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§0. Introduction

0.0 By the celebrated work of Beilinson-Bernstein of the vanishing theorem on the D-modules over flag varieties ([BB]), we can study representations of Lie group through the geometry of flag varieties. In this lecture, we review this and add what happens when the infinitesimal characters are not regular.

0.1 Let $G$ be a reductive group and $X$ its flag variety. Let $\mathfrak{g}$ be the Lie algebra of $G$, $\mathfrak{t}$ the Cartan algebra and $\Delta$ the root system. For $\lambda \in \mathfrak{t}^*$, let $\chi_\lambda$ be the corresponding character of the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$. We normalize this so that $\chi_\lambda = \chi_{w\lambda}$ for $w$ in the Weyl group $W$. For $\lambda \in \mathfrak{t}^*$, set $U_\lambda(\mathfrak{g}) = U(\mathfrak{g})/U(\mathfrak{g}) \text{Ker} \chi_\lambda$. 

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Then we can construct a twisted ring of differential operators $D_\lambda$ on $X$ such that $\Gamma(X; D_\lambda) = U_\lambda(g)$. Beilinson-Bernstein's achievements are summarized by the following three theorems (The last one is an easy consequence of the first two)

Theorem A  If $\lambda$ is regular and anti-dominant, any coherent $D_\lambda$-module is generated by global sections.

Theorem B  If $\lambda$ is anti-dominant, then any coherent $D_\lambda$-module $M$ satisfies $H^n(X; M) = 0$ for $n \neq 0$.

Theorem C  If $\lambda$ is anti-dominant and regular, the category of finitely generated $U_\lambda(g)$-modules are equivalent to the category of coherent $D_\lambda$-modules.

In [BK], Brylinski and Kashiwara proved these theorems in a very special case ($\chi_\lambda$ trivial, $M$ is $U$-equivariant) in an ad-hoc manner, in order to prove the Kazhdan-Lustzig conjecture.

0.2 Let $G_R$ be a real semisimple group, $K_R$ a maximal compact subgroup of $G_R$ and let $G$ and $K$ be their complexification. Let $g$ and $k$ be their Lie algebras. Then by Harish-Chandra [H], admissible representation $G_R$ is described by $(g, K)$-modules, so called Harish-Chandra module.

By Theorem C, $(g, K)$-module with infinitesimal character $\chi_\lambda$ is described by $K$-equivariant $D_\lambda$-module.

The structure of irreducible $K$-equivariant $D_\lambda$-module $M$ can be described by using the geometry of $K$-orbits. The crucial point here is that $X$ has only finite many $K$-orbits. First the support of $M$ is a closure of a $K$-orbit $S$. Assume, for the sake of simplicity, $\chi_\lambda$ is the trivial infinitesimal character. Then, $M$ determines a $K$-equivariant local system $F$ on $S$, and $M$ is completely described by the pair $(S, F)$. If $\chi_\lambda$ is not trivial, we have to replace $F$ with a twisted local system. When $\lambda$ is not regular, we have to put some auxiliary condition on $F$ (see §9).

0.3 Except the irregular case, the contents of this article are more or less known. In the appendix of the paper by Hecht, Milicic, Schmid and Wolf [HMSW], we can find also the review of the result
of Beilinson-Bernstein. Also see Ginsburg [G].

0.4 We did not include the following important topics concerning D-modules on the flag variety.

1. The derived category of $D_{\lambda}$-modules are equivalent to that of $D_{w\lambda}$-modules for any $w \in W$ and $\lambda \in t^*$. This is obtained by Beilinson-Bernstein ([BB]2).

2. There is a one-to-one correspondence between K-orbits of X and $G_\mathbb{R}$-orbits of X by Matsuki [M]. This gives the construction of representations of $G_\mathbb{R}$ corresponding to Harish-Chandra modules by W. Schmid - J. Wolf. See [SW], [K].

3. Relations with representation of the Weyl group, the affine Weyl group and their Hecke algebras, Hodge modules, invariant eigendistributions on the group.
§1 Vanishing theorem for cohomology groups of modules over \( \mathcal{O}_X \)-rings.

1.1 Let \((X, \mathcal{O}_X)\) be a commutative ringed space over a commutative ring \(k\). A \((k, \mathcal{O}_X)\)-ring is a sheaf of rings \( \mathcal{A} \) with a ring homomorphism \( \mathcal{O}_X \to \mathcal{A} \) such that the image of \( k + \mathcal{O}_X \to \mathcal{A} \) is contained in the center of \( \mathcal{A} \). We do not assume that the image of \( \mathcal{O}_X \to \mathcal{A} \) is contained in the center of \( \mathcal{A} \).

If there is no afraid of confusion, we simply call \( \mathcal{O}_X \)-ring for a \((k, \mathcal{O}_X)\)-ring. We shall study in this section the criterion for the vanishing of cohomology groups of modules over \( \mathcal{O}_X \)-rings.

1.2 Let us recall Serre's result on ample invertible sheaves. Let \(k\) be a commutative field and let \((X, \mathcal{O}_X)\) be a projective variety over \(k\).

**Definition-Theorem 1.2.1** Let \(L\) be an invertible \(\mathcal{O}_X\)-module. Then the following conditions are equivalent.

1. There exists an integer \(r > 0\) and a closed embedding \(j: X \hookrightarrow \mathbb{P}^N\) such that \(L^\otimes r = j^*\mathcal{O}_{\mathbb{P}^N}(1)\).

2. For any pair of distinct closed points \(x, y\) of \(X\), there exists \(r > 0\) and \(s \in \Gamma(X; L^\otimes r)\) such that \(s(x) = 0\) and \(s(y) \neq 0\).

3. For any coherent sheaf \(F\), \(F \otimes \mathcal{O}_X^\otimes r\) is generated by global sections for \(r \gg 0\) (i.e. \(\Gamma(X; F \otimes \mathcal{O}_X^\otimes r) \otimes \mathcal{O}_X \to F \otimes \mathcal{O}_X^\otimes r\) is surjective).

4. For any coherent \(\mathcal{O}_X\)-module \(F\), \(\Gamma(X; F \otimes \mathcal{O}_X^\otimes r) = 0\) for \(j \neq 0\) and \(r \gg 0\).

If these equivalent conditions are satisfied, we say that \(L\) is ample.

Here, for \(s \in \Gamma(X; L)\) and \(x \in X\), \(s(x)\) is the image of \(s\) in \((O_{X,x}/m_{X,x}) \otimes O_{X,x}\) with the maximal ideal \(m_{X,x}\) of \(O_{X,x}\).

1.3 Let \((X, \mathcal{O}_X)\) be a projective scheme over \(k\) and \(L\) an ample invertible \(\mathcal{O}_X\)-module. Let \(\mathcal{A}\) be an \(\mathcal{O}_X\)-ring. Throughout this section, we assume
1.3.1. $\mathcal{A}$ is quasi-coherent as a left $O_X$-module.

**Theorem 1.3.1** Under the condition (1.3.1), the following conditions are equivalent.

1. For any left $\mathcal{A}$-module $M$, quasi-coherent over $O_X$, $M$ is generated by global sections (i.e. $\mathcal{A} \otimes \Gamma(X;M) \to M$ is surjective).

2. For $n \gg 0$, $\mathcal{A} \otimes L^{\Theta(-n)}$ is generated by global sections.

**Proof.** (1) $\Rightarrow$ (2) trivial.

(2) $\Rightarrow$ (1) $M$ is a union of coherent sub-$O_X$-modules $F$. For such an $F$, there exists a surjective morphism $O_X^N \to F \otimes L^{\Theta n}$ for $n \gg 0$.

Hence $(\mathcal{A} \otimes L^{\Theta n})^N \to \mathcal{A} \otimes F$ is surjective. Since $\mathcal{A} \otimes L^{\Theta n}$ is generated by global sections, there exists $\mathcal{A}^N \to \mathcal{A} \otimes F$. Hence the image of $\mathcal{A} \otimes \Gamma(X;M) \to M$ contains $F$. This shows (2) $\Rightarrow$ (1).

1.4. Let $(X,O_X)$, $L$ and $\mathcal{A}$ be as in the preceding sections.

**Theorem 1.4.1** Under the condition (1.3.1), the following conditions are equivalent.

1. For any left $\mathcal{A}$-module $M$, quasi-coherent over $O_X$, $H^n(X;M) = 0$ for $n \neq 0$.

2. For $r \gg 0$,

$$\Gamma(X;\mathcal{A} \otimes L^{\Theta -r}) \otimes \Gamma(X;L^{\Theta r}) \to \Gamma(X;\mathcal{A})$$

is surjective.

3. For $r > 0$,

$$\mathcal{A} \otimes (L^{\Theta r} \otimes \Gamma(X;L^{\Theta r}*)) \otimes \mathcal{A}$$

has a cosection (i.e. a left inverse) as right $\mathcal{A}$-modules.

4. For $r \gg 0$,
has a section (i.e. right inverse) as a left $A$-module.

Remark For $r \geq 0$, $O_X \otimes \Gamma(X; L^0 r) \to L^0 r$ gives $O_X \otimes L^0 r \otimes \Gamma(X; L^0 r)^*$ and $L^0 r \otimes O_X \otimes \Gamma(X; L^0 r)^*$. The morphisms in (3) and (4) come from them.

Proof (3) $\Leftrightarrow$ (4) follows by the operation of the functor $\text{Hom}_A(*, A)$.

(2) $\Leftrightarrow$ (4) obvious.

(1) $\Rightarrow$ (2) follows from the exact sequence

$$0 \to M \to A \otimes (L^0 r \otimes \Gamma(X; L^0 r)) \to A \to 0 \text{ and } H^1(X; M) = 0$$

(3) $\Rightarrow$ (1).

We have

(1.4.1) $H^n(X; M) = \lim_{\to} H^n(X; F)$

where $F$ ranges over coherent sub-$O_X$-modules of $M$. For such an $F$, we shall show that $H^n(X; F) \to H^n(X; M)$ is the zero map for $n \neq 0$.

We have $H^n(X; F \otimes O^r) = 0$ for $n \neq 0, r > 0$.

Set $V = \Gamma(X; L^0 r)$. By letting $A \otimes M$ operate on $A \otimes (L^0 r \otimes V^*) \otimes A$,

$M \otimes (L^0 r \otimes V^*) \otimes M$ has a cosection by (3).

Now, letting $H^n(X; *)$ operate on a commutative diagram

$$\begin{array}{ccc}
F & \to & L^0 r \otimes V^* \otimes F \\
\downarrow & & \downarrow \\
M & \to & L^0 r \otimes V^* \otimes M
\end{array}$$

we obtain a commutative diagram
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\[ H^n(X;F) \longrightarrow H^n(X;L^{\Theta r} \otimes V^* \otimes F) \]
\[ \downarrow a \downarrow \]
\[ H^n(X;M) \longrightarrow H^n(L^{\Theta r} \otimes V^* \otimes M). \]

Since \( H^n(X;L^{\Theta r} \otimes V^* \otimes F) = 0 \) for \( n \neq 0 \), \( b \circ a = 0 \). Since \( b \) has a cosection \( a = 0 \). By (1.4.1), we have \( H^n(X;M) = 0 \). Q.E.D.

1.5 Let \( (X;O_X) \), \( L \) and \( A \) be as in \( \S 1.3 \). Set \( R = \Gamma(X;A) \).
Let \( \text{Mod}_{\text{qc}}(A) \) be the category of left \( A \)-modules quasi-coherent over \( O_X \) and \( \text{Mod}(R) \) the category of left \( R \)-modules. We define the functors
\[ \Gamma: \text{Mod}_{\text{qc}}(A) \to \text{Mod}(R) \]
and
\[ \Theta: \text{Mod}(R) \to \text{Mod}_{\text{qc}}(A) \]
by
\[ \Gamma: M \to \Gamma(X;M), \Theta: N \to A \otimes_R N. \]

Then \( \Theta \) and \( \Gamma \) are adjoint functors; i.e.
\[ \text{Hom}(N, \Gamma(M)) = \text{Hom}(\Theta(N), M). \]

**Proposition 1.5.1**
(a) If the equivalent conditions of Theorem 1.4.1 are satisfied, then \( \Gamma \) is an exact functor and \( \Gamma \circ \Theta = \text{id} \).
(b) If the equivalent conditions of Theorem 1.3.1 and those of Theorem 1.4.1 are satisfied, then \( \Theta \circ \Gamma = \text{id} \), \( \circ \Gamma = \text{id} \).

**Proof** (a) The first assertion is obvious. Let \( 0 \to M \to R^{(I)} \to R^{(J)} \) be a free resolution. Then we have \( 0 \to A \otimes M \to A^{(I)} \to A^{(J)} \).
Since \( \Gamma(X;*) \) is an exact functor, the rows of the following diagram
\[
\begin{array}{c}
0 \\ \uparrow \\ 0
\end{array}
\begin{array}{c}
\Gamma(X;A \otimes M) \\ \uparrow \|
\Gamma(X;A^{(I)}{}^+ \Gamma(X;A^{(J)})
\end{array}
\begin{array}{c}
M \\ \uparrow \\ A^{(I)} \\ \uparrow \\ A^{(J)}
\end{array}
\]
are exact. Hence \( M \cong \Gamma(X;A \otimes M) \).

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(b) The proof is similar as that of (a). For $M \in \text{Ob} \ \text{Mod}_{qc}(A)$, there exist an exact sequence

$$0 \rightarrow M \rightarrow A(I) \rightarrow A(J).$$

This gives the exact sequence

$$0 \rightarrow \Gamma(X;M) \rightarrow R(I) \rightarrow R(J).$$

Operating $\Theta$, we have

$$0 \rightarrow A\Theta\Gamma(X;M) \rightarrow A(I) \rightarrow A(J).$$

Hence $A \Theta \Gamma(X;M) \rightarrow M$ is an isomorphism.

**Proposition 1.5.2** Assume the equivalent conditions of Theorem 1.4.1. Let $E$ be the full subcategory of $\text{Mod}_{qc}(A)$ consisting of $M$ such that $M$ is generated by global sections and $M$ has no non-zero subobject $N$ such that $\Gamma(X;N) = 0$. Then $\Gamma: E \rightarrow \text{Mod}(R)$ is an equivalence of categories.

**Proof** We shall show first $\Gamma$ is fully faithful. For two objects $M_1$ and $M_2$ of $E$, $\varphi: \text{Hom}(M_1,M_2) \rightarrow \text{Hom}(\Gamma(M_1),\Gamma(M_2))$ is injective because $M_1$ is generated by global sections. Let $f: \Gamma(M_1) \rightarrow \Gamma(M_2)$ be a homomorphism. Since the kernel $N$ of $A\Theta_R\Gamma(M_1) \rightarrow M_2$ satisfies $\Gamma(N) = 0$, the composition of $N \rightarrow A\Theta_R\Gamma(M_1) \rightarrow A\Theta_R\Gamma(M_2) \rightarrow M_2$ is zero, and hence, this gives a homomorphism $g: M_1 \rightarrow M_2$. It is evident that $\varphi(g) = f$.

Let us show $\Gamma: E \rightarrow \text{Mod}_{qc}(R)$ is essentially surjective. For an $R$-module $N$, let $I$ be the set of subobjects $M$ of $A\Theta_RN$ such that $\Gamma(X;M) = 0$. Then $I$ is inductively ordered and the sum of any two subobjects in $I$ belongs again to $I$. Hence $I$ has the largest element $M_0$. Then $M = (A\Theta_RN)/M_0$ is an object of $E$ and satisfies $\Gamma(M) = N$.

**Corollary 1.5.3** Assume the equivalent conditions of Theorem 1.4.1. The set of isomorphic classes of the simple $R$-modules is isomorphic to the set of the isomorphic classes of the simple objects $M$ in $\text{Mod}_{qc}(A)$ satisfying $\Gamma(X;M) \neq 0$.
§2 Twisted ring of differential operators

2.0 Let \( X \) be a complex manifold. Let \( D_X \) be the ring of differential operators on \( X \). We shall call twisted ring of differential operators an \( O_X \)-ring locally isomorphic to \( D_X \). If \( L \) is an invertible \( O_X \)-module, \( L \otimes_{O_X} D_X \otimes_{O_X} L^{-1} \) gives such an example.

In this section, we shall study the properties of such \( O_X \)-rings.

