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Representation theory and D-modules on flag varieties

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§0. Introduction

0.0 By the celebrated work of Beilinson-Bernstein of the vanishing theorem on the D-modules over flag varieties ([BB]), we can study representations of Lie group through the geometry of flag varieties. In this lecture, we review this and add what happens when the infinitesimal characters are not regular.

0.1 Let G be a reductive group and X its flag variety. Let $\underline{g}$ be the Lie algebra of G, $\underline{t}$ the Cartan algebra and Δ the root system. For $\lambda \in \underline{t}^*$, let χ_λ be the corresponding character of the center $\underline{Z}(\underline{g})$ of the universal enveloping algebra $U(\underline{g})$. We normalize this so that $\chi_\lambda = \chi_{w\lambda}$ for w in the Weyl group W. For $\lambda \in \underline{t}^*$, set $U_\lambda(\underline{g}) = U(\underline{g})/U(\underline{g})\operatorname{Ker}\chi_\lambda$.

Then we can construct a twisted ring of differential operators $\underline{D}_\lambda$ on X such that $\Gamma(X;\underline{D}_\lambda) = U_\lambda(\underline{g})$. Beilinson-Bernstein's achievements are summarized by the following three theorems (The last one is an easy consequence of the first two)

Theorem A *If λ is regular and anti-dominant, any coherent $\underline{D}_\lambda$-module is generated by global sections.*

Theorem B *If λ is anti-dominant, then any coherent $\underline{D}_\lambda$-module $\underline{M}$ satisfies $H^n(X;\underline{M}) = 0$ for $n \neq 0$*

Theorem 0 *If λ is anti-dominant and regular, the category of finitely generated $U_\lambda(\underline{g})$-modules are equivalent to the category of coherent $\underline{D}_\lambda$-modules.*

In [BK], Brylinski and Kashiwara proved these theorems in a very special case (χ_λ trivial, $\underline{M}$ is U-equivariant) in an ad-hoc manner, in order to prove the Kazhdan-Lustzig conjecture.

0.2 Let G_R be a real semisimple group, K_R a maximal compact subgroup of G_R and let G and K be their complexification. Let $\underline{g}$ and $\underline{k}$ be their Lie algebras. Then by Harish-Chandra [H], admissible representaion of G_R is described by $(\underline{g},K)$-modules, so called Harish-Chandra module.

By Theorem 0, $(\underline{g},K)$-module with infinitesimal character χ_λ is described by K-equivariant $\underline{D}_\lambda$-module.

The structure of irreducible K-equivariant $\underline{D}_\lambda$-module $\underline{M}$ can be described by using the geometry of K-orbits. The crucial point here is that X has only finite many K-orbits. First the support of $\underline{M}$ is a closure of a K-orbit S. Assume, for the sake of simplicity, χ_λ is the trivial infinitesimal character. Then, $\underline{M}$ determines a K-equivariant local system F on S, and $\underline{M}$ is completely described by the pair (S,F). If χ_λ is not trivial, we have to replace F with a twisted local system. When λ is not regular, we have to put some auxiliary condition on F(see §9).

0.3 Except the irregular case, the contents of this article are more or less known. In the appendix of the paper by Hecht, Milicic, Schmid and Wolf [HMSW], we can find also the review of the result of Beilinson-Bernstein. Also see Ginsburg [G].

0.4 We did not include the following important topics concerning D-modules on the flag variety.

(1) The derived category of $\underline{D}_\lambda$-modules are equivalent to that of $\underline{D}_{w\lambda}$-modules for any $w \in W$ and $\lambda \in \underline{t}^*$. This is obtained by Beilinson-Bernstein ([BB]2).

(2) There is a one-to-one correspondence between K-orbits of X and $G_{\mathbb{R}}$-orbits of X by Matsuki [M]. This gives the construction of representations of $G_{\mathbb{R}}$ corresponding to Harish-Chandra modules by W. Schmid - J. Wolf. See [SW], [K].

(3) Relations with representation of the Weyl group, the affine Weyl group and their Hecke algebras, Hodge modules, invariant eigendistributions on the group.

§1 Vanishing theorem for cohomology groups of modules over $\underline{O}_X$-rings.

<u>1.1</u> Let $(X,\underline{O}_X)$ be a commutative ringed space over a commutative ring k. A <u>$(k,\underline{O}_X)$-ring</u> is a sheaf of rings $\underline{A}$ with a ring homomorphism $\underline{O}_X \to \underline{A}$ such that the image of $k \to \underline{O}_X \to \underline{A}$ is contained in the center of $\underline{A}$. We do not assume that the image of $\underline{O}_X \to \underline{A}$ is contained in the center of $\underline{A}$.

If there is no afraid of confusion, we simply call <u>$\underline{O}_X$-ring</u> for a $(k,\underline{O}_X)$-ring. We shall study in this section the criterian for the vanishing of cohomology groups of modules over $\underline{O}_X$-rings.

<u>1.2</u> Let us recall Serre's result on ample invertible sheaves. Let k be a commutative field and let $(X,\underline{O}_X)$ be a projective variety over k.

<u>Definition-Theorem 1.2.1 Let L be an invertible $\underline{O}_X$-module. Then the following conditions are equivalent.</u>

(1) <u>There exists an integer $r > 0$ and a closed embedding</u>

$j: X \hookrightarrow \mathbb{P}^N$ <u>such that</u> $L^{\otimes r} = j^*\underline{O}_X(1)$.

(2) <u>For any pair of distinct closed points x,y of X, there exists $r > 0$ and $s \in \Gamma(X;L^{\otimes r})$ such that $s(x)=0$ and $s(y)\neq 0$.</u>

(3) <u>For any coherent sheaf F, $F\otimes L^{\otimes r}$ is generated by global sections for $r \gg 0$ (i.e. $\Gamma(X;F\otimes L^{\otimes r}) \otimes \underline{O}_X \to F\otimes L^{\otimes r}$ is surjective),</u>

(4) <u>For any coherent $\underline{O}_X$-module F, $H^j(X;F\otimes L^{\otimes r}) = 0$ for $j\neq 0$ and $r \gg 0$.</u>

<u>If these equivalent conditions are satisfied, we say that L is ample.</u>

Here, for $s \in \Gamma(X;L)$ and $x \in X$, $s(x)$ is the image of s in $(\underline{O}_{X,x}/\underline{m}_x) \otimes_{\underline{O}_{X,x}} L_x$ with the maximal ideal $\underline{m}_x$ of $\underline{O}_{X,x}$.

<u>1.3</u> Let $(X,\underline{O}_X)$ be a projective scheme over k and L an ample invertible $\underline{O}_X$-module. Let $\underline{A}$ be an $\underline{O}_X$-ring. Throughout this section, we assume

(1.3.1) $\underline{A}$ is quasi-coherent as a left $\underline{O}_X$-module.

Theorem 1.3.1 *Under the condition* (1.3.1), *the following conditions are equivalent.*

(1) *For any left* $\underline{A}$-*module* $\underline{M}$, *quasi-coherent over* $\underline{O}_X$, $\underline{M}$ *is generated by global sections* (i.e. $\underline{A} \otimes \Gamma(X;\underline{M}) \to \underline{M}$ *is surjective*).

(2) *For* $n \gg 0$, $\underline{A} \underset{\underline{O}_X}{\otimes} L^{\otimes(-n)}$ *is generated by global sections.*

Proof. (1) $\Rightarrow$ (2) trivial.

(2) $\Rightarrow$ (1) $\underline{M}$ is a union of coherent sub-$\underline{O}_X$-modules F. For such an F, there exists a surjective morphism $\underline{O}_X{}^N \to F \otimes L^{\otimes n}$ for $n \gg 0$. Hence $(\underline{A} \underset{\underline{O}_X}{\otimes} L^{\otimes -n})^N \to \underline{A} \underset{\underline{O}_X}{\otimes} F$ is surjective. Since $\underline{A} \underset{\underline{O}_X}{\otimes} L^{\otimes -n}$ is generated by global section, there exists $\underline{A}^{N'} \twoheadrightarrow \underline{A} \underset{\underline{O}_X}{\otimes} F$. Hence the image of $\underline{A} \otimes \Gamma(X;\underline{M}) \to \underline{M}$ contains F. This shows (2) $\Rightarrow$ (1).

1.4 Let $(X,\underline{O}_X)$, L and $\underline{A}$ be as in the preceding sections.

Theorem 1.4.1 *Under the condition* (1.3.1), *the following conditions are equivalent.*

(1) *For any left* $\underline{A}$-*module* $\underline{M}$, *quasi-coherent over* $\underline{O}_X$, $H^n(X;\underline{M}) = 0$ *for* $n \neq 0$.

(2) *For* $r \gg 0$,

$$\Gamma(X;\underline{A} \underset{\underline{O}_X}{\otimes} L^{\otimes -r}) \otimes \Gamma(X;L^{\otimes r}) \to \Gamma(X;\underline{A})$$

is surjective.

(3) *For* $r \gg 0$,

$$\underline{A} \quad (L^{\otimes r} \underset{k}{\otimes} \Gamma(X;L^{\otimes r})^*) \underset{\underline{O}_X}{\otimes} \underline{A}$$

has a cosection (i.e. *a left inverse*) *as right* $\underline{A}$-*modules.*

(4) *For* $r \gg 0$,

$$\underline{A} \underset{\underline{O}_X}{\otimes} (L^{\otimes -r} \underset{k}{\otimes} \Gamma(X;L^{\otimes r})) \to \underline{A}$$

<u>has a section</u> (i.e. <u>right inverse</u>) <u>as a left A-module</u>.

<u>Remark</u> For $r \geq 0$, $\underline{O}_X \underset{k}{\otimes} \Gamma(X;L^{\otimes r}) \to L^{\otimes r}$ gives $\underline{O}_X \to L^{\otimes r} \otimes \Gamma(X;L^{\otimes r})^*$ and $L^{\otimes -r} \to \underline{O}_X \otimes \Gamma(X;L^{\otimes r})^*$. The morphisms in (3) and (4) come from them.

<u>Proof</u> (3) $\Leftrightarrow$ (4) follows by the operation of the functor $\underline{Hom}_{\underline{A}}(*,\underline{A})$.
(2) $\Leftrightarrow$ (4) obvious.
(1) $\Rightarrow$ (2) follows from the exact sequence

$$0 \to \underline{M} \to \underline{A} \otimes (L^{\otimes -r} \otimes \Gamma(X;L^{\otimes r})) \to \underline{A} \to 0 \quad \text{and} \quad H^1(X;\underline{M}) = 0$$

(3) $\Rightarrow$ (1).

We have

$$(1.4.1) \qquad H^n(X;\underline{M}) = \varinjlim H^n(X;F)$$

where F ranges over coherent sub-$\underline{O}_X$-modules of $\underline{M}$. For such an F, we shall show that $H^n(X;F) \to H^n(X;\underline{M})$ is the zero map for $n \neq 0$.

We have $H^n(X;F \otimes L^{\otimes r}) = 0$ for $n \neq 0$, $r \gg 0$.
Set $V = \Gamma(X;L^{\otimes r})$. By letting $\otimes_{\underline{A}} \underline{M}$ operate on $\underline{A} \to (L^{\otimes r} \underset{k}{\otimes} V^*) \underset{\underline{O}_X}{\otimes} \underline{A}$, $\underline{M} \to (L^{\otimes r} \underset{k}{\otimes} V^*) \underset{\underline{O}_X}{\otimes} \underline{M}$ has a cosection by (3).

Now, letting $H^n(X;*)$ operate on a commutative diagram

$$\begin{array}{ccc} F & \to & L^{\otimes r} \otimes V^* \otimes F \\ \downarrow & & \downarrow \\ \underline{M} & \to & L^{\otimes r} \otimes V^* \otimes \underline{M} \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} H^n(X;F) & \longrightarrow & H^n(X;L^{\otimes r} \otimes V^* \otimes F) \\ \big\downarrow a & & \big\downarrow \\ H^n(X;\underline{M}) & \xrightarrow{b} & H^n(L^{\otimes r} \otimes V^* \otimes \underline{M}). \end{array}$$

Since $H^n(X;L^{\otimes r} \otimes V^* \otimes F) = 0$ for $n \neq 0$, $b \circ a = 0$. Since b has a cosection $a = 0$. By (1.4.1), we have $H^n(X;\underline{M}) = 0$. Q.E.D.

<u>1.5</u> Let $(X;\underline{O}_X)$, L and $\underline{A}$ be as in §1.3. Set $R = \Gamma(X;\underline{A})$. Let $\mathrm{Mod}_{qc}(\underline{A})$ be the category of left $\underline{A}$-modules quasi-coherent over $\underline{O}_X$ and $\mathrm{Mod}(R)$ the category of left R-modules. We define the functors

$$\Gamma : \mathrm{Mod}_{qc}(\underline{A}) \to \mathrm{Mod}(R)$$

and

$$\otimes : \mathrm{Mod}(R) \to \mathrm{Mod}_{qc}(\underline{A})$$

by

$$\Gamma : \underline{M} \mapsto \Gamma(X;\underline{M}),\ \otimes : N \mapsto \underline{A}\otimes_R N.$$

Then $\otimes$ and Γ are adjoint functors; i.e.

$$\mathrm{Hom}(N, \Gamma(\underline{M})) \cong \mathrm{Hom}(\otimes(N),\underline{M}).$$

<u>Proposition 1.5.1</u> (a) <u>If the equivalent conditions of</u> Theorem 1.4.1 <u>are satisfied, then</u> Γ <u>is an exact functor and</u> $\Gamma\circ\otimes \cong \mathrm{id}$.

(b) <u>If the equivalent conditions of</u> Theorem 1.3.1 <u>and those of</u> Theorem 1.4.1 <u>are satisfied, then</u> $\Gamma\circ\otimes \cong \mathrm{id}$, $\otimes\circ\Gamma = \mathrm{id}$.

<u>Proof</u> (a) The first assertion is obvious. Let $0 \leftarrow M \leftarrow R^{(I)} \leftarrow R^{(J)}$ be a free resolution. Then we have $0 \leftarrow \underline{A}\otimes M \leftarrow \underline{A}^{(I)} \leftarrow \underline{A}^{(J)}$. Since $\Gamma(X;*)$ is an exact functor, the rows of the following diagram

$$\begin{array}{ccccccc} 0 & \leftarrow & \Gamma(X;\underline{A}\otimes M) & \leftarrow & \Gamma(X;\underline{A}^{(I)}) & \leftarrow & \Gamma(X;\underline{A}^{(J)}) \\ & & \uparrow & & \uparrow\wr & & \uparrow\wr \\ 0 & \longleftarrow & M & \longleftarrow & \underline{A}^{(I)} & \longleftarrow & \underline{A}^{(J)} \end{array}$$

are exact. Hence $M \xrightarrow{\sim} \Gamma(X;\underline{A}\otimes M)$.

(b) The proof is similar as that of (a). For $\underline{M} \in \mathrm{Ob}\,\mathrm{Mod}_{qc}(\underline{A})$, there exist an exact sequence

$$0 \leftarrow \underline{M} \leftarrow \underline{A}^{(I)} \leftarrow \underline{A}^{(J)}.$$

This gives the exact sequence

$$0 \leftarrow \Gamma(X;\underline{M}) \leftarrow R^{(I)} \leftarrow R^{(J)}.$$

Operating $\otimes$, we have

$$0 \leftarrow \underline{A}\otimes\Gamma(X;\underline{M}) \leftarrow \underline{A}^{(I)} \leftarrow \underline{A}^{(J)}.$$

Hence $\underline{A} \otimes \Gamma(X;\underline{M}) \to \underline{M}$ is an isomorphism.

Proposition 1.5.2 *Assume the equivalent conditions of* Theorem 1.4.1 *Let* E *be the full subcategory of* $\mathrm{Mod}_{qc}(\underline{A})$ *consisting of* M *such that* M *is generated by global sections and* $\underline{M}$ *has no non-zero subobject* $\underline{N}$ *such that* $\Gamma(X;\underline{N}) = 0$. *Then* $\Gamma\colon E \to \mathrm{Mod}(R)$ *is an equivalence of categories.*

Proof We shall show first Γ is fully faithful. For two objects $\underline{M}_1$ and $\underline{M}_2$ of E, $\varphi\colon \mathrm{Hom}(\underline{M}_1,\underline{M}_2) \to \mathrm{Hom}(\Gamma(\underline{M}_1),\Gamma(\underline{M}_2))$ is injective because $\underline{M}_1$ is generated by global sections. Let $f\colon \Gamma(\underline{M}_1) \to \Gamma(M_2)$ be a homomorphism. Since the kernel $\underline{N}$ of $\underline{A}\otimes_R\Gamma(\underline{M}_1) \to \underline{M}_1$ satisfies $\Gamma(\underline{N}) = 0$, the composition of $\underline{N} \to \underline{A}\otimes_R\Gamma(\underline{M}_1) \to \underline{A}\otimes_R\Gamma(\underline{M}_2) \to \underline{M}_2$ is zero, and hence, this gives a homomorphism $g\colon \underline{M}_1 \to \underline{M}_2$. It is evident that $\varphi(g) = f$.

Let us show $\Gamma\colon E \to \mathrm{Mod}_{qc}(R)$ is essentially surjective. For an R-module N, Let I be the set of subobjects $\underline{M}$ of $\underline{A}\otimes_R N$ such that $\Gamma(X;\underline{M}) = 0$. Then I is inductively ordered and the sum of any two subobjects in I belongs again to I. Hence I has the largest element $\underline{M}_0$. Then $\underline{M} = (\underline{A}\otimes_R N)/\underline{M}_0$ is an object of E and satisfies $\Gamma(\underline{M}) = N$.

Corollary 1.5.3 *Assume the equivalent conditions of* Theorem 1.4.1. *The set of isomorphic classes of the simple* R-*modules is isomorphic to the set of the isomorphic classes of the simple objects* M *in* $\mathrm{Mod}_{qc}(\underline{A})$ *satisfying* $\Gamma(X;M) \neq 0$.

§2 Twisted ring of differential operators

2.0 Let X be a complex manifold. Let $\underline{D}_X$ be the ring of differential operators on X. We shall call twisted ring of differential operators an $\underline{O}_X$-ring locally isomorphic to $\underline{D}_X$. If L is an invertible $\underline{O}_X$-module, $L\otimes_{\underline{O}_X}\underline{D}_X\otimes_{\underline{O}_X}L^{\otimes-1}$ gives such an example.

In this section, we shall study the properties of such $\underline{O}_X$-rings.

