FINITE-DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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Abstract. We present a conjecture on the irreducibility of the tensor products of fundamental representations of quantized affine algebras. This conjecture implies in particular that the irreducibility of the tensor products of fundamental representations is completely described by the poles of R-matrices. The conjecture is proved in the cases of type $A_n^{(1)}$ and $C_n^{(1)}$.

0. Introduction

In this paper we study finite-dimensional representations of quantum affine algebras. It is known that any finite-dimensional irreducible representation is isomorphic to the irreducible subquotient of a tensor product $\otimes_\nu V(\varpi_{i_\nu})_{a_\nu}$ containing the highest weight (Drinfeld [7], Chari-Pressley [2]). Here $V(\varpi_i)$ is the fundamental representation corresponding to the fundamental weight ϖ_i and a_ν are spectral parameters. Moreover $\{(\varpi_{i_\nu}; a_\nu)\}_\nu$ is uniquely determined up to permutation. This gives a parameterization of the isomorphic classes of finite-dimensional irreducible representations.

However it is not known for example what is the character of those irreducible representations except the complete result for $A_1^{(1)}$ ([2]) and some other results due to Chari-Pressley ([2, 3, 4]). We have even not known when $\otimes V(\varpi_{i_\nu})_{a_\nu}$ itself is irreducible.

In this paper we propose a conjecture on the irreducibility of $\otimes_\nu V(\varpi_{i_\nu})_{a_\nu}$ and prove this conjecture for $A_n^{(1)}$ and $C_n^{(1)}$.

For $x, y \in \mathbb{C}(q)$, let us denote $x \le y$ if x/y does not have a pole at $q = 0$. We denote by u_i the highest weight vector of $V(\varpi_i)$.

Conjecture 1.
(1) If $a_1 \le \cdots \le a_N$, then $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_N})_{a_N}$ is generated by $u_{i_1} \otimes \cdots \otimes u_{i_N}$ as a $U'_q(\mathfrak{g})$-module.
(2) If $a_1 \ge \cdots \ge a_N$, then any non-zero $U'_q(\mathfrak{g})$-submodule of $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_N})_{a_N}$ contains $u_{i_1} \otimes \cdots \otimes u_{i_N}$.

Here $U'_q(\mathfrak{g})$ is the quantum affine algebra without derivation (see §1.1).

This conjecture implies in particular the following consequences.

Claim 1. *If $a_1 \le a_2$, then the normalized R-matrix*

$$R^{\mathrm{nor}}_{i,j}(x,y) : V(\varpi_i)_x \otimes V(\varpi_j)_y \to V(\varpi_j)_y \otimes V(\varpi_i)_x$$

does not have a pole at $(x,y) = (a_1, a_2)$.

Here $R^{\mathrm{nor}}_{i,j}(x,y)$ is so normalized that it sends $u_i \otimes u_j$ to $u_j \otimes u_i$.

Claim 2. *$V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_N})_{a_N}$ is irreducible if and only if the R-matrix*

$$R^{\mathrm{nor}}_{i_\nu,i_\mu}(x,y) : V(\varpi_{i_\nu})_x \otimes V(\varpi_{i_\mu})_y \to V(\varpi_{i_\mu})_y \otimes V(\varpi_{i_\nu})_x$$

does not have a pole at $(x,y) = (a_\nu, a_\mu)$ for any $1 \le \nu$, $\mu \le N$ $(\nu \ne \mu)$.

Claim 3. *Assume that $R^{\mathrm{nor}}_{i_\nu,i_\mu}(x,y)$ has no pole at $(x,y) = (a_\nu, a_\mu)$ for any $1 \le \mu < \nu \le N$. Then the submodule generated by $u_{i_1} \otimes \cdots \otimes u_{i_N}$ is an irreducible submodule of $V(\varpi_{i_1})_{a_\nu} \otimes \cdots \otimes V(\varpi_{i_N})_{a_N}$. Conversely, any finite-dimensional irreducible integrable module is obtained in this way.*

Claim 4. *If M and M' are irreducible finite-dimensional integrable $U'_q(\mathfrak{g})$-modules, then $M \otimes M'_z$ is an irreducible $U'_q(\mathfrak{g})$-module except for finitely many z.*

The plan of the paper is as follows. In §1, we fix notations and explain the results used later. We announce non published results but they can be directly checked for the $A^{(1)}_n$ and $C^{(1)}_n$ cases. In §2, we announce the main conjecture and discuss its consequences. In §3, we reduce the main conjecture to another auxiliary conjecture, which will be proved in the case $A^{(1)}_n$ and $C^{(1)}_n$ in §4. In the appendix, we shall calculate the explicit form of the normalized R-matrices and the universal R-matrices between fundamental representations of $A^{(1)}_n$ and $C^{(1)}_n$.