2.1 Let \((X, O_X)\) be either a smooth algebraic variety over a field \( k \) of characteristic 0 or a complex manifold. The following discussions are almost same in the both cases. We shall recall the properties of the sheaf \( D_X \) of differential operators. Let \( \Theta_X \) be the sheaf of tangent vector fields. Let \( F_k(D_X) \) be the sheaf of differential operators of order at most \( k \). Then this gives an increasing filtration called the order filtration of \( D_X \) that satisfy the following properties.

\[
\begin{align*}
(2.1.1) & \quad F_m(D_X) = 0 \quad \text{for } m < 0 \\
(2.1.2) & \quad F_0(D_X) = O_X. \\
(2.1.3) & \quad F_m(D_X) = \{ p \in D_X : [p, O_X] \in F_{m-1}(D_X) \} \quad \text{for } m \geq 0. \\
(2.1.4) & \quad D_X = \bigcup F_m(D_X) \\
(2.1.5) & \quad F_{m_1}(D_X) \cdot F_{m_2}(D_X) \subseteq F_{m_1 + m_2}(D_X) \\
(2.1.6) & \quad [F_m(D_X), F_{m_2}(D_X)] \subseteq F_{m_1 + m_2 - 1}(D_X) \\
(2.1.7) & \quad \text{gr}^F_1(D_X) = F_1(D_X)/F_0(D_X) \cong \Theta_X \\
(2.1.8) & \quad S(\Theta_X) \cong \text{gr}^F D_X = \Theta \text{gr}^F D_X = \Theta F_m(D_X)/F_{m-1}(D_X)
\end{align*}
\]

where \( S(\Theta_X) \) is the symmetric algebra of \( \Theta_X \) over \( O_X \), and the arrow in (2.1.8) is given via (2.1.7).
Let $\sigma_\chi : \mathbb{F}_p(D_\chi) \to S(\Theta_\chi)$ be the homomorphism given by $S(\Theta_\chi) = \text{gr}_D F_{p}(D_\chi)$. Then for $a \in S_P(\Theta_\chi)$ and $b \in S_Q(\Theta_\chi)$ taking $P \in F_p(D_\chi)$ and $Q \in F_q(D_\chi)$, we define

$$(a, b) = \sigma_{p+q-1} ([P, Q]).$$

This does not depend on the choice of $P, Q$. We extend this by the linearity:

$$(a, b) : S(\Theta_\chi) \otimes S(\Theta_\chi) \to S(\Theta_\chi).$$

This is called Poisson bracket. This satisfies the following well-known properties:

1. (2.2.1) $\{a, b\} = -\{b, a\}$
2. (2.2.2) $\{a b, c\} = b \{a, c\} + a \{b, c\}$
3. (2.2.3) $\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0.$
4. (2.2.4) If $v \in \Theta_\chi$ and $a \in \Theta_\chi$, then $\{v, a\} = v(a)$.

The following properties are easily checked.

**Lemma 2.2.1** (2.2.1), (2.2.2) and (2.2.4) characterises $\{\, , \}$.

**Lemma 2.2.2** Let $x_i \in \Theta_\chi$ ($i = 1, \ldots, n = \dim X$) be sections such that $dx_i$ are linearly independent. Then for $m \geq 1$, and $a_i \in S_{m-1}(\Theta_\chi)$ with $\{a_i, x_j\} = \{a_j, x_i\}$, there exists unique $u \in S_m(\Theta_\chi)$ such that $\{a, x_i\} = a_i$.

**Proof** Let $\{v_i\}$ be the dual base of $\{dx_i\}$. Then $S(\Theta_\chi) = O_\chi[v_1, \ldots, v_n]$ and $f, x_j = \frac{\partial f}{\partial v_j}$. This shows immediately this lemma.

2.3 We shall study $O_\chi$-rings with the similar properties as $D_\chi$. Let $A$ be an $O_\chi$-ring with increasing filtration $F(A)$ satisfying

$$\{2.3.1\} \quad \bar{A} = \cup F_m(\bar{A})$$
(2.3.2) \[ O_\chi \cong F_0(A) \]

(2.3.3) \[ F_m(A) = 0 \quad \text{for} \quad m < 0 \]

(2.3.4) \[ F_{m_1}(A) \cdot F_{m_2}(A) \subset F_{m_1 + m_2 + 1}(A) \]

(2.3.5) \[ [F_{m_1}(A), F_{m_2}(A)] \subset F_{m_1 + m_2 - 1}(A). \]

Then \( \text{gr}^F(A) = \mathcal{O}(F_m(A)/F_{m-1}(A)) \) has the structure of commutative ring. Moreover \([*, *]: F_{m_1}(A) \otimes F_{m_2}(A) \to F_{m_1 + m_2 - 1}(A)\) gives the bracket \(\{ , \} \) on \(\text{gr}^F(A)\).

Associating to \( F \in F(A)\), the derivation \[ O_\chi \triangleright \mathbb{R} \times [p, a] \in F_0(A) = O_\chi \]

we obtain \( \text{gr}^F_1(A) \to \mathcal{O}_\chi \).

Assume further

(2.3.6) \( \text{gr}^F_1(A) \to \mathcal{O}_\chi \) is an isomorphism.

This gives a ring homomorphism \( S(\mathcal{O}_\chi) \to \text{gr}^F(A) \). This preserves the bracket \(\{ , \}\).

**Lemma 2.3.1** Under the conditions (2.3.1)-(2.3.6), \( S(\mathcal{O}_\chi) \to \text{gr}^F(A) \) is injective.

**Proof** We shall prove that, for \( m \geq 2 \) the injectivity of \( \varphi_{m-1} \):

\[ S_{m-1}(\mathcal{O}_\chi) \to \text{gr}_{m-1}(A) \]

implies the injectivity of \( \varphi_m: S_m(\mathcal{O}_\chi) \to \text{gr}_m(A) \).

Assume \( \mu \in S_{m}(\mathcal{O}_\chi) \) satisfies \( \varphi_m(\mu) = 0 \). Then for any \( a \in \mathcal{O}_\chi \),

\[ \varphi_{m-1}(\mu(a)) = \{ \varphi_m(\mu), a \} = 0, \]

and hence \( \mu(a) = 0 \). Then Lemma 2.2.2 implies \( \mu = 0 \).

**Proposition 2.3.2** Under the conditions (2.3.1)-(2.3.6), the following conditions are equivalent

(2.3.7) \( S(\mathcal{O}_\chi) \to \text{gr}^F(A) \) is an isomorphism.

(2.3.8) \( F_m(A) = F_{1}(A)F_{m-1}(A) \) for \( m \geq 1 \).
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(2.3.9) $F_m(A) = \{ p \in A; [p, a] \in F_{m-1}(A) \text{ for any } a \in O_X \} \text{ for } m \geq 0$.

(2.3.10) **The condition (2.3.9) holds for any** $m \geq 1$.

**Proof**

(2.3.7) $\iff$ (2.3.8) **clear by the preceding lemma.**

(2.3.7) $\implies$ (2.3.9) **It is enough to show**

$$F_m(A) = \{ p \in F_{m+1}(A); [p, a] \in F_{m-1}(A) \text{ for any } a \in O_X \} \text{ for } m \geq 0.$$  

This follows from

$$\{ u \in S_{m+1}(\theta_X); \{ u, O_X \} = 0 \} = 0 \text{ for } m \geq 0.$$  

(2.3.10) $\implies$ (2.3.7) **Assuming that** $S_j(\theta_X) + gr_j^{F}(\theta_X)$ **is an isomorphism for** $j \leq m$, **we shall show the surjectivity of** $S_m(\theta_X) + gr_m^{F}(\theta_X)$. **For** $j < m$, **let** $\sigma_j : F_j(A) \rightarrow S_j(\theta_X)$ **be the composition** $F_j(A) \rightarrow gr_j^{F}(\theta_X) = S_j(\theta_X)$. **Let** $x_1, \ldots, x_n \in O_X$ **be such that** $dx_1, \ldots, dx_n$ **forms a base of** $\Omega_X^1$. **For** $p \in F_m(A)$, **set** $u_i = \sigma_{m-1}([p, x_i])$. **Since** $[p, x_1], x_3 = [p, x_1], x_3$, $\{ u_i, x_i \} = \{ u_i, x_i \}$. **Hence there exists** $u \in S_m(\theta_X)$ **such that** $\{ u, x_i \} = u_i$. **Let** $q \in F_m^{F}(A)$ **be an element that gives the image of** $u$ **by** $S_m(\theta_X) + gr_m^{F}(\theta_X)$. **Replacing** $F$ **with** $F-Q$, **we may assume that** $[p, x_1] \in F_{m-2}(A)$ **for any** $i$. **Since** $\psi : a + [p, a]$ **is a derivation from** $O_X$ **to** $gr_{m-1}(A)$ **and** $\psi(x_i) = 0$, **we have** $\psi = 0$. **Hence, we have** $[p, O_X] \in F_{m-2}(A)$. **This shows** $p \in F_{m-1}(A)$.

**Q.E.D.**

**Definition 2.3.3** **An** $O_X$-**ring** $A$ **is called twisted ring of differential operators if it admits a filtration** $F(A)$ **satisfying (2.3.1)-(2.3.6) and the equivalent conditions (2.3.7)-(2.3.9).**

**Remark** that if $A$ **is a twisted ring of differential operators, then the filtration** $F(A)$ **is uniquely determined by (2.3.3) and (2.3.9). We call** $F(A)$ **the order filtration of** $A$.

**2.4** **Let** $A$ **be a twisted ring of differential operators. Let** $F_1(A)^*$ **be** $\overline{Hom}_{O_X}(F_1(A), O_X)$ **with the left** $O_X$-**module structure of** $F_1(A)$.
Then, similarly to the de Rham complex, we can define a complex:

\[ \Omega_X^2 \xrightarrow{d} F_1(\mathcal{A})^* \xrightarrow{d} \Lambda F_1(\mathcal{A})^* \xrightarrow{d} \Lambda^2 F_1(\mathcal{A})^* \rightarrow \cdots \]

Here, \( d: \Lambda^p F_1(\mathcal{A})^* \rightarrow \Lambda^{p+1} F_1(\mathcal{A})^* \) is defined by

\[
(d\xi)(P_0^{\cdot \cdot \cdot} P_p) = \sum (-1)^i \sigma_i(P_i)(f(P_0^{\cdot \cdot \cdot} P_{i-1} P_{i+1}^{\cdot \cdot \cdot} P_p))
\]

\[ + \sum_{i<j} (-1)^{i+j}f([P_i, P_j] P_0^{\cdot \cdot \cdot} P_{i-1} P_{i+1}^{\cdot \cdot \cdot} P_{j-1} P_{j+1}^{\cdot \cdot \cdot} P_p). \]

The exact sequence \( 0 \rightarrow \Omega_X \rightarrow F_1(\mathcal{A}) \rightarrow \Theta_X \rightarrow 0 \) gives \( 0 \rightarrow \Omega_X^1 \rightarrow F_1(\mathcal{A})^* \rightarrow \Omega_X^0 \) and we obtain a short exact sequence of complexes.

\[
0 \rightarrow \Omega_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \Omega_X^3 \rightarrow \cdots
\]

(2.4.1)

\[
0 \rightarrow \Omega_X \xrightarrow{d} F_1(\mathcal{A})^* \xrightarrow{d} \Lambda F_1(\mathcal{A})^* \xrightarrow{d} \Lambda^2 F_1(\mathcal{A})^* \rightarrow \cdots
\]

Assume that there exists \( i \in F_1(\mathcal{A})^* \) that is mapped to \( 1 \) by \( F_1(\mathcal{A})^* \) \( \Omega_X \) (such an \( i \) exists locally). Then \( \eta=d i \) belongs to \( \Omega_X^2 \) and satisfies \( d\eta=0 \). Remark that \( \eta \) corresponds to curvature form.

Take another section \( i' \) of \( F_1(\mathcal{A})^* \) satisfying the same property as \( i \) and set \( \eta'=di' \). Then \( \xi=i'-i \) is a 1-form and \( \eta'=\eta+d\xi \).

2.5 Conversely let \( \eta \) be a closed 2-form. Let us define an \( \Omega_X \)-ring \( \Lambda_\eta \) the \( \Omega_X \)-algebra generated by \( \Theta_X \) with the fundamental relation

(2.5.1) \( j: \Theta_X \rightarrow \Lambda_\eta \) is left \( \Omega_X \)-linear,

(2.5.2) \( [j(v_1), j(v_2)] = j([v_1, v_2]) - \langle \eta, v_1^* v_2^* \rangle \) for \( v_1, v_2 \in \Theta_X \).

(2.5.3) \( j(v), a = v(a) \) for \( v \in \Theta_X \), \( a \in \Omega_X \).
Then we can check easily that $\Lambda_\eta$ is a twisted ring of differential operators. 

If $A$, $i$, and $\eta$ are as in §2.4, then $A\cong \Lambda_\eta$.

If $\eta$ is a closed 2-form and $\xi$ is a 1-form then we have a canonical isomorphism $\Lambda_\eta \cong \Lambda_\eta \otimes \mathfrak{d}_\xi$ by $\Lambda_\eta \ni \theta \mapsto \nu \mapsto \nu \cdot \xi, \nu \cdot e^\Lambda_\eta \otimes \mathfrak{d}_\xi$.

**Proposition 2.5.1** If $X$ is a complex manifold, then a sheaf of twisted differential operators is locally isomorphic to $D_X$.

In fact any closed 2-form is locally the exterior derivative of a 1-form.

**2.6** Let $\Omega_X^*$ be the de Rham complex $\Omega_X^0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots$ and let $\sigma_{\geq 1}(\Omega_X)$ be its subcomplex $0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots$.

**Theorem 2.6.1** The set of isomorphic classes of twisted rings of differential operators is isomorphic to $H^2(X; \sigma_{\geq 1}(\Omega_X^*))$.

**Proof** We can calculate $H^2(X; \sigma_{\geq 1}(\Omega_X^*))$ by the Cech cohomology. Let $\mathcal{U} = \{U_i\}$ be an open covering. Then $H^2(\mathcal{U}; \sigma_{\geq 1}(\Omega_X^*))$ is given by

$$\eta_i \in \Gamma(U_i; \Omega_X^2), \, \xi_{ij} \in \Gamma(U_i \cap U_j; \Omega_X^1)$$

such that

$$(2.6.1) \quad d\eta_i = 0, \; \eta_i - \eta_j = d\xi_{ij} \quad \text{ on } U_i \cap U_j$$

$$(2.6.2) \quad \xi_{ij} + \xi_{jk} + \xi_{ki} = 0 \quad \text{ on } U_i \cap U_j \cap U_k.$$

Then we can patch twisted rings of differential operators $A_{\eta_i}$ on $U_i$ by

$$A_{\eta_i} |_{U_i \cap U_j} = A_{\eta_j} + d\xi_{ij} |_{U_i \cap U_j} = A_{\eta_j} |_{U_i \cap U_j}$$

and obtain a globally defined twisted ring of differential operators.
Conversely if \( A \) is a twisted ring of differential operator, then there exist an open covering \( \mathcal{U} = \{ U_j \} \) of \( X \) and a section \( i_j: \theta_X|U_j \to F_1(A)|U_j \) of \( \sigma_{\mathcal{O}_X}(\mathcal{U}) \). As in §2.4 \( i_j \) defines a closed 2-form \( \eta_j \), and \( i_j^{-1}k \) gives a 1-form \( \xi_{jk} \), so that (2.1.1) and (2.6.2) are satisfied. Hence they give an element of \( H^2(\mathcal{U};\sigma_{\mathcal{O}_X}(\mathcal{U})) \).

It is easy to see that they do not depend on the choices introduced there and these two correspondences are inverse to each other.

**Corollary 2.6.2** If \( X \) is a complex manifold, the set of the isomorphic classes of twisted rings of differential operators is isomorphic to \( H^1(X;\mathcal{O}_X) \).

In fact, \( \sigma_{\mathcal{O}_X}(\mathcal{U}) \) is quasi-isomorphic to \( \mathcal{D}_{\mathcal{O}_X}[-1] \).

**Remark 2.6.3** In an algebraic case, a twisted ring of differential operator is not locally isomorphic to \( \mathcal{D}_{\mathcal{O}_X} \) even in the etale topology. In fact, for a closed 2-form \( \eta \), \( A\eta \) is isomorphic to \( \mathcal{D}_{\mathcal{O}_X} \) if and only if \( \eta \) is a coboundary.

**Remark 2.6.4** Let \( A \) be a twisted ring of differential operators. Then

\[
\text{Aut}(A) = \text{End}(A) = H^1(X;\sigma_{\mathcal{O}_X}) \circ \text{Ker}(d: \Gamma(X;\mathcal{O}_X^1) \to \Gamma(X;\mathcal{O}_X^2)).
\]

Here \( \text{Aut} \) and \( \text{End} \) signify the sheaf of automorphisms and endomorphisms as \( \mathcal{O}_X \)-rings. For a closed 1-form \( \omega \), the associated automorphism of \( A \) is \( F_1(A) \ni P \mapsto P + \langle \sigma_1(P), \omega \rangle \in F_1(A) \).

**Remark 2.6.5** Let \( A \) be a twisted ring of differential operators and \( L \) an invertible \( \mathcal{O}_X \)-module. Then \( \mathcal{O}_X \otimes L^{-1} \) is also a twisted ring of differential operators. Then the cohomology class \( c(L \otimes \mathcal{O}_X) \in H^2(X;\sigma_{\mathcal{O}_X}(\mathcal{U})) \) corresponds to \( [L] + c(A) \). Here \( [L] \) is the image of the class of \( L \) in \( H^1(X;\mathcal{O}_X^*) \) by the homomorphism

\[
H^1(X;\mathcal{O}_X^*) \to H^2(X;\sigma_{\mathcal{O}_X}(\mathcal{U})) \text{ given by } \mathcal{O}_X^* \text{ dlog } \text{Ker}(d^1: \mathcal{O}_X^1 \to \mathcal{O}_X^2) \to \sigma_{\mathcal{O}_X}(\mathcal{U})[1].
\]

More generally, for any \( \lambda \in k \) (\( k \) is the base field when \( X \) is algebraic and \( k = \mathbb{C} \) when \( X \) is a complex manifold), we can

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define $\mu^\lambda_{\mathfrak A_0 \mathfrak L \langle O \rangle - \lambda}$ such that \(c(\mu^\lambda_{\mathfrak A_0 \mathfrak L \langle O \rangle - \lambda}) = c(\mathfrak A) + \lambda[L]\). In fact take an open covering \(\{U_i\}\) of \(X\) and \(s_i \in \Gamma(U_i, L)\) such that \(L|_{U_i} = O_{\mathfrak A_i}\).