2.1 Let $(X,\underline{O}_X)$ be either a smooth algebraic variety over a field k of characteristic 0 or a complex manifold. The following discussions are almost same in the both cases. We shall recall the properties of the sheaf $\underline{D}_X$ of differential operators. Let Θ_X be the sheaf of tangent vector fields. Let $F_k(\underline{D}_X)$ be the sheaf of differential operators of order at most k. Then this gives an increasing filtration called the order filtration of $\underline{D}_X$ that satisfy the following properties.

(2.1.1) $F_m(\underline{D}_X) = 0$ for $m<0$

(2.1.2) $F_0(\underline{D}_X) \cong O_X$.

(2.1.3) $F_m(\underline{D}_X) = \{P\in\underline{D}_X;\ [P,\underline{O}_X]\in F_{m-1}(\underline{D}_X)\}$ for $m\geq 0$.

(2.1.4) $\underline{D}_X = \cup F_m(\underline{D}_X)$

(2.1.5) $F_{m_1}(\underline{D}_X)\cdot F_{m_2}(\underline{D}_X) \subset F_{m_1+m_2}(\underline{D}_X)$

(2.1.6) $[F_{m_1}(\underline{D}_X),\ F_{m_2}(\underline{D}_X)] \subset F_{m_1+m_2-1}(\underline{D}_X)$

(2.1.7) $gr_1^F(\underline{D}_X) = F_1(\underline{D}_X)/F_0(\underline{D}_X) \xrightarrow{\sim} \Theta_X$

(2.1.8) $S(\Theta_X) \xrightarrow{\sim} gr^F\underline{D}_X = \oplus gr^F\underline{D}_X = \oplus F_m(\underline{D}_X)/F_{m-1}(\underline{D}_X)$

where $S(\Theta_X)$ is the symmetric algebra of Θ_X over $\underline{O}_X$, and the arrow in (2.1.8) is given via (2.1.7).

2.2 Let $\sigma_k: F_k(\underline{D}_X) \to S_k(\Theta_X)$ be the homomorphism given by $S(\Theta_X) \cong \mathrm{gr}^F \underline{D}_X$. Then for $a \in S_p(\Theta_X)$ and $b \in S_q(\Theta_X)$ taking $P \in F_p(\underline{D}_X)$ and $Q \in F_q(\underline{D}_X)$, we define

$$\{a,b\} = \sigma_{p+q-1}([P,Q]).$$

This does not depend on the choice of P,Q. We extend this by the linearity:

$$\{\ ,\ \}: S(\Theta_X) \otimes S(\Theta_X) \to S(\Theta_X).$$

This is called Poisson bracket. This satisfies the following well-known properties:

(2.2.1) $\{a,b\} = -\{b,a\}$

(2.2.2) $\{ab,c\} = b\{a,c\}+a\{b,c\}$

(2.2.3) $\{\{a,b\},c\}+\{\{b,c\},a\}+\{\{c,a\},b\} = 0.$

(2.2.4) If $v \in \Theta_X$ and $a \in \underline{O}_X$, then $\{v,a\}=v(a)$.

The following properties are easily checked.

Lemma 2.2.1 *(2.2.1), (2.2.2) and (2.2.4) characterises* $\{\ ,\ \}$.

Lemma 2.2.2 *Let* $x_i \in \underline{O}_X$ $(i=1,\cdots,n=\dim X)$ *be sections such that* dx_i *are linearly independent. Then for* $m \geq 1$, *and* $a_i \in S_{m-1}(\Theta_X)$ *with* $\{a_i,x_j\} = \{a_j,x_i\}$, *there exists unique* $u \in S_m(\Theta_X)$ *such that* $\{a,x_i\}=a_i$.

Proof Let $\{v_i\}$ be the dual base of $\{dx_i\}$. Then $S(\Theta_X)= \underline{O}_X[v_1,\cdots,v_n]$ and $\{f,x_j\} = \frac{\partial f}{\partial v_j}$. This shows immediately this lemma.

2.3 We shall study $\underline{O}_X$-rings with the similar properties as $\underline{D}_X$. Let $\underline{A}$ be an $\underline{O}_X$-ring with increasing filtration $F(\underline{A})$ satisfying

(2.3.1) $\underline{A} = \cup F_m(\underline{A})$

(2.3.2) $\underline{O}_X \xrightarrow{\sim} F_0(\underline{A})$

(2.3.3) $F_m(\underline{A})=0$ for $m<0$

(2.3.4) $F_{m_1}(\underline{A})\cdot F_{m_2}(\underline{A}) \subset F_{m_1+m_2}(\underline{A})$

(2.3.5) $[F_{m_1}(\underline{A}),\ F_{m_2}(\underline{A})] \subset F_{m_1+m_2-1}(\underline{A})$.

Then $gr^F(\underline{A})=\oplus(F_m(\underline{A})/F_{m-1}(\underline{A}))$ has the structure of commutative ring. Moreover $[*,*]:F_{m_1}(\underline{A})\otimes F_{m_2}(\underline{A}) \to F_{m_1+m_2-1}(\underline{A})$ gives the bracket $\{\ ,\ \}$ on $gr^F(\underline{A})$.

Associating to $P\in F_1(\underline{A})$, the derivation

$$\underline{O}_X \ni a \to [P,a]\in F_0(\underline{A}) \cong \underline{O}_X,$$

we obtain $gr_1^F(\underline{A}) \to \Theta_X$.

Assume further

(2.3.6) $gr_1^F(\underline{A}) \to \Theta_X$ is an isomorphism.

This gives a ring homomorphism $S(\Theta_X) \to gr^F(\underline{A})$. This preserves the bracket $\{\ ,\ \}$.

Lemma 2.3.1 Under the conditions (2.3.1)-(2.3.6), $S(\Theta_X) \to gr^F(\underline{A})$ is injective.

Proof We shall prove that, for $m\geq 2$ the injectivity of φ_{m-1}: $S_{m-1}(\Theta_X) \to gr_{m-1}^F(\underline{A})$ implies the injectivity of φ_m: $S_m(\Theta_X) \to gr_m^F(\underline{A})$. Assume $u\in S_m(\Theta_X)$ satisfies $\varphi_m(u)=0$. Then for any $a\in\underline{O}_X$, $\varphi_{m-1}(\{u,a\}) = \{\varphi_m(u),a\} = 0$, and hence $\{u,a\}=0$. Then Lemma 2.2.2 implies $u=0$.

Proposition 2.3.2 Under the conditions (2.3.1)-(2.3.6), the following conditions are equivalent

(2.3.7) $S(\Theta_X) \to gr^F(\underline{A})$ is an isomorphism.

(2.3.8) $F_m(\underline{A}) = F_1(\underline{A})F_{m-1}(\underline{A})$ for $m\geq 1$.

(2.3.9) $F_m(\underline{A})=\{P\in\underline{A};\ [P,a]\in F_{m-1}(\underline{A})$ for any $a\in\underline{O}_X\}$ for $m\geq 0$.

(2.3.10) The condition (2.3.9) holds for any $m\geq 1$.

Proof

(2.3.7) $\Longleftrightarrow$ (2.3.8) clear by the preceding lemma.

(2.3.7) $\Longrightarrow$ (2.3.9) It is enough to show

$$F_m(\underline{A})=\{P\in F_{m+1}(\underline{A});\ [P,a]\in F_{m-1}(\underline{A}) \text{ for any } a\in\underline{O}_X\} \text{ for } m\geq 0.$$

This follows from

$$\{u\in S_{m+1}(\Theta_X);\ \{u,\underline{O}_X\}=0\}=0 \quad \text{for } m\geq 0.$$

(2.3.10) $\Longrightarrow$ (2.3.7) Assuming that $S_j(\Theta_X)\to gr_j^F(\Theta_X)$ is an isomorphism for $j<m$, we shall show the surjectivity of $S_m(\Theta_X)\to gr_m^F(\underline{A})$. For $j<m$, let σ_j: $F_j(\underline{A})\to S_j(\Theta_X)$ be the composition $F_j(\underline{A})\to gr_j^F(\underline{A})\cong S_j(\Theta_X)$. Let $x_1,\dots,x_n\in\underline{O}_X$ be such that $dx_1,\dots,dx_n$ forms a base of Ω_X^1. For $P\in F_m(\underline{A})$, set $u_i=\sigma_{m-1}([P,x_i])$. Since $[[P,x_i],x_j]=[[Px_j],x_i]$, $\{u_i,x_j\}=\{u_j,x_i\}$. Hence there exists $u\in S_m(\Theta_X)$ such that $\{u,x_i\}=u_i$. Let $Q\in F_m(\underline{A})$ be an element that gives the image of u by $S_m(\Theta_X)\to gr_m^F(\underline{A})$. Replacing P with $P-Q$, we may assume that $[P,x_i]\in F_{m-2}(\underline{A})$ for any i. Since ψ: $a\to[P,a]$ is a derivation from $\underline{O}_X$ to $gr_{m-1}^F(\underline{A})$ and $\psi(x_i)=0$, we have $\psi=0$. Hence, we have $[P,\underline{O}_X]\in F_{m-2}(\underline{A})$. This shows $P\in F_{m-1}(\underline{A})$.

Q.E.D.

Definition 2.3.3 An $\underline{O}_X$-ring $\underline{A}$ is called twisted ring of differential operators if it admits a filtration $F(\underline{A})$ satisfying (2.3.1)-(2.3.6) and the equivalent conditions (2.3.7)-(2.3.9).

Remark that if $\underline{A}$ is a twisted ring of differential operators, then the filtration $F(\underline{A})$ is uniquely determined by (2.3.3) and (2.3.9). We call $F(\underline{A})$ the order filtration of $\underline{A}$.

2.4 Let $\underline{A}$ be a twisted ring of differential operators. Let $F_1(\underline{A})^*$ be $\underline{Hom}_{\underline{O}_X}(F_1(\underline{A}),\underline{O}_X)$ with the left $\underline{O}_X$-module structure of $F_1(\underline{A})$.

Then, similarly to the de Rham complex, we can define a complex:

$$\underline{O}_X \xrightarrow{d} F_1(\underline{A})^* \xrightarrow{d} \overset{2}{\wedge} F_1(\underline{A})^* \xrightarrow{d} \overset{3}{\wedge} F_1(\underline{A})^* \longrightarrow \cdots.$$

Here $d\colon \overset{p}{\wedge} F_1(\underline{A})^* \longrightarrow \overset{p+1}{\wedge} F_1(\underline{A})^*$ is defined by

$$(df)(P_0\wedge\cdots\wedge P_p)=\sum(-1)^i\sigma_1(P_i)(f(P_0\wedge\cdots\wedge P_{i-1}\wedge P_{i+1}\wedge\cdots\wedge P_p))$$

$$+\sum_{i<j}(-1)^{i+j}f([P_i,P_j]\wedge P_0\wedge\cdots\wedge P_{i-1}\wedge P_{i+1}\wedge\cdots\wedge P_{j-1}\wedge P_{j+1}\wedge\cdots\wedge P_p).$$

The exact sequence $0 \to \underline{O}_X \to F_1(\underline{A}) \to \Theta_X \to 0$ gives $0 \to \Omega_X^1 \to F_1(\underline{A})^* \to \underline{O}_X \to 0$ and we obtain a short exact sequence of complexes.

(2.4.1)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \underline{O}_X \arrow[r, "d"] \arrow[d] & \Omega_X^1 \arrow[r, "d"] \arrow[d] & \Omega_X^2 \arrow[r] \arrow[d] & \Omega_X^3 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \underline{O}_X \arrow[r, "d"] \arrow[d] & F_1(\underline{A})^* \arrow[r, "d"] \arrow[d] & \overset{2}{\wedge} F_1(\underline{A})^* \arrow[r] \arrow[d] & \overset{3}{\wedge} F_1(\underline{A})^* \arrow[r] \arrow[d] & \cdots \\
 & 0 \arrow[r] & \underline{O}_X \arrow[r, "d"] \arrow[d] & \Omega_X^1 \arrow[r] \arrow[d] & \Omega_X^2 \arrow[r] \arrow[d] & \cdots \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

Assume that there exists $i\in F_1(\underline{A})^*$ that is mapped to 1 by $F_1(\underline{A})^* \to \underline{O}_X$ (such an i exists locally). Then $\eta=di$ belongs to Ω_X^2 and satisfies $d\eta=0$. Remark that η corresponds to curvature form.

Take another section i' of $F_1(\underline{A})^*$ satisfying the same property as i and set $\eta'=di'$. Then $\xi=i'-i$ is a 1-form and $\eta'=\eta+d\xi$.

<u>2.5</u> Conversely let η be a closed 2-form. Let us define an $\underline{O}_X$-ring $\underline{A}_\eta$ the $\underline{O}_X$-algebra generated by Θ_X with the fundamental relation

(2.5.1) $\quad j\colon \Theta_X \to \underline{A}_\eta$ is left $\underline{O}_X$-linear,

(2.5.2) $\quad [j(v_1),j(v_2)]=j([v_1,v_2])-\langle\eta,v_1\wedge v_2\rangle$ for $v_1,v_2\in\Theta_X$.

(2.5.3) $\quad [j(v),a]=v(a)$ for $v\in\Theta_X$, $a\in\underline{O}_X$.

Then we can check easily that $\underline{A}_\eta$ is a twisted ring of differential operators.

If $\underline{A}$, i and η are as in §2.4, then $\underline{A} \cong \underline{A}_\eta$.

If η is a closed 2-form and ξ is a 1-form then we have a canonical isomorphism $\underline{A}_\eta \cong \underline{A}_{\eta+d\xi}$ by $\underline{A}_\eta \ni \Theta_X \ni v \longmapsto v - \langle \xi, v \rangle \in \underline{A}_{\eta+d\xi}$.

Proposition 2.5.1 If X is a complex manifold, then a sheaf of twisted differential operators is locally isomorphic to $\underline{D}_X$.

In fact any closed 2-form is locally the exterior derivative of a 1-form.

2.6 Let $\Omega_X^\cdot$ be the de Rham complex $\Omega_X^0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots$ and let $\sigma_{\geq 1}(\Omega_X^\cdot)$ be its subcomplex $0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots$

Theorem 2.6.1 The set of isomorphic classes of twisted rings of differential operators is isomorphic to $H^2(X; \sigma_{\geq 1}(\Omega_X^\cdot))$.

Proof We can calculate $H^2(X; \sigma_{\geq 1}(\Omega_X^\cdot))$ by the Cech cohomology. Let $\mathcal{U} = \{U_i\}$ be an open covering. Then $H^2(\mathcal{U}; \sigma_{\geq 1}(\Omega_X^\cdot))$ is given by

$$\eta_i \in \Gamma(U_i; \Omega_X^2), \quad \xi_{ij} \in \Gamma(U_i \cap U_j; \Omega_X^1)$$

such that

(2.6.1) $d\eta_i = 0, \quad \eta_i - \eta_j = d\xi_{ij}$ on $U_i \cap U_j$

(2.6.2) $\xi_{ij} + \xi_{jk} + \xi_{ki} = 0$ on $U_i \cap U_j \cap U_k$.

Then we can patch twisted rings of differential operators $\underline{A}_{\eta_i}$ on U_i by

$$\underline{A}_{\eta_i}|_{U_i \cap U_j} = \underline{A}_{\eta_j + d\xi_{ij}}|_{U_i \cap U_j} \cong \underline{A}_{\eta_j}|_{U_i \cap U_j}$$

and obtain a globally defined twisted ring of differential operators.

Conversly if $\underline{A}$ is a twisted ring of differential operator, then there exist an open covering $\mathcal{U} = \{U_j\}$ of X and a section i_j: $\Theta_X|_{U_j} \to F_1(\underline{A})|_{U_j}$ of $\sigma_1|_{U_j}$. As in §2.4 i_j defines a closed 2-form η_j, and $i_j - i_k$ gives a 1-form ξ_{jk}, so that (2.1.1) and (2.6.2) are satisfied. Hence they give an element of $H^2(\mathcal{U}; \sigma_{\geq 1}(\Omega_X^{\cdot}))$.

It is easy to see that they do not depend on the choices introduced there and these two correspondences are inverse to each other.

<u>Corollary 2.6.2</u> <u>If</u> X <u>is a complex manifold, the set of the isomorphic classes of twisted rings of differential operators is isomorphic to</u> $H^1(X; d\,\underline{O}_X)$.

<u>In fact,</u> $\sigma_{\geq 1}(\Omega_X^{\cdot})$ <u>is quasi-isomorphic to</u> $d\underline{O}_X[-1]$.

<u>Remark 2.6.3</u> In an algebraic case, a twisted ring of differential operator is not locally isomorphic to $\underline{D}_X$ even in the etale topology. In fact, for a closed 2-form η, $\underline{A}\eta$ is isomorphic to $\underline{D}_X$ if and only if η is a coboundary.

<u>Remark 2.6.4</u> Let $\underline{A}$ be a twisted ring of differential operators. Then

$$\underline{\mathrm{Aut}}(\underline{A}) = \underline{\mathrm{End}}(\underline{A}) \cong H^1(X; \sigma_{\geq 1}\Omega_X^{\cdot}) \cong \mathrm{Ker}(d: \Gamma(X;\Omega_X^1) \to \Gamma(X;\Omega_X^2)).$$

Here $\underline{\mathrm{Aut}}$ and $\underline{\mathrm{End}}$ signify the sheaf of automorphisms and endomorphisms as $\underline{O}_X$-rings. For a closed 1-form ω, the associated automorphism of $\underline{A}$ is $F_1(\underline{A}) \ni P \longmapsto P + \langle\sigma_1(P), \omega\rangle \in F_1(\underline{A})$.

