and $j + 2t + t_0 \le l$.

Note that $b' = \tilde{f}_n^{2t+t_0}\cdots\tilde{f}_1^{2t+t_0}b_0$, where $b_0 = (b_{0,k})_{k=1}^{s(b')}$ is an element of B such that $b_{0,k} = 1$ for all $k = 1,\cdots,j+2t+t_0$. Then we have

$$\begin{aligned} \psi_n(b) &= \psi_0(b') = \psi_0(\tilde{f}_n^{2t+t_0}\cdots\tilde{f}_1^{2t+t_0}b_0) \\ &= \tilde{f}_n^{2t+t_0}\cdots\tilde{f}_1^{2t+t_0}\psi_0(b_0) = \tilde{f}_n^{2t+t_0}\cdots\tilde{f}_1^{2t+t_0}\psi_n(b_0) \\ &= \tilde{f}_n^{2t+t_0}\cdots\tilde{f}_2^{2t+t_0}\psi_n(\tilde{f}_1^{2t+t_0}b_0). \end{aligned} \tag{6.10.14}$$

Suppose that $2t+t_0>0$. Observe that

$$\tilde{f}_0^{l-j-(2t+t_0)+1}\psi_n(b)=\psi_n(\tilde{f}_0^{l-j-(2t+t_0)+1}b)\neq 0.$$

On the other hand,

$$\begin{aligned}\tilde{f}_0^{l-j-(2t+t_0)+1}\psi_0(b')&=\tilde{f}_0^{l-j-(2t+t_0)+1}\tilde{f}_n^{2t+t_0}\cdots\tilde{f}_2^{2t+t_0}\psi_n(\tilde{f}_1^{2t+t_0}b_0)\\&=\tilde{f}_n^{2t+t_0}\cdots\tilde{f}_2^{2t+t_0}\tilde{f}_0^{l-j-(2t+t_0)+1}\psi_n(\tilde{f}_1^{2t+t_0}b_0)\\&=\tilde{f}_n^{2t+t_0}\cdots\tilde{f}_2^{2t+t_0}\psi_n(\tilde{f}_0^{l-j-(2t+t_0)+1}\tilde{f}_1^{2t+t_0}b_0)\\&=0,\end{aligned}$$

which is a contradiction. Therefore $2t+t_0=0$, and hence $b'=b$. □

Proof of Theorem 6.10.2.
Let b be a highest element of $\iota'(B^{1,l})$. Then, by Proposition 6.10.1, $\bar{x}_1(b)=0$, $x_i(b)=\bar{x}_i(b)=0$ for $i=2,\cdots,n-1$. Set $x_1(b)=t_1$, $x_n(b)=t_n$, $\bar{x}_n(b)=\bar{t}_n$, and $x_0(b)=t_0$. It is clear that $t_1+t_n+t_0+\bar{t}_n\leq l$, and

$$wt(b)=-2t_1\Lambda_0+t_1\Lambda_1+(\bar{t}_n-t_n)\Lambda_{n-1}+2(t_n-\bar{t}_n)\Lambda_n.$$

By the same argument of Lemma 6.10.3, $\psi_n(b)=\psi_0(b')$ for some highest weight element b' of $\iota'^*(B^{1,l})$ with the same highest weight. Set $x_1(b')=s_1$, $x_n(b')=s_n$, $\bar{x}_n(b')=\bar{s}_n$, and $x_0(b')=s_0$. Then $s_1=t_1$ and $\bar{s}_n-s_n=\bar{t}_n-t_n$.

Note that $b'=\tilde{f}_n^{2\bar{s}_n+s_0}\tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n}\cdots\tilde{f}_1^{s_n+s_0+\bar{s}_n}b_0$, where $b_0=(b_{0,k})_{k=1}^{s(b')}$ is an element of $B^{1,l}$ with $b_{0,k}=1$ for all $k=1,\cdots,s(b')$. Therefore we have

$$\begin{aligned}\psi_n(b)&=\psi_0(b')\\&=\psi_0(\tilde{f}_n^{2\bar{s}_n+s_0}\tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n}\cdots\tilde{f}_1^{s_n+s_0+\bar{s}_n}b_0)\\&=\tilde{f}_n^{2\bar{s}_n+s_0}\tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n}\cdots\tilde{f}_1^{s_n+s_0+\bar{s}_n}\psi_0(b_0)\\&=\tilde{f}_n^{2\bar{s}_n+s_0}\tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n}\cdots\tilde{f}_1^{s_n+s_0+\bar{s}_n}\psi_n(b_0)\\&=\tilde{f}_n^{2\bar{s}_n+s_0}\tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n}\cdots\tilde{f}_2^{s_n+s_0+\bar{s}_n}\psi_n(\tilde{f}_1^{s_n+s_0+\bar{s}_n}b_0).\end{aligned}\tag{6.10.15}$$