Then we can patch \(\mathfrak A|_{U_i}\) and \(\mathfrak A|_{U_j}\) by \((\mathfrak A|_{U_i})|_{U_i \cap U_j} \overset{\mathfrak P}{\longrightarrow} (s_i/s_j)^{\lambda} \mathfrak P(s_i/s_j)^{-\lambda} \in (\mathfrak A|_{U_j})|_{U_i \cap U_j}\). Remark that for any \(a \in \mathfrak A_X, \mathfrak P \mapsto a^\lambda \mathfrak P a^{-\lambda}\) is a well-defined automorphism of \(\mathfrak A\) (See Remark 2.6.4). Hence if \(s\) is an invertible section of \(L\) and \(\mathfrak P\) is a section of \(\mathfrak A, s^\lambda \mathfrak P \mathfrak P s^{-\lambda}\) gives a section of \(\mu^\lambda_{\mathfrak A_0 \mathfrak L \langle O \rangle - \lambda}\).

**Remark 2.6.5** The map from the set of the isomorphic classes of twisted rings of differential operators to \(H^2(X; \sigma_{\geq 1}(\Omega_X))\) is also given as follows.

Let us consider the diagram (2.4.1). Since the columns are exact, it defines a morphism in the derived category \([\mathcal O_X + \Omega_X^1 + \cdots] \rightarrow \sigma_{\geq 1}(\Omega_X^1)[2]\). Hence we obtain \(H^0(X; \Omega_X^1) + H^2(X; \sigma_{\geq 1}(\Omega_X^1))\). The image of \(1 \in H^0(X; \Omega_X^1) \subset \Gamma(X; \mathcal O_X)\) gives the corresponding class \(c(\mathfrak A) \in H^2(X; \sigma_{\geq 1}(\Omega_X^1))\).

### 2.7 If \(\mathfrak A\) is a twisted ring of differential operators, then its opposite ring \(\mathfrak A^{\text{op}}\) is also a twisted ring of differential operators. If \(c(\mathfrak A) \in H^2(X; \sigma_{\geq 1}(\Omega_X^1))\) denotes the corresponding cohomology class, then \(c(\mathfrak A^{\text{op}}) = [\Omega_X^{\dim X}] - c(\mathfrak A)\). Here \([\Omega_X^{\dim X}] \in H^2(X; \sigma_{\geq 1}(\Omega_X^1))\) is the one given in Remark 2.6.4. We omit its proof. We just remark that it follows from the following fact:

(2.7.1) If we define \(\varphi: v \mapsto \varphi^\theta - 1 \varphi \eta + \eta^\varphi - 1 \varphi \eta \rightarrow \mathcal L \sigma_1^{-1}(v)\), then \(\varphi\) defines a left \(\mathcal O_X\)-linear isomorphism \(F_1(\mathfrak A^{\text{op}}) \rightarrow F_1(\omega_X^{\varphi - 1} \mathfrak A \omega_X)\) \(\cdot \omega_X^{\varphi - 1} \mathfrak A \omega_X\) and \(\eta \in \omega_X\). The diagram

\[
\begin{array}{ccc}
\mathcal O_X & \longrightarrow & F_1(\mathfrak A^{\text{op}}) \\
\downarrow & & \sigma_1 \\
\mathcal O_X & \longrightarrow & F_1(\omega_X^{\varphi - 1} \mathfrak A \omega_X)
\end{array}
\]

commutes. Moreover, \(\psi([v_1, v_2]) = [\varphi(v_1), \varphi(v_2)]\) for \(v_1, v_2 \in F_1(\mathfrak A^{\text{op}})\).
This shows $c(\Lambda^{op}) = -c(\omega_\Lambda^{-1} \otimes \Lambda \otimes \omega_\Lambda)$ by the construction given by Remark 2.6.5.

2.8 Let $f: X \to Y$ be a morphism of smooth algebraic varieties or complex manifolds. Let $A_Y$ be a twisted ring of differential operators on $Y$. Let $f^*(A_Y)$ be $\mathcal{O}_X \otimes_{f^{-1} \mathcal{O}_Y} f^{-1} A_Y$. Then $f^*(A_Y)$ is a right $f^{-1}A_Y$-module. Let $\text{End}_{f^*(A_Y)}$ be the ring of right $f^{-1}A_Y$-linear endomorphisms of $f^*(A_Y)$. Let us define subsheaves $F_m$ of $\text{End}_{f^*(A_Y)}$ inductively by

(2.8.1) $F_m = 0$ for $m < 0$

(2.8.2) $F_m = \{ P \in \text{End}_{f^*(A_Y)} : (P, \mathcal{O}_X) \in F_{m-1} \}$ for $m \geq 0$

Set $f^#A_Y = \bigcup F_m$.

Proposition 2.8.1 $f^#A_Y$ is a twisted ring of differential operators with $\mathcal{O}_X(f^#A_Y) = F_0$, and we have a Cartesian diagram

(2.8.3) \[
\begin{array}{ccc}
F_1(f^#A_Y) & \longrightarrow & f^*F_1(A_Y) \\
\downarrow & & \downarrow \\
\mathcal{O}_X & \longrightarrow & f^*\mathcal{O}_Y
\end{array}
\]

Proof It is enough to check $F_0 = \mathcal{O}_X$ and (2.8.3) by Proposition (2.3.2). The other properties are easily derived by the definition of $F_m$.

Lemma 2.8.2 \(\{ P \in f^*(A_Y) : (P, a) \in f^*F_{m-1}(A_Y) \}$ for any $a \in \mathcal{O}_Y = f^*F_m(A_Y)$ for $m \geq 0$.

Proof Take $y_1, \ldots, y_n \in \mathcal{O}_Y$ such that $dy_1, \ldots, dy_n$ forms a base, and $v_1, \ldots, v_n \in \mathcal{O}_Y$ be its dual base. Then $P \mapsto [P, y_i]$ gives a homomorphism from $f^*S_m(\mathcal{O}_Y) \to f^*S_{m-1}(\mathcal{O}_Y)$. If we identify $f^*S(\mathcal{O}_Y) = \mathcal{O}_X \otimes \mathcal{O}_X(1)_{v_1, \ldots, v_n}$, then $P \mapsto [P, y_i]$ is given by $\mathcal{O}_X \otimes \mathcal{O}_X(1)_{v_i}$. Hence for $m \geq 1$, $\{ P \in f^*S_m(\mathcal{O}_Y) : (P, y_i) = 0 \}$ for any $i \neq 0$. This shows \(\{ P \in f^*F_m(A_Y) : (P, y_i) \in f^*F_{m-1}(A_Y) \} \). The lemma follows.
Proof of Proposition 2.8.1 (continued) If \( \varphi \in F_0 \), then for \( a \in O_\mathcal{X} \), \( [\varphi(\text{ld})], a] = 0 \). Hence \( \varphi(\text{ld}) \in O_\mathcal{X} \) by the preceding lemma. Hence \( \varphi(a \varphi) = a \varphi(\text{ld}) P = \varphi(\text{ld}) a \varphi P \) for \( a \in O_\mathcal{X} \) and \( P \in \mathcal{A}_\mathcal{Y} \). Thus \( \varphi \in O_\mathcal{X} \).

Assume \( \varphi \in F_1 \). Then for \( a \in O_\mathcal{X} \), \([\varphi, a] \in F_0 \). Hence \( a \mapsto [\varphi, a] \in O_\mathcal{X} \) gives a derivation of \( O_\mathcal{X} \). If we denote it \( \nu \), then \([\varphi(\text{ld})], a] = \nu(a) \in O_\mathcal{X} \).

Hence \( \varphi(\text{ld}) \in f^*F_1(\mathcal{A}_\mathcal{Y}) \) and its image on \( f^*O_\mathcal{Y} \) coincides with the image of \( \nu \). Hence we have \( F_1 + f^*F_1(\mathcal{A}_\mathcal{Y}) \times f^*O_\mathcal{Y}^* \). It is easy to check that this an isomorphism.

2.9 Let \( f : X \to Y \) and \( \mathcal{A}_\mathcal{Y} \) be as in the preceding section. Then \( f^*\mathcal{A}_\mathcal{Y} \) has a structure of \((f^*\mathcal{A}_\mathcal{Y}, f^{-1}\mathcal{A}_\mathcal{Y})\)-bimodule. If \( M \) is a left \( \mathcal{A}_\mathcal{Y} \)-module, then

\[
f^*M = O_X \otimes_{f^{-1}O_Y} f^{-1}M = f^*\mathcal{A}_\mathcal{Y} \otimes_{f^{-1}\mathcal{A}_\mathcal{Y}} f^{-1}M
\]

has a structure of left \( f^*\mathcal{A}_\mathcal{Y} \)-module.

2.10 Let \( f : X \to Y \) and \( g : Y \to Z \) be two morphisms of smooth varieties and let \( \mathcal{A}_Z \) be a twisted ring of differential operators on \( Z \). Then we have a canonical isomorphism

\[f^*g^*\mathcal{A}_Z = (gof)^*\mathcal{A}_Z.\]

In fact, \( g^*\mathcal{A}_Z \) is a left \( g^*\mathcal{A}_Z \)-module. Hence \( f^*g^*\mathcal{A}_Z = (gof)^*\mathcal{A}_Z \) is a left \( f^*g^*\mathcal{A}_Z \)-module. Hence we obtain \( f^*g^*\mathcal{A}_Z \to \text{End}((gof)^*\mathcal{A}_Z) \). It is easy to prove that this gives an isomorphism from \( f^*g^*\mathcal{A}_Z \) to the subring \( (gof)^*\mathcal{A}_Z \) of \( \text{End}((gof)^*\mathcal{A}_Z) \).

2.11 We have the following lemma, whose proof is left to the reader.

**Lemma 2.11.1** Let \( f : X \to Y \) be a morphism of smooth varieties, and \( \mathcal{A}_Y \) a twisted ring of differential operators on \( Y \). Then

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\[
\oline{f^\#(A_Y^{OP})}^{OP} = \omega_{X/Y}^{\Theta_{O_X} f^\#(A_Y) \otimes_{O_X} \omega_{X/Y}^{\Theta-1}}
\]

where \( \omega_{X/Y} = \Omega_X^{\dim X} \otimes \Omega_Y^{\dim Y, \Theta-1} \).

Since \( \oline{f^*}(A_Y^{OP}) \) is a right \( \oline{f^\#(A_Y^{OP})}^{OP} \)-module, \( \oline{f^*}(A_Y^{OP}) \otimes_{O_X} \omega_{X/Y} \) is a right \( \oline{f^\#(A_Y)} \) module by this lemma. Together with the right module structure on \( \oline{f^*}(A_Y^{OP}) \), gives a \( (f^{-1}A_Y, f^\#A_Y) \)-bimodule structure on \( \oline{f^*}(A_Y^{OP}) \otimes_{O_X} \omega_{X/Y} \). We set

\[
(2.11.1) \quad A_{Y+X} = \oline{f^*}(A_Y^{OP}) \otimes_{O_X} \omega_{X/Y} = f^{-1}A_Y \otimes_{O_Y} f^{-1} \omega_{X/Y}.
\]

Then for a left \( \oline{f^\#A_Y} \) module \( M \), \( f_* (A_{Y+X} \otimes_{O_X} \oline{f^\#(A_Y)} M) \) is a left \( A_Y \)-module.
§3 Twisted sheaves and regular holonomic modules over twisted rings of differential operators

3.0 We know that the derived category of $\mathcal{D}_X$-modules with regular holonomic $\mathcal{D}_X$-modules as cohomology groups is equivalent to the derived category of $\mathcal{E}_X$-modules with constructible cohomologies. In the case of twisted rings of differential operators, we have the similar theories. However, we have to introduce the notion of twisted sheaves that we are going to discuss in this chapter.

3.1 Let $(X, \mathcal{O}_X)$ be a smooth algebraic variety defined over a field $k$ of characteristic 0 or a complex manifold. The notion of regular holonomic system can be generalized in the case of twisted rings of differential operators.

3.2 Let $A$ be a twisted ring of differential operators on $X$ and let $F(A)$ be the order filtration of $A$.

3.3 For a coherent $A$-module $M$, a filtration $F(M)$ over $F(A)$ (i.e. $F_m(A)F_k(M) \subseteq F_{m+k}(M)$) is called a good filtration if there exists locally a finite number of sections $\{u_i\}$ of $M$ and integers $m_i$ such that $F_k(M) = \bigcap_{k-m_i} F_k(A)u_i$. Such a filtration exists always at least locally.

3.4 If $F(M)$ is a good filtration, then $\text{gr}^F M$ is a coherent $(\text{gr}^F A)$-module. If we denote $\pi: T^*X \to X$, the cotangent bundle of $X$, then we have a ring homomorphism

\[ \text{gr}^F A \to \pi_* \mathcal{O}_{T^*X}. \]

In the algebraic case, (3.4.1) is an isomorphism. We set $\text{Ch}(M) = \text{supp}(\mathcal{O}_{T^*X} \otimes \pi^{-1} \text{gr}^F M)$ and call this the characteristic variety of $M$. Since this is independent from the choice of $F(M)$, this is a well-defined closed subset of $T^*X$. 

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3.5 We have

Proposition 3.5.1 Ch $M$ is always involutive. (i.e. the ideal
defining Ch $M$ is closed under the Poisson bracket).

In particular the codimension of Ch $M$ is $\leq \dim X$ at any point of
Ch $M$.

Definition 3.5.2 A coherent $\mathcal{A}$-module is called holonomic if
codim Ch $M = \dim X$.

Let $M$ be a holonomic $\mathcal{D}_X$-module and $\mathcal{A} = \text{Ch } M$. If there exists
a good filtration $F(M)$ such that $f|_{\text{gr}^F_M} = 0$ for any $f \in \text{gr}^F_A$
with $f|_{\mathcal{A}} = 0$, then we call $M$ regular holonomic.

3.6 If $X$ is an open subset of $\mathbb{C}$ and if $M = D/DP$ with a non-
zero differential operator $P, M$ is always holonomic. Moreover $M$
is regular holonomic on a neighborhood of $x=0$, if and only if $0$ is
a regular point of the equation $Pu=0$ in the classical sense; that
is, if we set $P = \sum \frac{a_m}{m!}a_j(x)\partial^j$, with $a_m \neq 0$, ord $a_j \leq \text{ord } a_m - (m-j)$.
Here ord is the order of zero at the origin.

3.7 Since any twisted sheaf of differential operators is locally
isomorphic to $\mathcal{D}_X$ (in the complex case), many properties of regular
holonomic $\mathcal{D}_X$-modules are valid for those over $\mathcal{A}$. Here are some of
their properties.

Proposition 3.7.1 (i) A coherent submodule and a coherent quotient
of regular holonomic module is regular.
(ii) If $M' \rightarrow M \rightarrow M''$ is an exact sequence of coherent modules and
if $M'$ and $M''$ are regular holonomic, then so is $M$.

3.8 In this section, we assume $X$ is a smooth algebraic variety,
and we work in the algebraic category. Let $j: X \hookrightarrow \bar{X}$ be an embedding
into a proper smooth variety $\bar{X}$. For any holonomic $\mathcal{D}_X$-module $M$,
$j_*M$ is always holonomic. If $j_*M$ is regular holonomic, we say $M$ is completely regular. This property does not depend on the embedding $j$.

Regular holonomic has the following functorial properties.

Proposition 3.8.1 Let $f : X \to Y$ be a morphism.

(i) If $M$ is a (completely) regular holonomic $D_Y$-module, then

\[ \text{Tor}^f_j(D_Y \cdot (f^*D_Y, M)) \text{ is a (completely) regular holonomic } D_X\text{-module}. \]

(ii) If $M$ is a completely regular holonomic $D_X$-module, then

\[ R^jf_*(D_Y \cdot \otimes_{D_X}^L M) \text{ is a completely regular holonomic } D_X\text{-module}. \]

Proposition 3.8.2 Let $f : X \to Y$ be a surjective map of smooth varieties $X, Y$. Let $M$ be a holonomic $D_Y$-module. Then $M$ is completely regular if and only if $\text{Tor}^j_j(D_Y \cdot (f^*D_Y, M))$ is completely regular for any $j$.

3.9 Let $D(D_X)$ be the derived category of the abelian category of $D_X$-modules and let $D_{rh}(D_X)$ be the full subcategory of $D(D_X)$ consisting of bounded complexes with regular holonomic cohomology groups.

3.10 Assume $X$ complex analytic. Let $D(\mathcal{E}_X)$ be the derived category of sheaves of $\mathbb{C}$-vector spaces and let $D_c(\mathcal{E}_X)$ be its full subcategory consisting of bounded complexes whose cohomology groups are constructible. Recall that a sheaf $F$ is called constructible if there exists a complex analytic stratification on whose strata $F$ is locally constant of finite rank.

3.11 Now the Riemann-Hilbert correspondence says

Theorem 3.11.1 Let $X$ be a complex manifold

\[ \mathbb{H}om_{D_X}(\mathcal{O}_X, \ast) : D_{rh}(D_X) \to D_c(\mathcal{E}_X) \]

is an equivalence of categories.
An object $\mathcal{F} \in D_c(\mathbb{C}_X)$ is called perverse, if $\text{codim} \text{ Supp} H^j(\mathcal{F}) \geq j$ and $\text{codim} \text{ Supp} \text{ Ext}^j(\mathcal{F}, \mathbb{C}_X) \geq j$ for any $j$. Let $\text{RH}(D_X)$ be the category of regular holonomic $D_X$-modules and $\text{Perv}(\mathbb{C}_X)$ the full subcategory of $D_c(\mathbb{C}_X)$ consisting of perverse objects. Then

**Theorem 3.11.2** $\mathbb{D}_{D_X}^\ast(O_X^n) : \text{RH}(D_X) \rightarrow \text{Perv}(\mathbb{C}_X)$ is an equivalence of categories.