<u>Remark 2.6.5</u> Let $\underline{A}$ be a twisted ring of differential operators and L an invertible $\underline{O}_X$-module. Then $L \otimes_{\underline{O}_X} \underline{A} \otimes_{\underline{O}_X} L^{\otimes -1}$ is also a twisted ring of differential operators. Then the cohomology class $c(L\otimes\underline{A}\otimes L^{\otimes -1}) \in H^2(X; \sigma_{\geq 1}\Omega^{\cdot})$ corresponds to $[L] + c(\underline{A})$. Here $[L]$ is the image of the class of L in $H^1(X; \underline{O}_X^*)$ by the homomorphism

$H^1(X;\underline{O}_X^*) \to H^2(X; \sigma_{\geq 1}\Omega^{\cdot})$ given by $\underline{O}_X^* \xrightarrow{\mathrm{dlog}} \mathrm{Ker}(d^1: \Omega_X^1 \to \Omega_X^2) \to \sigma_{\geq 1}\Omega^{\cdot}[1]$. More generally, for any $\lambda \in k$ (k is the base field when X is algebraic and $k = \mathbb{C}$ when X is a complex manifold), we can define $L^{\otimes\lambda}\otimes\underline{A}\otimes L^{\otimes-\lambda}$ such that $c(L^{\otimes\lambda}\otimes\underline{A}\otimes L^{\otimes-\lambda})=c(\underline{A})+\lambda[L]$. In fact take an open covering $\{U_i\}$ of X and $s_i\in\Gamma(U_i,L)$ such that $L=\underline{O}_X s_i$. Then we can patch $\underline{A}|_{U_i}$ and $\underline{A}|_{U_j}$ by $(\underline{A}|_{U_i})|_{U_i\cap U_j}\ni P\longmapsto (s_i/s_j)^{\lambda}P(s_i/s_j)^{-\lambda}\in(\underline{A}|_{U_j})|_{U_i\cap U_j}$. Remark that for any $a\in\underline{O}_X^{\times}$, $P\mapsto a^{\lambda}Pa^{-\lambda}$ is a well-defined automorphism of $\underline{A}$ (See Remark 2.6.4). Hence if s is an invertible section of L and P is a section of $\underline{A}$, $s^{\lambda}\otimes P\otimes s^{-\lambda}$ gives a section of $L^{\otimes\lambda}\otimes\underline{A}\otimes L^{\otimes-\lambda}$.

Remark 2.6.5 The map from the set of the isomorphic classes of twisted rings of differential operators to $H^2(X;\sigma_{\geq 1}(\Omega_X^{\cdot}))$ is also given as follows.

Let us consider the diagram (2.4.1). Since the columns are exact, it defines a morphism in the derived category $[\underline{O}_X\to\Omega_X^1\to\cdots]\longrightarrow\sigma_{\geq 1}(\Omega_X^{\cdot})[2]$. Hence we obtain $H^0(X;\Omega_X^{\cdot})\to H^2(X;\sigma_{\geq 1}(\Omega_X^{\cdot}))$. The image of $1\in H^0(X;\Omega_X^{\cdot})\subset\Gamma(X;\underline{O}_X)$ gives the corresponding class $c(\underline{A})\in H^2(X;\sigma_{\geq 1}(\Omega_X^{\cdot}))$.

2.7 If $\underline{A}$ is a twisted ring of differential operators, then its opposite ring $\underline{A}^{op}$ is also a twisted ring of differential operators. If $c(\underline{A})\in H^2(X;\sigma_{\geq 1}(\Omega_X^{\cdot}))$ denotes the corresponding cohomology class, then $c(\underline{A}^{op})=[\Omega_X^{\dim X}]-c(\underline{A})$. Here $[\Omega_X^{\dim X}]\in H^2(X;\sigma_{\geq 1}\Omega_X^{\cdot})$ is the one given in Remark 2.6.4. We omit its proof. We just remark that it follows from the following fact:

(2.7.1) If we define φ: $v\mapsto -\eta^{\otimes-1}\otimes v\otimes\eta+\eta^{\otimes-1}\otimes\dfrac{L_{\sigma_1(v)}(\eta)}{\eta}\otimes\eta$, then φ defines a left $\underline{O}_X$-linear isomorphism $F_1(\underline{A}^{op})\to F_1(\omega_X^{\otimes-1}\otimes\underline{A}\otimes\omega_X)$ where $\omega_X=\Omega_X^{\dim X}$ and $\eta\in\omega_X$. The diagram

$$\begin{array}{ccccc}
\underline{O}_X & \longrightarrow & F_1(\underline{A}^{op}) & \xrightarrow{\ \sigma_1\ } & \Theta_X\\
\Big\downarrow{\scriptstyle -1} & & \Big\downarrow & & \Big\| \\
\underline{O}_X & \longrightarrow & F_1(\omega_X^{\otimes-1}\otimes\underline{A}\otimes\omega_X) & \xrightarrow[\ \sigma_1\ ]{} & \Theta_X
\end{array}$$

commutes. Moreover, $\varphi([v_1,v_2])=[\varphi(v_1),\varphi(v_2)]$ for $v_1,v_2\in F_1(\underline{A}^{op})$.

This shows $c(\underline{A}^{op}) = -c(\omega_X^{\otimes -1}\otimes\underline{A}\otimes\omega_X)$ by the construction given by Remark 2.6.5.

2.8 Let $f: X \to Y$ be a morphism of smooth algebraic varieties or complex manifolds. Let $\underline{A}_Y$ be a twisted ring of differential operators on Y. Let $f^*(\underline{A}_Y)$ be $\underline{O}_X \otimes_{f^{-1}\underline{O}_Y} f^{-1}\underline{A}_Y$. Then $f^*(\underline{A}_Y)$ is a right $f^{-1}\underline{A}_Y$-module. Let $\underline{\mathrm{End}}_{\underline{A}_Y}(f^*\underline{A}_Y)$ be the ring of right $f^{-1}\underline{A}_Y$-linear endomorphisms of $f^*\underline{A}_Y$. Let us define subsheaves F_m of $\underline{\mathrm{End}}_{\underline{A}_Y}(f^*\underline{A}_Y)$ inductively by

(2.8.1) $F_m = 0$ for $m < 0$

(2.8.2) $F_m = \{P \in \underline{\mathrm{End}}_{\underline{A}_Y}(f^*\underline{A}_Y);\ [P,\underline{O}_X] \in F_{m-1}\}$ for $m \geq 0$

Set $f^{\#}\underline{A}_Y = \bigcup F_m$.

Proposition 2.8.1 $f^{\#}\underline{A}_Y$ is a twisted ring of differential operators with $F_m(f^{\#}\underline{A}_Y) = F_m$, and we have a Cartesian diagram

(2.8.3)
$$\begin{tikzcd} F_1(f^{\#}\underline{A}_Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & f^*F_1(\underline{A}_Y) \arrow[d] \\ \Theta_X \arrow[r] & f^*\Theta_Y \end{tikzcd} .$$

Proof It is enough to check $F_0 = \underline{O}_X$ and (2.8.3) by Proposition (2.3.2). The other properties are easily derived by the definition of F_m.

Lemma 2.8.2 $\{P \in f^*(\underline{A}_Y);\ [P,a] \in f^*F_{m-1}(\underline{A}_Y)$ for any $a \in \underline{O}_Y\} = f^*F_m(\underline{A}_Y)$

for $m \geq 0$.

Proof Take $y_1,\cdots,y_n \in \underline{O}_Y$ such that $dy_1,\cdots,dy_n$ forms a base, and $v_1,\cdots,v_n \in \Theta_Y$ be its dual base. Then $P \longmapsto [P,y_i]$ gives a homomorphism from $f^*S_m(\Theta_Y) \to f^*S_{m-1}(\Theta_Y)$. If we identify $f^*S(\Theta_Y) = \underline{O}_X \otimes k[v_1,\cdots,v_n]$, then $P \mapsto [P,y_i]$ is given by $\partial/\partial v_i$. Hence for $m \geq 1$, $\{P \in f^*S_m(\Theta_Y);\ [P,y_i] = 0$ for any $i\} = 0$. This shows $\{P \in f^*F_m(\underline{A}_Y);\ [P,y_i] \in f^*F_{m-2}(\underline{A}_Y)\} \subset f^*F_{m-1}(\underline{A}_Y)$. The lemma follows immediately from this.

Proof of Proposition 2.8.1 (continued) If $\varphi\in F_0$, then for $a\in \underline{O}_Y$, $[\varphi(1\otimes 1),a]=0$. Hence $\varphi(1\otimes 1)\in \underline{O}_X$ by the preceding lemma. Hence $\varphi(a\otimes P)=a\varphi(1\otimes 1)P=\varphi(1\otimes 1)a\otimes P$ for $a\in\underline{O}_X$ and $P\in\underline{A}_Y$. Thus $\varphi\in\underline{O}_X$. Assume $\varphi\in F_1$. Then for $a\in\underline{O}_X$, $[\varphi,a]\in F_0$. Hence $a\longmapsto[\varphi,a]\in\underline{O}_X$ gives a derivation of Θ_X. If we denote it v, then $[\varphi(1\otimes 1),a]=v(a)\in\underline{O}_X$. Hence $\varphi(1\otimes 1)\in f^*F_1(\underline{A}_Y)$ and its image on $f^*\Theta_Y$ coincides with the image of v. Hence we have $F_1\to f^*F_1(\underline{A}_Y)\times_{f^*\Theta_Y}\Theta_X$. It is easy to check that this an isomorphism.

2.9 Let $f\colon X\to Y$ and $\underline{A}_Y$ be as in the preceding section. Then $f^*\underline{A}_Y$ has a structure of $(f^{\#}\underline{A}_Y,f^{-1}\underline{A}_Y)$-bimodule. If $\underline{M}$ is a left $\underline{A}_Y$-module, then

$$f^*\underline{M}=\underline{O}_X\otimes_{f^{-1}\underline{O}_Y}f^{-1}\underline{M}\cong f^*\underline{A}_Y\otimes_{f^{-1}\underline{A}_Y}f^{-1}\underline{M}$$

has a structure of left $f^{\#}\underline{A}_Y$-module.

2.10 Let $f\colon X\to Y$ and $g\colon Y\to Z$ be two morphisms of smooth varieties and let $\underline{A}_Z$ be a twisted ring of differential operators on Z. Then we have a canonical isomorphism

$$f^{\#}g^{\#}\underline{A}_Z\cong(g\circ f)^{\#}\underline{A}_Z. \tag{2.10.1}$$

In fact, $g^*\underline{A}_Z$ is a left $g^{\#}\underline{A}_Z$-module. Hence $f^*g^*\underline{A}_Z=(g\circ f)^*\underline{A}_Z$ is a left $f^{\#}g^{\#}\underline{A}_Z$-module. Hence we obtain $f^{\#}g^{\#}\underline{A}_Z\to\underline{\mathrm{End}}((g\circ f)^*\underline{A}_Z)$. It is easy to prove that this gives an isomorphism from $f^{\#}g^{\#}\underline{A}_Z$ to the subring $(g\circ f)^{\#}\underline{A}_Z$ of $\underline{\mathrm{End}}((g\circ f)^*\underline{A}_Z)$.

2.11 We have the following lemma, whose proof is left to the reader.

Lemma 2.11.1 Let $f\colon X\to Y$ be a morphism of smooth varieties, and $\underline{A}_Y$ a twisted ring of differential operators on Y. Then

$$f^{\#}(\underline{A}_Y^{op})^{op} \cong \omega_{X/Y} \otimes_{\underline{O}_X} f^{\#}(\underline{A}_Y) \otimes_{\underline{O}_X} \omega_{X/Y}^{\otimes -1}$$

<u>where</u> $\omega_{X/Y} = \Omega_X^{\dim X} \otimes (\Omega_Y^{\dim Y})^{\otimes -1}$.

Since $f^*(\underline{A}_Y^{op})$ is a right $f^{\#}(\underline{A}_Y^{op})^{op}$-module, $f^*(\underline{A}_Y^{op}) \otimes_{\underline{O}_X} \omega_{X/Y}$ is a right $f^{\#}(\underline{A}_Y)$ module by this lemma. Together with the right module structure on $f^*(\underline{A}_Y^{op})$, gives a $(f^{-1}\underline{A}_Y,\ f^{\#}\underline{A}_Y)$-bimodule structure on $f^*(\underline{A}_Y^{op}) \otimes \omega_{X/Y}$. We set

(2.11.1) $$\underline{A}_{Y \leftarrow X} = f^*(\underline{A}_Y^{op}) \otimes \omega_{X/Y} = f^{-1}\underline{A}_Y \otimes_{f^{-1}\underline{O}_Y} \omega_{X/Y}.$$

Then for a left $f^{\#}\underline{A}_Y$ module $\underline{M}$, $f_*(\underline{A}_{Y \leftarrow X} \otimes_{f^{\#}(\underline{A}_Y)} \underline{M})$ is a left $\underline{A}_Y$-module.

§3 Twisted sheaves and regular holonomic modules over twisted rings of differential operators

3.0 We know that the derived category of $\underline{D}_X$-modules with regular holonomic $\underline{D}_X$-modules as cohomology groups is equivalent to the derived category of $\mathbb{C}_X$-modules with constructible cohomologies. In the case of twisted rings of differential operators, we have the similar theories. However, we have to introduce the notion of twisted sheaves that we are going to discuss in this chapter.

3.1 Let $(X,\underline{O}_X)$ be a smooth algebraic variety defined over a field k of characteristic 0 or a complex manifold. The notion of regular holonomic system can be generalized in the case of twisted rings of differential operators.

3.2 Let $\underline{A}$ be a twisted ring of differential operators on X and let $F(\underline{A})$ be the order filtration of $\underline{A}$.

3.3 For a coherent $\underline{A}$-module $\underline{M}$, a filtration $F(\underline{M})$ over $F(\underline{A})$ (i.e. $F_m(\underline{A})F_k(\underline{M}) \subset F_{m+k}(\underline{M})$) is called a good filtration if there exists locally a finite number of sections $\{u_i\}$ of $\underline{M}$ and integers m_i such that $F_k(\underline{M}) = \sum F_{k-m_i}(\underline{A})u_i$. Such a filtration exists always at least locally.

3.4 If $F(\underline{M})$ is a good filtration, then $gr^F\underline{M}$ is a coherent $(gr^F\underline{A})$-module. If we denote $\pi\colon T^*X \to X$, the cotangent bundle of X, then we have a ring homomorphism

$$gr^F\underline{A} \to \pi_*\underline{O}_{T^*X}. \tag{3.4.1}$$

In the algebraic case, (3.4.1) is an isomorphism. We set $Ch(\underline{M}) = \mathrm{supp}(\underline{O}_{T^*X} \otimes_{\pi^{-1}gr^F\underline{A}} \pi^{-1}gr^F\underline{M})$ and call this the characteristic variety of $\underline{M}$. Since this is independent from the choice of $F(\underline{M})$, this is a well-defined closed subset of T^*X.

3.5 We have

Proposition 3.5.1 Ch $\underline{M}$ is always involutive. (i.e. the ideal defining Ch $\underline{M}$ is closed under the Poisson bracket).

In particular the codimension of Ch $\underline{M}$ is $\leq$ dim X at any point of Ch $\underline{M}$.

Definition 3.5.2 A coherent $\underline{A}$-module is called holonomic if codim Ch $\underline{M}$ = dim X.

Let $\underline{M}$ be a holonomic $\underline{D}_X$-module and Λ = Ch $\underline{M}$. If there exists a good filtration $F(\underline{M})$ such that $f|\mathrm{gr}^F\underline{M} = 0$ for any $f \in \mathrm{gr}^F\underline{A}$ with $f|_\Lambda = 0$, then we call $\underline{M}$ regular holonomic.

3.6 If X is an open subset of $\mathbb{C}$ and if $\underline{M} = \underline{D}/\underline{D}P$ with a non-zero differential operator P, $\underline{M}$ is always holonomic. Moreover $\underline{M}$ is regular holonomic on a neighborhood of x=0, if and only if 0 is a regular point of the equation Pu=0 in the classical sense; that is, if we set $P=\sum_{j=0}^{m} a_j(x)\partial^j$, with $a_m \not\equiv 0$, ord $a_j \geq$ ord $a_m-(m-j)$.
Here ord is the order of zero at the origin.

3.7 Since any twisted sheaf of differential operators is locally isomorphic to $\underline{D}_X$ (in the complex case), many properties of regular holonomic $\underline{D}_X$-modules are valid for those over $\underline{A}$. Here are some of their properties.

Proposition 3.7.1 (i) A coherent submodule and a coherent quotient of regular holonomic module is regular.
(ii) If $\underline{M}' \to \underline{M} \to \underline{M}''$ is an exact sequence of coherent modules and if $\underline{M}'$ and $\underline{M}''$ are regular holonomic, then so is $\underline{M}$.

3.8 In this section, we assume X is a smooth algebraic variety, and we work in the algebraic category. Let $j: X \hookrightarrow \bar{X}$ be an embedding into a proper smooth variety $\bar{X}$. For any holonomic $\underline{D}_X$-module $\underline{M}$,

$j_*\underline{M}$ is always holonomic. If $j_*\underline{M}$ is regular holonomic, we say $\underline{M}$ is <u>completely regular</u>. This property does not depend on the embedding j.

Regular holonomicity has the following functorial properties.

<u>Proposition 3.8.1</u> <u>Let $f: X \to Y$ be a morphism</u>

(i) <u>If $\underline{M}$ is a (completely) regular holonomic $\underline{D}_Y$-module, then $\mathcal{T}or_j^{f^{-1}\underline{D}_Y}(f^*\underline{D}_Y, \underline{M})$ is a (completely) regular holonomic $\underline{D}_X$-module</u>.

(ii) <u>If $\underline{M}$ is a completely regular holonomic $\underline{D}_X$-module, then $R^j f_*(\underline{D}_{Y\leftarrow X} \overset{L}{\otimes}_{\underline{D}_X} \underline{M})$ is a completely regular holonomic $\underline{D}_Y$-module</u>.

<u>Proposition 3.8.2</u> <u>Let $f: X \to Y$ be a surjective map of smooth varieties X, Y. Let $\underline{M}$ be a holonomic $\underline{D}_Y$-module. Then $\underline{M}$ is completely regular if and only if $\mathcal{T}or_j^{\underline{D}_Y}(f^*\underline{D}_Y, \underline{M})$ is completely regular for any j.</u>

<u>3.9</u> Let $D(\underline{D}_X)$ be the derived category of the abelian category of $\underline{D}_X$-modules and let $D_{rh}(\underline{D}_X)$ be the full subcategory of $D(\underline{D}_X)$ consisting of bounded complexes with regular holonomic cohomology groups.

<u>3.10</u> Assume X complex analytic. Let $D(\underline{\mathbb{C}}_X)$ be the derived category of sheaves of $\mathbb{C}$-vector spaces and let $D_c(\underline{\mathbb{C}}_X)$ be its full subcategory consisting of bounded complexes whose cohomology groups are constructible. Recall that a sheaf F is called <u>constructible</u> if there exists a complex analytic stratification on whose strata F is locally constant of finite rank.

<u>3.11</u> Now the Riemann-Hilbert correspondence says

<u>Theorem 3.11.1</u> <u>Let X be a complex manifold</u>

$$\mathbb{R}\mathcal{H}om_{\underline{D}_X}(\underline{O}_X, *): D_{rh}(\underline{D}_X) \to D_c(\underline{\mathbb{C}}_X)$$

<u>is an equivalence of categories</u>.