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1. Notations

1.1. **Quantized affine algebras.** Let $(a_{ij})_{i,j\in I}$ be a generalized Cartan matrix of affine type. We choose a $\mathbb{Q}$-vector space $\mathfrak{t}$ of dimension $\sharp I + 1$ and simple roots $\alpha_i \in \mathfrak{t}^*$ and simple coroots $h_i \in \mathfrak{t}$ such that $\langle h_i, \alpha_j \rangle = a_{ij}$. We assume further that α_i and h_i are linearly independent. Set $Q = \sum_i \mathbb{Z}\alpha_i$ and $Q^\vee = \sum_i \mathbb{Z} h_i$. Let $\delta = \sum a_i \alpha_i$ be the smallest positive imaginary root and let $c = \sum a_i^\vee h_i \in Q^\vee$ be the center. Set $\mathfrak{t}^*_{\mathrm{cl}} = \mathfrak{t}^*/\mathbb{Q}\delta$ and let $\mathrm{cl} : \mathfrak{t}^* \to \mathfrak{t}^*_{\mathrm{cl}}$ be the projection. We set $\mathfrak{t}^{*0} = \{\lambda \in \mathfrak{t}^*; \langle c, \lambda \rangle = 0\}$ and $\mathfrak{t}^{*0}_{\mathrm{cl}} = \mathrm{cl}(\mathfrak{t}^{*0})$.

We take a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{t}^*$ such that

$$\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any } i \in I \text{ and } \lambda \in \mathfrak{t}^*.$$

We normalize it by

$$\langle c, \lambda\rangle = (\delta, \lambda) \quad \text{for any } \lambda \in \mathfrak{t}^*. \tag{1.1}$$

We identify sometimes $\mathfrak{t}$ and $\mathfrak{t}^*$ by this symmetric form.

Let us take a (weight) lattice $P \subset \mathfrak{t}^*$ such that $\alpha_i \in P$ and $h_i \in P^*$ for every $i \in I$. We assume further that P contains Λ_i satisfying $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ and that $P \cap \mathbb{Q}\delta = \mathbb{Z}\delta$. We set $P_{\mathrm{cl}} = P/\mathbb{Z}\delta \subset \mathfrak{t}^*_{\mathrm{cl}}$, $P^0 = \{\lambda \in P\,; \langle c, \lambda\rangle = 0\} \subset \mathfrak{t}^{*0}$, and $P^0_{\mathrm{cl}} = \mathrm{cl}(P^0) \subset \mathfrak{t}^{*0}_{\mathrm{cl}}$. Note that the dual lattice of $Q^\vee$ coincides with $P_{\mathrm{cl}} \cong \oplus_{i\in I}\mathbb{Z}\mathrm{cl}(\Lambda_i)$.

Let γ be the smallest positive integer such that

$$\gamma(\alpha_i, \alpha_i)/2 \in \mathbb{Z} \quad \text{for any } i \in I. \tag{1.2}$$

Then the quantized affine algebra $U_q(\mathfrak{g})$ is the algebra over $k = \mathbb{Q}(q^{1/\gamma})$ generated by the symbols e_i, $f_i (i \in I)$ and $q(h)\,(h \in \gamma^{-1}P^*)$ satisfying the following defining relations.

(1) $q(h) = 1$ for $h = 0$.
(2) $q(h_1)q(h_2) = q(h_1 + h_2)$ for $h_1, h_2 \in \gamma^{-1}P^*$.
(3) For any $i \in I$ and $h \in \gamma^{-1}P^*$,

$$\begin{aligned} q(h)e_iq(h)^{-1} &= q^{\langle h,\alpha_i\rangle}e_i \quad \text{and} \\ q(h)f_iq(h)^{-1} &= q^{-\langle h,\alpha_i\rangle}f_i\,. \end{aligned}$$

(4) $[e_i, f_j] = \delta_{ij}\dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$. Here $q_i = q^{(\alpha_i,\alpha_i)/2}$ and $t_i = q(\frac{(\alpha_i,\alpha_i)}{2}h_i)$.
(5) (Serre relations) For $i \neq j$,

$$\sum_{k=0}^{b}(-1)^k e_i^{(k)}e_je_i^{(b-k)} = \sum_{k=0}^{b}(-1)^k f_i^{(k)}f_jf_i^{(b-k)} = 0.$$

Here $b = 1 - \langle h_i, \alpha_j\rangle$ and

$$\begin{aligned} e_i^{(k)} &= e_i^k/[k]_i!\,, & f_i^{(k)} &= f_i^k/[k]_i!\,, \\ [k]_i &= (q_i^k - q_i^{-k})/(q_i - q_i^{-1})\,, & [k]_i! &= [1]_i\cdots[k]_i\,. \end{aligned}$$

We denote by $U'_q(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, $f_i\,(i \in I)$ and $q(h)\,(h \in \gamma^{-1}Q^\vee)$.