**Remark 3.11.3** Let $X$ be a proper smooth algebraic variety defined over $\mathbb{C}$, and let $X_{\text{an}}$ be the underlying complex manifold. Then by GAGA, we have $D_{\text{rh}}(D_X) \cong D_{\text{rh}}(D_{X_{\text{an}}})$ and $\text{RH}(D_X) \cong \text{RH}(D_{X_{\text{an}}})$. This is also true in twisted cases.

3.12 We shall generalize the Riemann-Hilbert correspondence in the twisted case.

3.13 Let $(X; A)$ be a commutative ringed space. Let us take an open covering $\{U_i\}_{i \in I}$ of $X$, invertible $A|_{U_i \cap U_j}$-modules $L_{ij}$ and $A$-linear isomorphism $\varphi_{ijk} : (L_{ij} \otimes L_{jk})|_{U_1 \cap U_j \cap U_k} \rightarrow L_{ik}|_{U_1 \cap U_j \cap U_k}$ which satisfies

\[(3.13.1)\quad L_{ii} = A.\]
\[(3.13.2)\quad \varphi_{ij} = \text{id}_{L_{ij}}, \quad \varphi_{jj} = \text{id}_{L_{ij}}.\]
\[(3.13.3)\quad \text{For } i, j, k, \ell \in I, \text{ we have a commutative diagram of morphisms of } A|_{U_1 \cap U_j \cap U_k \cap U_\ell} \text{-modules:}\]

\[
\begin{array}{ccc}
L_{ij} \otimes L_{jk} \otimes L_{k\ell} & \xrightarrow{\varphi_{ijk}} & L_{ik} \otimes L_{k\ell} \\
\downarrow \varphi_{j\ell} & & \downarrow \varphi_{ik\ell} \\
L_{ij} \otimes L_{j\ell} & \xrightarrow{\varphi_{ij\ell}} & L_{i\ell}
\end{array}
\]

In this case, we say $((U_i)_{i \in I}, \{L_{ij}\}, \{\varphi_{ijk}\})$ a **twisting data**.
Remark that (3.13.1) and (3.13.2) are consequences of (3.13.3).

3.14 Let $T=\{\{U_i\}_{i\in I}, \{L_{ij}\}, \{\gamma_{ijk}\}\}$ be a twisting data. For an open set $\Omega$ of $X$, a twisted sheaf $F$ on $\Omega$ with twist $T$ is data $F=\{F_i, \rho_{ij}\}$ with

(3.14.1) $F_i$ is an $A|_{\Omega\cap U_i}$-module,

(3.14.2) $\rho_{ij}: (L_{ij}\otimes F_j)|_{\Omega\cap U_i\cap U_j} \cong F_i|_{\Omega\cap U_i\cap U_j}$,

such that

(3.14.2.1) $\rho_{ii} = 1$.

(3.14.2.2) For $i, j, k$, on $U_i \cap U_j \cap U_k \cap \Omega$

\[
\begin{array}{ccc}
L_{ij}\otimes L_{jk}\otimes F_k & \rightarrow & L_{ik}\otimes F_k \\
\downarrow \rho_{jk} & & \downarrow \rho_{ik} \\
L_{ij}\otimes F_j & \rightarrow & F_i
\end{array}
\]

commutes.

Then the category $\mathcal{M}(\Omega; T)$ of twisted sheaves on $\Omega$ with twist $T$ form an abelian category. If $\Omega \subset U_i$ for some $U_i$, then $\mathcal{M}(\Omega; T)$ is equivalent to the category of $(A|_{\Omega})$-modules.

Moreover it is a champs in the sense of Giraud $[G]$, i.e.

i) For $F, F' \in \mathcal{M}(\Omega; T)$, $U \rightarrow \text{Hom}_{\mathcal{M}(U, T)}(F|_U, F'|_U)$ is a sheaf on $\Omega$.

ii) Let $\Omega = U_j$ be an open covering and let $F_j \in \mathcal{M}(\Omega_j, T)$. If $\gamma_{jk}: F_k|_{\Omega_j \cap \Omega_k} \cong F_j|_{\Omega_j \cap \Omega_k}$ is given so that

(3.14.3) $\gamma_{ii} = \text{id}$

(3.14.4) $\gamma_{ij} \gamma_{jk} = \gamma_{ik}$.

Then there exists $F \in \mathcal{M}(\Omega; T)$ and that $\alpha_i: F|_{\Omega_i} \rightarrow F_i$ with $\alpha_i \alpha_j^{-1} = \gamma_{ij}$.

3.15 Remark that a twisting data $T$ gives an element $c(T)$ of
$H^2(X; A^\times)$. If two twisting data $T_1, T_2$ satisfy $c(T_1) = c(T_2)$, then $M(\mathfrak{g}; T_1)$ and $M(\mathfrak{g}; T_2)$ are equivalent (as a champs). But this equivalence is not unique. In fact the ambiguity is given by $\Theta L$ for a twisted invertible $A$-module $L$. Also, note that for any $c \in H^2(X; A^\times)$, there exists a twisting data $T$ with $c(T) = c$.

For a twisting data $T = \{L_{ij}\}$, we denote by $T^{-1}$ the twisting data $\{L_{ij}^{\Theta^{-1}}\}$.

3.16 Let $X$ be a complex manifold and $A$ a twisted ring of differential operators. Since $A$ is locally isomorphic to $D_X$, there exists an open covering $X = \bigcup U_i$ of $X$ and an $A|_{U_i}$-module $L_i$ which is an invertible $O_{U_i}$-module. Set

$$L_{ij} = \text{Hom}_A(L_i|_{U_{ij}}, L_j|_{U_{ij}}).$$

Then $L_{ij}$ is an invertible $\mathcal{T}_{U_{ij}}$-module. Moreover $L_{ij} \circ L_{jk} \simeq L_{ik}$ canonically. Thus $\{L_{ij}\}$ defines a twisting data $T$ on $X$. Then we have

$$L_{ji} \circ L_{ij} \simeq L_i|_{U_{ij}}.$$

Hence $L = \{L_{ij}\}$ is a twisted sheaf with twist $T^{-1}$. Moreover $A \to \text{End}(L)$ defines a structure of $A$-module on $L$. Then we can define

$$DR(M) = \mathbb{R}\text{Hom}_A(L, M)$$

for an $A$-module. This gives a functor from the derived category of $A$-modules to the derived category $D(T)$ of twisted sheaves with twist $T$.

Similarly to $D_X$, we have the following Riemann-Hilbert correspondence in the twisted case. Let us define $D_{rh}(A)$ and $D_C(T)$ just as $D_{rh}(D_X)$ and $D_C(C_X)$.

**Theorem 3.16.1** $D_{rh}(A)$ is equivalent to $D_C(T)$.

**Theorem 3.16.2** The category of regular holonomic $A$-modules is equivalent to the category of twisted perverse sheaves with twist $T$.

3.17 Let $X$ be a complex manifold and $A$ a twisted ring of differential operators on $X$. Let $Y$ be a closed analytic set. Let $M$ be a
regular holonomic $\mathcal{A}|_{X \setminus Y}$-module which can be extended to a holonomic $\mathcal{A}$-module defined on $X$. Then there exists a regular holonomic $\mathcal{A}$-module $\mathcal{M}$ defined on $X$ satisfying

\[(3.17.1) \quad \mathcal{M}|_{X \setminus Y} = \mathcal{M}\]

\[(3.17.2) \quad \mathcal{M} \text{ has no non-zero coherent submodule supported in } Y \text{ nor non-zero coherent quotient supported on } Y.\]

This $\mathcal{M}$ is unique and called the minimal extension of $\mathcal{M}$.

3.18 This can be generalized into an algebraic case. Let $X$ be a smooth algebraic variety, $\mathcal{A}$ a twisted ring of differential operators. Let $\mathcal{M}$ be a holonomic $\mathcal{A}$-module defined on an open set $U$ of $X$. Then there exists a holonomic $\mathcal{A}$-module $\mathcal{M}$ defined on $X$ satisfying (3.17.1) and (3.17.2). Such a $\mathcal{M}$ is unique.

3.19 Let $X$ be a complex manifold and $\mathcal{A}$ a twisted ring of differential operators on $X$.

**Theorem 3.19.1** The set of the isomorphic classes of irreducible regular holonomic $\mathcal{A}$-modules is isomorphic to the set of pairs $(S,F)$ where $S$ is a Zariski locally closed non-singular connected subset of $X$ and $F$ is an irreducible twisted locally constant sheaf of finite rank on $S$ with twist $T$. Here $(S,F)=(S',F')$ if $S \cap S'$ is open dense in both $S$ and $S'$ and if $F|_{S \cap S'} = F'|_{S \cap S'}$.

Let $\mathcal{M}$ be an irreducible regular holonomic. Then $\text{Supp } \mathcal{M}$ must be irreducible. Let $S'$ be a non-singular locus of $\text{Supp } \mathcal{M}$. Then $\text{Ext}_\mathcal{A}^k(L,M)|_S$ vanishes for $k \neq \text{codim } S$ and when $k = \text{codim } S$, this is a twisted local system on some Zariski open subset $S$ of $S'$ with twist $T$. Conversely, for $(S,F)$, there exists a regular holonomic $\mathcal{A}$-module $\mathcal{M}$ defined on $X \setminus \mathcal{S}$ such that $\mathcal{E} \text{Hom}_{\mathcal{A}}(L,M) \cong \mathbb{F}[-\text{codim } S]$.

Then we associate to $(S,F)$ the minimal extension of $\mathcal{M}$ onto $X$.

3.20 Let us give an example of twisting data

**Example 3.20.1** $X = \mathbb{P}^1 = \mathbb{P}^1 \setminus \{0\}$, $U_0 = \mathbb{P}^1 \setminus \{0\}$, $U_1 = \mathbb{P}^1 \setminus \{0\}$. For $\lambda$, let $\mathcal{C}_\lambda$ be the invertible $\mathcal{C}_U$-module with the monodromy $e^{2\pi i \lambda}$.

$\mathcal{U}_0 \cup \mathcal{U}_1$

Then $\mathcal{T} = \{(U_0, U_1); \mathcal{C}_\lambda\}$ defines a twisting data on $X$. If $e^{2\pi i \lambda} \neq 1$, there is no twisted local system on $X$. 

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§4 Equivariant twisted rings of differential operators

4.1 Let $X$ be a complex manifold or a smooth algebraic variety defined over $\mathbb{C}$. Let $G$ be a complex analytic group or algebraic group acting on $X$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathcal{O}_X$ the sheaf of vector field on $X$. Then the infinitesimal action induces a Lie algebra homomorphism

$$D : \mathfrak{g} \to \Gamma(X, \mathcal{O}_X)$$

4.2 Let $\mathcal{O}_X(\mathfrak{g})$ be the ring generated by $\mathcal{O}_X$ and $\mathfrak{g}$ with the fundamental relation:

(4.2.1) $\mathcal{O}_X \overset{\mathfrak{g}}{\cdot} \mathcal{O}_X(\mathfrak{g})$ is a ring homomorphism,

(4.2.2) $\mathfrak{g} \overset{i}{\cdot} \mathcal{O}_X(\mathfrak{g})$ is a Lie algebra homomorphism,

(4.2.3) $[j(A), i(a)] = i(D(A)(a))$ for $A \in \mathfrak{g}$ and $a \in \mathcal{O}_X$.

Then $\mathcal{O}_X(\mathfrak{g}) \cong \mathcal{O}_X \otimes U(\mathfrak{g})$, where $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$. The multiplication rule of $\mathcal{O}_X \otimes U(\mathfrak{g})$ is given as follows: $\mathfrak{g}$ acts on $\mathcal{O}_X$ and $U(\mathfrak{g})$ (by the left multiplication) and hence we have $\mathfrak{g} \to \text{End}(\mathcal{O}_X \otimes U(\mathfrak{g}))$, which extends to $U(\mathfrak{g}) \to \text{End}(\mathcal{O}_X \otimes U(\mathfrak{g}))$. Moreover $\mathcal{O}_X$ acts on $\mathcal{O}_X \otimes U(\mathfrak{g})$ and we obtain $\mathcal{O}_X \otimes U(\mathfrak{g}) \to \text{End}(\mathcal{O}_X \otimes U(\mathfrak{g}))$. This gives the left multiplication of sections of $\mathcal{O}_X \otimes U(\mathfrak{g})$ on $\mathcal{O}_X \otimes U(\mathfrak{g})$. This gives the ring structure on $\mathcal{O}_X \otimes U(\mathfrak{g})$. We can easily prove that $\mathcal{O}_X \otimes U(\mathfrak{g})$ is isomorphic to $\mathcal{O}_X(\mathfrak{g})$.

4.3 Let $\mathfrak{q}$ be the kernel of $\mathcal{O}_X \otimes U(\mathfrak{g}) \to \mathcal{O}_X$. Then we have $[\mathfrak{g}, \mathfrak{q}] \subseteq \mathfrak{q}$ (in $\mathcal{O}_X(\mathfrak{g})$). If $G$ acts transitively on $X$, $\mathfrak{q}$ is a vector sub-bundle of $\mathcal{O}_X \otimes \mathfrak{g}$.

4.4 Let us recall the notion of $G$-equivariant $\mathcal{O}_X$-modules. Let us consider

$$\frac{p_1}{p_2} \quad \frac{p_1}{p_3} \quad \mu \quad \frac{1}{pr}$$

(4.4.1) $G \times G \times X \xrightarrow{\mu} G \times X \xrightarrow{\frac{1}{pr}} X$
where $\text{pr}$ is the projection, $\mu$ the multiplication map $:(g,x) \mapsto gx$, $i(x) = (1,x)$ and the $p_j$ are given by

$$p_1(g_1,g_2,x) = (g_1,g_2^x), \quad p_2(g_1,g_2,x) = (g_1g_2,x),$$

$$p_3(g_1,g_2,x) = (g_2,x).$$

Then we have $\mu \circ p_1 = \mu \circ p_2 = \text{pr \circ p_3}$, $\mu \circ i = \text{pr \circ i = id}$. An $O_X$-module $F$ is called $G$-equivariant if an $O_{G \times X}$-linear isomorphism $\alpha : \mu \circ F \cong \text{pr \circ F}$ is given such that it satisfies:

$$(4.4.2) \quad \begin{array}{ccc}
i \mu F & \xrightarrow{i \alpha} & i \circ \text{pr} \circ F \\
\downarrow & & \downarrow \\
F & \xrightarrow{id} & F
\end{array} \quad \text{commutes.}$$

$$(4.4.3) \quad \begin{array}{c}
p_2^\# \mu F \\
\downarrow \\
p_1^\# \mu F \xrightarrow{p_1^\#(\alpha)} p_1^\# \circ \text{pr} \circ F \quad \cong \quad p_3^\# \mu F \xrightarrow{p_3^\#(\alpha)} p_3^\# \circ \text{pr} \circ F
\end{array} \quad \text{commutes.}$$

4.5 For a $G$-equivariant $O_X$-module $F$ and for $g \in G$, let $\nu_g : X \rightarrow X$ be the map $x \mapsto gx$. Then we have $\nu_g \circ F \cong F$. Let $T_g$ be an inverse homomorphism. Then setting $A \cdot u = \frac{d}{dt}(T_g t A u)|_{t=0}$ for $A \in g$ and $u \in F$, we obtain a Lie algebra homomorphism $D : \mathfrak{g} \rightarrow \mathfrak{end}_G(F)$, which satisfies

$$D(A)u = a D(A)u + D(A)(a)u$$

and hence it extends to a ring homomorphism $U_X(g) \rightarrow \mathfrak{end}_G(F)$. Thus $F$ has a structure of left $U_X(g)$-module.

4.6 Similarly to $G$-equivariant $O_X$-modules, we shall define the notion of equivariant twisted rings of differential operators. Let $\mathcal{A}$ be a twisted ring of differential operators on $X$. We say that $\mathcal{A}$ is $G$-equivariant if an $O_X$-ring isomorphism $\alpha : \mu \circ \mathcal{A} \cong \text{pr} \circ \mathcal{A}$ is given.
satisfying the following property:

\[ (4.6.1) \quad \begin{array}{ccc}
\mu^\# A & \xrightarrow{i^\#(a)} & \mu^\# A \\
\| & \| & \| \\
\frac{\Lambda}{\Lambda} & \xrightarrow{id} & \frac{\Lambda}{\Lambda}
\end{array} \quad \text{commutes.} \]

\[ (4.6.2) \quad \begin{array}{cccc}
p_2^\# \mu^\# A & \xrightarrow{p_2^\#(a)} & p_2^\# \mu^\# A \\
\| & \| & \| \\
p_1^\# \mu^\# A & \xrightarrow{p_1^\#(a)} & p_1^\# \mu^\# A & \xrightarrow{p_3^\#(a)} & p_3^\# \mu^\# A & \xrightarrow{p_3^\#(a)} & p_3^\# \mu^\# A
\end{array} \quad \text{commutes.} \]

Let \( A \) be a \( G \)-equivariant twisted ring of differential operators. Since \( \mu^* A \) is a \( \mu^* A \)-module, we have \( \text{pr}^\# A \simeq \mu^* A + \mu^* A \) by operating on \( 1 \in \mu^* A \). Hence we obtain \( p^* D_G \to \text{pr}^\# A + \mu^* A \), where \( p : G \times X \to G \) is the projection. Thus we obtain \( i^* p^* D_G + i^* \mu^* A \). This gives \( g \to A \). This extends to an \( O_X \)-ring homomorphism \( U_X(g) \to A \). Note that the composition \( g \to p_1(A) \to O_X \) coincides with \( D \).