An object $F \in D_c(\underline{\mathbb{C}}_X)$ is called perverse, if codim Supp $H^j(F) \geq j$ and codim Supp $\underline{\mathrm{Ext}}^j(F, \underline{\mathbb{C}}_X) \geq j$ for any j. Let $\mathrm{RH}(\underline{D}_X)$ be the category of regular holonomic $\underline{D}_X$-modules and $\mathrm{Perv}\,(\underline{\mathbb{C}}_X)$ the full subcategory of $D_c(\underline{\mathbb{C}}_X)$ consisting of perverse objects. Then

Theorem 3.11.2 $\mathbb{R}\underline{\mathrm{Hom}}_{D_X}(\underline{O}_X, *) : \mathrm{RH}(D_X) \to \mathrm{Perv}(\underline{\mathbb{C}}_X)$ is an equivalence of categories.

Remark 3.11.3 Let X be a proper smooth algebraic variety defined over $\underline{\mathbb{C}}$, and let X_{an} be the underlying complex manifold. Then by GAGA, we have $D_{rh}(\underline{D}_X) \cong D_{rh}(\underline{D}_{X_{an}})$ and $\mathrm{RH}(\underline{D}_X) \cong \mathrm{RH}(\underline{D}_{X_{an}})$. This is also true in twisted cases.

3.12 We shall generalize the Riemann-Hilbert correspondence in the twisted case.

3.13 Let $(X;A)$ be a commutative ringed space. Let us take an open covering $\{U_i\}_{i\in I}$ of X, invertible $A|_{U_i \cap U_j}$-modules L_{ij} and A-linear isomorphism $\varphi_{ijk}: (L_{ij} \otimes L_{jk})|_{U_i \cap U_j \cap U_k} \xrightarrow{\sim} L_{ik}|_{U_i \cap U_j \cap U_k}$ which satisfies

(3.13.1) $\quad L_{ii} \cong A$.

(3.13.2) $\quad \varphi_{iij} = \mathrm{id}_{L_{ij}}, \quad \varphi_{ijj} = \mathrm{id}_{L_{ij}}$.

(3.13.3) For $i,j,k,\ell \in I$, we have a commutative diagram of morphisms of $A|_{U_i \cap U_j \cap U_k \cap U_\ell}$-modules:

$$\begin{array}{ccc} L_{ij} \otimes L_{jk} \otimes L_{k\ell} & \xrightarrow{\varphi_{ijk}} & L_{ik} \otimes L_{k\ell} \\ \downarrow{\scriptstyle \varphi_{jk\ell}} & & \downarrow{\scriptstyle \varphi_{ik\ell}} \\ L_{ij} \otimes L_{j\ell} & \xrightarrow{\varphi_{ij\ell}} & L_{i\ell} \end{array} .$$

In this case, we say $((U_i)_{i\in I}, \{L_{ij}\}, \{\varphi_{ijk}\})$ a twisting data.

Remark that (3.13.1) and (3.13.2) are consequences of (3.13.3).

<u>3.14</u> Let $\underline{T}=(\{U_i\}_{i\in I}, \{L_{ij}\}, \{\varphi_{ijk}\})$ be a twisting data. For an open set Ω of X, a <u>twisted sheaf</u> F on Ω with twist $\underline{T}$ is data $F=\{F_i, \rho_{ij}\}$ with

(3.14.1) F_i is an $A|_{\Omega\cap U_i}$-module,

(3.14.2) $\rho_{ij}: (L_{ij}\otimes F_j)|_{\Omega\cap U_i\cap U_j} \xrightarrow{\sim} F_i|_{\Omega\cap U_i\cap U_j}$,

such that

(3.14.2.1) $\rho_{ii}=1$.

(3.14.2.2) For i,j,k, on $U_i\cap U_j\cap U_k\cap\Omega$

$$\begin{tikzcd} L_{ij}\otimes L_{jk}\otimes F_k \arrow[r, "\varphi_{ijk}"'] \arrow[d, "\rho_{jk}"] & L_{ik}\otimes F_k \arrow[d, "\rho_{ik}"] \\ L_{ij}\otimes F_j \arrow[r, "\rho_{ij}"] & F_i \end{tikzcd}$$

commutes.

Then the category $\underline{M}(\Omega;\underline{T})$ of twisted sheaves on Ω with twist $\underline{T}$ form an abelian category. If $\Omega\subset U_i$ for some U_i, then $\underline{M}(\Omega;\underline{T})$ is equivalent to the category of $(A|_\Omega)$-modules.

Moreover it is a champs in the sense of Giraud [G], i.e.

i) For $F,F'\in M(\Omega;\underline{T})$, $U \mapsto \mathrm{Hom}_{M(U,\underline{T})}(F|_U,F'|_U)$ is a sheaf on Ω.

ii) Let $\Omega=\cup\Omega_j$ be an open covering and let $F_j\in M(\Omega_j,\underline{T})$. If $\psi_{jk}: F_k|_{\Omega_j\cap\Omega_k} \xrightarrow{\sim} F_j|_{\Omega_j\cap\Omega_k}$ is given so that

(3.14.3) $\varphi_{ii} = \mathrm{id}$

(3.14.4) $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$.

Then there exists $F\in M(\Omega;\underline{T})$ and that $\alpha_i:F|_{\Omega_i} \to F_i$ with $\alpha_i\alpha_j^{-1}=\varphi_{ij}$.

<u>3.15</u> Remark that a twisting data $\underline{T}$ gives an element $c(\underline{T})$ of

$H^2(X;A^\times)$. If two twisting data $\underline{T}_1$, T_2 satisfy $c(\underline{T}_1)=c(\underline{T}_2)$, then $M(\Omega;\underline{T}_1)$ and $M(\Omega;\underline{T}_2)$ are equivalent (as a champs). But this equivalence is not unique. In fact the ambiguity is given by $\otimes L$ for a twisted invertible A-module L. Also, note that for any $c \in H^2(X;A^\times)$, there exists a twisting data $\underline{T}$ with $c(\underline{T})=c$.

For a twisting data $\underline{T}=\{L_{ij}\}$, we denote by $\underline{T}^{-1}$ the twisting data $\{L_{ij}^{\otimes -1}\}$.

<u>3.16</u> Let X be a complex manifold and $\underline{A}$ a twisted ring of differential operators. Since $\underline{A}$ is locally isomorphic to $\underline{D}_X$, there exists an open covering $X=\cup U_i$ of X and an $\underline{A}|_{U_i}$-module $\underline{L}_i$ which is an invertible $\underline{O}_{U_i}$-module. Set

$$L_{ij} = \underline{\mathrm{Hom}}_{\underline{A}}(\underline{L}_i|_{U_{ij}}, \underline{L}_j|_{U_{ij}}).$$

Then L_{ij} is an invertible $\mathbb{C}_{U_{ij}}$-module. Moreover $L_{ij}\otimes L_{jk} \xrightarrow{\sim} L_{ik}$ canonically. Thus $\{L_{ij}\}$ defines a twisting data $\underline{T}$ on X. Then we have

$$L_{ji}\otimes \underline{L}_j|_{U_{ij}} \xrightarrow{\sim} \underline{L}_i|_{U_{ij}}.$$

Hence $\underline{L}=\{\underline{L}_j\}$ is a twisted sheaf with twist $\underline{T}^{-1}$. Moreover $\underline{A} \to \underline{\mathrm{End}}(\underline{L})$ defines a structure of $\underline{A}$-module on $\underline{L}$. Then we can define

$$DR(\underline{M}) = \mathbb{R}\underline{\mathrm{Hom}}_{\underline{A}}(\underline{L},\underline{M})$$

for an $\underline{A}$-module. This gives a functor from the derived category of $\underline{A}$-modules to the derived category $D(\underline{T})$ of twisted sheaves with twist $\underline{T}$. Similarly to $\underline{D}_X$, we have the following Riemann-Hilbert correspondence in the twisted case. Let us define $D_{rh}(\underline{A})$ and $D_c(\underline{T})$ just as $D_{rh}(\underline{D}_X)$ and $D_c(\mathbb{C}_X)$.

<u>Theorem 3.16.1</u> $D_{rh}(\underline{A})$ <u>is equivalent to</u> $D_c(\underline{T})$.

<u>Theorem 3.16.2</u> <u>The category of regular holonomic</u> $\underline{A}$<u>-modules is equivalent to the category of twisted perverse sheaves with twist</u> $\underline{T}$.

<u>3.17</u> Let X be a complex manifold and $\underline{A}$ a twisted ring of differential operators on X. Let Y be a closed analytic set. Let $\underline{M}$ be a

regular holonomic $\underline{A}|_{X\setminus Y}$-module which can be extended to a holonomic $\underline{A}$-module defined on X. Then there exists a regular holonomic $\underline{A}$-module ${}^{\pi}\underline{M}$ defined on X satisfying

(3.17.1) $\quad {}^{\pi}\underline{M}_{X\setminus Y} \cong \underline{M}$

(3.17.2) $\quad {}^{\pi}\underline{M}$ has no non-zero coherent submodule supported in Y nor non-zero coherent quotient supported on Y.

This ${}^{\pi}\underline{M}$ is unique and called the minimal extension of $\underline{M}$.

3.18 This can be generalized into an algebraic case. Let X be a smooth algebraic variety, $\underline{A}$ a twisted ring of differential operators. Let $\underline{M}$ be a holonomic $\underline{A}$-module defined on an open set U of X. Then there exists a holonomic $\underline{A}$-module ${}^{\pi}\underline{M}$ defined on X satisfying (3.17.1) and (3.17.2). Such a ${}^{\pi}\underline{M}$ is unique.

3.19 Let X be a complex manifold and $\underline{A}$ a twisted ring of differential operators on X.

Theorem 3.19.1 The set of the isomorphic classes of irreducible regular holonomic $\underline{A}$-modules is isomorphic to the set of pairs (S,F) where S is a Zariski locally closed non-singular connected subset of X and F is an irreducible twisted locally constant sheaf of finite rank on S with twist $\underline{T}$. Here $(S,F)=(S',F')$ if $S\cap S'$ is open dense in both S and S' and if $F|_{S\cap S'} \cong F'|_{S\cap S'}$.

Let $\underline{M}$ be an irreducible regular holonomic. Then $\operatorname{Supp}\underline{M}$ must be irreducible. Let S' be a non-singular locus of $\operatorname{Supp}\underline{M}$. Then $\underline{\mathrm{Ext}}^k_{\underline{A}}(\underline{L},\underline{M})|_S$ vanishes for $k\neq \operatorname{codim} S$ and when $k=\operatorname{codim} S$, this is a twisted local system on some Zariski open subset S of S' with twist $\underline{T}$. Conversely, for (S,F), there exists a regular holonomic $\underline{A}$-module $\underline{M}$ defined on $X\setminus\partial S$ such that $\mathbb{R}\underline{\mathrm{Hom}}_{\underline{A}}(\underline{L},\underline{M})\cong F[-\operatorname{codim} S]$. Then we associate to (S,F) the minimal extension of $\underline{M}$ onto X.

3.20 Let us give an example of twisting data

Example 3.20.1 $X=\mathbb{P}^1=U_0\cup U_1$ with $U_0=\mathbb{P}^1\setminus\{\infty\}$, $U_1=\mathbb{P}^1\setminus\{0\}$. For λ, let $\mathbb{C}_\lambda$ be the invertible $\mathbb{C}_{U_0\cap U_1}$-module with the monodromy $e^{2\pi i\lambda}$. Then $\underline{T} = \{(U_0,U_1);\mathbb{C}_\lambda\}$ defines a twisting data on X. If $e^{2\pi i\lambda}\neq 1$, there is no twisted local system on X.

§4 Equivariant twisted rings of differential operators

4.1 Let X be a complex manifold or a smooth algebraic variety defined over $\mathbb{C}$. Let G be a complex analytic group or algebraic group acting on X. Let $\underline{g}$ be the Lie algebra of G and Θ_X the sheaf of vector field on X. Then the infinitesimal action induces a Lie algebra homomorphism

$$D : \underline{g} \to \Gamma(X; \Theta_X)$$

4.2 Let $\underline{U}_X(g)$ be the ring generated by $\underline{O}_X$ and $\underline{g}$ with the fundamental relation:

(4.2.1) $\underline{O}_X \xrightarrow{i} \underline{U}_X(\underline{g})$ is a ring homomorphism,

(4.2.2) $\underline{g} \xrightarrow{j} U_X(\underline{g})$ is a Lie algebra homomorphism,

(4.2.3) $[j(A), i(a)] = i(D(A)(a))$ for $A \in \underline{g}$ and $a \in \underline{O}_X$.

Then $\underline{U}_X(\underline{g}) \cong \underline{O}_X \otimes U(\underline{g})$, where $U(\underline{g})$ is the enveloping algebra of $\underline{g}$. The multiplication rule of $\underline{O}_X \otimes U(\underline{g})$ is given as follows: $\underline{g}$ acts on $\underline{O}_X$ and $U(\underline{g})$ (by the left multiplication) and hence we have $\underline{g} \to \mathrm{End}(\underline{O}_X \otimes U(\underline{g}))$, which extends to $U(\underline{g}) \to \mathrm{End}(\underline{O}_X \otimes U(\underline{g}))$. Moreover $\underline{O}_X$ acts on $\underline{O}_X \otimes U(\underline{g})$ and we obtain $\underline{O}_X \otimes U(\underline{g}) \to \mathrm{End}(\underline{O}_X \otimes U(\underline{g}))$. This gives the left multiplication of sections of $\underline{O}_X \otimes U(\underline{g})$ on $\underline{O}_X \otimes U(\underline{g})$. This gives the ring structure on $\underline{O}_X \otimes U(\underline{g})$. We can easily prove that $\underline{O}_X \otimes U(\underline{g})$ is isomorphic to $\underline{U}_X(\underline{g})$.

4.3 Let $\tilde{\underline{g}}$ be the kernel of $\underline{O}_X \otimes \underline{g} \to \Theta_X$. Then we have $[\underline{g}, \tilde{\underline{g}}] \subset \tilde{\underline{g}}$ (in $\underline{U}_X(\underline{g})$). If G acts transitively on X, $\tilde{\underline{g}}$ is a vector sub-bundle of $\underline{O}_X \otimes \underline{g}$.

4.4 Let us recall the notion of G-equivariant $\underline{O}_X$-modules. Let us consider

$$(4.4.1) \qquad G \times G \times X \overset{p_1}{\underset{p_3}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{p_2}}}}} G \times X \overset{\mu}{\underset{pr}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{i}}}}} X$$

where pr is the projection, μ the multiplication map $:(\underline{g},x) \to gx$, $i(x) = (1,x)$ and the p_j are given by

$$p_1(g_1,g_2,x) = (g_1,g_2x) , \quad p_2(g_1,g_2,x) = (g_1g_2,x),$$

$$p_3(g_1,g_2,x) = (g_2,x).$$

Then we have $\mu\circ p_1=\mu\circ p_2$ $pr\circ p_2=pr\ p_3$, $\mu\circ p_3=pr\circ p_1$ and $\mu\circ i=pr\circ i=id$. An $\underline{O}_X$-module F is called G-equivariant if an $\underline{O}_{G\times X}$-linear isomorphism $\alpha : \mu^*F \xrightarrow{\sim} pr^*F$ is given such that it satisfies:

(4.4.2)
$$\begin{tikzcd} i^*\mu^*F \arrow[r, "i^*\alpha"] \arrow[d, phantom, "\wr\|"] & i^*pr^*F \arrow[d, phantom, "\wr\|"] \\ F \arrow[r, "id"] & F \end{tikzcd}$$
commutes.

(4.4.3)
$$\begin{tikzcd} p_2^*\mu^*F \arrow[rrr, "p_2^*(\alpha)"] \arrow[d, phantom, "\wr\|"] & & & p_2^*pr^*F \arrow[d, phantom, "\wr\|"] \\ p_1^*\mu^*F \arrow[r, "p_1^*(\alpha)"] & p_1^*pr^*F \cong p_3\mu^*F \arrow[rr, "p_3^*(\alpha)"] & & p_3^*pr^*F \end{tikzcd}$$

commutes.

4.5 For a G-equivariant $\underline{O}_X$-module F and for $g\in G$, let $\mu_g:X\to X$ be the map $x \mapsto gx$. Then we have $\mu_g^*F \xrightarrow{\sim} F$. Let T_g be an inverse homomorphism. Then setting $A\cdot u= \frac{d}{dt}(T_{e^{tA}}u)|_{t=0}$ for $A\in\underline{g}$ and $u\in F$, we obtain a Lie algebra homomorphism $D:\underline{g} \to \underline{End}_{\mathbb{C}}(F)$, which satisfies

$$D(A)au = aD(A)u + D(A)(a)\ u$$

and hence it extends to a ring homomorphism $\underline{U}_X(\underline{g}) \to \underline{End}_{\mathbb{C}}(F)$. Thus F has a structure of left $\underline{U}_X(\underline{g})$-module.

4.6 Similarly to G-equivariant $\underline{O}_X$-modules, we shall define the notion of equivariant twisted rings of differential operators. Let $\underline{A}$ be a twisted ring of differential operators on X. We say that $\underline{A}$ is G-equivariant if an $\underline{O}_X$-ring isomorphism $\alpha : \mu^\#\underline{A} \xrightarrow{\sim} pr^\#\underline{A}$ is given satisfiying the following property:

(4.6.1)
$$\begin{array}{ccc} i^{\#}\mu^{\#}\underline{A} & \xrightarrow{i^{\#}(\alpha)} & i^{\#}pr^{\#}\underline{A} \\ \wr\| & & \wr\| \\ \underline{A} & \xrightarrow{id} & \underline{A} \end{array}$$
commutes.

(4.6.2)
$$\begin{array}{c} p_2^{\#}\mu^{\#}\underline{A} \xrightarrow{\qquad\qquad p_2^{\#}(\alpha)\qquad\qquad} p_2^{\#}pr^{\#}\underline{A} \\ p_1^{\#}\mu^{\#}\underline{A} \xrightarrow{p_1^{\#}(\alpha)} p_1^{\#}pr^{\#}\underline{A} \cong p_3^{\#}\mu^{\#}\underline{A} \xrightarrow{p_3^{\#}(\alpha)} p_3^{\#}pr^{\#}\underline{A} \end{array}$$

commutes.