In this paper we consider only $U'_q(\mathfrak{g})$. A $U'_q(\mathfrak{g})$-module M is called *integrable* if M has the weight decomposition $M = \oplus_{\lambda\in P_{\mathrm{cl}}}M_\lambda$ where $M_\lambda = \{u \in M; q(h)u = q^{\langle h,\lambda\rangle}u\}$, and if M is $U_q(\mathfrak{g})_i$-locally finite (i.e. $\dim U_q(\mathfrak{g})_iu < \infty$ for every $u \in M$) for every $i \in I$. Here $U_q(\mathfrak{g})_i$ is the subalgebra generated by e_i, f_i and t_i.

We use the coproduct Δ of $U_q(\mathfrak{g})$ given by

$$\Delta(q(h)) = q(h)\otimes q(h)\,, \tag{1.3}$$
$$\Delta(e_i) = e_i\otimes t_i^{-1}+1\otimes e_i\,, \tag{1.4}$$
$$\Delta(f_i) = f_i\otimes 1+t_i\otimes f_i\,, \tag{1.5}$$

so that the lower crystal bases behave well under the corresponding tensor products ([12]).

1.2. **Finite-dimensional representations.** Let $W\subset \mathrm{Aut}(\mathfrak{t}^*)$ be the Weyl group, and let $l: W\to\mathbb{Z}$ be the length function. Since δ is invariant by W, we have the group homomorphism $\mathrm{cl}_0: W\to \mathrm{Aut}(\mathfrak{t}_{\mathrm{cl}}^{*0})$. Let $W_{\mathrm{cl}}\subset \mathrm{Aut}(\mathfrak{t}_{\mathrm{cl}}^{*0})$ be the image of W by cl_0. Then W_{cl} is a finite group. Let us take $i_0\in I$ such that W_{cl} is generated by $\mathrm{cl}_0(s_i)$ ($i\in I_0=I\setminus\{i_0\}$) and that $a_{i_0}^\vee=1$. Such an i_0 is unique up to Dynkin diagram automorphism. Hereafter we write 0 instead of i_0. We have $(\alpha_0,\alpha_0)=2$.

Let us denote by W_0 the subgroup of W generated by s_i ($i\in I_0=I\setminus\{0\}$). Then W_0 is isomorphic to W_{cl}. The kernel of $W\to W_{\mathrm{cl}}$ is the commutative group $\{t(\xi);\xi\in Q_{\mathrm{cl}}\cap Q_{\mathrm{cl}}^\vee\}$. Here $Q_{\mathrm{cl}}=\mathrm{cl}(Q)=\sum_{i\in I}\mathbb{Z}\mathrm{cl}(\alpha_i)$ and $Q_{\mathrm{cl}}^\vee=\mathrm{cl}(Q^\vee)=\sum_{i\in I_0}\mathbb{Z}\mathrm{cl}(h_i)$ and $t(\xi)$ is the automorphism of $\mathfrak{t}^*$ given by

$$t(\xi)(\lambda)=\lambda+(\delta,\lambda)\xi'-(\xi',\lambda)\delta-\frac{(\xi',\xi')}{2}(\delta,\lambda)\delta$$

for $\xi'\in\mathfrak{t}^*$ such that $\mathrm{cl}(\xi')=\xi$.

The following lemma is well-known.

Lemma 1.1. *Let $\xi\in Q_{\mathrm{cl}}\cap Q_{\mathrm{cl}}^\vee$ and $w\in W_0$.*
(i) *If ξ is dominant (with respect to I_0), then we have*

$$l(w\circ t(\xi))=l(w)+l(t(\xi)).$$

(ii) *If ξ is regular and dominant, then we have*

$$l(t(\xi)\circ w)=l(t(\xi))-l(w).$$

Let us choose i_1 such that W_{cl} is generated by $\mathrm{cl}_0(s_i)$ ($i\in I\setminus\{i_1\}$) and that $a_{i_1}=1$. For any $z\in k\setminus\{0\}$, let $\psi(z)$ be the automorphism of $U_q'(\mathfrak{g})$ given by

$$\begin{aligned}\psi(z)(e_i) &= z^{\delta_{i,i_1}}e_i\,,\\ \psi(z)(f_i) &= z^{-\delta_{i,i_1}}f_i\,,\\ \psi(z)(q(h)) &= q(h)\,.\end{aligned}$$

For a $U_q'(\mathfrak{g})$-module M, let M_z be the $U_q'(\mathfrak{g})$-module with M as its underlying k-vector space and with $U_q'(\mathfrak{g})\xrightarrow{\psi(z)}U_q'(\mathfrak{g})\longrightarrow\mathrm{End}\,(M)$ as the action of $U_q'(\mathfrak{g})$. Then $M\mapsto M_z$ is a functor satisfying $(M\otimes N)_z\cong M_z\otimes N_z$. This definition extends to the case $z\in K\setminus\{0\}$ for a field extension $K\supset k$.