4.7 Let \( A \) be a \( G \)-equivariant twisted ring of differential operators. Then \( \text{pr}^\# A = D_G \circ A \), and hence \( \text{pr}^* A \subset \text{pr}^\# A \) becomes a subring. A left \( A \)-module \( M \) is called \( G \)-equivariant if \( \beta : \mu^* M \simeq \text{pr}^* M \) gives a structure of equivariant \( O_X \)-modules and \( \beta \) is \( \text{pr}^\# A \)-linear (through \( \mu^* \frac{\Lambda}{\Lambda} \simeq \text{pr}^\# A \) and the \( \mu^* A \)-module structure on \( \mu^* M \)). If \( \beta \) is only \( \text{pr}^* A \)-linear, we call \( M \) \( \text{quasi-G-equivariant} \).

If \( N \) is a \( G \)-module (see §4.8), then \( A \otimes_A N \) has a structure of \( G \)-equivariant \( A \)-module.

4.8 We shall investigate the description of \( G \)-equivariant twisted rings of differential operators and quasi-\( G \)-equivariant modules when \( X \) is a homogeneous space. Let \( x \) be a point of \( X \). Let \( H \) be the isotropic subgroup of \( X \) at \( x \) and let \( \mathfrak{h} \) be its Lie algebra. We assume \( G/H \simeq X \). An \( H \)-module of finite dimension is, by definition, a finite-dimensional vector space \( V \) with a group morphism \( H \to GL(V) \) and we assume that this is algebraic in the algebraic case and holomorphic in the complex analytic case. An \( H \)-module is a vector
space with $H$-action, which is a union of finite-dimensional $H$-modules. The following is well-known.

**Theorem 4.8.1** The category of $G$-equivariant $O_X$-modules is equivalent to the category of $H$-modules by $M \mapsto M(x)$, where $M(x) = \mathbb{C} \otimes O_X^X$.

Let $V_X$ be the inverse functor of $M \mapsto M(x)$. Then in the analytic case (and in the algebraic case with suitable interpretation), for an $H$-module $V$, we have for an open set $U$ of $X$:

(4.8.1) $\Gamma(U; V_X(V)) = \{\phi; V$-valued function on $p^{-1}U \text{ such that } f(gh) = h^{-1}f(g) \text{ for } g \in p^{-1}U \text{ and } h \in H\}.$

Here $p: G + X$ is the projection $g \mapsto gx$. Note that

(4.8.2) $\tilde{\phi} = V_X(h)$ (see §4.3)

Also note that

(4.8.3) If $V$ is a $G$-module and $W$ is an $H$-module,

$$V_X(V \otimes W) \cong V_X(W) \otimes \mathbb{C}.$$

4.9 Let $\lambda$ be an $H$-invariant element of $h^*$. Then $\lambda([h, h]) = 0$ and hence $\lambda$ gives a 1-dimensional representation $\mathbb{C}_\lambda = \mathbb{C} \cdot 1_\lambda$ of $h$ by $A \cdot 1_\lambda = \lambda(A) 1_\lambda$ for $A \in h$. On the other hand, $\lambda$ gives an $H$-linear homomorphism from $h$ to $\mathbb{C}$ and hence a $G$-equivariant homomorphism $V_X(h) = \tilde{\phi}$ to $V_X(\mathbb{C}) = O_X$. Then we can easily check that

$$\sum_{A \in \mathbb{C}} U_X(g)(A-\lambda(A))$$

is a both-sided ideal. We set

(4.9.1) $\mathbb{A}_X(\lambda) = U_X(g)/ \sum_{A \in \mathbb{C}} U_X(g)(A-\lambda(A)).$

**Theorem 4.9.2** (i) $\mathbb{A}_X(\lambda)$ is a $G$-equivariant twisted ring of differential operators.

(ii) Any $G$-equivariant twisted ring of differential operators is isomorphic to $\mathbb{A}_X(\lambda)$ (for a unique $\lambda$).
We shall give only a sketch of the proof. Let $\mathbb{A}$ be a $G$-equivariant twisted ring of differential operators.

As in §4.6, we have a ring homomorphism $U_X(\mathfrak{g}) \rightarrow \mathbb{A}$. Since $X$ is a homogeneous space, this is surjective, and $\mathfrak{g}$ is mapped into $F_0(\mathbb{A}) = O_X$. Since this is $H$-linear, it comes from some $H$-invariant $\lambda \in \mathfrak{h}^*$ and we obtain $A_X(\lambda) \rightarrow \mathbb{A}$, which is an isomorphism.

4.10 In order to describe quasi-$G$-equivariant $A_X(\lambda)$-modules, we shall introduce the notion of twisted $(g, H)$-module. Let $\lambda \in \mathfrak{h}^*$ be an $H$-invariant form.

Definition 4.10.1 A twisted $(g, H)$-module $M$ with twist $\lambda$ is a $g$-module $M$ with a structure of $H$-module on $\mathcal{C}_\lambda \otimes M$ such that

\[(4.1.1)\] Two $H$-module structures on $\mathcal{C}_\lambda \otimes M$ which come from the $g$-module structure on $M$ and the $H$-module structure on $\mathcal{C}_\lambda \otimes M$ coincide.

\[(4.1.2)\] $g \otimes (\mathcal{C}_\lambda \otimes M) \rightarrow \mathcal{C}_\lambda \otimes M$ given by $A \otimes l_\lambda \otimes u \mapsto l_\lambda \otimes Au$ is $H$-linear.

If $M$ is an $H$-module, then $U(g) \otimes (C_{-\lambda} \otimes M)$ is a twisted $(g, H)$-module with twist $\lambda$. Here the action of $H$ on $\mathcal{C}_\lambda \otimes U(g) \otimes (C_{-\lambda} \otimes M)$ is given by $H \mapsto h : l_\lambda \otimes p \otimes l_{-\lambda} \otimes u \mapsto l_\lambda \otimes Ad(h)p \otimes l_{-\lambda} \otimes hu$.

Theorem 4.10.2 (i) The category of quasi-$G$-equivariant $A_X(\lambda)$-modules is equivalent to the category of twisted $(g, H)$-modules with twist $\lambda$.

(ii) For a twisted $(g, H)$-module $M$ with twist $\lambda$, the corresponding quasi-$G$-equivariant $A_X(\lambda)$-module is isomorphic, as a $G$-equivariant $O_X$-module, to $V_X(C_{\lambda} \otimes M)$.

We shall give here only the sketch of the proof.

Let $M$ be a quasi-$G$-equivariant $A_X(\lambda)$-module. Then $M$ has two actions of $\mathfrak{g}$ on $M$ which comes from the $A_X(\lambda)$-module structure

In general (not nec. $X$-homog.) the same const. gives a $D_X$-linear homomorphism $\text{End}_{D_X}(M)$.

$M$ is $(D_X, U(\mathfrak{g}))$-module
and the structure of G-equivariant $O_X$-module (see § 4.5). Let $\alpha$ be the first action and $\beta$ the last action. Then $\gamma = \beta - \alpha$ is $O_X$-linear since $[\alpha(A),a]=[\beta(A),a]=D(A)(a)$. Since $g \otimes M \cap M$ via $\alpha$ is $g$-linear with respect to the $\beta$-action, we have

\[(4.10.3) \quad [\beta(A),\alpha(A')] = \alpha([A,A']). \]

This implies $\gamma : g \to \text{End}_{O_X}(M)$ is a Lie algebra homomorphism. Hence we obtain $\gamma : g \to \text{End}_{O_X}(M(x_0))$. For $A \in h$, $\gamma(A) = \beta(A) - \alpha(A) = \beta(A) - \lambda(A)$ we have $\beta(A) = \gamma(A) + \lambda(A)$. Since the infinitesimal action of $H$ on $M(x_0)$ coincides with $\beta$, the $h$-module structure of $M(x_0)$ by $\gamma$ is isomorphic to $\otimes_{-\lambda}^h M(x_0)$. Therefore $\otimes_{-\lambda}^h M(x_0)$ is a twisted $(g,h)$-module with twist $\lambda$. Conversely let $M$ be a twisted $(g,h)$-module with twist $\lambda$. $\otimes_{-\lambda}^h M$ is an $H$-module. Let $M = V_X(\otimes_{-\lambda}^h M)$ be the corresponding G-equivariant $O_X$-modules. The morphism (4.10.2) gives a $g$-action $\gamma : g \to \text{End}_{O_X}(M)$ and the G-equivariant structure defines $\beta : g \to \text{End}_{O_X}(M)$. Then $\alpha = \beta - \gamma$ defines and $A_X(\lambda)$-module structure on $M$.

4.11 If moreover $M$ is G-equivariant, then we have $\beta = \alpha$.

Therefore $\gamma = 0$ and the $g$-module structure on $\otimes_{-\lambda}^h M(x_0)$ is trivial. The converse is also true and we obtain the following proposition.

Proposition 4.11.1 The category of G-equivariant $A_X(\lambda)$-modules is equivalent to the category of $H$-modules $M$ such that $H$ acts trivially on $\otimes_{-\lambda}^h M$.

4.12 We have

\[(4.12.1) \quad A_X(\lambda) = V_X(\otimes_{-\lambda}^h E(\theta(g) \otimes E(\lambda))). \]

For a twisted $(g,h)$-module $M$ with twist $\lambda$ and a $G$-module $V$, $M \otimes V$ has canonically a structure of twisted $(g,h)$-module and

\[(4.12.2) \quad V_X(\otimes_{-\lambda}^h (M \otimes V)) = V_X(\otimes_{-\lambda}^h M) \otimes_{E} V. \]
4.13 In a complex analytic case, we can describe $A_X(\lambda)$ as follows. Let $p : G \to X$ be the projection $g \mapsto gx_0$. Let $F$ be the sheaf on $G$ defined by

\begin{equation}
F = \{ \varphi \in \mathcal{O}_G ; R_A \varphi = -\lambda(A)\varphi \text{ for any } A \in \mathfrak{h} \}.
\end{equation}

Here $R_A \varphi(g) = \frac{d}{dt} \varphi(ge^{tA})|_{t=0}$.

Then $F$ is locally constant along fiber of $p$ with the monodromy corresponding to $\lambda$, and $F$ has a structure of $p^{-1}\mathcal{O}_X$-module. Then $\mathcal{G}$ acts on $F$ through the left action of $G$ on $G$. Then $A_X(\lambda)$ is the subring of $p_! \text{End}_G(F)$ generated by $\mathcal{O}_X$ and $\mathcal{G}$.

4.14 Let $G'$ be another Lie group and $H'$ its subgroup. Let $\varphi : G' \to G$ be a group morphism such that $\varphi(H') \subset H$. Set $X' = G'/H'$, $X = G/H$. Then $\varphi$ induces the map $f : X' \to X$. Let $\mathfrak{h}', \mathfrak{h}''$ be the Lie algebra of $H$ and $\mathfrak{h}'$. Let $\lambda \in \mathfrak{h}^*$ be an $H$-invariant form. Then, we can easily prove

**Proposition 4.14.1**

(i) $f^#A_X(\lambda) = A_{X'}(\lambda|_{\mathfrak{h}'})$.

(ii) For a twisted $(\mathfrak{g}, H)$-module $M$, we have $f^*V_X(\mathfrak{t}, \mathfrak{OM}) \cong V_{X'}(\mathfrak{t}, \mathfrak{OM})$ as $A_{X'}(\lambda|_{\mathfrak{h}'})$-module.

4.15 For a homogeneous space $X$ with the isotropy subgroup $H$, we have the following proposition.

**Proposition 4.15.1** $A_X(\lambda)^{Op} \cong A_X(2\rho - \lambda)$, where $\rho \in \mathfrak{h}^*$ is given by $\rho(A) = \frac{1}{2} \text{tr} q^H(\text{ad} A)$ for $A \in \mathfrak{h}$.

This follows from the following observation. By $q \Theta A \mapsto -A \in \mathfrak{q}$, we have an anti-isomorphism $\mathfrak{g}$ of $\mathcal{O}_X$-ring $U_X(q)$ onto itself. Then, we have $\varphi(A) = -A + 2\rho(A)$ for $A \in \mathfrak{g}$. Here $\rho : \mathfrak{g} \to \mathcal{O}_X$ is the $G$-equivariant homomorphism given by $\rho \in \mathfrak{h}^*$.

4.16 Even in algebraic category, any $G$-equivariant twisted ring $A$ of differential operators on homogeneous space of $G$ is locally isomorphic to $D_X$ in the Zariski topology. In fact, if $p : G \to X$
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is a G-equivariant projection, then \( p^\# A = D_G \). Hence if \( p \) has a section \( i \), then \( A \cong i^\# p^\# A \cong D_X \). Since \( G \times X \) has a section locally in the étale topology, \( A \) is locally isomorphic to \( D_X \) in the étale topology. Hence there exists a non empty set \( U \) and an étale map \( f: U \to X \) such that \( f^\# A \) is locally isomorphic to \( D_U \). There exists an open set \( \Omega \) of \( X \) such that \( f^{-1}\Omega \to \Omega \) is finite and étale. Now, \( A \) is isomorphic to \( A_\eta \) for some closed 2-form \( \eta \) defined on \( \Omega \), by shrinking \( \Omega \) if necessary. Since \( f^\# A \) is locally isomorphic to \( D_U \), \( f^*\eta = d\omega \) for some 1-form \( \omega \). Hence \( \eta = d(f^*\omega) / n \) where \( n \) is the number of sheets of \( f^{-1}\Omega \to U \). Hence \( A_\eta \cong D_X \) on \( \Omega \). Since \( A \) is G-equivariant, \( A \) is locally isomorphic to \( D_X \) on the G-translates of \( \Omega \), which cover \( X \).
§5 Flag variety

5.1 We shall review about flag varieties. Let $G$ be a connected algebraic reductive group defined over $\mathbb{T}$. The set of Borel group forms an algebraic variety and called the flag variety of $G$. We shall denote it by $X$. Then $G$ acts on $X$ transitively. For $x \in X$, the isotropy subgroup $B(x)$ at $x$ coincides with the Borel subgroup corresponding to $x \in X$ and $G/B(x) \to X$ $(g \mapsto gx)$ gives an isomorphism.

Let $\mathfrak{b}(x)$ denote the Lie algebra of $\mathfrak{b}(x)$ and $\mathfrak{n}(x) = [\mathfrak{b}(x),\mathfrak{b}(x)]$ the nilpotent part of $\mathfrak{b}(x)$. Then $x \mapsto \mathfrak{b}(x)$ and $x \mapsto \mathfrak{n}(x)$ form $G$-equivariant vector bundles on $X$. Note that $x \mapsto \mathfrak{b}(x)/\mathfrak{n}(x)$ is the trivial bundle, because the isotropy subgroup $B(x)$ acts on $\mathfrak{b}(x)/\mathfrak{n}(x)$ trivially.

5.2 Let us fix $x_0 \in X$, $B = B(x_0)$, and let $U$ denote the unipotent part of $B$. Let us take a Cartan subgroup $T$ of $B$. Then $T = B/U$. Let us denote by $g,B,U$, and $T$ the Lie algebra of $G,B,U$, and $T$, respectively. Let $\Delta$ be the root system of $(g,t)$ and $\Delta_+$ the set of positive roots consisting of roots appearing as weight of $\mathfrak{b}$. For $\alpha \in \Delta$, let $h_\alpha^t$ the coroot of $\alpha$ and $s_\alpha$ the simple reflection corresponding to $\alpha$, i.e. $\mathfrak{t}^* \ni \lambda \mapsto \lambda - \langle h_\alpha, \lambda \rangle \alpha$. Let $W$ be the Weyl group, i.e. the group generated by $s_\alpha$'s. Recall that we have $W = N_G(T)/T$ and we have the Bruhat decomposition:

\begin{align}
5.2.1 & \quad G = \bigcup_{w \in W} BwB \\
5.2.2 & \quad X = \bigcup_{w \in W} Bwx_0 \\
5.2.3 & \quad X \times X = \bigcup_{w \in W} G(wx_0,x_0).
\end{align}

Here $w$ in the right hand side is an element of $N_G(T)$ which gives $w$ by taking mod $T$. Let $Q \subseteq \mathfrak{t}^*$ be the $\mathfrak{g}$-module generated by $\Delta$. Set

\begin{align}
5.2.4 & \quad Q_+ = \bigoplus_{\alpha \in \Delta_+} \mathbb{Z}_+ \alpha.
\end{align}

Here $\mathbb{Z}_+$ is the set of non-negative integers.

We say $\lambda \in \mathfrak{t}^*$ is \textbf{anti-dominant} (resp. regular) if $\langle h_\alpha, \lambda \rangle \neq 1,2,3,...$
(resp. \( <h_\alpha, \lambda> \neq 0 \)) for any \( \alpha \in \Delta_+ \). The following lemmas are well-known.

**Lemma 5.2.1** The following conditions are equivalent.

(i) \( \lambda \) is anti-dominant.