Let $\underline{A}$ be a G-equivariant twisted ring of differential operators. Since $\mu^*\underline{A}$ is a $\mu^{\#}\underline{A}$-module, we have $pr^{\#}A \xrightarrow{\sim} \mu^{\#}A \to \mu^*\underline{A}$ by operating on $1\otimes 1 \in \mu^*\underline{A}$. Hence we obtain $p^*D_G \to pr^{\#}A \to \mu^*\underline{A}$, where $p\colon G \times X \to G$ is the projection. Thus we obtain $i^*p^*D_G \to i^*\mu^*\underline{A}$. This gives $\underline{g} \to \underline{A}$. This extends to an $\underline{O}_X$-ring homomorphism $\underline{U}_X(\underline{g}) \to \underline{A}$. Note that the composition $\underline{g} \to F_1(\underline{A}) \to \Theta_X$ coincides with D.

<u>4.7</u> Let $\underline{A}$ be a G-equivariant twisted ring of differential operators. Then $pr^{\#}\underline{A} \cong \underline{D}_G \otimes \underline{A}$, and hence $pr^*\underline{A} \subset pr^{\#}\underline{A}$ becomes a subring. A left $\underline{A}$-module $\underline{M}$ is called G-<u>equivariant</u> if $\beta : \mu^*\underline{M} \xrightarrow{\sim} pr^*\underline{M}$ gives a structure of equivariant $\underline{O}_X$-modules and β is $pr^{\#}\underline{A}$-linear (through $\mu^{\#}\underline{A} \xrightarrow{\sim} pr^{\#}\underline{A}$ and the $\mu^{\#}\underline{A}$-module structure on $\mu^*\underline{M}$). If β is only $pr^*\underline{A}$-linear, we call $\underline{M}$ <u>quasi</u>-G-<u>equivariant</u>.

If N is a G-module (see §4.8), then $\underline{A}\otimes_{\underline{g}} N$ has a structure of G-equivariant $\underline{A}$-module.

<u>4.8</u> We shall investigate the description of G-equivariant twisted rings of differential operators and quasi-G-equivariant modules when X is a homogeneous space. Let x be a point of X. Let H be the isotropic subgroup of X at x and let $\underline{h}$ be its Lie algebra. We assume $G/H \xrightarrow{\sim} X$. An H-module of finite dimension is, by definition, a finite-dimensional vector space V with a group morphism $H \to GL(V)$ and we assume that this is algebraic in the algebraic case and holomorphic in the complex analytic case. An <u>H-module</u> is a vector

space with H-action, which is a union of finite-dimensional H-modules. The following is well-known.

Theorem 4.8.1 The category of G-equivariant $\underline{O}_X$-modules is equivalent to the category of H-modules by $\underline{M} \mapsto \underline{M}(x)$, where $\underline{M}(x) = \mathbb{C} \otimes_{\underline{O}_{X,x}} \underline{M}_x$.

Let $\underline{V}_X$ be the inverse functor of $\underline{M} \mapsto \underline{M}(x)$. Then in the analytic case (and in the algebraic case with suitable interpretation), for an H-module V, we have for an open set U of X:

(4.8.1) $\Gamma(U;\underline{V}_X(V)) = \{f;$V-valued function on $p^{-1}U$ such that $f(gh) = h^{-1}f(g)$ for $g \in p^{-1}U$ and $h \in H\}$.

Here $p:G \to X$ is the projection $g \mapsto gx$. Note that

(4.8.2) $\tilde{\underline{g}} = \underline{V}_X(\underline{h})$ (see §4.3)

Also note that

(4.8.3) If V is a G-module and W is an H-module,

$$\underline{V}_X(V\otimes W) \cong \underline{V}_X(W) \underset{\mathbb{C}}{\otimes} V.$$

4.9 Let λ be an H-invariant element of $\underline{h}^*$. Then $\lambda([\underline{h},\underline{h}])=0$ and hence λ gives a 1-dimensional representation $\mathbb{C}_\lambda = \mathbb{C}\cdot 1_\lambda$ of $\underline{h}$ by $A\cdot 1_\lambda = \lambda(A)1_\lambda$ for $A \in \underline{h}$. On the other hand, λ gives an H-linear homomorphism from $\underline{h}$ to $\mathbb{C}$ and hence a G-equivariant homomorphism $\underline{V}_X(h) \cong \tilde{\underline{g}}$ to $\underline{V}_X(\mathbb{C}) = \underline{O}_X$. Then we can easily check that $\sum_{A\in\tilde{\underline{g}}} \underline{U}_X(\underline{g})(A-\lambda(A))$ is a both-sided ideal. We set

(4.9.1) $\underline{A}_X(\lambda) = \underline{U}_X(\underline{g})/\sum_{A\in\tilde{\underline{g}}} \underline{U}_X(\underline{g})(A-\lambda(A))$.

Theorem 4.9.2 (i) $\underline{A}_X(\lambda)$ is a G-equivariant twisted ring of differential operators.

(ii) Any G-equivariant twisted ring of differential opetators is isomorphic to $\underline{A}_X(\lambda)$ (for a unique λ).

We shall give only a sketch of the proof. Let $\underline{A}$ be a G-equivariant twisted ring of differential operators.

As in §4.6, we have a ring homomorphism $\underline{U}_X(\underline{g}) \to \underline{A}$. Since X is a homogeneous space, this is surjective, and $\tilde{\underline{g}}$ is mapped into $F_0(\underline{A}) \cong \underline{O}_X$. Since this is H-linear, it comes from some H-invariant $\lambda \in \underline{h}^*$ and we obtain $\underline{A}_X(\lambda) \to \underline{A}$, which is an isomorphism.

4.10 In order to describe quasi-G-equivariant $\underline{A}_X(\lambda)$-modules, we shall introduce the notion of twisted $(\underline{g},H)$-module. Let $\lambda \in \underline{h}^*$ be an H-invariant form.

Definition 4.10.1 A twisted $(\underline{g},H)$-module M with twist λ *is a g-module* $\underline{M}$ *with a structure of* H-*module on* $\mathbb{C}_\lambda \otimes \underline{M}$ *such that*

(4.10.1) *Two h-module structures on* $\mathbb{C}_\lambda \otimes \underline{M}$ *which come from the g-module structure on* $\underline{M}$ *and the* H-*module structure on* $\mathbb{C}_\lambda \otimes \underline{M}$ coincide.

(4.10.2) $\underline{g} \otimes (\mathbb{C}_\lambda \otimes \underline{M}) \to \mathbb{C}_\lambda \otimes \underline{M}$ *given by* $A \otimes 1_\lambda \otimes u \mapsto 1_\lambda \otimes Au$ *is* H-*linear.*

If $\underline{M}$ is an H-module, then $U(\underline{g}) \underset{\underline{h}}{\otimes} (\mathbb{C}_{-\lambda} \otimes \underline{M})$ is a twisted $(\underline{g},H)$-module with twist λ. Here the action of H on $\mathbb{C}_\lambda \otimes U(\underline{g}) \underset{\underline{h}}{\otimes} (\mathbb{C}_{-\lambda} \otimes \underline{M})$ is given by $H \ni h : 1_\lambda \otimes P \otimes 1_{-\lambda} \otimes u \mapsto 1_\lambda \otimes \mathrm{Ad}(h)P \otimes 1_{-\lambda} \otimes hu$.

Theorem 4.10.2 (i) *The category of quasi-*G-*equivariant* $\underline{A}_X(\lambda)$-*modules is equivalent to the category of twisted* $(\underline{g},H)$-*modules with twist* λ.

(ii) *For a twisted* $(\underline{g},H)$-*module* $\underline{M}$ *with twist* λ, *the corresponding quasi-*G-*equivariant* $\underline{A}_X(\lambda)$-*module is isomorphic, as a* G-*equivariant* $\underline{O}_X$-*module, to* $\underline{V}_X(\mathbb{C}_\lambda \otimes \underline{M})$.

We shall give here only the sketch of the proof.

Let $\underline{M}$ be a quasi-G-equivariant $\underline{A}_X(\lambda)$-module. Then $\underline{M}$ has two actions of $\underline{g}$ on $\underline{M}$ which comes from the $\underline{A}_X(\lambda)$-module structure

In general (not nec. X homog.) the same const. gives a D_X-linear homo.
$g \xrightarrow{r} \mathrm{End}_{D_X}(M)$

$\therefore$ M is $(D_X, U(g))$-module and the structure of G-equivariant $\underline{O}_X$-module (see § 4.5). Let α the first action and β the last action. Then $\gamma = \beta - \alpha$ is $\underline{O}_X$-linear since $[\alpha(A), a] = [\beta(A), a] = D(A)(a)$. Since $\underline{g} \otimes \underline{M} \to \underline{M}$ via α is $\underline{g}$-linear with respect to the β-action, we have

$$(4.10.3) \qquad [\beta(A), \alpha(A')] = \alpha([A, A']).$$

γ: $[\beta(A), v] = [\alpha(A), v]$ for $\forall v \in \mathcal{D}_X(\lambda)$

This implies $\gamma : \underline{g} \to \mathrm{End}_{\underline{O}_X}(\underline{M})$ $\mathrm{End}_{\mathcal{D}_X}(M)$ is a Lie algebra homomorphism. Hence we obtain $\gamma : \underline{g} \to \mathrm{End}_{\mathbb{C}}(\underline{M}(x_0))$. For $A \in \underline{h}$, $\gamma(A) = \beta(A) - \alpha(A) = \beta(A) - \lambda(A)$ we have $\beta(A) = \gamma(A) + \lambda(A)$. Since the infinitesimal action of H on $\underline{M}(x_0)$ coincides with β, the $\underline{h}$-module structure of $\underline{M}(x_0)$ by γ is isomorphic to $\mathbb{C}_{-\lambda} \otimes \underline{M}(x_0)$. Therefore $\mathbb{C}_{-\lambda} \otimes \underline{M}(x_0)$ is a twisted $(\underline{g}, H)$-module with twist λ. Conversely let M be a twisted $(\underline{g}, H)$-module with twist λ. $\mathbb{C}_{\lambda} \otimes \underline{M}$ is an H-module. Let $\underline{M} = \underline{V}_X(\mathbb{C}_{\lambda} \otimes \underline{M})$ be the corresponding G-equivariant $\underline{O}_X$-modules. The morphism (4.10.2) gives a $\underline{g}$-action $\gamma : \underline{g} \to \mathrm{End}_{\underline{O}_X}(\underline{M})$ and the G-equivariant structure defines $\beta : \underline{g} \to \mathrm{End}(\underline{O}_X)$. Then $\alpha = \beta - \gamma$ defines and $\underline{A}_X(\lambda)$-module structure on $\underline{M}$.

<u>4.11</u> If moreover $\underline{M}$ is G-equivariant, then we have $\beta = \alpha$. Therefore $\gamma = 0$ and the $\underline{g}$-module structure on $\mathbb{C}_{-\lambda} \otimes \underline{M}(x_0)$ is trivial. The converse is also true and we obtain the following proposition.

<u>Proposition 4.11.1</u> <u>The category of G-equivariant $\underline{A}_X(\lambda)$-modules is equivalent to the category of H-modules $\underline{M}$ such that $\underline{h}$ acts trivially on $\mathbb{C}_{-\lambda} \otimes \underline{M}$.</u>

<u>4.12</u> We have

$$(4.12.1) \qquad \underline{A}_X(\lambda) \cong \underline{V}_X(\mathbb{C}_{\lambda} \otimes (U(\underline{g}) \underset{\underline{h}}{\otimes} \mathbb{C}_{-\lambda})).$$

For a twisted $(\underline{g}, H)$-module M with twist λ and a G-module V, $M \underset{\mathbb{C}}{\otimes} V$ has canonically a structure of twisted $(\underline{g}, H)$-module and

$$(4.12.2) \qquad \underline{V}_X(\mathbb{C}_{\lambda} \otimes (M \underset{\mathbb{C}}{\otimes} V)) \cong \underline{V}_X(\mathbb{C}_{\lambda} \otimes M) \underset{\mathbb{C}}{\otimes} V$$

4.13 In a complex analytic case, we can describe $\underline{A}_X(\lambda)$ as follows. Let $p\colon G \to X$ be the projection $g \mapsto gx_0$. Let F be the sheaf on G defined by

$$F = \{\varphi \in \underline{O}_G;\ R_A\varphi = -\lambda(A)\varphi \ \text{ for any } A \in \underline{h}\}. \tag{4.13.1}$$

Here $R_A\varphi(g) = \frac{d}{dt}\varphi(ge^{tA})|_{t=0}$.

Then F is locally constant along fiber of p with the monodromy corresponding to λ, and F has a structure of $p^{-1}\underline{O}_X$-module. Then $\underline{g}$ acts on F through the left action of G on G. Then $\underline{A}_X(\lambda)$ is the subring of $p_*\underline{End}_{\mathbb{C}}(F)$ generated by $\underline{O}_X$ and $\underline{g}$.

4.14 Let G' be another Lie group and H' its subgroup. Let $\varphi\colon G' \to G$ be a group morphism such that $\varphi(H') \subset H$. Set $X' = G'/H'$, $X = G/H$. Then φ induces the map $f\colon X' \to X$. Let $\underline{h}, \underline{h}'$ be the Lie algebra of H and H'. Let $\lambda \in \underline{h}^*$ be an H-invariant form. Then, we can easily prove

Proposition 4.14.1 (i) $f^{\#}\underline{A}_X(\lambda) = \underline{A}_{X'}(\lambda|\underline{h}')$.

(ii) *For a twisted $(\underline{g},H)$-module $\underline{M}$, we have* $f^*\underline{V}_X(\mathbb{C}_\lambda \otimes \underline{M}) \cong \underline{V}_{X'}(\mathbb{C}_\lambda \otimes \underline{M})$ *as $\underline{A}_{X'}(\lambda|\underline{h}')$-module.*

4.15 For a homogeneous space X with the isotropy subgroup H, we have the following proposition.

Proposition 4.15.1 $\underline{A}_X(\lambda)^{op} \cong \underline{A}_X(2\rho-\lambda)$, *where* $\rho \in \underline{h}^*$ *is given by* $\rho(A) = -\frac{1}{2}\mathrm{tr}_{\underline{g}/\underline{h}}(\mathrm{ad}A)$ *for* $A \in \underline{h}$.

This follows from the following observation. By $\underline{g} \ni A \mapsto -A \in \underline{g}$, we have an anti-isomorphism φ of $\underline{O}_X$-ring $\underline{U}_X(\underline{g})$ onto it self. Then, we have $\varphi(A) = -A + 2\rho(A)$ for $A \in \tilde{\underline{g}}$. Here $\rho\colon \tilde{\underline{g}} \to \underline{O}_X$ is the G-equivariant homomorphism given by $\rho \in \underline{h}^*$.

4.16 Even in algebraic category, any G-equivariant twisted ring $\underline{A}$ of differential operators on homogeneous space of G is locally isomorphic to $\underline{D}_X$ in the Zariski topology. In fact, if $p\colon G \to X$

is a G-equivariant projection, then $p^{\#}\underline{A} \cong \underline{D}_G$. Hence if p has a secton i, then $\underline{A} \cong i^{\#}p^{\#}\underline{A} \cong D_X$. Since $G \to X$ has a section locally in the étale topology, $\underline{A}$ is locally isomorphic to D_X in the étale topology. Hence there exists a non empty set U and an étale map $f: U \to X$ such that $f^{\#}\underline{A}$ is locally isomorphic to D_U. There exists an open set Ω of X such that $f^{-1}\Omega \to \Omega$ is finite and étale. Now, $\underline{A}$ is isomorphic to $\underline{A}_\eta$ for some closed 2-form η defined on Ω, by shrinking Ω if necessary. Since $f^{\#}\underline{A}$ is locally isomorphic to $\underline{D}_U$, $f^*\eta = d\omega$ for some 1-form ω. Hence $\eta = d(f_*\omega)/n$ where n is the number of sheets of $f^{-1}\Omega \to U$. Hence $\underline{A}_\eta \cong \underline{D}_X$ on Ω. Since $\underline{A}$ is G-equivariant, $\underline{A}$ is locally isomorphic to D_X on the G-translates of Ω, which cover X.

§5 Flag variety

5.1 We shall review about flag varieties. Let G be a connected algebraic reductive group defined over $\mathbb{C}$. The set of Borel group forms an algebraic variety and called the flag variety of G. We shall denote it by X. Then G acts on X transitively. For $x \in X$, the isotropy subgroup $B(x)$ at x coincides with the Borel subgroup corresponding to $x \in X$ and $G/B(x) \to X$ $(g \mapsto gx)$ gives an isomorphism.

Let $\underline{b}(x)$ denote the Lie algebra of $\underline{b}(x)$ and $\underline{n}(x) = [\underline{b}(x),\underline{b}(x)]$ the nilpotent part of $\underline{b}(x)$. Then $x \mapsto \underline{b}(x)$ and $x \mapsto \underline{n}(x)$ form G-equivariant vector budles on X. Note that $x \mapsto \underline{b}(x)/\underline{n}(x)$ is the trivial bundle, because the isotropy subgroup $B(x)$ acts on $\underline{b}(x)/\underline{n}(x)$ trivially.

5.2 Let us fix $x_0 \in X$, $B = B(x_0)$, and let U denote the unipotent part of B. Let us take a Cartan subgroup T of B. Then $T \simeq B/U$. Let us denote by $\underline{g},\underline{b},\underline{n}$, and $\underline{t}$ the Lie algebra of G,B,U and T, respectively. Let Δ be the root system of $(\underline{g},t)$ and Δ_+ the set of positive roots consisting of roots appearing as weight of $\underline{b}$. For $\alpha \in \Delta$, let $h_\alpha \in t$ the coroot of α and s_α the simple reflection corresponding to α, i.e. $t^* \ni \lambda \mapsto \lambda - \langle h_\alpha, \lambda \rangle \alpha$. Let W be the Weyl group, i.e. the group generated by s_α's. Recall that we have $W \cong N_G(T)/T$ and we have the Bruhat decomposition:

(5.2.1) $$G = \bigcup_{w \in W} BwB$$

(5.2.2) $$X = \bigcup_{w \in W} Bwx_0$$

(5.2.3) $$X \times X = \bigcup_{w \in W} G(wx_0, x_0).$$

Here w in the right hand side is an element of $N_G(T)$ which gives w by taking mod T. Let $Q \subset \underline{t}^*$ be the $\mathbb{Z}$-module generated by Δ. Set

(5.2.4) $$Q_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z}_+ \alpha.$$

Here $\mathbb{Z}_+$ is the set of non-negative integers.