If M is a finite-dimensional integrable $U_q'(\mathfrak{g})$-module, then the weights of M are contained in P^0_{cl}.

1.3. **Fundamental representations.** We set $\varpi_i = \mathrm{cl}(\Lambda_i - a_i^\vee \Lambda_0)$ for $i \in I_0$. Then $(\varpi_i)_{i\in I_0}$ forms a basis of P^0_{cl}. We call ϖ_i a *fundamental weight* (of level 0).

For $i \in I_0$, there exists an irreducible integrable $U_q'(\mathfrak{g})$-module $V(\varpi_i)$ satisfying the following properties.

(1) The weights of $V(\varpi_i)$ are contained in the convex hull of $W_{\mathrm{cl}}\varpi_i$.
(2) $\dim V(\varpi_i)_{\varpi_i} = 1$.
(3) For any $\mu \in W_{\mathrm{cl}}\varpi_i \subset P^0_{\mathrm{cl}}$, we can associate a non-zero vector u_μ of weight μ such that

$$u_{s_i\mu} = \begin{cases} f_i^{(\langle h_i,\mu\rangle)} u_\mu & \text{if } \langle h_i,\mu\rangle \ge 0, \\ e_i^{(-\langle h_i,\mu\rangle)} u_\mu & \text{if } \langle h_i,\mu\rangle \le 0. \end{cases}$$

Then $V(\varpi_i)$ is unique up to an isomorphism. Moreover $V(\varpi_i)$ has a global crystal base. We call $V(\varpi_i)$ a *fundamental representation.* Then $V(\varpi_i)$ has a non-degenerate symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$ such that ${}^t e_i = f_i$ and ${}^t q(h) = q(h)$. Hence the duality is given as follows. Let w_0 be the longest element of W_0. Then for $i \in I_0$ there exists $i^* \in I_0$ such that

$$\varpi_{i^*} = -w_0 \varpi_i.$$

(Remark that $i \mapsto i^*$ with $0^* = 0$ gives a Dynkin diagram automorphism.) Then the right dual of $V(\varpi_i)$ is $V(\varpi_{i^*})_p$ with the duality morphisms:

$$k \to V(\varpi_{i^*})_{p^*} \otimes V(\varpi_i) \quad \text{and} \quad V(\varpi_i) \otimes V(\varpi_{i^*})_{p^*} \to k \tag{1.6}$$

with $p^* = (-1)^{\langle \rho^\vee, \delta\rangle} q^{(\rho,\delta)}$. Here ρ and $\rho^\vee$ are defined by: $\langle h_i, \rho\rangle = 1$ and $\langle \rho^\vee, \alpha_i\rangle = 1$ for every $i \in I$. Usually $(\rho,\delta) = \sum_{i\in I} a_i^\vee$ is called the dual Coxeter number and $\langle \rho^\vee, \delta\rangle = \sum_{i\in I} a_i$ the Coxeter number.

Let m_i be a positive integer such that

$$W(\Lambda_i - a_i^\vee \Lambda_0) = (\Lambda_i - a_i^\vee\Lambda_0) + \mathbb{Z} m_i \delta\,.$$

We have $m_i = (\alpha_i,\alpha_i)/2$ in the case where $\mathfrak{g}$ is the dual of an untwisted affine algebra, and $m_i = 1$ in the other cases.

Then for z, $z' \in K^*$, we have

$$V(\varpi_i)_z \cong V(\varpi_i)_{z'} \quad \text{if and only if } z^{m_i} = z'^{m_i}. \tag{1.7}$$

Hence we set

$$V(\varpi_i; z^{m_i}) = V(\varpi_i)_z\,.$$

The following theorem is announced by Drinfeld ([7]) in Yangian case, and its proof is given by Chari-Pressley ([3, 4]).