(ii) For any \( w \in \tilde{W}, \lambda - w \lambda \notin Q_+ \backslash \{0\} \).

**Lemma 5.2.2** The following conditions are equivalent.

(i) \( \lambda \) is regular and anti-dominant.

(ii) For any \( w \in \tilde{W} \) with \( w \neq 1 \), we have \( \lambda - w \lambda \notin Q_+ \).

5.3 Let \( P \) be the lattice of weights of \( T \). We regard \( P \subseteq t^* \) and for \( \lambda \in P \), let \( b \mapsto b^{\lambda} \) denote the character of \( B \) given by \( B \to T + \mathfrak{c}^* \), where the last arrow is the character given by \( \lambda \).

Set

\[(5.3.1) \quad P_+ = \{ \lambda \in P; \pm<\lambda, h_\alpha> \geq 0 \text{ for any } \alpha \in \Delta_+ \} \]

and

\[(5.3.2) \quad P_{\pm} = \{ \lambda \in P; \pm<\lambda, h_\alpha> > 0 \text{ for any } \alpha \in \Delta_+ \}. \]

5.4 For \( \lambda \in P \) let us denote by \( Q^*_X(\lambda) \) the \( G \)-equivariant line bundle corresponding to the character \( B \ni b \mapsto b^{\lambda} \).

Let \( p : G \to X \) be the projection \( g \mapsto gx_0 \). Then by the definition, for any open set \( U \) of \( X \)

\[(5.4.1) \quad \Gamma(U; Q^*_X(\lambda)) = \{ f \in \Gamma(p^{-1}U; Q^*_G); f(gb) = b^{-\lambda}f(g) \text{ for } (g,b) \in p^{-1}U \times B \}. \]

The following results are well-known.

**Proposition 5.4.1** If \( \lambda \notin P_- \), \( \Gamma(X; Q^*_X(\lambda)) = 0 \) and if \( \lambda \in P_- \), \( \Gamma(X; Q^*_X(\lambda)) \) is an irreducible representation of \( G \) with lowest weight \( \lambda \).

**Proposition 5.4.2** If \( \lambda \in P_{--} \), then \( Q^*_X(\lambda) \) is ample.
Proof. We shall use the criterion of Definition-Theorem 1.2.1 (2). Let $V_\lambda$ be an irreducible representation of $G$ with lowest weight $\lambda$ and let $v$ be a lowest weight vector and $u \in (V_\lambda)^*$ be a highest weight vector. Then $f = \langle v, gu \rangle$ gives a section of $O_X(\lambda)$. We have $f(1) = \langle v, u \rangle \neq 0$ and $f(w) = \langle v, wu \rangle = 0$ for any $w \neq 1$, because the weight of $wu$ is not $-\lambda$. Hence the corresponding section $s$ of $O_X(\lambda)$ satisfies $s(x_0) \neq 0$ and $s(wx_0) = 0$ for $w \neq 1, w \in W$. Since $\bigcup \bigcup \bigcup \bigcup G(wx_0, x_0) = X \times X \setminus \{\text{the diagonal set}\}$, for $x \neq y \in G,$ $w \in W \setminus \{1\}$ there is $g \in G$ such that $g^{-1}x = x_0, g^{-1}y = wx_0$. Hence $(g^*s)(x) \neq 0,$ $(g^*s)(y) = 0$. Hence $O_X(\lambda)$ satisfies the condition (2) of Definition-Theorem 1.2.1. Q.E.D.

5.5 Let $U(g)$ be the universal enveloping algebra of $g$ and let $Z(g)$ be the center of $g$. By Harish-Chandra's result, we have

\begin{equation}
(5.5.1) \quad \chi : Z(g) \cong \mathbb{C}[t^*]^W
\end{equation}

Let us recall how the isomorphism (5.5.1) is defined. For $P \in Z(g)$, there exists a unique $f \in U(t) \cong S(t) \cong \mathbb{C}[t^*]$ such that $P - f \in U(g)g$. Then we set $\chi_{\lambda}(P) = f(\lambda - \rho)$ for $\lambda \in t^*$. Here $\rho = \left( \sum_{\alpha \in \Delta^+} \right) / 2$. Then $\chi_{\lambda}(P)$ is $W$-invariant polynomial in $\lambda \in t^*$, and $\chi_{\lambda}(P)$ gives the isomorphism (5.5.1).

If we denote by $*$ the anti-isomorphism of $U(g)$ given by $g \mapsto A \mapsto -A \in g$, then we have

\begin{equation}
(5.5.2) \quad \chi_{\lambda}(P^*) = \chi_{-\lambda}(P) \quad \text{for} \quad P \in Z(g).
\end{equation}
§6 Twisted rings of differential operators on the flag variety

6.0 The notations are as in §5.

6.1 We shall study $G$-equivariant twisted ring of differential operators on the flag variety $X$. In order to do this, we shall apply Theorem 4.9.2. Since $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{t}$ and $B$ acts trivially on $\mathfrak{t}$, the isomorphic classes of equivariant twisted ring of differential operators are parametrized by $\mathfrak{t}^*$. For $\lambda \in \mathfrak{t}^*$, let us denote by $D^\lambda$ the twisted ring of differential operators $A^\lambda_X(\lambda+\rho)$ corresponding to the character $b + t \lambda + \rho \in \mathfrak{c}$. By Prop. 4.1.5., we have

\begin{equation}
D^\lambda = D - \lambda
\end{equation}

The shift $\rho$ is added so that (6.1.1) holds. Hence the ring of differential operators is $D - \rho$. For $\mu \in \mathfrak{p}$, we have

\begin{equation}
o_X(\mu) \otimes D^\lambda \otimes o_X(-\mu) = D^\lambda + \mu
\end{equation}

6.2 By 4.6 and 4.10, we have a Lie algebra homomorphism $\mathfrak{g} \rightarrow \Gamma(X; D^\lambda)$, which extends to a ring homomorphism:

\begin{equation}
U(\mathfrak{g}) \rightarrow \Gamma(X; D^\lambda).
\end{equation}

Lemma 6.2.2 Ker $\chi_\lambda$ is contained in the kernel of (6.2.1).

Proof Since (6.2.1) is $G$-equivariant it is enough to show that

$$\text{Ker } \chi_\lambda = C_{x_0} \otimes_o D^\lambda$$

is the zero map, where $C_{x_0} = O^*_\mathfrak{x}_0/x_0/m(x_0)$ with the maximal ideal $m(x_0)$ of $O^*_\mathfrak{x}_0$. Note that $D^\lambda = U^\mathfrak{x}(\mathfrak{g})/ \sum_{\lambda+\rho \in \mathfrak{h}} \sum_{A \in \mathfrak{g}} (A-\lambda+\rho, A)U(\mathfrak{g})$ where $\mathfrak{g}$ is the kernel of $O^* \otimes \mathfrak{g} \rightarrow O^*_X$. (See § 4.10.) Hence we have

$$C_{x_0} \otimes_o D^\lambda = U(\mathfrak{g})/ \sum_{A \in \mathfrak{p}} (A-\lambda+\rho, A)U(\mathfrak{g})$$

For $\mathfrak{p} \in \mathfrak{z}(\mathfrak{g})$, we have
with \( f \in U_0(g) + f \) and \( \chi_{\lambda}(P) = f(\lambda + \rho) \). Hence we obtain \( P \in \bigcap_{\lambda \in b} (\Lambda - (\lambda + \rho))U(g) \) if \( \chi_{\lambda}(P) = 0 \).

We define

\[(6.2.2) \quad U_0(g) = U(g)U(g)(\text{Ker}(\chi_{\lambda}; \mathbb{Z}(g) + C)).\]

**Proposition 6.2.3** \( U_0(g) \to \Gamma(X; D_{\lambda}) \) is an isomorphism.

**Proof** Let \( F(U(g)) \) be the filtration given by \( F_m(U(g)) = F_1(U(g))F_{m-1}(U(g)), F_1(U(g)) = g \Theta C, F_0(U(g)) = C \). Then \( \text{gr}^F U(g) \cong S(g) \).

Let \( F(U_0(g)) \) be the induced filtration. Then we have

\[\text{gr}^F U_0(g) = S_0^0(S(g)I_+) \]

where \( I_+ = (gS(g))^G \). Now, we have the following lemma.

**Lemma 6.2.4** \( \Gamma(T^*X; O_{T^*X}) = S(g)/S(g)I_+ \).

For \( x \in X \), the infinitesimal action of \( g \) on \( X \) gives \( g \to T_xX \).

Taking the dual, we obtain \( T^*_xX \to g^* \). This gives \( \rho : T^*X \to g^* \).

If we identify \( g \) with its dual by \( G \)-invariant non-degenerate symmetric form, \( \rho(T^*X) \) coincides with the set \( N \) of nilpotent elements. Then \( N \) is normal and

\[\Gamma(N; O_N) \cong S(g)/S(g)I_+ \]

Since \( \rho \) is birational and proper, \( \Gamma(T^*X; O_{T^*X}) \cong \Gamma(N; O_N) \).

Q.E.D.

Hence \( \Gamma(X; \text{gr}^F D_{\lambda}) \cong \Gamma(T^*X; O_{T^*X}) \cong S(g)/S(g)I_+ \cong \text{gr}^F U_{\lambda} \).

Therefore we have a diagram

\[
\begin{array}{cccccc}
0 & \to & F_{m-1}(U_{\lambda}) & \to & F_m(U_{\lambda}) & \to & \text{gr}^F U_{\lambda} & \to & 0 \\
& & \downarrow{\alpha_{m-1}} & & \downarrow{\alpha_m} & & \{ & & \\
0 & \to & \Gamma(X; F_{m-1}(D_{\lambda})) & \to & \Gamma(X; F_m(D_{\lambda})) & \to & \Gamma(X; \text{gr}^F D_{\lambda}) & \to & 0
\end{array}
\]

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Therefore, if $\alpha_{m-1}$ is bijective, $\alpha_m$ is bijective. Thus by the induction, $\alpha_m$ is bijective for every $m$.

**Remark 6.2.5** In the course of the proof, we used the fact that $\rho(T^*X)$ is normal. This is not true if $X$ is a generalized flag manifold (i.e. a projective homogeneous space of $G$), and $\Gamma(X;A_X(\lambda)) \to U(g)$ is not necessarily surjective (See [BoB]).

### 6.3
We shall prove the following theorem.

**Theorem 6.3.1** Assume that $\lambda$ is anti-dominant. Then for any $D_\lambda$-module $M$ quasi-coherent over $O_X$, we have

$$H^k(X;M) = 0 \text{ for } k \neq 0.$$  

**Proof** If $\mu$ is in $\mathcal{P}_{++}$, then $O(\mu)$ is ample. Hence by Theorem 1.4.1, it is enough to show that

$$(6.3.1) \quad D_\lambda \otimes O_X(-\mu) \otimes \Gamma(X;O_X(\mu)) \to D_\lambda$$

splits. Set $V_\mu = \Gamma(X;O_X(\mu))$. Then (6.3.1) corresponds to a morphism of twisted $(g,B)$-modules

$$(6.3.2) \quad U(g) \otimes (C_{-\lambda-\rho} \otimes V_\mu) \to U(g) \otimes C_{-\lambda-\rho}.$$  

Hence it is enough to show (6.3.2) splits. Let us take a filtration of $V_\mu$ by $B$-modules:

$$(6.3.3) \quad V_\mu = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N \supseteq V_{N+1} = 0$$

such that

$$(6.3.4) \quad V_0/V_1 = C_\mu$$

$$(6.3.5) \quad V_j/V_{j+1} = C_{\mu_j} \quad \text{for some } j.$$  

Hence $\mu_0, \mu_1, \cdots, \mu_N$ are weights of $V_\mu$. Hence we have

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(6.3.6) \( \mu_j - \mu \in \mathbb{Q}_+ \)

(6.3.7) \( \mu_j - \mu \in \mathbb{Q}_+ \setminus \{0\} \) for \( j \neq 0 \).

Set \( M_j = U(g) \otimes_B (C_{-\lambda - \mu - \rho \otimes V_j}) \). Then we have

(6.3.8) \( M_j / M_{j+1} = U(g) \otimes_B C_{-\lambda - \rho - \mu + \nu_j} \)

and (6.3.2) is given by \( M_0 \to M_0 / M_1 \). Hence \( M_j / M_{j+1} \) has an infinitesimal character \( \chi_{-\lambda - \mu + \nu_j} \).

**Lemma 6.3.2** \( \chi_{-\lambda - \mu + \nu_j} \neq \chi_{-\lambda} \) for \( \nu_j \neq 0 \).

Admitting this lemma for a while, we shall complete the proof of Theorem 6.3.1. We have

\[
\left( \chi_{-\lambda} \right)^{M_0} \rightarrow (M_0 / M_1).
\]

Here \( M_0 = \{ u \in M_0 ; P \in \chi_{-\lambda}(P)u \text{ for any } P \in \mathbb{P}(g) \} \).

Hence, \( M_0 \to M_0 / M_1 \) splits. Q.E.D.

**Proof of Lemma 6.3.2** Assume \( \chi_{-\lambda - \mu + \nu_j} = \chi_{-\mu} \). Then there exists \( w \in W \) such that \( -\lambda - \mu + \nu_j = -w\lambda \). Hence \( \mu - \nu_j = \lambda - \omega \lambda \in \mathbb{Q}_+ \). Since \( \lambda \) is anti-dominant, \( \mu - \nu_j = 0 \). This is a contradiction.

**Theorem 6.3.3** If \( \lambda \) is regular and anti-dominant, then for any \( D_\lambda \)-module \( M \) quasi-coherent over \( O_X \), \( M \) is generated by global sections.

**Proof** By Theorem 1.3.1, it is enough to show that \( D_\lambda \otimes O_X(-\mu) \) is generated by global sections for \( \mu \in \mathbb{P}_- \). In order to see this, it is enough to show the morphism

\[
D_\lambda \otimes \Gamma(X; O_X(\mu)) \rightarrow D_\lambda \otimes O_X(-\mu)
\]

splits. Consider the corresponding morphism of twisted \((g,B)\)-modules

(6.3.9) \( U(g) \otimes_B (C_{-\lambda - \rho \otimes V}) \rightarrow U(g) \otimes_B C_{-\lambda - \rho - \mu} \).

Here \( V = \Gamma(X; O_X(\mu)) \) is an irreducible representation with highest weight \( -\mu \). Take a filtration of \( V \) by \( B \)-modules:
such that

\begin{equation}
0 = V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_N = V
\end{equation}

(6.3.10) \quad V_0 = \mathbb{C}_{-\mu}

(6.3.11) \quad V_j/V_{j-1} \cong \mathbb{C}_{\mu_j}.

Then \( \mu_0 = -\mu, \mu_1, \ldots, \mu_N \) are weights on \( V \). Then
\( M_j = U(g) \otimes_{\mathbb{C}_{-\lambda-\rho}} \mathbb{C}_{-\mu_j} \)
gives a filtration of \( M = U(g) \otimes_{\mathbb{C}_{-\lambda-\rho}} \mathbb{C}_{-\mu_j} \) and \( M_j/M_{j-1} = U(g) \otimes_{\mathbb{C}_{-\lambda-\rho}} \mathbb{C}_{-\mu_j} \).

The last module has an infinitesimal character \( \chi_{-\lambda+\mu_j} \). If we have

\begin{equation}
\chi_{-\lambda+\mu_j} \neq \chi_{-\lambda-\mu} \quad \text{for any } j \neq 0,
\end{equation}

then we have \( M/\mathbb{C}_{-\lambda-\mu}M_0 = M_0 \) and \( M_0 \) is a direct summand of \( M \). Thus
(6.3.9) splits.

Finally, we shall prove (6.3.12). If \( \chi_{-\lambda+\mu_j} = \chi_{-\lambda-\mu} \), there exists \( w \in W \) such that \( w(-\lambda+\mu_j) = -\lambda-\mu \). Hence we have \( -\mu - w\mu_j = -\lambda - w\lambda \).

Since \( w\mu_j \) is a weight of \( V \), \( -\mu - w\mu_j \in \mathbb{Q}_+ \). Since \( \lambda \) is regular and anti-dominant \( \lambda = w\lambda \) and \( w = 1 \). Hence \( \mu_j = -\mu \).

Remark 6.3.4 In the situation of Theorem 6.3.3, \( M \) is generated by global sections not only as a \( \mathbb{D}_\lambda \)-module but as an \( \mathbb{O}_X \)-module because so is \( \mathbb{D}_\lambda \).

6.4 Thus we can apply the result of 1.5.

Theorem 6.4.1 If \( \lambda \) is anti-dominant and regular, then the category of \( \mathbb{D}_\lambda \)-modules quasi-coherent over \( \mathbb{O}_X \) is equivalent to the category of \( U_\lambda (g) \)-modules.

Theorem 6.4.2 If \( \lambda \) is anti-dominant, the category of \( U_\lambda (g) \)-modules are equivalent to the category of \( \mathbb{D}_\lambda \)-modules \( M \) quasi-coherent over \( \mathbb{O}_X \) satisfying the following properties
(a) \( M \) is generated by global sections.
(b) If a \( \mathbb{D}_\lambda \)-submodule \( N \) of \( M \) which is quasi-coherent over \( \mathbb{O}_X \) satisfies \( \Gamma(X; N) = 0 \), then \( N = 0 \).