If $s_n+s_0+\bar{s}_n>t_n+t_0+\bar{t}_n$, we have

$$\tilde{f}_0^{l-(s_1+s_n+s_0+\bar{s}_n)+1}\psi_n(b)=\psi_n(\tilde{f}_0^{l-(s_1+s_n+s_0+\bar{s}_n)+1}b)\neq 0.$$

On the other hand,

$$\begin{aligned}&\tilde{f}_0^{l-(s_1+s_n+s_0+\bar{s}_n)+1}\psi_0(b')\\&=\tilde{f}_0^{l-(s_1+s_n+s_0+\bar{s}_n)+1}\tilde{f}_n^{2\bar{s}_n+s_0}\tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n}\cdots\tilde{f}_2^{s_n+s_0+\bar{s}_n}\psi_n(\tilde{f}_1^{s_n+s_0+\bar{s}_n}b_0)\\&=\tilde{f}_n^{2\bar{s}_n+s_0}\tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n}\cdots\tilde{f}_2^{s_n+s_0+\bar{s}_n}\tilde{f}_0^{l-(s_1+s_n+s_0+\bar{s}_n)+1}\psi_n(\tilde{f}_1^{s_n+s_0+\bar{s}_n}b_0)\\&=\tilde{f}_n^{2\bar{s}_n+s_0}\tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n}\cdots\tilde{f}_2^{s_n+s_0+\bar{s}_n}\psi_n(\tilde{f}_0^{l-(s_1+s_n+s_0+\bar{s}_n)+1}\tilde{f}_1^{s_n+s_0+\bar{s}_n}b_0)\\&=0,\end{aligned}\tag{6.10.16}$$

which is a contradiction.

If $s_n + s_0 + \bar{s}_n < t_n + t_0 + \bar{t}_n$, we have

$$\tilde{f}_0^{l-(t_1+t_n+t_0+\bar{t}_n)+1} \psi_n(b) = \psi_n(\tilde{f}_0^{l-(t_1+t_n+t_0+\bar{t}_n)+1} b) = 0.$$

But, since $\tilde{f}_0^s \tilde{f}_1^t b_0 = \tilde{f}_1^t \tilde{f}_0^s b_0$ for $0 \le s \le l - s(b_0)$, $0 \le t \le s(b_0)$,

$$\begin{aligned} &\tilde{f}_0^{l-(t_1+t_n+t_0+\bar{t}_n)+1} \psi_0(b') \\ (6.10.17) &= \tilde{f}_n^{2\bar{s}_n+s_0} \tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n} \cdots \tilde{f}_2^{s_n+s_0+\bar{s}_n} \psi_n(\tilde{f}_0^{l-(t_1+t_n+t_0+\bar{t}_n)+1} \tilde{f}_1^{s_n+s_0+\bar{s}_n} b_0) \\ &= \tilde{f}_n^{2\bar{s}_n+s_0} \tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n} \cdots \tilde{f}_2^{s_n+s_0+\bar{s}_n} \psi_n(\tilde{f}_1^{s_n+s_0+\bar{s}_n} b''), \end{aligned}$$

where $b'' = (b''_k)_{k=1}^{l+s(b')-s(b)+1}$ is a highest weight element of $\iota'^*(B^{1,l})$ such that $b''_k = 1$ for all k. Hence (6.10.17) is the same as

$$\begin{aligned} & \tilde{f}_n^{2\bar{s}_n+s_0} \tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n} \cdots \tilde{f}_1^{s_n+s_0+\bar{s}_n} \psi_n(b'') \\ (6.10.18) \qquad &= \tilde{f}_n^{2\bar{s}_n+s_0} \tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n} \cdots \tilde{f}_1^{s_n+s_0+\bar{s}_n} \psi_0(b'') \\ &= \psi_0(\tilde{f}_n^{2\bar{s}_n+s_0} \tilde{f}_{n-1}^{s_n+s_0+\bar{s}_n} \cdots \tilde{f}_1^{s_n+s_0+\bar{s}_n} b'') \neq 0, \end{aligned}$$

which also gives a contradiction.