Theorem 1.2. *Let $K \supset k$ be an algebraically closed field and let M be an irreducible finite-dimensional $U_q'(\mathfrak{g})_K$-module. Then there exist $i_1, \ldots, i_N \in I_0$ and $z_1, \ldots z_N \in K \setminus \{0\}$ such that M is isomorphic to a unique irreducible subquotient of $V(\varpi_{i_1}; z_1) \otimes \cdots \otimes V(\varpi_{i_N}; z_N)$ containing the weight $\sum_{\nu=1}^{N} \varpi_{i_\nu}$. Moreover, $\{(i_1; z_1), \ldots, (i_N; z_N)\}$ is unique up to permutations.*

Definition 1.3. We call $V(\varpi_{i_\nu}; z_\nu)$ a component of M.

1.4. **Extremal vectors.** We say that a crystal B over $U_q'(\mathfrak{g})$ is *a regular crystal* if, for any $J \subsetneq I$, B is isomorphic to the crystal associated with an integrable $U_q(\mathfrak{g}_J)$-module. Here $U_q(\mathfrak{g}_J)$ is the subalgebra of $U_q'(\mathfrak{g})$ generated by e_i, f_i and t_i $(i \in J)$. This condition is equivalent to saying that the same assertion holds for any $J \subsetneq I$ with two elements (see [15, Proposition 2.4.4]).

By [14], the Weyl group W acts on any regular crystal. This action S is given by

$$S_{s_i} b = \begin{cases} \tilde{f}_i^{\langle h_i, \mathrm{wt}(b) \rangle} b & \text{if } \langle h_i, \mathrm{wt}(b) \rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, \mathrm{wt}(b) \rangle} b & \text{if } \langle h_i, \mathrm{wt}(b) \rangle \leq 0. \end{cases}$$

A vector b of a regular crystal B is called i-extremal if $\tilde{e}_i b = 0$ or $\tilde{f}_i b$=0. We call b an *extremal* vector if $S_w b$ is i-extremal for any $w \in W$ and $i \in I$.

Lemma 1.4. *For any λ, $\mu \in \mathfrak{t}_{\mathrm{cl}}^{*0}$ in the same W_{cl}-orbit, we can find $i_1, \ldots, i_N \in I$ such that*

$$\mu = s_{i_N} \cdots s_{i_1} \lambda,$$
$$\langle h_{i_k}, s_{i_{k-1}} \cdots s_{i_1} \lambda \rangle > 0 \quad \textit{for any } 1 \leq k \leq N.$$

Proof. It is enough to prove the statement above for a regular integral anti-dominant (with respect to I_0) weight λ and the dominant weight $\mu \in W\lambda$. We may assume further $\lambda \in Q_{\mathrm{cl}} \cap Q_{\mathrm{cl}}^{\vee}$. Let w_0 be the longest element of W_0.
By Lemma 1.1, we have

$$\begin{aligned} l(t(\lambda)) &= l(t(-\lambda)) \\ &= l(w_0) + l(t(-\lambda) w_0) \\ &= l(w_0) + l(w_0 t(\lambda)). \end{aligned}$$

Take a reduced expression $w_0 t(\lambda) = s_{i_N} \cdots s_{i_1}$. Then for $1 \leq k \leq N$ we have $l(t(\lambda) s_{i_1} \cdots s_{i_k}) = l(t(\lambda)) - k$ and hence $t(\lambda) s_{i_1} \cdots s_{i_{k-1}} \alpha_k$ is a negative root. Since it is equal to $s_{i_1} \cdots s_{i_{k-1}} \alpha_k - (\lambda, s_{i_1} \cdots s_{i_{k-1}} \alpha_k) \delta$ and $s_{i_1} \cdots s_{i_{k-1}} \alpha_k$ is a positive root, we conclude

$$(\lambda, s_{i_1} \cdots s_{i_{k-1}} \alpha_k) > 0.$$

On the other hand we have the equality $s_{i_N} \cdots s_{i_1} \lambda = w_0 t(\lambda) \lambda = w_0 \lambda$ in $\mathfrak{t}_{\mathrm{cl}}^{*0}$. Hence it is equal to μ. $\square$

For a regular crystal B, $b \in B$ and $i \in I$, let us denote by $\tilde{e}_i^{\max} b$ the i-highest weight vector in the i-string containing i. Namely we have

$$\tilde{e}_i^{\max} b = \tilde{e}_i^{\varepsilon_i(b)} b.$$

Lemma 1.5. *Let B be a finite regular crystal with level* 0 (*with weight in P_{cl}^0*).