We say $\lambda \in \underline{t}^*$ is <u>anti-dominant</u> (resp. regular) if $\langle h_\alpha, \lambda \rangle \neq 1,2,3,\ldots$

(resp. $\langle h_\alpha,\lambda\rangle\neq 0$) for any $\alpha\in\Delta_+$. The following lemmas are well-known.

Lemma 5.2.1 The following conditions are equivalent.

(i) λ is anti-dominant.

(ii) For any $w\in W$, $\lambda-w\lambda\notin Q_+\setminus\{0\}$.

Lemma 5.2.2 The following conditions are equivalent.

(i) λ is regular and anti-dominant.

(ii) For any $w\in W$ with $w\neq 1$, we have $\lambda-w\lambda\notin Q_+$.

5.3 Let P be the lattice of weights of T. We regard $P\subset t^*$ and for $\lambda\in P$, let $b\mapsto b^\lambda$ denote the character of B given by $B\to T\to \mathbb{C}^*$, where the last arrow is the character given by λ.

Set

$$P_\pm=\{\lambda\in P;\ \pm\langle\lambda,h_\alpha\rangle\geqq 0\quad\text{for any}\quad\alpha\in\Delta_+\}\tag{5.3.1}$$

and

$$P_{\pm\pm}=\{\lambda\in P;\pm\langle\lambda,h_\alpha\rangle>0\quad\text{for any}\quad\alpha\in\Delta_+\}.\tag{5.3.2}$$

5.4 For $\lambda\in P$ let us denote by $\underline{O}_X(\lambda)$ the G-equivariant line bundle corresponding to the character $B\ni b\mapsto b^\lambda$.

Let $p: G\to X$ be the projection $g\to gx_0$. Then by the definition, for any open set U of X

$$\Gamma(U;\underline{O}_X(\lambda))=\{f\in\Gamma(p^{-1}U;\underline{O}_G);\ f(gb)=b^{-\lambda}f(g)\quad\text{for}\quad(g,b)\in p^{-1}U\times B\}.\tag{5.4.1}$$

The following results are well-known.

Proposition 5.4.1 If $\lambda\notin P_-$, $\Gamma(X;\underline{O}_X(\lambda))=0$ and if $\lambda\in P_-$, $\Gamma(X;\underline{O}_X(\lambda))$ is an irreducible representation of G with lowest weight λ.

Proposition 5.4.2 If $\lambda\in P_{--}$, then $\underline{O}_X(\lambda)$ is ample.

Proof We shall use the criterian of Definition-Theorem 1.2.1 (2). Let V_λ be an irreducible representation of G with lowest weight λ and let v be a lowest weight vector and $u \in (V_\lambda)^*$ be a highest weight vector. Then $f = \langle v, gu \rangle$ gives a section of $\underline{O}_X(\lambda)$. We have $f(1) = \langle v,u \rangle \neq 0$ and $f(w) = \langle v,wu \rangle = 0$ for any $w \neq 1$, because the weight of wu is not $-\lambda$. Hence the corresponding section s of $\underline{O}_X(\lambda)$ satisfies $s(x_0) \neq 0$ and $s(wx_0) = 0$ for $w \neq 1$, $w \in W$. Since $\bigcup_{w \in W \setminus \{1\}} G(wx_0, x_0) = X \times X \setminus \{\text{the diagonal set}\}$, for $x \neq y \in G$, there is $g \in G$ such that $g^{-1}x = x_0$, $g^{-1}y = wx_0$. Hence $(g^*s)(x) \neq 0$, $(g^*s)(y) = 0$. Hence $\underline{O}_X(\lambda)$ satisfies the condition (2) of Definition-Theorem 1.2.1. Q.E.D.

5.5 Let $U(\underline{g})$ be the universal enveloping algebra of $\underline{g}$ and let $Z(\underline{g})$ be the center of $\underline{g}$. By Harish-Chandra's result, we have

$$(5.5.1) \qquad \chi : Z(\underline{g}) \xrightarrow{\sim} \mathbb{C}[\underline{t}^*]^W$$

Let us recall how the isomorphism (5.5.1) is defined. For $P \in Z(\underline{g})$, there exists a unique $f \in U(\underline{t}) \cong S(\underline{t}) \cong \mathbb{C}[t^*]$ such that $P - f \in U(\underline{g})\underline{n}$. Then we set $\chi_\lambda(P) = f(\lambda - \rho)$ for $\lambda \in \underline{t}^*$. Here $\rho = (\sum_{\alpha \in \Delta_+} \alpha)/2$. Then $\chi_\lambda(P)$ is W-invariant polynomial in $\lambda \in \underline{t}^*$, and gives the isomorphism (5.5.1).

If we denote by * the anti-isomorphism of $U(\underline{g})$ given by $\underline{g} \ni A \mapsto -A \in \underline{g}$, then we have

$$(5.5.2) \qquad \chi_\lambda(P^*) = \chi_{-\lambda}(P) \quad \text{for} \quad P \in Z(\underline{g}).$$

§6 Twisted rings of differential operators on the flag variety

6.0 The notations are as in §5.

6.1 We shall study G-equivariant twisted ring of differential operators on the flag variety X. In order to do this, we shall apply Theorem 4.9.2. Since $\underline{b}/[\underline{b},\underline{b}]=\underline{t}$ and B acts trivially on $\underline{t}$, the isomorphic classes of equivariant twisted ring of differential operators are parametrized by $\underline{t}^*$. For $\lambda\in\underline{t}^*$, let us denote by $\underline{D}_\lambda$ the twisted ring of differential operators $\underline{A}_X(\lambda+\rho)$ corresponding to the character $\underline{b}\to\underline{t}\xrightarrow{\lambda+\rho}\mathbb{C}$. By Prop.4.1.5., we have

(6.1.1) $\quad \underline{D}_\lambda^{op}\cong\underline{D}_{-\lambda}.$

The shift ρ is added so that (6.1.1) holds. Hence the ring of differential operators is $\underline{D}_{-\rho}$. For $\mu\in P$, we have

(6.1.2) $\quad \underline{O}_X(\mu)\otimes\underline{D}_\lambda\otimes\underline{O}_X(-\mu)\cong\underline{D}_{\lambda+\mu}.$

6.2 By 4.6 and 4.10, we have a Lie algebra homomorphism $\underline{g}\to\Gamma(X;\underline{D}_\lambda)$, which extends to a ring homomorphism:

(6.2.1) $\quad U(\underline{g})\to\Gamma(X;\underline{D}_\lambda).$

Lemma 6.2 2 Ker χ_λ *is contained in the kernel of* (6.2.1).

Proof Since (6.2.1) is G-equivariant it is enough to show that

$$\operatorname{Ker}\chi_\lambda\to\mathbb{C}_{x_0}\otimes_{\underline{O}_X}\underline{D}_\lambda$$

is the zero map, where $\mathbb{C}_{x_0}=\underline{O}_{X,x_0}/m(x_0)$ with the maximal ideal $m(x_0)$ of $\underline{O}_{X,x_0}$. Note that $\underline{D}_\lambda=\underline{U}_X(\underline{g})/\sum_{A\in\tilde{\underline{g}}}(A-\langle\lambda+\rho,A\rangle)U(\underline{g})$ where $\tilde{\underline{g}}$ is the kernel of $\underline{O}_X\otimes\underline{g}\to\Theta_X$ (See § 4.10). Hence we have

$$\mathbb{C}_{x_0}\otimes\underline{D}_\lambda=U(\underline{g})/\sum_{A\in\underline{b}}(A-\langle\lambda+\rho,A\rangle)U(\underline{g}).$$

For $P\in Z(\underline{g})$, we have

$$P \in \underline{n}U(\underline{g})+f$$

with $f \in U(\underline{t})$ and $\chi_\lambda(P)=f(\lambda+\rho)$. Hence we obtain $P \in \sum_{A \in \underline{b}} \{A-(\lambda+\rho)A\}U(\underline{g})$ if $\chi_\lambda(P)=0$. Q.E.D.

We define

$$(6.2.2) \qquad U_\lambda(\underline{g}) = U(\underline{g})/U(\underline{g})(\mathrm{Ker}(\chi_\lambda\colon Z(\underline{g}) \to \mathbb{C})).$$

<u>Proposition 6.2.3</u> $U_\lambda(\underline{g}) \to \Gamma(X;\underline{D}_\lambda)$ <u>is an isomorphism.</u>

<u>Proof</u> Let $F(U(\underline{g}))$ be the filtration given by $F_m(U(\underline{g})) = F_1(U(\underline{g}))\cdot F_{m-1}(U(\underline{g}))$, $F_1(U(\underline{g})) = \underline{g} \oplus \mathbb{C}$, $F_0(U(\underline{g}))=\mathbb{C}$. Then $\mathrm{gr}^F U(\underline{g}) \tilde{\to} S(\underline{g})$. Let $F(U_\lambda(\underline{g}))$ be the induced filtration. Then we have

$$\mathrm{gr}^F U_\lambda(\underline{g}) = S(\underline{g})/S(\underline{g})I_+$$

where $I_+=(\underline{g}S(\underline{g}))^G$. Now, we have the following lemma.

<u>Lemma 6.2.4</u> $\Gamma(T^*X;\ \underline{O}_{T^*X}) = S(\underline{g})/S(\underline{g})I_+$.

For $x \in X$, the infinitesimal action of $\underline{g}$ on X gives $\underline{g} \to T_xX$. Taking the dual, we obtain $T_x^*X \to \underline{g}^*$. This gives $\rho\colon T^*X \to \underline{g}^*$. If we identify $\underline{g}$ with its dual by G-invariant non-degenerate symmetric form, $\rho(T^*X)$ coincides with the set $\underline{N}$ of nilpotent elements. Then $\underline{N}$ is normal and

$$\Gamma(\underline{N};\ \underline{O}_{\underline{N}}) \cong S(\underline{g})/S(\underline{g})I_+.$$

Since ρ is birational and proper, $\Gamma(T^*X;\underline{O}_{T^*X}) \tilde{\to} \Gamma(\underline{N};\underline{O}_{\underline{N}})$.

Q.E.D.

Hence $\Gamma(X;\mathrm{gr}^F\underline{D}_\lambda) \cong \Gamma(T^*X;\underline{O}_{T^*X}) \cong S(\underline{g})/S(\underline{g})I_+ \cong \mathrm{gr}^F U_\lambda$.
Therefore we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{m-1}(U_\lambda) \arrow[r] \arrow[d, "\alpha_{m-1}"] & F_m(U_\lambda) \arrow[r] \arrow[d, "\alpha_m"] & \mathrm{gr}_m^F U_\lambda \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \Gamma(X;F_{m-1}(\underline{D}_\lambda)) \arrow[r] & \Gamma(X;F_m(\underline{D}_\lambda)) \arrow[r] & \Gamma(X;\mathrm{gr}_m^F(\underline{D}_\lambda)) &
\end{tikzcd}$$

Therefore, if α_{m-1} is bijective, α_m is bijective. Thus by the induction, α_m is bijective for every m.

Remark 6.2.5 In the course of the proof, we used the fact that $\rho(T^*X)$ is normal. This is not true if X is a generalized flag manifold (i.e. a projective homogeneous space of G), and $\Gamma(X;\underline{A}_X(\lambda)) \leftarrow U(\underline{g})$ is not necessarily surjective (See [BoB]).

6.3 We shall prove the following theorem.

Theorem 6.3.1 Assume that λ is anti-dominant. Then for any $\underline{D}_\lambda$-module $\underline{M}$ quasi-coherent over $\underline{O}_X$, we have

$$H^k(X;\underline{M})=0 \quad \underline{\text{for}} \quad k\neq 0.$$

Proof If μ is in P_{++}, then $\underline{O}(\mu)$ is ample. Hence by Theorem 1.4.1, it is enough to show that

$$\underline{D}_\lambda \otimes \underline{O}_X(-\mu) \otimes \Gamma(X;\underline{O}_X(\mu)) \to \underline{D}_\lambda \tag{6.3.1}$$

splits. Set $V_\mu = \Gamma(X;\underline{O}_X(\mu))$. Then (6.3.1) corresponds to a morphism of twisted $(\underline{g},B)$-modules

$$U(\underline{g}) \otimes_{\underline{b}} (\mathbb{C}_{-\lambda-\rho-\mu} \otimes V_\mu) \to U(\underline{g}) \otimes \mathbb{C}_{-\lambda-\rho}. \tag{6.3.2}$$

Hence it is enough to show (6.3.2) splits. Let us take a filtration of V_μ by B-modules:

$$V_\mu = V_0 \supset V_1 \supset \cdots \supset V_N \supset V_{N+1} = 0 \tag{6.3.3}$$

such that

$$V_0/V_1 = \mathbb{C}_\mu \tag{6.3.4}$$

$$V_j/V_{j+1} \cong \mathbb{C}_{\mu_j} \quad \text{for some } j. \tag{6.3.5}$$

Hence $\mu_0=\mu, \mu_1, \cdots, \mu_N$ are weights of V_μ.

Hence we have

(6.3.6) $\quad \mu_j - \mu \in Q_+$

(6.3.7) $\quad \mu_j - \mu \in Q_+ \setminus \{0\}$ for $j \neq 0$.

Set $M_j = U(\underline{g}) \otimes_{\underline{b}} (\mathbb{C}_{-\lambda-\mu-\rho} \otimes V_j)$. Then we have

(6.3.8) $\quad M_j / M_{j+1} \cong U(\underline{g}) \otimes_{\underline{b}} \mathbb{C}_{-\lambda-\rho-\mu+\mu_j}$,

and (6.3.2) is given by $M_0 \to M_0/M_1$. Hence M_j/M_{j+1} has an infinitesimal character $\chi_{-\lambda-\mu+\mu_j}$.

Lemma 6.3.2 $\quad \chi_{-\lambda-\mu+\mu_j} \neq \chi_{-\lambda}$ for $\mu_j \neq \mu$.

Admitting this lemma for a while, we shall complete the proof of Theorem 6.3.1. We have

$$M_0^{(\chi_{-\lambda})} \xrightarrow{\sim} (M_0/M_1).$$

Here $M_0^{(\chi_{-\lambda})} = \{u \in M_0;\ Pu = \chi_{-\lambda}(P)u$ for any $P \in Z(\underline{g})\}$.

Hence, $M_0 \to M_0/M_1$ splits. Q.E.D.

Proof of Lemma 6.3.2 Assume $\chi_{-\lambda-\mu+\mu_j} = \chi_{-\mu}$. Then there exists $w \in W$ such that $-\lambda-\mu+\mu_j = -w\lambda$. Hence $\mu - \mu_j = \lambda - w\lambda \in Q_+$. Since λ is anti-dominant, $\mu - \mu_j = 0$. this is a contradiction.

Theorem 6.3.3 If λ is regular and anti-dominant, then for any $\underline{D}_\lambda$-module M quasi-coherent over $\underline{O}_X$, M is generated by global sections.

Proof By Theorem 1.3.1, it is enough to show that $\underline{D}_\lambda \otimes \underline{O}_X(-\mu)$ is generated by global sections for $\mu \in P_-$. In order to see this, it is enough to show the morphism

$$\underline{D}_\lambda \otimes \Gamma(X; \underline{O}_X(\mu))^* \longleftarrow \underline{D}_\lambda \otimes \underline{O}_X(-\mu)$$

splits. Consider the corresponding morphism of twisted $(\underline{g}, B)$-modules

(6.3.9) $\quad U(\underline{g}) \otimes_{\underline{b}} (\mathbb{C}_{-\lambda-\rho} \otimes V) \longleftarrow U(\underline{g}) \otimes_{\underline{b}} \mathbb{C}_{-\lambda-\rho-\mu}$.

Here $V = \Gamma(X;\ \underline{O}_X(\mu))^*$ is an irreducible representation with highest weight $-\mu$. Take a filtration of V by B-modules:

$$0=V_{-1}\subset V_0\subset V_1\subset\cdots\subset V_N=V$$

such that

(6.3.10) $\quad V_0 = \mathbb{C}_{-\mu}$

(6.3.11) $\quad V_j/V_{j-1} \cong \mathbb{C}_{\mu j}.$

Then $\mu_0=-\mu,\mu_1,\cdots,\mu_N$ are weights on V. Then $M_j=U(\underline{g})\otimes_{\underline{b}}\mathbb{C}_{-\lambda-\rho}\otimes V_j$ gives a filtration of $M=U(\underline{g})\otimes_{\underline{b}}\mathbb{C}_{-\lambda-\rho}\otimes V$ and $M_j/M_{j-1}\cong U(\underline{g})\otimes_{\underline{b}}\mathbb{C}_{-\lambda-\rho+\mu_j}$. The last module has an infinitesimal character $\chi_{-\lambda+\mu_j}$. If we have

(6.3.12) $\quad \chi_{-\lambda+\mu_j}\neq\chi_{-\lambda-\mu}$ for any $j\neq 0$,

then we have $M^{(\chi_{-\lambda-\mu})}=M_0$ and M_0 is a direct summand of M. Thus (6.3.9) splits.

Finally, we shall prove (6.3.12). If $\chi_{-\lambda+\mu_j}=\chi_{-\lambda-\mu}$, there exists $w\in W$ such that $w(-\lambda+\mu_j)=-\lambda-\mu$. Hence we have $-\mu-w\mu_j=\lambda-w\lambda$. Since $w\mu_j$ is a weight of V, $-\mu-w\mu_j\in Q_+$. Since λ is regular and anti-dominant $\lambda=w\lambda$ and $w=1$. Hence $\mu_j=-\mu$.

Remark 6.3.4 In the situation of Theorem 6.3.3, M is generated by global sections not only as a $\underline{D}_\lambda$-module but as an $\underline{O}_X$-module because so is $\underline{D}_\lambda$.

6.4 Thus we can apply the result of 1.5.

Theorem 6.4.1 *If λ is anti-dominant and regular, then the category of $\underline{D}_\lambda$-modules quasi-coherent over $\underline{O}_X$ is equivalent to the category of $U_\lambda(\underline{g})$-modules.*

Theorem 6.4.2 *If λ is anti-dominant, the category of $U_\lambda(\underline{g})$-modules are equivalent to the category of $\underline{D}_\lambda$-modules $\underline{M}$ quasi-coherent over $\underline{O}_X$ satisfying the following properties*

(a) *$\underline{M}$ is generated by global sections.*

(b) *If a $\underline{D}_\lambda$-submodule $\underline{N}$ of $\underline{M}$ which is quasi-coherent over $\underline{O}_X$ satisfies $\Gamma(X;\underline{N})=0$, then $\underline{N}=0$.*

Remark that finitely generated $U_\lambda(\underline{g})$-modules corresponds to coherent $\underline{D}_\lambda$-modules.