Therefore we must have $s_n + s_0 + \bar{s}_n = t_n + t_0 + \bar{t}_n$, which implies $b = b'$. Thus ψ_0 and ψ_n coincide with each other on every highest weight element of $\iota'^*(B^{1,l})$, and hence on every element of $B^{1,l}$, which completes the proof. □

In Section 5, we proved that there exists a finite dimensional representation V of $U_q(D_{n+1}^{(2)})$ with crystal base (L, B) such that

$$\iota_0^*(B) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

and

$$\iota_n^*(B) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

as crystals for $U_q(B_n)$. Therefore by Theorem 6.10.2, there is a unique isomorphism $B^{1,l} \cong B$ as crystals for $U_q(D_{n+1}^{(2)})$. Now, we have completed the proof of Proposition 1.9.1. □

Proposition 6.10.4. *The crystal $B^{1,l} \otimes B^{1,l}$ is connected.*

Proof. Since each element is connected to a highest weight element of $\iota_0^*(B^{1,l} \otimes B^{1,l})$, it suffices to prove that all the highest weight elements of $\iota_0^*(B^{1,l} \otimes B^{1,l})$ are connected to each other. Let b_0 be the element of $B^{1,l}$ given by

$$\begin{gathered} x_0(b_0) = 0, \quad x_1(b_0) = l, \quad \bar{x}_1(b_0) = 0, \\ x_i(b_0) = \bar{x}_i(b_0) = 0 \quad \text{for } i = 2, \cdots, n. \end{gathered}$$

We will show that all the highest weight elements of $\iota_0^*(B^{1,l} \otimes B^{1,l})$ are connected to $b_0 \otimes b_0$.

By (2.2.17) in [KMN^2], the highest weight elements of $\iota_0^*(B^{1,l}\otimes B^{1,l})$ are of the form $b_1 \otimes b_2$, where b_1 and b_2 satisfy

$$\begin{aligned}
&x_1(b_1) = j \ \text{ for some } j = 0, 1, \cdots, l,\\
&x_0(b_1) = 0, \quad \bar{x}_1(b_1) = 0,\\
&x_i(b_1) = \bar{x}_i(b_1) = 0 \ \text{ for } i = 2, \cdots, n,\\
&x_0(b_2) = 0, \quad \bar{x}_2(b_2) = 0,\\
&x_i(b_2) = \bar{x}_i(b_2) = 0 \ \text{ for } i = 3, \cdots, n.
\end{aligned}$$

Let $x_1(b_2) = t_1$, $x_2(b_2) = t_2$, and $\bar{x}_1(b_2) = \bar{t}_1$. Then we have

$$\begin{aligned}
b_0 \otimes b_0 = {}& \tilde{f}_0^{j+s(b_2)+2l}\,\tilde{f}_1^{j+t_1+t_2} \cdots \tilde{f}_{n-1}^{j+t_1+t_2}\,\tilde{f}_n^{2(j+t_1+t_2)}\,\tilde{f}_{n-1}^{j+t_1+t_2}\\
&\cdots \tilde{f}_2^{j+t_1+t_2}\,\tilde{f}_1^{j+t_1+t_2}(b_1 \otimes b_2),
\end{aligned}$$

which completes the proof. □

Proof of Theorem 1.9.2 and Theorem 1.9.3.

We first show that $\langle c, \varepsilon(b)\rangle \geq l$ for all $b \in B^{1,l}$. Since $c = h_0 + 2h_1 + \cdots + 2h_{n-1} + h_n$, we have from (6.10.5.), (6.10.6), and (6.10.13)

$$\begin{aligned}
\text{(6.10.19)}\quad \langle c, \varepsilon(b)\rangle = {}& l - s(b) + 2(x_1(b) - \bar{x}_1(b))_+\\
&+ 2\sum_{i=1}^{n-1}(\bar{x}_i(b) + (x_{i+1}(b) - \bar{x}_{i+1}(b))_+) + 2\bar{x}_n(b) + x_0(b).
\end{aligned}$$