(1) *For $b \in B$, there are $i_1, \cdots, i_N \in I$ such that $\tilde{e}_{i_N}^{\max} \cdots \tilde{e}_{i_1}^{\max} b$ is an extremal vector.*
(2) *Any vector in the W-orbit of an extremal vector b of B is written in the form $\tilde{e}_{i_N}^{\max} \cdots \tilde{e}_{i_1}^{\max} b$.*

Proof. Let us set $F_l = \{\tilde{e}_{i_l}^{\max} \cdots \tilde{e}_{i_1}^{\max} b\,;\, i_1, \cdots, i_l \in I\}$, $F = \bigcup_{l \geq 0} F_l$. Replacing b with $b' \in F$ with maximal $(\mathrm{wt}(b'), \mathrm{wt}(b'))$, we may assume from the beginning that $(\mathrm{wt}(b'), \mathrm{wt}(b')) \leq (\mathrm{wt}(b), \mathrm{wt}(b))$ for any $b' \in F$. Since $(\mathrm{wt}(b'), \mathrm{wt}(b')) \geq (\mathrm{wt}(b), \mathrm{wt}(b))$, we have $(\mathrm{wt}(b'), \mathrm{wt}(b')) = (\mathrm{wt}(b), \mathrm{wt}(b))$ for any $b' \in F$, and hence any $b' \in F$ is i-extremal for every $i \in I$. Moreover the weight of b' is in the W_{cl}-orbit of $\mathrm{wt}(b)$. Then for any weight μ of F and i such that $\langle h_i, \mu \rangle \leq 0$, S_{s_i} sends injectively F_μ to $F_{s_i\mu}$. Hence $\sharp(F_\mu) \leq \sharp(F_{s_i\mu})$, and Lemma 1.4 asserts that they must be equal. Therefore $S_i : F_\mu \to F_{s_i\mu}$ is bijective. This shows that F is stable by all S_{s_i}. Thus we have (1) and (2). □

Lemma 1.6. *Let B_1 and B_2 be two finite regular crystals. Let b_1 and b_2 be vectors in B_1 and B_2, respectively.*

(1) *If b_1 and b_2 are extremal vectors and if their weights are in the same Weyl chamber, then $b_1 \otimes b_2$ is extremal.*
(2) *Conversely if $b_1 \otimes b_2$ is extremal, then b_1 and b_2 are extremal vectors and their weights are in the same Weyl chamber.*

Proof. (1) is obvious because $S_w(b_1 \otimes b_2) = S_w b_1 \otimes S_w b_2$ under this condition.

We shall prove (2). Since $\tilde{e}_{i_1}^{\max} \cdots \tilde{e}_{i_N}^{\max}(b_1 \otimes b_2) = \tilde{e}_{i_1}^{\max} \cdots \tilde{e}_{i_N}^{\max} b_1 \otimes b_2'$ for some $b_2' \in B_2$, the preceding lemma implies that b_1 is extremal. Similarly b_2 is extremal. It remains to prove that $\mathrm{wt}(b_1)$ and $\mathrm{wt}(b_2)$ are in the same Weyl chamber. Let us show first that $\mathrm{wt}(b_1 \otimes b_2)$ and $\mathrm{wt}(b_1)$ are in the same Weyl chamber. We may assume without loss of generality that $\mathrm{wt}(b_1 \otimes b_2)$ is dominant (with respect to I_0). Then $\tilde{e}_i(b_1 \otimes b_2) = 0$ for every $i \in I_0$. Hence $\tilde{e}_i b_1 = 0$. Hence $\mathrm{wt}(b_1)$ is dominant. Hence $\mathrm{wt}(b_1 \otimes b_2)$ and $\mathrm{wt}(b_1)$ are in the same Weyl chamber. Similarly $\mathrm{wt}(b_1 \otimes b_2)$ and $\mathrm{wt}(b_2)$ are in the same Weyl chamber. Thus $\mathrm{wt}(b_1)$ and $\mathrm{wt}(b_2)$ are in the same Weyl chamber. □

Definition 1.7. We say that a finite regular crystal B is simple if B satisfies

(1) There exists $\lambda \in P_{\mathrm{cl}}^0$ such that the weights of B are in the convex hull of $W_{\mathrm{cl}}\lambda$.
(2) $\sharp(B_\lambda) = 1$.
(3) The weight of any extremal vector is in $W_{\mathrm{cl}}\lambda$.

Proposition 1.8. *The crystal graph of the fundamental representations is simple.*

The proof will be given elsewhere. However we can easily check this for the $A_n^{(1)}$ and $C_n^{(1)}$ cases.

Lemma 1.9. *A simple crystal B is connected.*

Proof. In fact, any vector is connected with an extremal vector by Lemma 1.5. ☐

Lemma 1.10. *The tensor product of simple crystals is also simple.*

Proof. This immediately follows from Lemma 1.6. ☐

Proposition 1.11. *Let M be a finite-dimensional integrable $U'_q(\mathfrak{g})$-module with a crystal base (L, B). Assume the following conditions.*
(1.8) *B is connected.*
(1.9) *There exists a weight $\lambda \in P^0_{\mathrm{cl}}$ such that $\dim(M_\lambda) = 1$.*
Then M is irreducible.