Remark that finitely generated \( U_\lambda (g) \)-modules corresponds to coherent \( \mathbb{D}_\lambda \)-modules.
§7 SL₂-case

7.1 We shall exhibit the results in the preceding section in the case of SL₂. Set $G=SL_2$, $g=sl_2$. Take a base of $g$

(7.1.1) $h=(1 \ -1)$, $e=(0 \ 1 \ 0)$, $f=(0 \ 0 \ 1)$.

The flag manifold $X$ can be identified with $\mathbb{P}^1$. Set $U_0=\mathbb{P}^1\setminus\{0\}$, $U_1=\mathbb{P}^1\setminus\{0\}$ and take coordinates $x$ of $U_0$ and $y$ of $U_1$ related by $xy=1$. The action of $G$ on $X$ is given by

(7.1.2) $g=(a \ b; c \ d): x \mapsto \frac{ax+b}{cx+d}, y \mapsto \frac{dy+c}{by+a}$.

Take $x_0=\infty$ (i.e. $y=0$ in $U_1$). Then $B=(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix})$ and $z=ch\otimes e$.

Take $t=ch$. Then $\Delta=(a,-a)$ with $a(h)=2$. We have $\rho=a/2$ and $\rho(h)=1$. The center of $U(g)$ is generated by

$\Delta=(h-1)^2+4ef=(h+1)^2+4fe$.

7.2 The infinitesimal action $g \to \Theta_X$ is given by

$$h \mapsto -2x_+_x = 2y_+y$$

$$e \mapsto -x = y^2_+y$$

$$f \mapsto x^2_+y = -y$$

and hence the kernel $\mathfrak{g}$ of $\mathcal{O}_X \to \Theta_X$ is generated by $h-2xe$ and $f+xh-x^2e$ (on $U_0$) and $\rho: \mathfrak{g} \to \mathcal{O}_X$ is given by $\rho(f+xh-x^2e)=0$, $\rho(h-2xe)=-1$.

7.3 For $\lambda \in \mathbb{C}^*$, set $c=\lambda(h)$. Hence $\lambda=cp$. If $c$ is an integer

$\mathcal{O}_X(1)=\mathcal{O}_X(-c\infty)$ where $\mathcal{O}_X(-c\infty)$ is the sheaf of meromorphic functions with pole of degree $-c$ at $\infty$.

7.4 For $\lambda=c\rho \in \mathbb{C}^*$, $D_\lambda$ is given as follows

(7.4.1) $i_0: D_\lambda|_{U_0} \cong D_{U_0}$, $i_1: D_\lambda|_{U_1} \cong D_{U_1}$
and \( i_{\nu_0}^{-1} \mid_{U_0} \nu_1 \rightarrow D_{U_0} \nu_1 \rightarrow D_{U_0} \nu_1 \) is given by \( p \rightarrow x^{1+c_{P_x}^{-1}} \). The homomorphism \( \alpha: g \rightarrow \Gamma(X; D_\lambda) \) is given by

\[
\begin{align*}
i_{\nu_0} \circ \alpha: h & \mapsto -2x \partial_x - (c+1) \\
i_{\nu_1} \circ \alpha: h & \mapsto 2y \partial_y + (c+1) \\
e & \mapsto -\partial_x \\
e & \mapsto y^2 \partial_y + (c+1)y \\
f & \mapsto x^2 \partial_x + (c+1)x \\
f & \mapsto -\partial_y.
\end{align*}
\]

7.5 For example, let \( M \) be a \( D_\lambda \)-module given by

\[
M \mid_{U_1} = D_\lambda / D_\lambda y \quad \text{and} \quad M \mid_{U_0} = 0.
\]

Then \( \text{supp } M = \{x_0\} \), and if we denote by \( \delta \) the generator \( 1 \mod D_\lambda y \), then

\[
\Gamma(X; M) = M_{x_0} = \mathbb{C}[\partial_y] \delta = \mathbb{C}[f] \delta
\]

with the relation \( h \delta = (2y \partial_y + c-1) \delta = (c-1) \delta, e \delta = (y \partial_y + c)y \delta = 0 \). Thus we have \( \Gamma(X; M) \) is isomorphic to the Verma module \( U(g) / U(g)e + U(g)(h - (c-1)) \).

7.6 If \( \lambda \) is not anti-dominant (i.e. \( c=1,2,3,\ldots \)) then \( O_X(\lambda+p) \rightarrow O_X(-(c+1)e) \) is a \( D_\lambda \)-module. Since \( H^1(X; O_X(-(c+1)e)) \approx \mathbb{C} \), the vanishing theorem for \( D_\lambda \)-modules does not hold in this case.

7.7 We shall investigate the case where \( \lambda \) is anti-dominant and not regular, i.e. \( \lambda = 0 \). In this case, for a non-zero \( D_\lambda \)-module \( M \), \( \Gamma(X; M) \) may vanish. In fact \( O_X(\lambda+p) \rightarrow O_X(-e) \) gives such an example. The following proposition asserts that essentially this is the only case.

Lemma 7.7.1 Let \( M \) be a coherent \( D_0 \)-module. Then the following conditions are equivalent

(i) As a \( D_0 \)-module, \( M \) is isomorphic to the direct sum of copies of \( O_X(p) \).

(ii) \( \Gamma(X; M) = 0 \)

and in this case, \( g \) acts trivially on \( \Gamma(X; O_X(-p) \otimes M) \) and \( O_X(p) \otimes \Gamma(X; O_X(-p) \otimes M) \approx M \).
Proof  The last statement follows from (i).

(i) ⇒ (ii) trivial.

(ii) ⇒ (i) Since $O_X(-\rho)\Theta M$ is a $D_{-\rho}$-module, it is generated by global sections. Hence we have

$$D_{-\rho}^m \rightarrow O_X(-\rho)\Theta M.$$

Tensoring $O_X(\rho)$ from the left we have

$$(D_0 \Theta O_X(\rho))^m \rightarrow M \rightarrow 0.$$

Since $\Gamma(X; M) = 0$, we have

$$(D_0 \Theta O_X(\rho)/D_0 \Gamma(X; D_0 \Theta O_X(\rho)))^m \rightarrow M.$$

Hence it is enough to show

$$(7.7.1) \quad D_0 \Theta O_X(\rho)/D_0 \Gamma(X; D_0 \Theta O_X(\rho)) \hookrightarrow O_X(\rho).$$

In fact, any submodule of $O_X(\rho)^m$ has also the same type. We have an exact sequence

$$U(\mathfrak{g})\Theta_{\mathfrak{g}}^2 \rightarrow U(\mathfrak{g})\Theta_{\mathfrak{g}}^1 \rightarrow 0$$

of twisted $(g,B)$-modules with twist $\rho$. Here $\mathfrak{g}^2$ is the fundamental representation of $G$. Correspondingly, we have

$$D_0 \Theta \mathfrak{g}^2 \rightarrow D_0 \Theta O_X(\rho) \rightarrow O_X(\rho) \rightarrow 0.$$

This shows the existence of (7.7.1).

Corollary 7.7.2  For any coherent $D_0$-module, the kernel and the cokernel of

$$D_0 \Theta \Gamma(X; M) \rightarrow M$$

are isomorphic to a direct product of copies of $O_X(\rho)$.
§8. Singular case

8.1 For a simple root \( \alpha \), let \( P_{\alpha} \) be the parabolic subgroup such that \( \text{Lie}(P_{\alpha}) = t \oplus g_{-\alpha} \oplus \bigoplus_{\beta \in \Delta_+} g_{\beta} \). Let \( X_\alpha \) be the set of conjugate subgroups of \( P_{\alpha} \). Then \( X_\alpha \cong G/P_{\alpha} \). Let \( P_\alpha : X \rightarrow X_\alpha \) be the projection. Then \( P_\alpha \) is a \( \mathbb{P}^1 \)-bundle. For \( \lambda \in t^* \) with \( \langle \lambda, h_\alpha \rangle = 0 \), \( \lambda \) defines a character \( \text{Lie}(P_{\alpha}) \rightarrow t \rightarrow \mathbb{C} \), and this defines a \( G \)-equivariant twisted ring of differential operators \( A_{X_\alpha}(\lambda) \) on \( X_\alpha \).

By Proposition 4.14.1, we have

**Proposition 8.1.1** \( p^*_\alpha A_{X_\alpha}(\lambda) \cong D^\lambda_{-\rho} \).

**Corollary 8.1.2** If \( \lambda - \rho \) is anti-dominant, then for any coherent \( A_{X_\alpha}(\lambda) \)-module \( M \), we have \( H^j(X_\alpha; M) = 0 \) for \( j \neq 0 \).

**Proof** We have \( H^j(X_\alpha; p^*_\alpha M) = 0 \) for \( j \neq 0 \) since \( p^*_\alpha M \) is a \( D^\lambda_{-\rho} \)-module. The relations \( R^k p^*_\alpha p^*_\alpha M = M \) for \( k = 0 \) and \( = 0 \) for \( k \neq 0 \) imply

\[
H^j(X_\alpha; p^*_\alpha M) = H^j(X_\alpha; M).
\]

**Remark 8.1.3** Corollary is true for any parabolic subgroup other than \( P_{\alpha} \).

8.2 Let \( \lambda \in t^* \) be such that \( \langle \lambda, h_\alpha \rangle = 0 \). We assume

\[
(8.2.1) \quad \rho \in P.
\]

This is not a strict condition because we can replace \( G \) with a covering group of it. The flag varieties \( X \) and \( X_\alpha \) do not change after this replacement.

**Proposition 8.2.1** Let \( M \) be a coherent \( D^\lambda \)-module.

(i) \( R^k p^*_\alpha M = 0 \) for \( k \neq 0 \).

(ii) The following conditions are equivalent.

(a) \( p^*_\alpha M = 0 \).

(b) There exists a coherent \( A_{X_\alpha}(\lambda) \)-module \( N \) such that \( M \cong O_{X}(\rho) \otimes p^*_\alpha N \) as an \( D^\lambda \)-module.
(c) \[ M \cong O_X(\rho) \otimes_{\alpha^*} \alpha^*(O_X(-\rho) \otimes M) \]

Proof. If (b) is satisfied, then

\[ p_{\alpha^*}(O_X(-\rho) \otimes M) = p_{\alpha^*} \otimes_{\alpha^*} N = N. \]

Hence \( N \) is uniquely determined. Thus, these properties are local in \( X \). Locally in \( \alpha \), we have

(8.2.2) \( X = \mathbb{P}^1 \times \alpha \)

(8.2.3) \( D_\lambda = D_0 \otimes D_\alpha \).

Hence we can reduce them to Lemma 7.7.1.

8.3 Let \( \Sigma \) be the set of simple roots. For \( \lambda \in \mathfrak{t}^* \), we set

(8.3.1) \[ \Delta_\lambda = \{ \alpha \in \Delta; \langle \alpha', \lambda \rangle = 0 \}, \]
\[ W_\lambda = \{ w \in W; w \lambda = \lambda \}, \]
\[ \Sigma_\lambda = \Sigma \cap \Delta_\lambda. \]

Then \( \Delta_\lambda \) is also a root system and \( W_\lambda \) is the Weyl group for \( \Delta_\lambda \); i.e. \( W_\lambda \) is generated by the \( s_\alpha \) (\( \alpha \in \Delta_\lambda \)).

Let us consider the conditions

(8.3.2) \( \Delta_\lambda \) is the lattice generated by \( \Sigma_\lambda \), or equivalently \( \Sigma_\lambda \) is the set of simple roots for \( \Delta_\lambda \).

Then (8.3.2) implies that

(8.3.3) \( W_\lambda \) is generated by \( s_\alpha \) (\( \alpha \in \Sigma_\lambda \)).

Theorem 8.3.1 We assume that \( \lambda \) is anti-dominant and satisfies (8.3.2). Then there exists a sequence \( \alpha_1, \ldots, \alpha_N \) in \( \Sigma_\lambda \) such that for any coherent \( D_\lambda \)-module \( M \) the following conditions are equivalent.

(8.3.4) \( \Gamma(X; M) = 0. \)

(8.3.5) There exists a filtration \( M = M_0 \supset M_1 \supset \cdots \supset M_N = 0 \) by coherent \( D_\lambda \)-module such that \( p_{\alpha_j^*}(M_{j-1}/M_j) = 0 \) for \( j = 1, 2, \ldots, N \).
Proof (8.3.5) implies (8.3.4) because
\[ \Gamma(X; M_{j-1}/M_j) = \Gamma(X_{\alpha_j}, p_{\alpha_j}^*(M_{j-1}/M_j)) = 0. \]

We shall prove the inverse implications. Since \( \lambda - \rho \) is anti-dominant and regular, \( O_X(-\rho) \otimes M \) is generated by global sections. Hence there exists a sequence
\[ D_{\lambda - \rho}^n \rightarrow O_X(-\rho) \otimes M. \]

Tensoring \( O_X(\rho) \) we obtain
\[ (D_{\lambda} \otimes O_X(\rho))^n \rightarrow M. \]

Hence, setting
\[ \mathcal{M} = D_{\lambda} \otimes O_X(\rho)/D_{\lambda} \Gamma(X; D_{\lambda} \otimes O_X(\rho)) \]
we have \( \mathcal{M}^n \rightarrow \mathcal{M} \). Since for a coherent \( D_{\lambda} \)-module \( N \), the relation \( p_{\alpha_j}^*(N) = 0 \) is invariant by taking coherent quotients of \( N \), it is enough to show (8.3.5) for \( \mathcal{M} \) for some \( \alpha_1, \ldots, \alpha_N \in L_{\lambda} \).

Note that
\[ D_{\lambda} \otimes O_X(\rho) = V_X(\mathbb{C}_{\lambda} + \rho) \otimes U(\mathfrak{g}) \otimes \mathbb{C}_{-(\lambda + \rho) + \rho}. \]

Set \( M_0 = U(\mathfrak{g}) \otimes \mathbb{C}_{-\lambda} \).

Since \( \rho \) is regular and integral with respect to \( \Lambda_{\lambda} \), there exists \( \alpha_1, \ldots, \alpha_N \in L_{\lambda} \) such that, setting \( \mu_0 = \rho \), \( s_{\alpha_j}(\mu_{j-1}) = \mu_j \), \( \mu_N \) is anti-dominant with respect to \( L_{\lambda} \), and \( \langle h_{\alpha_j}, \mu_{j-1} \rangle = 1, 2, 3, \ldots \). The last property implies \( M_j \subset M_{j-1} \), where \( M_j = V_X(\mathbb{C}_{\lambda} \otimes (M_{j-1}/M_j)) \).

It is easy to see that \( M_{j-1}/M_j \) is a twisted \((\mathfrak{g}, p_{\alpha_j})\)-module with twist \( \lambda \). Hence if we set \( \mathcal{N}_j = V_{X_{\alpha_j}}(\mathbb{C}_{\lambda} \otimes (M_{j-1}/M_j)) \) it is an \( A_{X_{\alpha_j}}(\lambda) \)-module.

Set \( M_j = V_X(\mathbb{C}_{\lambda} + \rho \otimes M_j). \)

Then \( M_{j-1}/M_j = O_X(\rho) \otimes p_{\alpha_j}^*(\mathcal{N}_j). \) Hence it is enough to show that
is generated by global sections. In fact, then there is a surjective morphism $M_0/M_N \to D_\lambda \mathcal{O}_{\mathfrak{g}}(\rho)/D_\lambda \mathcal{O}(X; D_\lambda \mathcal{O}_{\mathfrak{g}}(\rho))$.

Let $V$ be an irreducible representation with highest weight $\rho$. In order to see that $M_N$ is generated by global sections, it is enough to construct a surjective morphism

$$U(\mathfrak{g}) \otimes_{\mathfrak{g}} V \to U(\mathfrak{g}) \otimes_{\mathfrak{g}} V_{\lambda - \rho + \mu_N}^\cdot,$$

For $\xi \in \mathfrak{p}$, let $V_{\xi}$ be the weight space of $V$ with weight $\xi$.

Set $V' = \bigoplus \limits_{\lambda - \rho + \mu_N \in \mathfrak{q}_+} \xi \in \mathfrak{q}_+$. Then, $V'$ is a $\mathfrak{b}$-module. Set $V'' = V/V'$. Then the weight $\xi$ of $V''$ satisfies $\mu_N - \xi \in \mathfrak{q}_+$. Moreover $V''/V'$ is a sub-$\mathfrak{b}$-module of $V''$.

**Lemma 8.3.2** If $\xi$ is a weight of $V''$ different from $\mu_N$, then $\chi - \lambda + \xi \neq \chi - \lambda + \mu_N$.

If this lemma is shown, then $U(\mathfrak{g}) \otimes_{\mathfrak{g}} (\mathfrak{g} - \mathfrak{g}) \otimes_{\mathfrak{g}} \mathfrak{g}$ is a direct summand of $U(\mathfrak{g}) \otimes (\mathfrak{g} - \mathfrak{g}) \otimes V''$. Hence we obtain a surjective homomorphism

$$U(\mathfrak{g}) \otimes_{\mathfrak{g}} V \to U(\mathfrak{g}) \otimes (\mathfrak{g} - \mathfrak{g}) \otimes \mathfrak{g} \otimes V'' + U(\mathfrak{g}) \otimes (\mathfrak{g} - \mathfrak{g}) \otimes \mathfrak{g} \otimes V''.$$

This completes the proof of Theorem 8.3.1.