Set $S_0 = \{j \in J \mid x_j(b) = \bar{x}_j(b)\}$, $S_1 = \{j \in J \mid x_j(b) > \bar{x}_j(b)\}$, and $S_2 = \{j \in J \mid x_j(b) < \bar{x}_j(b)\}$. Then (6.10.19) becomes

$$\begin{aligned}
\langle c, \varepsilon(b)\rangle &= l - s(b) + 2\sum_{j\in S_0}\bar{x}_j(b) + 2\sum_{j\in S_1}x_j(b)\\
&\quad + 2\sum_{j\in S_2}\bar{x}_j(b) + x_0(b)\\
\text{(6.10.20)}\qquad &\geq l - s(b) + \sum_{j\in S_0}(x_j(b) + \bar{x}_j(b)) + \sum_{j\in S_1}(x_j(b) + \bar{x}_j(b))\\
&\quad + \sum_{j\in S_2}(x_j(b) + \bar{x}_j(b)) + x_0(b)\\
&= l - s(b) + s(b) = l.
\end{aligned}$$

Now let $\Lambda = \sum_{i=0}^{n} k_i\Lambda_i$ be a dominant integral weight of level l, i.e.,

$$\text{(6.10.21)}\qquad \langle \Lambda, c\rangle = k_0 + 2k_1 + \cdots + 2k_{n-1} + k_n = l.$$

We will show that there exists a unique element $b \in B^{1,l}$ such that $\varepsilon_i(b) = k_i$ for all $i = 0, \cdots, n$. For existence, we take $b \in B^{1,l}$ with

$$\begin{aligned}
&x_0(b) = 0 \quad \text{if } k_n \text{ is even},\\
& = 1 \quad \text{if } k_n \text{ is odd},\\
\text{(6.10.22)}\qquad &x_i(b) = \bar{x}_i(b) = k_i \quad \text{for } i = 1, \cdots, n-1,\\
&x_n(b) = \bar{x}_n(b) = \frac{k_n - x_0(b)}{2}.
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}\varepsilon_i(b) &= \bar{x}_i(b) = k_i \text{ for } i = 1, \cdots, n-1,\\ \varepsilon_n(b) &= 2\bar{x}_n(b) + x_0(b) = k_n.\end{aligned}$$

Moreover, we see from (6.10.21) and (6.10.22) that

$$\begin{aligned}\varepsilon_0(b) = l - s(b) &= l - (\sum_{i=1}^{n} x_i(b) + \sum_{i=1}^{n} \bar{x}_i(b) + x_0(b))\\ &= l - (2\sum_{i=1}^{n-1} k_i + (k_n - x_0(b)) + x_0(b)) = l - (2\sum_{i=1}^{n-1} k_i + k_n)\\ &= k_0.\end{aligned}$$

For the uniqueness, let b' be an element of $B^{1,l}$ such that $\varepsilon_i(b') = k_i$ for all $i = 0, 1, \cdots, n$. Then $\langle c, \varepsilon(b')\rangle = l$. In (6.10.20), the equality holds if and only if $S_1 = S_2 = \phi$, i.e., $x_i(b') = \bar{x}_i(b')$ for all $i = 1, \cdots, n$. Hence

$$\begin{aligned}\varepsilon_0(b') &= l - s(b') = k_0,\\ \varepsilon_i(b') &= x_i(b') = \bar{x}_i(b') = k_i \quad \text{for } i = 1, \cdots, n-1,\\ \varepsilon_n(b') &= 2x_n(b') + x_0(b') = 2\bar{x}_n(b') + x_0(b') = k_n.\end{aligned}$$

Thus we have

$$\begin{aligned}x_i(b') &= \bar{x}_i(b') = k_i \quad \text{for } i = 1, \cdots, n-1,\\ x_n(b') &= \bar{x}_n(b') = \frac{k_n - x_0(b')}{2},\end{aligned}$$

which implies

$$\begin{aligned}x_0(b') &= 0 \quad \text{if } k_n \text{ is even},\\ &= 1 \quad \text{if } k_n \text{ is odd}.\end{aligned}$$

Hence $b = b'$. Now, we have completed the proof of Theorem 1.9.3 and then that of Theorem 1.9.2 by the arguments above, Remark 1.9.4 and Proposition 6.10.4. □

Proposition 6.10.5. *Let Λ and b be as in the proof of Theorem 1.9.2. Then we have*

$$\Lambda + af(wt(b)) = \Lambda.$$

Thus the ground-state path of weight Λ is the sequence $(b, b, b, b, \cdots)$.