Proof. We shall show first that M_λ generates M. Set $N = U'_q(\mathfrak{g})M_\lambda$ and $\bar{N} = (L \cap N)/(qL \cap N) \subset L/qL$. Then $\bar{N}$ is invariant by $\tilde{e}_i$ and $\tilde{f}_i$. Hence $\bar{N}$ contains B, and Nakayama's lemma asserts that $N = M$. By duality, any non-zero submodule of M contains M_λ. Therefore M is irreducible. ☐

Corollary 1.12. *A finite-dimensional $U'_q(\mathfrak{g})$-module with a simple crystal base is irreducible.*

Corollary 1.13. *For $i_1, \dots, i_N \in I_0$, $V(\varpi_{i_1}) \otimes \cdots \otimes V(\varpi_{i_N})$ is irreducible.*

We define similarly an extremal vector of an integrable $U'_q(\mathfrak{g})$-module.

Definition 1.14. Let v be a weight vector of an integrable $U'_q(\mathfrak{g})$-module. We call v *extremal* if the weights of $U'_q(\mathfrak{g})v$ are contained in the convex hull of $W\mathrm{wt}(v)$.

When the weight of v is of level 0 and dominant (with respect to I_0), v is extremal if and only if $\mathrm{wt}(U'_q(\mathfrak{g})v) \subset \mathrm{wt}(v) + \sum_{i \in I_0} \mathbb{Z}_{\le 0}\mathrm{cl}(\alpha_i)$. In this case, we call v a *dominant extremal vector*.

Since the following proposition is not used in this paper, the proof will be given elsewhere.

Proposition 1.15. *Let v be a weight vector of an integrable $U'_q(\mathfrak{g})$-module. The following two conditions are equivalent.*

(1) *v is an extremal vector.*
(2) *We can associate a vector v_w of weight $w\mathrm{wt}(v)$ to each $w \in W$ satisfying the following properties:*

(a) $v_w = v$ *if* $w = e$,
(b) *If* $i \in I$ *and* $w \in W$ *satisfy* $\langle h_i, w\mathrm{wt}(v)\rangle \geq 0$, *then* $e_i v_w = 0$ *and* $v_{s_i w} = f_i^{(\langle h_i, w\mathrm{wt}(v)\rangle)} v_w$,
(c) *If* $i \in I$ *and* $w \in W$ *satisfy* $\langle h_i, w\mathrm{wt}(v)\rangle \leq 0$, *then* $f_i v_w = 0$ *and* $v_{s_i w} = e_i^{-(\langle h_i, w\mathrm{wt}(v)\rangle)} v_w$.

The implication (1)⇒(2) is obvious.

Let us denote by $U_q'(\mathfrak{b})$ the subalgebra of $U_q'(\mathfrak{g})$ generated by t_i and e_i ($i \in I$).

Proposition 1.16. *Let M be a finite-dimensional integrable $U_q'(\mathfrak{g})$-module. Then any $U_q'(\mathfrak{b})$-submodule of M is a $U_q'(\mathfrak{g})$-submodule.*

Proof. Let N be a $U_q'(\mathfrak{b})$-submodule. For any pair of weights λ and μ conjugate by W_{cl}, there exist $i_1, \ldots, i_l$ such that $m_k = -\langle h_{i_k}, s_{i_{k-1}} \cdots s_{i_1}\lambda\rangle > 0$ and $\mu = s_{i_l} \cdots s_{i_1}\lambda$ by Lemma 1.4. Then $e_{i_l}^{m_l} \cdots e_{i_1}^{m_1}$ sends injectively N_λ to N_μ. Hence we have $\dim N_\lambda \leq \dim N_\mu$. Thus we obtain $\dim N_\lambda = \dim N_\mu$. Then the proposition follows from the following lemma. □

Lemma 1.17. *Let M be a finite-dimensional integrable $U_q(\mathfrak{sl}_2)$-module and let N be a vector subspace of M stable by e and t. If $\dim N_\lambda = \dim N_{s(\lambda)}$ for any λ (s is the simple reflection), then N is a $U_q(\mathfrak{sl}_2)$-submodule.*

Proof. Any $u \in N_\lambda$ can be written

$$u = \sum_n f^{(n)} v_n$$

with $ev_n = 0$. Here n ranges over $\{n \in \mathbb{Z}_{\geq 0}; n + \langle h, \lambda\rangle \geq 0\}$.