**Proof of Lemma 8.3.2** Assume that $\chi - \lambda + \xi = \chi - \lambda + \mu_N$ for a weight $\xi$ of $V''$. Then

$$\lambda - \xi \in W(\lambda - \mu_N) = W(\lambda - \rho).$$

Therefore there exists $w$ such that $w(\lambda - \xi) = \lambda - \rho$, or $\lambda - w\lambda = \rho - w\xi$. Since $w\xi$ is a weight of $V$, $\rho - w\xi \in \mathfrak{q}_+$. Since $\lambda$ is anti-dominant, we have $w\lambda = \lambda$. Thus we have $\xi \in W_\lambda \rho = W_\lambda \mu_N$. Since $\mu_N$ is regular anti-dominant with respect to $\mathfrak{a}$, we have $\xi - \mu_N \in \sum \limits_{\alpha \in \mathfrak{a}^+} \mathbb{Z}_{+} \alpha \subset \mathfrak{q}_+$. Since $\mu_N - \xi \in \mathfrak{q}_+$, we have $\xi = \mu_N$. Q.E.D.

**Remark 8.3.3** For any $\lambda \in \mathfrak{t}^*$, there exists $w \in W$ such that $w\lambda$ is anti-dominant and satisfies (8.3.2). Hence these two conditions are not severe.
9 Harish-Chandra modules

9.1 Let $G$, $\xi$, $\Delta$, $X$, ... be as in §5. Let $H$ be an affine algebraic group with a group morphism $f : H \to G$. Let $\mathfrak{h}$ be the Lie algebra of $H$.

Proposition 9.1.1 If $M$ is a $(g, H)$-module, then $\mathcal{D}_\lambda g_1^1 M$ is an $H$-equivariant $\mathcal{D}_\lambda$-module. Conversely, if $M$ is an $H$-equivariant $\mathcal{D}_\lambda$-module, then $\mathcal{F}(X; M)$ is a $(g, H)$-module. This follows from §4.7.

9.2 Hence if $\lambda$ is regular and anti-dominant, the category of finitely generated $(g, H)$-modules with infinitesimal character $\chi_\lambda$ is equivalent to that of $H$-equivariant $\mathcal{D}_\lambda$-modules. When $\lambda$ is not regular, we need the modification as in Theorem 6.4.2, that we discuss later more precisely.

9.3 Let us assume further

(9.3.1) The flag variety $X$ of $G$ has finitely many $H$-orbits.

Theorem 9.3.1 Under the condition (9.3.1), for any $\lambda \in t^*$, any $H$-equivariant coherent $\mathcal{D}_\lambda$-module is regular holonomic.

Remark 9.3.2 The following statement is false: let $X$ be a projective algebraic smooth variety and $G$ an affine algebraic group acting on $X$. If $X$ has finitely many $G$-orbits, then any $G$-equivariant coherent module over any $G$-equivariant twisted ring of differential operators is regular holonomic.

When $G$ is reductive, I have no counterexample.

9.4 Proof of Theorem 9.3.1 Let $Z = G/U$ and let $p : Z \to X = G/B$ be the canonical projection. Then $p$ is the principal fiber bundle with the structure group $T$. Then $p^*\mathcal{D}_\lambda = \mathcal{D}_Z$ as $G$-equivariant twisted ring of differential operators. Let $M$ be an $H$-equivariant coherent $\mathcal{D}_\lambda$-module. Then $N = p^*M$ is $(H \times T)$-equivariant. It is enough to show that $N$ is regular holonomic by Proposition 3.8.2. We shall prove by the induction of the number of $(H \times T)$-orbits in $\text{Supp } N$. Let $S$ be an open $(H \times T)$-orbits of $\text{Supp } N$. Let $j : S \to Z$ be an embedding. Then there
exists an \((H \times T)\)-equivariant \(D_S\)-module \(L\) such that \(N|_S \cong j^*(D_Z \otimes_{D_S} L)|_S\). Let \(q:H \times T \times S\) be an \((H \times T)\)-equivariant map. Then \(q\) is surjective and smooth. Since \(q^*L\) is \((H \times T)\)-equivariant, it is isomorphic to the direct sum of finite copies of \(O_{H \otimes D_{N'}} \sum_{A \in T} (A^{<\lambda+\rho,A>}\). Hence \(q^*L\) is completely regular. Therefore \(L\) is completely regular by Proposition 3.8.2. Hence \(N''=j^*(D_Z \otimes_{D_S} L)\cong H^0_S(N)\) is regular holonomic. Thus we obtain an \((H \times T)\)-equivariant \(D_Z\)-modules

\[0 \longrightarrow N' \longrightarrow N \longrightarrow N''.\]

Since \(\text{Supp } N' \subset \text{Supp } N \setminus S, N'\) is regular holonomic by the hypothesis of the induction. Hence \(N\) is also regular holonomic.

### 9.5 Let \(M\) be an irreducible \(H\)-equivariant coherent \(D_\lambda\)-module (i.e. there is no proper \(H\)-equivariant coherent sub-\(D_\lambda\)-module). Then \(\text{Supp } M\) is the closure of an \(H\)-orbit \(S\). In fact, \(M \hookrightarrow H^0_S(M)\) must be injective. Furthermore \(M\) must be the minimal extension of \(M|_X \setminus S\).

Here \(\delta S=S\setminus S\). Let \(j:S \hookrightarrow X\) be the embedding. Then there exists an \(H\)-equivariant \(j^!D_\lambda\)-module \(N\) such that \(M|_X \setminus S\cong j^!(D_\lambda, x \in S \otimes_{D_\lambda} N)|_X \setminus S\).

Since \(N\) is an \(H\)-equivariant module, it is described as in §4.11. Namely, take an \(x \in S\) and let \(H_X\) be the isotropy subgroup. Then we obtain \(H_X \times B(x) \rightarrow T\) and representing map \(\text{Lie}(H_X) \rightarrow \mathfrak{t}\). Then \(N\) is described by \(H_X\)-module such that its infinitesimal representation is \(\lambda+\rho\).

Let \(S'(H, \lambda)\) be the set of isomorphic classes of the triplets \((S, x, M)\), where \(S\) is an \(H\)-orbit of \(X, x \in S\) and \(M\) is an irreducible \(H_X\)-module such that its infinitesimal representation \(\text{Lie}(H_X) \rightarrow \text{End}(M)\) coincides with \(\text{Lie}(H_X) \rightarrow \mathfrak{t} \rightarrow \mathfrak{c} \subset \text{End}(M)\). Here, \((S, x, M) \cong (S', x', M')\) if \(S=S', x'=h_0 x\) for some \(h_0 \in H\) and there exists \(\psi:M \rightarrow M'\) such that \(\psi(hu)=(h_0 h h_0 ^{-1})u\) for \(h \in H_X\) and \(u \in M\).

**Remark 9.5.1** If \(H \subset G\), then \(M\) must be one-dimensional representation. In fact, if we denote by \(U(x)\) the unipotent part of \(B(x)\), then \(H \cap U(x)\) is connected and its infinitesimal action on \(M\) is trivial. Hence \(M\) is a representation of \(H_X/H \cap U(x) \subset B(x)/U(x) \cong T\).

**Theorem 9.5.2** The set of the isomorphic classes of irreducible \(H\)-
equivariant coherent $D_{\lambda}$-modules is isomorphic to $S'(H, \lambda)$.

9.6 As the Corollary of Theorem 9.5.2 and §9.2, we obtain the following theorem.

Theorem 9.6.1 Assume that $X$ has finitely many $H$-orbits and let $\lambda \in \mathfrak{h}^*$ be regular anti-dominant. Then the set of the isomorphic classes of irreducible $(g, H)$-modules is isomorphic to $S'(H, \lambda)$.

Remark 9.6.2 As seen in §3, the category of regular holonomic $D_{\lambda}$-modules is equivalent to the category of twisted perverse sheaves with the twist $\mathbb{T}$ corresponding to $D_{\lambda}$. Incidentally, $S'(H, \lambda)$ is isomorphic to the category of the pairs $(S, F)$ of $H$-orbits $S$ and irreducible $H$-equivariant twisted sheaves $F$ on $S$ with twist $\mathbb{T}$.

9.7 Now, we shall investigate the case when $\lambda$ is anti-dominant and satisfies the condition (8.3.2). We shall use the notations $E_\lambda, p_\alpha : X \rightarrow X_\alpha$ as in §8. In this case, irreducible $(g, H)$-module is obtained as the global sections of a unique irreducible $H$-equivariant $D_{\lambda}$-module $M$ such that $\Gamma(X; M) \neq 0$.

We shall interpret the condition $\Gamma(X; M) = 0$ in terms of $(S, x, M) \in S'(H, \lambda)$. If $\Gamma(X; M) = 0$, then by Theorem 8.3.1, there exists $\alpha \in E_\lambda$ and non-zero coherent submodule $N$ of $M$ such that $p_\alpha^*(N) = 0$. The largest $N$ among such $N$'s must be $H$-equivariant and hence $M = N$. Thus $p_\alpha^* M = 0$. Let us take a connected covering group $\tilde{G} \supset G$ such that $\rho$ is a weight of $\tilde{G}$. Let $\tilde{H}$ be the fiber product of $\tilde{G}$ and $H$ over $G$. By Proposition 8.2.1, $p_\alpha^* M = 0$ is equivalent to the existence of $A_{\lambda}^X(\rho)$-module $N$ such that $M = O_{\lambda}(\rho) \otimes p_\alpha^* N$. Hence $N$ is an $H$-equivariant $A_{\lambda}^X(\rho)$-module. Let $S$ be an open $H$-orbit of $\text{Supp } M$. Then $\text{Supp } M = \mathfrak{s}$, $\text{Supp } N = p_\alpha^* (\mathfrak{s})$ and $\mathfrak{s} = p_\alpha^{-1} p_\alpha (\mathfrak{s})$. Take $x \in S$ and set $y = p_\alpha(x)$. Then $S \cap p_\alpha^{-1}(y) = p_\alpha^{-1}(y)$. Since $p_\alpha^{-1}(y) \supset \mathfrak{g}^*$, $S \cap p_\alpha^{-1}(y)$, which is an orbit of $H_y$, must be either $\mathfrak{g}^*$, $\mathfrak{c}$ or $\mathfrak{c}^*$. Moreover the condition $M = O_{\lambda}(\rho) \otimes p_\alpha^* N$ is equivalent to saying that $M|_{S \cap p_\alpha^{-1}(y)}$ is isomorphic to $O_{\lambda}(\rho)^m$ for some $m$. When $S \cap p_\alpha^{-1}(y) = \mathfrak{c}$ or $\mathfrak{g}^*$, this is simply connected and hence it is true. The remaining case is the case $S \cap p_\alpha^{-1}(y) = \mathfrak{c}^*$. Let $(S, x, M) \in S'(H, \lambda)$.
correspond to $M$. Let $\varphi: H_x \to \text{Aut}(M)$ be the action of $H_x$ on $M$.
Similarly let $(p_\alpha(S), y, N)$ corresponds to $N$. Then $N$ is a representation of $\tilde{H}_y$ whose infinitesimal action is by $\lambda$. Here the suffix signifies the isotropy subgroup at that point and

$$\lambda: \text{Lie}(\tilde{H}_y) \to \mathbb{C}$$

is given by

$$\text{Lie}(\tilde{H}_y) \to \text{Lie}(\tilde{G}_y) \to \text{Lie}(\text{the reductive part of } \tilde{G}_y) \to \mathbb{C}.\tag{9.7.1}$$

Note that $\langle h_\alpha, \lambda \rangle = 0$ because $a \in \Sigma$. Moreover $M = C_{-\rho} \otimes N$ as an $\tilde{H}_x$-module. Hence the condition $p_{\alpha} \cdot M = 0$ is interpreted to the condition:

$C_{-\rho} \otimes \varphi: \tilde{H}_x \longrightarrow \text{Aut}(C_{-\rho} \otimes M)$ extends to $\psi: \tilde{H}_y \longrightarrow \text{Aut}(C_{-\rho} \otimes M)$ such that $d\psi = \lambda$. Now, we have, as $\tilde{H}_y = (\tilde{H}_y)^{\circ}, x,$

$$(\tilde{H}_y)^{\circ} = (\tilde{H}_y)^{\circ} \cdot \tilde{H}_x.$$  

Here $\circ$ signifies the connected component containing 1. Since $\psi(\tilde{H}_y)$ is in the center of $\text{Aut}(C_{-\rho} \otimes M)$, in order to extend $C_{-\rho} \otimes \varphi$ onto $\tilde{H}_y$, it is enough to extend $\varphi$ to $\psi^0: (\tilde{H}_y)^{\circ} \to \text{Aut}(C_{-\rho} \otimes M)$ with $d\psi^0 = \lambda$. Let $x_1$ be one of the points in $p_{\alpha}^{-1}(y) \setminus S$. Then $(\tilde{H}_{x_1})^{\circ} = (\tilde{H}_y)^{\circ}.$

Since $C_{\rho}$ is a representation of $(\tilde{H}_{x_1})^{\circ}$, it is enough to extend $\tilde{H}_x \to \text{Aut}(M)$ to $(\tilde{H}_{x_1})^{\circ} \to \text{Aut} M$. Since Ker($H^*H$) acts identically on $M$, $(\tilde{H}_{x_1})^{\circ} \to \text{Aut} M$ factors through $(\tilde{H}_x)^{\circ} \to \text{Aut} M$. Summing up, we obtain the following.

Let $S(H, \lambda)$ be the subset of $S'(H, \lambda)$ such that $(S, x, M) \in S(H, \lambda) \setminus S'(H, \lambda)$ if and only if $(S, x, M)$ satisfies the following two conditions for some $a \in \Sigma$.

$$(9.7.2) \quad S \cap p_{\alpha}^{-1} p_{\alpha}(x) \equiv H_{p_{\alpha}(x)}/H_x \text{ is not a finite set.}$$

$$(9.7.3) \quad \text{If } S \cap p_{\alpha}^{-1} p_{\alpha}(x) \equiv \mathbb{C}^*, \text{ then }$$

$$H_x \cap H_{p_{\alpha}(x)}^{\circ} \to \text{Aut}(M)$$

extends to $\psi: H_{p_{\alpha}(x)}^{\circ} \to \text{Aut}(M)$ such that $d\psi = \lambda + \rho$. Here $\text{Lie}(H_{p_{\alpha}(x)}) \overset{\lambda + \rho}{\to} \mathbb{C}$ is given as follows: taking $x_1 \in p_{\alpha}^{-1} p_{\alpha}(x) \setminus S,$

$$\text{Lie}(H_{p_{\alpha}(x)}) \overset{\lambda + \rho}{\to} \text{Lie}(B(x_1)) \overset{t}{\to} \overset{\lambda + \rho}{\to} \mathbb{C}.\quad \text{107}$$
Remark. Similarly to the case of $\tilde{H}_{\alpha}(x)$, if $H_\kappa \cap (x_{1})^\circ + \text{Aut}(M)$ extends onto $(x_{1})^\circ H_{\alpha}(x)$ as in (9.7.3), then $H_\kappa \cap H_{x_{1}} + \text{Aut}(M)$ extends to $\psi \cdot H_{x_{1}} + \text{Aut}(M)$ with $d\psi = \lambda + \rho$.

Theorem 9.7.1. Assume that $X$ has finitely many $H$-orbits and that $\lambda$ is anti-dominant and satisfies (8.3.2). Then the set of the isomorphic classes of irreducible $(q,H)$-modules is equal to $S(H,\lambda)$.

Example 9.7.2. Let us take $G = \text{SL}_2$ as in §7. Let us take as $H$ the torus $\{e^a\cdot 1; a \in \mathbb{C}^*\}$. Then the isomorphic classes of $(q,H)$-modules corresponds to the irreducible representations of $\text{SL}_2(\mathbb{R})$.

Now $X$ has three $H$-orbits, namely, $\{0\}$, $\{\infty\}$ and $S_0 = X \backslash \{0, \infty\}$. Then the isotropy subgroups are given by, $H_0 = H_\infty = H$ and $H_1 = \{\pm 1\}$.

If $\lambda \notin \mathbb{Z} \rho$, then the infinitesimal representation $\lambda + \rho$ of $\text{Lie}(H)$ cannot extend to representation of $H$. Hence

\begin{equation}
S(H,\lambda) = \{(S_0,1,M_+), (S_0,1,M_-)\} \quad \text{for } \lambda \notin \mathbb{Z} \rho.
\end{equation}

Here $M_+$ is the trivial representation of $H_1$ and $M_-$ is the other one-dimensional representation of $H_1$. If $\lambda = -m \rho$ with a positive integer $m$, then

\begin{equation}
S(H,\lambda) = (S_0,1,M_+), (S_0,1,M_-), (\{0\}, 0, *) (\{\infty\}, *, *)
\end{equation}

Here $*$ corresponds to the representation of $H$ corresponding to $\lambda + \rho$.

If $\lambda = 0$, then $(S_0,1,M_-) \notin S(H,\lambda)$ because $M_-$ extends to the representation of $H$ with infinitesimal representation of $H$ with infinitesimal character $\rho$. Hence

\begin{equation}
S(H,\lambda) = \{(S_0,1,M_+), \{0\}, \{\infty\}\} \quad \text{for } \lambda = 0.
\end{equation}

This coincides with the well-known classification of irreducible representation of $\text{SL}_2(\mathbb{R})$. The data $(S_0,1,M_+)$ correspond to the principal series (when $\lambda \notin \mathbb{Z} \rho$), and $(\{0\}, 0, *), (\{\infty\}, 0, *)$ correspond to discrete series.
Bibliography


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