6.11. $(A_{2n}^{(2)}, B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)\,(n \geq 2))$

First note that the proof of Proposition 1.10.1 is similar to that of Proposition 1.9.1.

We shall use the notations in 1.10. For $i = 1, \cdots, n$, the rule of drawing i-arrow on $\tilde{B}$ is given in [KN]. From that rule, for $i = 1, \cdots, n-1$, we observe that

$$\begin{aligned} \varepsilon_i(b) &= \bar{x}_i(b) + (x_{i+1}(b) - \bar{x}_{i+1}(b))_+, \\ \varphi_i(b) &= x_i(b) + (\bar{x}_{i+1}(b) - x_{i+1}(b))_+, \end{aligned} \tag{6.11.1}$$

and

$$\varepsilon_n(b) = \bar{x}_n(b), \quad \varphi_n(b) = x_n(b). \tag{6.11.2}$$

Let $B' = B((l - 2\left[\frac{l}{2}\right])\Lambda_1) \oplus \cdots \oplus B((l-2)\Lambda_1) \oplus B(l\Lambda_1)$ be the direct sum of crystals with highest weight for $U_q(B_n)$. Set $K' = \{1, \cdots, n, 0, \bar{n}, \cdots, \bar{1}\}$ and consider the ordering on K' given by

$$1 < 2 < \cdots < n < 0 < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

Then the elements of B' are labeled by $b' = (b'_k)_{k=1}^{j}$, where $b'_k \in K'$, $b'_k \leq b'_{k+1}$ for all k, and $0 \leq j \leq l$. Here we write $b' = \phi$ when $j = 0$. Let $x_0(b') = \sharp\{k | \ b'_k = 0\}$, $x_i(b') = \sharp\{k | \ b'_k = i\}$, $\bar{x}_i(b') = \sharp\{k | \ b'_k = \bar{i}\}$, and let $s(b') = \sum x_i(b') + x_0(b') + \sum \bar{x}_i(b')$. Note that $x_0(b') = 0$ or 1. For $i = 1, \cdots, n$, the rule of drawing i-arrow is given in [KN].

We define a bijection $\sigma : \tilde{B} \to B'$ as follows. Let $b = (b_k)_{k=1}^{j} \in B(j\Lambda_1)$. For $i = 1, \cdots, n-1$, we define

$$\begin{aligned} x_i(\sigma(b)) &= (\bar{x}_{n-i+1}(b) - x_{n-i+1}(b))_+ + \min(x_{n-i}(b), \bar{x}_{n-i}(b)), \\ \bar{x}_i(\sigma(b)) &= (x_{n-i+1}(b) - \bar{x}_{n-i+1}(b))_+ + \min(x_{n-i}(b), \bar{x}_{n-i}(b)). \end{aligned} \tag{6.11.3}$$

We also define

$$\begin{aligned} x_0(\sigma(b)) &= 0 \quad \text{if } l - s(b) \text{ is even}, \\ &= 1 \quad \text{if } l - s(b) \text{ is odd}, \end{aligned} \tag{6.11.4}$$

$$\begin{aligned} x_n(\sigma(b)) &= \left[\frac{l - s(b)}{2}\right] + (\bar{x}_1(b) - x_1(b))_+, \\ \bar{x}_n(\sigma(b)) &= \left[\frac{l - s(b)}{2}\right] + (x_1(b) - \bar{x}_1(b))_+. \end{aligned} \tag{6.11.5}$$

Note that $s(\sigma(b)) = l - t(b)$, where $t(b) = 2\min(x_n(b), \bar{x}_n(b))$. Hence $\sigma(b) \in B'$.