§7 SL_2-case

7.1 We shall exhibit the results in the preceding section in the case of SL_2. Set $G=SL_2$, $\underline{g}=sl_2$. Take a base of $\underline{g}$

(7.1.1) $h=\begin{pmatrix}1 & \\ & -1\end{pmatrix}$, $e=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $f=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$.

The flag manifold X can be identified with $\mathbb{P}^1$. Set $U_0=\mathbb{P}^1\setminus\{\infty\}$, $U_1=\mathbb{P}^1\setminus\{0\}$ and take coordinates x of U_0 and y of U_1 related by $xy=1$. The action of G on X is given by

(7.1.2) $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$: $x \longmapsto \dfrac{ax+b}{cx+d}$, $y \longmapsto \dfrac{dy+c}{by+a}$.

Take $x_0=\infty$ (i.e. $y=0$ in U_1). Then $B=\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\}$ and $\underline{b}=\mathbb{C}h\oplus\mathbb{C}e$. Take $\underline{t}=\mathbb{C}h$. Then $\Delta=\{\alpha,-\alpha\}$ with $\alpha(h)=2$. We have $\rho=\alpha/2$ and $\rho(h)=1$. The center of $U(\underline{g})$ is generated by

$$\Delta=(h-1)^2+4ef=(h+1)^2+4fe.$$

7.2 The infinitesimal action $\underline{g}\to\Theta_X$ is given by

$$h \longmapsto -2x\partial_x=2y\partial_y$$

$$e \longmapsto -\partial_x=y^2\partial_y$$

$$f \longmapsto x^2\partial_x=-\partial_y$$

and hence the kernel $\tilde{\underline{g}}$ of $\underline{O}_X\otimes\underline{g}\longrightarrow\Theta_X$ is generated by $h-2xe$ and $f+xh-x^2e$ (on U_0) and $\rho:\tilde{\underline{g}}\to\underline{O}_X$ is given by $\rho(f+xh-x^2e)=0$, $\rho(h-2xe)=-1$.

7.3 For $\lambda\in\underline{t}^*$, set $c=\lambda(h)$. Hence $\lambda=c\rho$. If c is an integer $\underline{O}_X(\lambda)\cong\underline{O}_X(-c\infty)$ where $\underline{O}_X(-c\infty)$ is the sheaf of meromorphic functions with pole of degree $-c$ at ∞.

7.4 For $\lambda=c\rho\in\underline{t}^*$, $\underline{D}_\lambda$ is given as follows

(7.4.1) i_0: $\underline{D}_\lambda|_{U_0}\xrightarrow{\sim}\underline{D}_{U_0}$, i_1: $\underline{D}_\lambda|_{U_1}\xrightarrow{\sim}\underline{D}_{U_1}$

and $i_1 i_0^{-1}|_{U_0 \cap U_1}$: $\underline{D}_{U_0\cap U_1}$ $\underline{D}_{U_0\ U_1}$ is given by $P \mapsto x^{1+c} P x^{-1-c}$. The homomorphism $\alpha\colon \underline{g} \to \Gamma(X;\underline{D}_\lambda)$ is given by

$$i_0\circ\alpha\colon \quad h \longmapsto -2x\partial_x-(c+1) \qquad i_1\circ\alpha\colon \quad h \longmapsto 2y\partial_y+(c+1)$$

$$e \longmapsto -\partial_x \qquad\qquad e \longmapsto y^2\partial_y+(c+1)y$$

$$f \longmapsto x^2\partial_x+(c+1)x \qquad\qquad f \longmapsto -\partial_y.$$

7.5 For example, let $\underline{M}$ be a $\underline{D}_\lambda$-module given by

$$\underline{M}|_{U_1} \cong \underline{D}_\lambda/\underline{D}_\lambda y \quad \text{and} \quad \underline{M}|_{U_0}=0.$$

Then $\operatorname{supp}\underline{M} = \{x_0\}$, and if we denote by δ the generator 1 mod $\underline{D}_\lambda y$, then

$$\Gamma(X;\underline{M}) \cong \underline{M}_{x_0} \cong \mathbb{C}[\partial_y]\delta \cong \mathbb{C}[f]\delta$$

with the relation $h\delta=(2\partial_y y+c-1)\delta=(c-1)\delta$, $e\delta=(y\partial_y+c)y\delta=0$. Thus we have $\Gamma(X;\underline{M})$ is isomorphic to the Verma module $U(\underline{g})/U(\underline{g})e+U(\underline{g})(h-(c-1))$.

7.6 If λ is not anti-dominant (i.e. $c=1,2,3,\cdots$) then $\underline{O}_X(\lambda+\rho)\cong\underline{O}_X(-(c+1)\infty)$ is a $\underline{D}_\lambda$-module. Since $H^1(X;\underline{O}_X(-(c+1)\infty))\cong\mathbb{C}^c$, the vanishing theorem for $\underline{D}_\lambda$-modules does not hold in this case.

7.7 We shall investigate the case where λ is anti-dominant and not regular, i.e. $\lambda=0$. In this case, for a non-zero $\underline{D}_\lambda$-module $\underline{M}$, $\Gamma(X;\underline{M})$ may vanish. In fact $\underline{O}_X(\lambda+\rho)\cong\underline{O}_X(-\infty)$ gives such an example. The following proposition asserts that essentially this is the only case.

Lemma 7.7.1 *Let $\underline{M}$ be a coherent $\underline{D}_0$-module. Then the following conditions are equivalent*

(i) *As a $\underline{D}_0$-module, $\underline{M}$ is isomorphic to the direct sum of copies of $\underline{O}_X(\rho)$.*

(ii) $\Gamma(X;\underline{M})=0$

and in this case, $\underline{g}$ acts trivially on $\Gamma(X;\underline{O}_X(-\rho)\otimes\underline{M})$ and $\underline{O}_X(\rho)\otimes\Gamma(X;\underline{O}_X(-\rho)\otimes\underline{M}) \xrightarrow{\sim} \underline{M}$.

Proof The last statement follows from (i).

(i) $\Rightarrow$ (ii) trivial.

(ii) $\Rightarrow$ (i) Since $\underline{O}_X(-\rho)\otimes\underline{M}$ is a $\underline{D}_{-\rho}$-module, it is generated by global sections.
Hence we have

$$\underline{D}_{-\rho}{}^m \twoheadrightarrow \underline{O}_X(-\rho)\otimes\underline{M}.$$

Tensoring $\underline{O}_X(\rho)$ from the left we have

$$(\underline{D}_0\otimes\underline{O}_X(\rho))^m \to \underline{M} \to 0.$$

Since $\Gamma(X;\underline{M})=0$, we have

$$(\underline{D}_0\otimes\underline{O}_X(\rho)/\underline{D}_0\Gamma(X;\underline{D}_0\otimes\underline{O}_X(\rho)))^m \twoheadrightarrow \underline{M}.$$

Hence it is enough to show

$$D_0\otimes\underline{O}_X(\rho)/\underline{D}_0\Gamma(X;\underline{D}_0\otimes\underline{O}_X(\rho)) \xrightarrow{\sim} \underline{O}_X(\rho). \tag{7.7.1}$$

In fact, any submodule of $\underline{O}_X(\rho)^m$ has also the same type. We have an exact sequence

$$U(\underline{g})\otimes_{\underline{b}}\mathbb{C}_{-\rho}\otimes \mathbb{C}^2 \to U(\underline{g})\otimes_{\underline{b}}\mathbb{C}_0 \to \mathbb{C} \to 0$$

of twisted $(\underline{g},B)$-modules with twist ρ. Here $\mathbb{C}^2$ is the fundamental representation of G. Correspondingly, we have

$$\underline{D}_0\otimes\mathbb{C}^2 \to \underline{D}_0\otimes\underline{O}_X(\rho) \to \underline{O}_X(\rho) \to 0.$$

This shows the existence of (7.7.1).

Corollary 7.7.2 For any coherent $\underline{D}_0$-module, the kernel and the cokernel of

$$\underline{D}_0\otimes\Gamma(X;\underline{M}) \to \underline{M}$$

are isomorphic to a direct product of copies of $\underline{O}_X(\rho)$.

§8. Singular case

<u>8.1</u> For a simple root α, let P_α be the parabolic subgroup such that $\mathrm{Lie}(P_\alpha) = \underline{t} \oplus \underline{g}_{-\alpha} \oplus \sum_{\beta\in\Delta_+} \underline{g}_\beta$. Let X_α be the set of conjugate subgroups of P_α. Then $X_\alpha \cong G/P_\alpha$. Let $p_\alpha : X \to X_\alpha$ be the projection. Then p_α is a $\mathbb{P}^1$-bundle. For $\lambda \in t^*$ with $\langle\lambda, h_\alpha\rangle=0$, λ defines a character $\mathrm{Lie}(P_\alpha) \to \underline{t} \to \mathbb{C}$, and this defines a G-equivariant twisted ring of differential operators $\underline{A}_{X_\alpha}(\lambda)$ on X_α.

By Proposition 4.14.1, we have

<u>Proposition 8.1.1</u> $p_\alpha^{\#}\underline{A}_{X_\alpha}(\lambda) \cong \underline{D}_{\lambda-\rho}$.

<u>Corollary 8.1.2</u> <u>If</u> $\lambda-\rho$ <u>is anti-dominant, then for any coherent</u> $\underline{A}_{X_\alpha}(\lambda)$<u>-module</u> $\underline{M}$, <u>we have</u> $H^j(X;\underline{M}) = 0$ for $j\neq 0$.

<u>Proof</u> We have $H^j(X;p_\alpha^*\underline{M}) = 0$ for $j\neq 0$ since $p_\alpha^*\underline{M}$ is a $\underline{D}_{\lambda-\rho}$-module. The relations $R^k p_{\alpha *}p_\alpha^*\underline{M} = \underline{M}$ for $k=0$ and $=0$ for $k\neq 0$ imply

$$H^j(X;p_\alpha^*\underline{M}) = H^j(X_\alpha;\underline{M}).$$

<u>Remark 8.1.3</u> Corollary is true for any parabolic subgroup other than P_α.

<u>8.2</u> Let $\lambda\in t^*$ be such that $\langle\lambda,h_\alpha\rangle = 0$. We assume

$$\rho\in P. \tag{8.2.1}$$

This is not a strict condition because we can replace G with a covering group of it. The flag varieties X and X_α do not change after this replacement.

<u>Proposition 8.2.1</u> <u>Let</u> $\underline{M}$ <u>be a coherent</u> $\underline{D}_\lambda$<u>-module</u>.

(i) $R^k p_{\alpha *}\underline{M} = 0$ <u>for</u> $k\neq 0$.

(ii) <u>The following conditions are equivalent</u>.

  (a) $p_{\alpha *}\underline{M} = 0$.

  (b) <u>There exists a coherent</u> $\underline{A}_{X_\alpha}(\lambda)$<u>-module</u> $\underline{N}$ <u>such that</u> $\underline{M} \cong \underline{O}_X(\rho)\otimes p_\alpha^*\underline{N}$ <u>as an</u> $\underline{D}_\lambda$<u>-module</u>.

(c) $\underline{M} \xrightarrow{\sim} \underline{O}_X(\rho)\otimes p_\alpha^* p_{\alpha *}(\underline{O}_X(-\rho)\otimes\underline{M})$

Proof If (b) is satisfied, then

$$p_{\alpha *}(\underline{O}_X(-\rho)\otimes\underline{M}) = p_{\alpha *}p_\alpha^*\underline{N} \cong \underline{N}.$$

Hence $\underline{N}$ is uniquely determined. Thus, these properties are local in X. Locally in X_α, we have

(8.2.2) $X \cong \mathbb{P}_1 \times X_\alpha$

(8.2.3) $\underline{D}_\lambda \cong \underline{D}_0 \boxtimes \underline{D}_{X_\alpha}$.

Hence we can reduce them to Lemma 7.7.1.

8.3 Let $\sum$ be the set of simple roots. For $\lambda\in t^*$, we set

(8.3.1)
$$\Delta_\lambda = \{\alpha\in\Delta;\ \langle h_\alpha,\lambda\rangle = 0\},$$
$$W_\lambda = \{w\in W;\ w\lambda = \lambda\},$$
$$\textstyle\sum_\lambda = \sum \cap \Delta_\lambda.$$

Then Δ_λ is also a root system and W_λ is the Weyl group for Δ_λ; i.e. W_λ is generated by the s_α $(\alpha\in\Delta_\lambda)$.

Let us consider the conditions

(8.3.2) Δ_λ is the lattice generated by $\sum_\lambda$, or equivalently $\sum_\lambda$ is the set of simple roots for Δ_λ.

Then (8.3.2) implies that

(8.3.3) W_λ is generated by s_α $(\alpha\in\sum_\lambda)$.

Theorem 8.3.1 We assume that λ is anti-dominant and satisfies (8.3.2). Then there exists a sequence $\alpha_1,\dots,\alpha_N$ in $\sum_\lambda$ such that for any coherent $\underline{D}_\lambda$-module $\underline{M}$ the following conditions are equivalent.

(8.3.4) $\Gamma(X;\underline{M}) = 0$.

(8.3.5) There exists a filtration $\underline{M} = \underline{M}_0 \supset \underline{M}_1 \supset \dots \supset \underline{M}_N = 0$ by coherent $\underline{D}_\lambda$-module such that $p_{\alpha_j *}(\underline{M}_{j-1}/\underline{M}_j) = 0$ for $j=1,2,\dots,N$.

Proof (8.3.5) implies (8.3.4) because

$$\Gamma(X;\ \underline{M}_{j-1}/\underline{M}_j) = \Gamma(X_{\alpha_j}, p_{\alpha_j *}(\underline{M}_{j-1}/\underline{M}_j)) = 0.$$

We shall prove the inverse implications. Since $\lambda-\rho$ is anti-dominant and regular, $\underline{O}_X(-\rho)\otimes\underline{M}$ is generated by global sections. Hence there exists a sequence

$$\underline{D}_{\lambda-\rho}{}^{n} \twoheadrightarrow \underline{O}_X(-\rho)\otimes\underline{M}.$$

Tensoring $\underline{O}_X(\rho)$ we obtain

$$(\underline{D}_\lambda\otimes\underline{O}_X(\rho))^n \twoheadrightarrow \underline{M}.$$

Hence, setting

$$\tilde{\underline{M}} = \underline{D}_\lambda\otimes\underline{O}_X(\rho)/\underline{D}_\lambda\Gamma(X;\underline{D}_\lambda\otimes\underline{O}_X(\rho))$$

we have $\tilde{\underline{M}}^n \twoheadrightarrow \underline{M}$. Since for a coherent $\underline{D}_\lambda$-module $\underline{N}$, the relation $p_{\alpha_j *}(\underline{N}) = 0$ is invariant by taking coherent quotients of $\underline{N}$, it is enough to show (8.3.5) for $\tilde{\underline{M}}$ for some $\alpha_1,\dots,\alpha_N\in\sum_\lambda$.

Note that

$$\underline{D}_\lambda\otimes\underline{O}_X(\rho) = \underline{V}_X(\mathbb{C}_{\lambda+\rho}\underset{\mathbb{C}}{\otimes}U(\underline{g})\underset{\underline{b}}{\otimes}\mathbb{C}_{-(\lambda+\rho)+\rho}).$$

Set $M_0 = U(\underline{g})\underset{\underline{b}}{\otimes}\mathbb{C}_{-\lambda}$.

Since ρ is regular and integral with respect to Δ_λ, there exists $\alpha_1,\dots,\alpha_N\in\sum_\lambda$ such that, setting $\mu_0=\rho$, $s_{\alpha_j}(\mu_{j-1}) = \mu_j$, μ_N is anti-dominant with respect to $\sum_\lambda$, and $\langle h_{\alpha_j},\mu_{j-1}\rangle = 1,2,3,\dots$ The last property implies $M_j\subset M_{j-1}$, where $M_j = U(\underline{g})\underset{\underline{b}}{\otimes}\mathbb{C}_{-\lambda-\rho+\mu_j}$.

It is easy to see that M_{j-1}/M_j is a twisted $(\underline{g},P_{\alpha_j})$-module with twist λ. Hence if wet set $\underline{N}_j = \underline{V}_{X_{\alpha_j}}(\mathbb{C}_\lambda\otimes(M_{j-1}/M_j))$ it is an $\underline{A}_{X_{\alpha_j}}(\lambda)$-module.

Set $\underline{M}_j = \underline{V}_X(\mathbb{C}_{\lambda+\rho}\otimes M_j)$.

Then $\underline{M}_{j-1}/\underline{M}_j \cong \underline{O}_X(\rho)\otimes p^*_{\alpha_j}(\underline{N}_j)$. Hence it is enough to show that

$\underline{M}_N$ is generated by global sections. In fact, then there is a surjective morphism $\underline{M}_0/\underline{M}_N \to \underline{D}_\lambda \otimes \underline{O}_X(\rho)/\underline{D}_\lambda \Gamma(X;\underline{D}_\lambda \otimes \underline{O}_X(\rho))$.

Let V be an irreducible representation with highest weight ρ. In order to see that $\underline{M}_N$ is generated by global sections, it is enough to construct a surjective morphism

$$U(\underline{g}) \underset{\underline{b}}{\otimes} \mathbb{C}_{-\lambda-\rho} \otimes V \to U(\underline{g}) \underset{\underline{b}}{\otimes} \mathbb{C}_{-\lambda-\rho+\mu_N}.$$

For $\xi \in P$, let V_ξ be the weight space of V with weight ξ.

Set $V' = \bigoplus_{\mu_N \notin \xi + Q_+} V_\xi$. Then, V' is a B-module. Set $V'' = V/V'$. Then the weight ξ of V'' satisfies $\mu_N - \xi \in Q_+$. Moreover V_{μ_N} is a sub-B-module of V''.

Lemma 8.3.2 If ξ is a weight of V'' different from μ_N, then $\chi_{-\lambda+\xi} \neq \chi_{-\lambda+\mu_N}$.

If this lemma is shown, then $U(\underline{g}) \underset{\underline{b}}{\otimes} (\mathbb{C}_{-\lambda-\rho} \otimes \mathbb{C}_{\mu_N})$ is a direct summand of $U(\underline{g}) \underset{\underline{b}}{\otimes} (\mathbb{C}_{-\lambda-\rho} \otimes V'')$. Hence we obtain a surjective homomorphism

$$U(\underline{g}) \underset{\underline{b}}{\otimes} \mathbb{C}_{-\lambda-\rho} \otimes V \to U(\underline{g}) \underset{\underline{b}}{\otimes} (\mathbb{C}_{-\lambda-\rho} \otimes \mathbb{C}_{\mu_N}).$$

This completes the proof of Theorem 8.3.1.