We define a map $\tau : B' \to \tilde{B}$ in the same principle. More precisely, for $i = 1, \cdots, n-1$, we define

$$\begin{aligned} x_i(\tau(b')) &= (\bar{x}_{n-i+1}(b') - x_{n-i+1}(b'))_+ + \min(x_{n-i}(b'), \bar{x}_{n-i}(b')), \\ \bar{x}_i(\tau(b')) &= (x_{n-i+1}(b') - \bar{x}_{n-i+1}(b'))_+ + \min(x_{n-i}(b'), \bar{x}_{n-i}(b')), \end{aligned} \tag{6.11.6}$$

and

$$\begin{aligned} x_n(\tau(b')) &= \left[\frac{l - s(b')}{2}\right] + (\bar{x}_1(b') - x_1(b'))_+, \\ \bar{x}_n(\tau(b)) &= \left[\frac{l - s(b')}{2}\right] + (x_1(b') - \bar{x}_1(b'))_+. \end{aligned} \tag{6.11.7}$$

It is straightforward to see that $\tau\sigma = id_{\tilde{B}}$ and $\sigma\tau = id_{B'}$.

Now we define the rule of 0-arrow by

$$\tilde{f}_0(b) = \sigma \tilde{f}_n \sigma(b). \tag{6.11.8}$$

That is, $b \xrightarrow{0} b'$ if and only if $\sigma(b) \xrightarrow{n} \sigma(b')$. Then we have

$$\begin{aligned} \varepsilon_0(b) &= l - s(b) + 2(x_1(b) - \bar{x}_1(b))_+, \\ \varphi_0(b) &= l - s(b) + 2(\bar{x}_1(b) - x_1(b))_+. \end{aligned} \tag{6.11.9}$$

Now, let us denote by $B^{1,l}$ the crystal $\tilde{B}$ endowed with 0-arrows defined as above.

Proposition 6.11.1. *Let $J' = \{1, \cdots, n-1\}$ be the index set for the simple roots for $U_q(\mathfrak{sl}(n))$, and define a map $\iota' : J' \to I$ by $\iota'(j) = j$ for $j \in J'$. Then $\iota'^*(B^{1,l})$ splits into a direct sum of crystals for $U_q(\mathfrak{sl}(n))$ with highest weight*

$$-2t_1\Lambda_0 + t_1\Lambda_1 + (\bar{t}_n - t_n)\Lambda_{n-1} + (t_n - \bar{t}_n)\Lambda_n,$$

where t_1, t_n, $\bar{t}_n$ are nonnegative integers such that $t_n \le \bar{t}_n$, $t_1 + t_n + \bar{t}_n \le l$.

Proof. Similar to Proposition 6.10.1. □

Proof of Proposition 1.10.1.
Now we have the uniqueness of $B^{1,l}$.

Theorem 6.11.2. *Let B be a crystal for $U_q(A_{2n}^{(2)})$ such that*

$$\iota_0^*(B) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

as crystals for $U_q(C_n)$ and

$$\iota_n^*(B) \cong B((l - 2\left[\frac{l}{2}\right])\Lambda_1) \oplus \cdots \oplus B((l-2)\Lambda_1) \oplus B(l\Lambda_1)$$

as crystals for $U_q(B_n)$. Then there exists a unique isomorphism $\psi : B^{1,l} \to B$ as crystals for $U_q(A_{2n}^{(2)})$.

Proof. Similar to Theorem 6.10.2. □

In Section 5, we proved that there exists a finite dimensional representation V of $U_q(A_{2n}^{(2)})$ with crystal base (L, B) such that

$$\iota_0^*(B) \cong B(0) \oplus B(\Lambda_1) \oplus \cdots \oplus B(l\Lambda_1)$$

as crystals for $U_q(C_n)$ and

$$\iota_n^*(B) \cong B((l - 2\left[\frac{l}{2}\right])\Lambda_1) \oplus \cdots \oplus B((l-2)\Lambda_1) \oplus B(l\Lambda_1)$$

as crystals for $U_q(B_n)$. Therefore by Theorem 6.11.2, there is a unique isomorphism $B^{1,l} \cong B$ as crystals for $U_q(A_{2n}^{(2)})$. Now, we have completed the proof of Proposition 1.10.1. □

Proposition 6.11.3. *The crystal $B^{1,l} \otimes B^{1,l}$ is connected.*

Proof. Similar to Proposition 6.10.4. □

Proof of Theorem 1.10.2 and Theorem 1.10.3.