Proof of Lemma 8.3.2 Assume that $\chi_{-\lambda+\xi} = \chi_{-\lambda+\mu_N}$ for a weight ξ of V''. Then

$$\lambda - \xi \in W(\lambda - \mu_N) = W(\lambda - \rho).$$

Therefore there exists w such that $w(\lambda-\xi) = \lambda-\rho$, or $\lambda - w\lambda = \rho - w\xi$. Since $w\xi$ is a weight of V, $\rho - w\xi \in Q_+$. Since λ is anti-dominant, we have $w\lambda = \lambda$. This implies $w \in W_\lambda$. Thus we have $\xi \in W_\lambda \rho = W_\lambda \mu_N$. Since μ_N is regular anti-dominant with respect to $\sum_\lambda$, we have $\xi - \mu_N \in \sum_{\alpha \in \Delta_{+\lambda}} \mathbb{Z}_+ \alpha \subset Q_+$. Since $\mu_N - \xi \in Q_+$, we have $\xi = \mu_N$. Q.E.D.

Remark 8.3.3 For any $\lambda \in t^*$, there exists $w \in W$ such that $w\lambda$ is anti-dominant and satisfies (8.3.2). Hence these two conditions are not severe.

§9 Harish-Chandra modules

9.1 Let G, $\underline{t}$, Δ, $X,\cdots$ be as in §5. Let H be an affine algebraic group with a group morphism $f: H \to G$. Let $\underline{h}$ be the Lie algebra of H.

Proposition 9.1.1 *If M is a $(\underline{g},H)$-module, then $\underline{D}_\lambda \otimes_{\underline{g}} M$ is an H-equivariant $\underline{D}_\lambda$-module. Conversely, if $\underline{M}$ is an H-equivariant $\underline{D}_\lambda$-module, then $\Gamma(\underline{X};\underline{M})$ is a $(\underline{g},H)$-module.*

This follows from §4.7.

9.2 Hence if λ is regular and anti-dominant, the category of finitely generated $(\underline{g},H)$-modules with infinitesimal character χ_λ is equivalent to that of H-equivariant $\underline{D}_\lambda$-modules. When λ is not regular, we need the modification as in Theorem 6.4.2, that we discuss later more precisely.

9.3 Let us assume further

(9.3.1) *The flag variety X of G has finitely many H-orbits.*

Theorem 9.3.1 *Under the condition (9.3.1), for any $\lambda \in \underline{t}^*$, any H-equivariant coherent $\underline{D}_\lambda$-module is regular holonomic.*

Remark 9.3.2 The following statement is false: let X be a projective algebraic smooth variety and G an affine algebraic group acting on X. If X has finitely many G-orbits, then any G-equivariant coherent module over any G-equivariant twisted ring of differential operators is regular holonomic.

When G is reductive, I have no counterexample.

9.4 Proof of Theorem 9.3.1 Let $Z=G/U$ and let $p:Z \to X=G/B$ be the canonical projection. Then p is the principal fiber bundle with the structure group T. Then $p^{\#}\underline{D}_\lambda \cong \underline{D}_Z$ as G-equivariant twisted ring of differential operators. Let $\underline{M}$ be an H-equivariant coherent $\underline{D}_\lambda$-module. Then $\underline{N}=p^*\underline{M}$ is $(H\times T)$-equivariant. It is enough to show that $\underline{N}$ is regular holonomic by Proposition 3.8.2. We shall prove by the induction of the number of $(H\times T)$-orbits in $\operatorname{Supp} \underline{N}$. Let S be an open $(H\times T)$-orbits of $\operatorname{Supp} \underline{N}$. Let $j: S \hookrightarrow Z$ be an embedding. Then there exists an $(H\times T)$-equivariant $\underline{D}_S$-module $\underline{L}$ such that $\underline{N}|_S \cong j_*(\underline{D}_{Z\hookleftarrow S}\otimes_{\underline{D}_S}\underline{L})|_S$. Let $q:H\times T\to S$ be an $H\times T$-equivariant map. Then q is surjective and smooth. Since $q^*\underline{L}$ is $H\times T$-equivariant, it is isomorphic to the direct sum of finite copies of $\underline{O}_H\boxtimes \underline{D}_T/\sum_{A\in \underline{t}}\underline{D}_T(A-\langle\lambda+\rho,A\rangle)$. Hence $q^*\underline{L}$ is completely regular. Therefore $\underline{L}$ is completely regular by Proposition 3.8.2. Hence $\underline{N}"=j_*(\underline{D}_{Z\hookleftarrow S}\otimes_{\underline{D}_S}\underline{L})\cong H^0_S(\underline{N})$ is regular holonomic. Thus we obtain an (H T)-equivariant $\underline{D}_Z$-modules

$$0 \longrightarrow \underline{N}' \longrightarrow \underline{N} \longrightarrow \underline{N}".$$

Since $\operatorname{Supp}\underline{N}'\subset \operatorname{Supp}\underline{N}\backslash S$, $\underline{N}'$ is regular holonomic by the hypothesis of the induction. Hence $\underline{N}$ is also regular holonomic.

9.5 Let $\underline{M}$ be an irreducible H-equivariant coherent $\underline{D}_\lambda$-module (i.e. there is no proper H-equivariant coherent sub-$\underline{D}_\lambda$-module). Then $\operatorname{Supp}\underline{M}$ is the closure of an H-orbit S. In fact, $\underline{M}\to \underline{H}^0_S(M)$ must be injective. Furthermore $\underline{M}$ must be the minimal extension of $\underline{M}|_{X\setminus\partial S}$. Here $\partial S=\bar{S}\backslash S$. Let $j: S\hookrightarrow X$ be the embedding. Then there exists an H-equivariant $j^{\#}\underline{D}_\lambda$-module $\underline{N}$ such that $\underline{M}|_{X\setminus\partial S}\cong j_*(\underline{D}_{\lambda,X\hookleftarrow S}\otimes_{j^{\#}\underline{D}_\lambda}\underline{N})|_{X\setminus\partial S}$.

Since $\underline{N}$ is an H-equivariant module, it is described as in §4.11. Namely, take an $x\in S$ and let H_x be the isotropy subgroup. Then we obtain $H_x\to B(x)\to T$ and corresponding map $\operatorname{Lie}(H_x)\to \underline{t}$. Then $\underline{N}$ is described by H_x-module such that its infinitesimal representation is $\lambda+\rho$.

Let $S'(H,\lambda)$ be the set of isomorphic classes of the triplets (S,x,M), where S is an H-orbit of X, $x\in S$ and M is an irreducible H_x-module such that its infinitesimal representation $\operatorname{Lie}(H_x)\to \operatorname{End}(M)$ coincides with $\operatorname{Lie}(H_x)\to \underline{t}\xrightarrow{\lambda+\rho}\mathbb{C}\subset\operatorname{End}(M)$. Here, $(S,x,M)\cong(S',x',M')$ if $S=S'$, $x'=h_0x$ for some $h_0\in H$ and there exists $\psi:M\xrightarrow{\sim}M'$ such that $\psi(hu)=(h_0hh_0^{-1})u$ for $h\in H_x$ and $u\in M$.

Remark 9.5.1 If $H\hookrightarrow G$, then M must be one-dimensional representation. In fact, if we denote by $U(x)$ the unipotent part of $B(x)$, then $H\cap U(x)$ is connected and its infinitesimal action on M is trivial. Hence M is a representation of $H_x/H\cap U(x)\subset B(x)/U(x)\cong T$.

Theorem 9.5.2 The set of the isomorphic classes of irreducible H-equivariant coherent $\underline{D}_\lambda$-modules is isomorphic to $S'(H,\lambda)$.

9.6 As the Corollary of Theorem 9.5.2 and § 9.2, we obtain the following theorem.

Theorem 9.6.1 Assume that X has finitely many H-orbits and let $\lambda \in \underline{t}^*$ be regular anti-dominant. Then the set of the isomorphic classes of irreducible $(\underline{g},H)$-modules is isomorphic to $S'(H,\lambda)$.

Remark 9.6.2 As seen in §3, the category of regular holonomic $\underline{D}_\lambda$-modules is equivalent to the category of twisted perverse sheaves with the twist $\underline{T}$ corresponding to $\underline{D}_\lambda$. Incidentally, $S'(H,\lambda)$ is isomorphic to the category of the pairs (S,F) of H-orbits S and irreducible H-equivariant twisted sheaves F on S with twist $\underline{T}$.

9.7 Now, we shall investigate the case when λ is anti-dominant and satisfies the condition (8.3.2). We shall use the notations Σ_λ, $p_\alpha : X \to X_\alpha$ as in §8. In this case, irreducible $(\underline{g},H)$-module is obtained as the global sections of a unique irreducible H-equivariant $\underline{D}_\lambda$-module $\underline{M}$ such that $\Gamma(X;\underline{M}) \neq 0$.

We shall interpret the condition $\Gamma(X;\underline{M})=0$ in terms of $(S,x,M) \in S'(H,\lambda)$. If $\Gamma(X;\underline{M})=0$, then by Theorem 8.3.1, there exists $\alpha \in \Sigma_\lambda$ and non-zero coherent submodule $\underline{N}$ of $\underline{M}$ such that $p_{\alpha *}(\underline{N})=0$. The largest $\underline{N}$ among such $\underline{N}$'s must be H-equivariant and hence $\underline{M}=\underline{N}$. Thus $p_{\alpha *}\underline{M}=0$. Let us take a connected covering group $\tilde{G} \to G$ such that ρ is a weight of $\tilde{G}$. Let $\tilde{H}$ be the fiber product of $\tilde{G}$ and H over G. By Proposition 8.2.1, $p_{\alpha *}\underline{M}=0$ is equivalent to the existence of $\underline{A}_{X_\alpha}(\lambda)$-module $\underline{N}$ such that $\underline{M} \cong \underline{O}_X(\rho) \otimes p_\alpha^* \underline{N}$. Hence $\underline{N}$ is an $\tilde{H}$-equivariant $\underline{A}_{X_\alpha}(\lambda)$-module. Let S be an open H-orbit of $\operatorname{Supp} \underline{M}$. Then $\operatorname{Supp} \underline{M} = \bar{S}$, $\operatorname{Supp} \underline{N} = p_\alpha(\bar{S})$ and $\bar{S} = p_\alpha^{-1} p_\alpha(\bar{S})$. Take $x \in S$ and set $y = p_\alpha(x)$. Then $\overline{S \cap p_\alpha^{-1}(y)} = p_\alpha^{-1}(y)$. Since $p_\alpha^{-1}(y) \cong \mathbb{P}^1$, $S \cap p_\alpha^{-1}(y)$, which is an orbit of H_y, must be either $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{C}^*$. Moreover the condition $\underline{M} \cong \underline{O}_X(\rho) \otimes p_\alpha^* \underline{N}$ is equivalent to saying that $\underline{M}|_{S \cap p_\alpha^{-1}(y)}$ is isomorphic to $\underline{O}_X(\rho)^{\oplus m}$ for some m. When $S \cap p_\alpha^{-1}(y) \cong \mathbb{C}$ or $\mathbb{P}^1$, this is simply connected and hence it is true. The remaining case is the case $S \cap p_\alpha^{-1}(y) \cong \mathbb{C}^*$. Let $(S,x,M) \in S'(H,\lambda)$

correspond to $\underline{M}$. Let $\varphi: H_x \to \mathrm{Aut}(M)$ be the action of H_x on M. Similarly let $(p_\alpha(S), y, N)$ corresponds to $\underline{N}$. Then N is a representation of $\tilde{H}_y$ whose infinitesimal action is by λ. Here the suffix signifies the isotropy subgroup at that point and $\lambda: \mathrm{Lie}(\tilde{H}_y) \to \mathbb{C}$ is given by

$$\mathrm{Lie}(\tilde{H}_y) \to \mathrm{Lie}(\tilde{G}_y) \to \mathrm{Lie}(\text{the reductive part of } \tilde{G}_y) \to \underline{t}/\mathbb{C}h_\alpha \xrightarrow{\lambda} \mathbb{C}.$$

Note that $\langle h_\alpha, \lambda\rangle = 0$ because $\alpha \in \Sigma_\lambda$. Moreover $M = \mathbb{C}_{-\rho} \otimes N$ as an $\tilde{H}_x$-module. Hence the condition $p_{\alpha *}\underline{M} = 0$ is interpreted to the condition: $\mathbb{C}_{-\rho}\otimes\varphi: \tilde{H}_x \longrightarrow \mathrm{Aut}(\mathbb{C}_{-\rho}\otimes M)$ extends to $\psi: \tilde{H}_y \longrightarrow \mathrm{Aut}(\mathbb{C}_{-\rho}\otimes M)$ such that $d\psi = \lambda$. Now, we have, as $\tilde{H}_y x = (\tilde{H}_y)^\circ x$,

$$\text{(9.7.1)} \qquad \tilde{H}_y = (\tilde{H}_y)^\circ \cdot \tilde{H}_x.$$

Here $^\circ$ signifies the connected component containing 1. Since $\psi(\tilde{H}_y)$ is in the center of $\mathrm{Aut}(\mathbb{C}_{-\rho}\otimes M)$, in order to extend $\mathbb{C}_{-\rho}\otimes\varphi$ onto $\tilde{H}_y$, it is enough to extend φ to $\psi^\circ: (\tilde{H}_y)^\circ \to \mathrm{Aut}(C_{-\rho}\otimes M)$ with $d\psi^\circ = \lambda$. Let x_1 be one of the points in $p_\alpha^{-1}(y)\setminus S$. Then $(\tilde{H}_{x_1})^\circ = (\tilde{H}_y)^\circ$. Since $\mathbb{C}_\rho$ is a representation of $(\tilde{H}_{x_1})^\circ$, it is enough to extend $\tilde{H}_x \to \mathrm{Aut}(M)$ to $(\tilde{H}_{x_1})^\circ \to \mathrm{Aut}\, M$. Since $\mathrm{Ker}(\tilde{H}\to H)$ acts identically on M, $(\tilde{H}_{x_1})^\circ \to \mathrm{Aut}\, M$ factors through $(H_{x_1})^\circ \to \mathrm{Aut}\, M$. Summing up, we obtain the following.

Let $S(H,\lambda)$ be the subset of $S'(H,\lambda)$ such that $(S,x,M) \in S(H,\lambda)\setminus S'(H,\lambda)$ if and only if (S,x,M) satisfies the following two conditions for some $\alpha \in \Sigma_\lambda$.

$$\text{(9.7.2)} \qquad S \cap p_\alpha^{-1} p_\alpha(x) \cong H_{p_\alpha(x)}/H_x \text{ is not a finite set.}$$

(9.7.3) If $S \cap p_\alpha^{-1} p_\alpha(x) \cong \mathbb{C}^*$, then

$$H_x \cap H_{p_\alpha(x)}{}^\circ \to \mathrm{Aut}(M)$$

extends to $\psi: H_{p_\alpha(x)}{}^\circ \to \mathrm{Aut}(M)$ such that $d\psi = \lambda + \rho$.

Here $\mathrm{Lie}(H_{p_\alpha(x)}) \xrightarrow{\lambda+\rho} \mathbb{C}$ is given as follows: taking $x_1 \in p_\alpha^{-1} p_\alpha(x)\setminus S$,

$$\mathrm{Lie}(H_{p_\alpha(x)}) = \mathrm{Lie}(H_{x_1}) \to \mathrm{Lie}(B(x_1)) \to \underline{t} \xrightarrow{\lambda+\rho} \mathbb{C}.$$

Remark Similarly to the case of $\tilde{H}_{p_\alpha}(x)$, if $H_x \cap (H_{x_1})^\circ \to \mathrm{Aut}(M)$ extends onto $(H_{x_1})^\circ = H_{p_\alpha}(x)^\circ$ as in (9.7.3), then $H_x \cap H_{x_1} \to \mathrm{Aut}(M)$ extends to $\psi \cdot H_{x_1} \to \mathrm{Aut}(M)$ with $d\psi = \lambda + \rho$.

Theorem 9.7.1 Assume that X has finitely many H-orbits and that λ is anti-dominant and satisfies (8.3.2). Then the set of the isomorphic classes of irreducible $(\mathfrak{g}, H)$-modules is equal to $S(H,\lambda)$.

Example 9.7.2 Let us take $G=SL_2$ as in §7. Let us take as H the torus $\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix};\ a \in \mathbb{C}^*\}$. Then the isomorphic classes of $(\mathfrak{g},H)$-modules corresponds to the irreducible representations of $SL_2(\mathbb{R})$.

Now X has three H-orbits, namely, $\{0\}$, $\{\infty\}$ and $S_0 = X \setminus \{0,\infty\}$. Then the isotropy subgroups are given by, $H_0 = H_\infty = H$ and $H_1 = \{\pm 1\}$.

If $\lambda \notin \mathbb{Z}\rho$, then the infinitesimal representation $\lambda+\rho$ of $\mathrm{Lie}(H)$ cannot extend to representation of H. Hence

$$(9.7.4) \quad S(H,\lambda) = \{(S_0,1,M_+),(S_0,1,M_-)\} \quad \text{for} \quad \lambda \notin \mathbb{Z}\rho.$$

Here M_+ is the trivial representation of H_1 and M_- is the other one-dimentional representation of H_1. If $\lambda = -m\rho$ with a positive integer m, then

$$(9.7.5) \quad S(H,\lambda) = (S_0,1,M_+),\ (S_0,1,M_-),\ (\{0\},0,*)(\{\infty\},\infty,*)\}\ .$$

Here $*$ corresponds to the representation of H corresponding to $\lambda+\rho$.

If $\lambda=0$, then $(S_0,1,M_-) \notin S(H,\lambda)$ because M_- extends to the representation of H with infinitesimal representation of H with infinitesimal character ρ. Hence

$$(9.7.6) \quad S(H,\lambda) = \{(S_0,1,M_+),\ \{0\},\{\infty\}\} \quad \text{for } \lambda=0.$$

This coincides with the well-known classification of irreducible representation of $SL_2(\mathbb{R})$. The date $(S_0,1,M_\pm)$ correspond to the principal series (when $\lambda \notin \mathbb{Z}\rho$), and $(\{0\},0,*)$, $(\{\infty\},0,*)$ correspond to discrete series.

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