We first show that $\langle c, \varepsilon(b)\rangle \geq l$ for all $b \in B^{1,l}$. Since $c = h_0 + 2h_1 + \cdots + 2h_{n-1} + 2h_n$, we have from (6.11.1), (6.11.2), and (6.11.9)

$$\langle c, \varepsilon(b)\rangle = l - s(b) + 2(x_1(b) - \bar{x}_1(b))_+ + 2\sum_{i=1}^{n-1}(\bar{x}_i(b) + (x_{i+1}(b) - \bar{x}_{i+1}(b))_+) + 2\bar{x}_n(b). \tag{6.11.10}$$

Set $S_0 = \{j \in J |\ x_j(b) = \bar{x}_j(b)\}$, $S_1 = \{j \in J |\ x_j(b) > \bar{x}_j(b)\}$, and $S_2 = \{j \in J |\ x_j(b) < \bar{x}_j(b)\}$. Then by (6.11.10), we have

$$\begin{aligned} \langle c, \varepsilon(b)\rangle &= l - s(b) + 2\sum_{j\in S_0} \bar{x}_j(b) + 2\sum_{j\in S_1} x_j(b) \\ &\quad + 2\sum_{j\in S_2} \bar{x}_j(b) \\ &\geq l - s(b) + \sum_{j\in S_0}(x_j(b) + \bar{x}_j(b)) \qquad (6.11.11) \\ &\quad + \sum_{j\in S_1}(x_j(b) + \bar{x}_j(b)) + \sum_{j\in S_2}(x_j(b) + \bar{x}_j(b)) \\ &= l - s(b) + s(b) = l. \end{aligned}$$

Now let $\Lambda = \sum_{i=0}^{n} k_i\Lambda_i$ be a dominant integral weight of level l, i.e.,

$$\langle \Lambda, c\rangle = k_0 + 2k_1 + \cdots + 2k_{n-1} + 2k_n = l. \tag{6.11.12}$$

We will show that there exists a unique element $b \in B^{1,l}$ such that $\varepsilon_i(b) = k_i$ for all $i = 0, \cdots, n$. For existence, we take $b \in B^{1,l}$ with

$$x_i(b) = \bar{x}_i(b) = k_i \quad \text{for } i = 1, \cdots, n. \tag{6.11.13}$$

Then it is easy to see that

$$\varepsilon_i(b) = \bar{x}_i(b) = k_i \text{ for } i = 1, \cdots, n.$$

Moreover, we see from (6.11.12) and (6.11.13) that

$$\begin{aligned} \varepsilon_0(b) = l - s(b) &= l - (\sum_{i=1}^{n} x_i(b) + \sum_{i=1}^{n} \bar{x}_i(b)) \\ &= l - 2\sum_{i=1}^{n} k_i = k_0. \end{aligned}$$

The proof of the uniqueness is similar to the argument in the proof of Theorem 1.9.2. Hence, we have completed the proof of Theorem 1.10.3 and then that of Theorem 1.10.2 by the arguments above, Remark 1.10.4 and Proposition 6.11.3. □

Proposition 6.11.4. *Let Λ and b be as in the proof of Theorem 1.10.2. Then we have*

$$\Lambda + af(wt(b)) = \Lambda.$$

Thus the ground-state path of weight Λ is the sequence $(b, b, b, b, \cdots)$.

Acknowledgements
S. J. K and K. C. M thank their colleagues and the secretaries at RIMS, Kyoto University for their hospitality during their participation to RIMS91, Infinite Analysis.

References

[ABF] Andrews, G.E., Baxter, R.J. and Forrester, P.J. Eight-Vertex SOS Model and Generalized Rogers-Ramanujan-Type Identities, J. Statistical Physics, Vol. 35, Nos. 3/4 (1984).

[B] Baxter, R. J., Exactly Solved Models in Statistical Mechanics, Academic, 1982.

[D] Drinfeld, V. G., Hopf algebra and the Yang-Baxter equation, Soviet Math. Dokl. **32** (1985), 254-258.

[J1] Jimbo, M., A q-difference analogue of $U_q(\mathfrak{g})$ and the Yang-Baxter equation, Lett. Math. Phys. **10** (1985), 63-69.

[J2] Jimbo, M., Introduction to the Yang-Baxter equation, Inter. J. Mod. Phys. A, 4 (15) (1989), 3759-3777.

[JMMO] Jimbo, M., Misra, K. C., Miwa, T. and Okado, M., Combinatorics of representations of $U_q(\hat{sl}(n))$ at $q = 0$, Commun. Math. Phys. **136** (1991), 543-566.

[K1] Kashiwara, M., Crystalizing the q-analogue of universal enveloping algebras, Commun. Math. Phys. **133** (1990), 249-260.

[K2] Kashiwara, M., Bases cristallines, C. R. Acad. Sci. Paris, t. **311**, Série I (1990), 277-280.

[K3] Kashiwara, M., Crystallizing the q-analogue of universal enveloping algebras, Proc. ICM Kyoto (1991).

[K4] Kashiwara, M., On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. **63** (1991), 465-516.

[Kac] Kac, V. G., Infinite dimensional Lie algebras, Cambridge Univ.Press. (1990).

[KMN^2] Kang, S. J., Kashiwara, M., Misra, K. C., Miwa, T., Nakashima, T. and Nakayashiki, A., Affine crystals and Vertex models, RIMS-828, to appear in *Int. J. Mod. Phys. A* and Infinite Analysis, (Proc. RIMS Project 91, World Scientific 1992).

Vertex models and crystals, to appear in C. R. Acad. Sci. Paris.

[KN] Kashiwara, M. and Nakashima, T., Crystal graphs for representations of the q-analogue of classical Lie algebras, RIMS-767.

[L] Lusztig, G. I., Quantum deformation of certain simple modules over enveloping algebras, Adv. in Math. **70** (1988), 237-249.

[MM] Misra, K. C. and Miwa, T., Crystal base for the basic representation of $U_q(\hat{sl}(n))$, Commun. Math. Phys. **134** (1990), 79-88.

[R] Rosso, M., Finite dimensional representations of the quantum analogue of the enveloping algebra of a complex semisimple Lie algebra, Commun. Math. Phys. **117** (1988), 581-593.

Figure 1

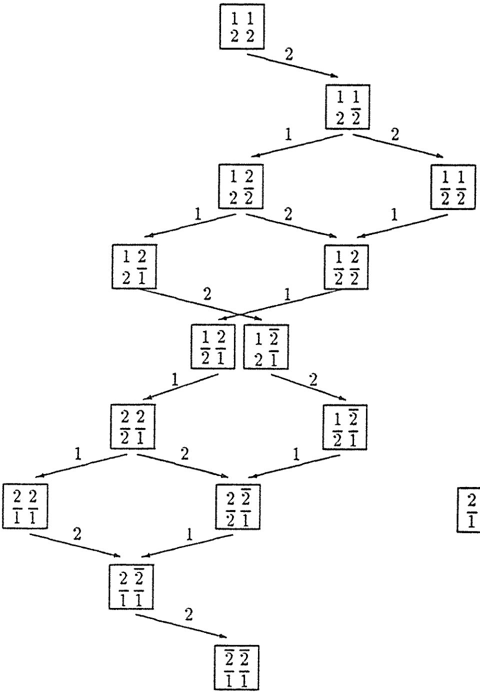

Figure 2

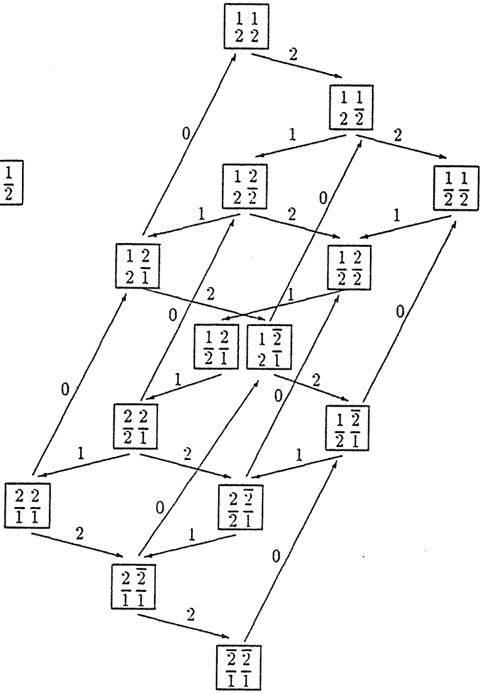

Figure 3

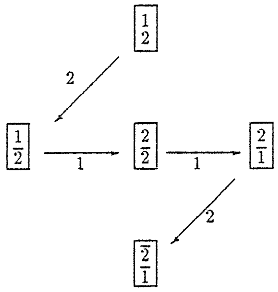

Figure 4