Let us prove $U_q(\mathfrak{sl}_2)v_n \subset N$ by the descending induction on $c = \langle h, \lambda\rangle$. We have $eu = \sum_n [1 + c + n] f^{(n-1)} v_n$. Hence the induction hypothesis implies $U_q(\mathfrak{sl}_2)v_n \subset N$ for $n > 0$. Hence we may assume that $eu = 0$, and then $c \geq 0$. The surjectivity of $e^c : N_{s\lambda} \to N_\lambda$ implies the existence of $w \in N_{s\lambda}$ such that $u = e^{(c)}w$. Then $fw = 0$ and $U_q(\mathfrak{sl}_2)u = U_q(\mathfrak{sl}_2)w$ is generated by $\{e^n w; n \geq 0\} \subset N$. □

Lemma 1.18. *Let M_1 and M_2 be finite-dimensional $U_q'(\mathfrak{g})$-modules and let v_1 and v_2 be non-zero weight vectors of M_1 and M_2. If $v_1 \otimes v_2$ is extremal, then v_1 and v_2 are extremal and their weights are in the same Weyl chamber (in $\mathfrak{t}_{\mathrm{cl}}^{*0}$).*

Proof. We may assume that $\mathrm{wt}(v_1 \otimes v_2)$ is dominant. Then for any $P \in U'(\mathfrak{b})$, we have

$$P(v_1 \otimes v_2) = v_1 \otimes Pv_2 + \cdots .$$

Hence the weights of $U_q'(\mathfrak{b})v_2$ is contained in $\mathrm{wt}(v_2) + Q_-$. Since $U_q'(\mathfrak{b})v_2 = U_q'(\mathfrak{g})v_2$ by Prop 1.16, v_2 is an extremal vector with a dominant weight. Similary so is v_1. □

2. Conjecture

We denote $\bigcup_{n>0}\mathbb{C}((q^{1/n}))$ by $\bar{k}$ and $\bigcup_{n>0}\mathbb{C}[[q^{1/n}]]$ by $\bar{A}$. Hence $\bar{k}$ is an algebraically closed field and $\bar{A}$ is a local ring. For a, $b \in \bar{k}^{\times} = \bar{k} \setminus \{0\}$, we write $a \le b$ if $a/b \in \bar{A}$.

For $i \in I_0$, let u_i denote the dominant extremal vector of $V(\varpi_i)$.

Conjecture 1. Let $i_1, \dots, i_l$ be elements of I_0 and $a_1, \dots, a_l$ non-zero elements of $\bar{k}$.
(1) If $a_1 \le \cdots \le a_l$, then $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_l})_{a_l}$ is generated by $u_{i_1} \otimes \cdots \otimes u_{i_l}$ as a $U_q'(\mathfrak{g})_{\bar{k}}$-module.
(2) If $a_1 \ge \cdots \ge a_l$, then any non-zero $U_q'(\mathfrak{g})_{\bar{k}}$-submodule of $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_l})_{a_l}$ contains $u_{i_1} \otimes \cdots \otimes u_{i_l}$.

Note that (1) and (2) are dual statements and therefore they are equivalent. One can compare (1) to the case of Verma modules and (2) to the case of the dual of Verma modules.

Let us discuss several consequences of this conjecture.

For i,$j \in I_0$, there is an intertwiner

$$R^{\mathrm{nor}}_{ij}(x, y) : V(\varpi_i)_x \otimes V(\varpi_j)_y \to V(\varpi_j)_y \otimes V(\varpi_i)_x \tag{2.1}$$

We normalize this such that R sends $u_i \otimes u_j$ to $u_j \otimes u_i$. Then we regard it as a rational function in (x, y). Since it is homogeneous, its pole locus has the form $y/x =$ constant. We call it the *normalized R-matrix.* By Corollary 1.13 such an $R^{\mathrm{nor}}_{ij}(x, y)$ is unique,

Corollary 2.1. *If $a_1 \le a_2$, the normalized R-matrix $R^{\mathrm{nor}}_{i,j}(x, y)$ does not have a pole at $(x, y) = (a_1, a_2)$.*

Proof. Suppose that $R^{\mathrm{nor}}_{i,j}(x, y)$ has a pole at $(x, y) = (a_1, a_2)$. Let R' be the non-zero $U_q'(\mathfrak{g})$-linear map $V(\varpi_i)_{a_1} \otimes V(\varpi_j)_{a_2} \to V(\varpi_j)_{a_2} \otimes V(\varpi_i)_{a_1}$ obtained after cancelling the poles of $R^{\mathrm{nor}}_{i,j}(x, y)$. Then $R'(u_i \otimes u_j) = 0$, and hence $\mathrm{Im}(R')$ does not have weight $\varpi_i + \varpi_j$. On the other hand, Conjecture 1 (2) implies that $\mathrm{Im}(R')$ contains $u_j \otimes u_i$, which is a contradiction. Hence $R^{\mathrm{nor}}_{ij}(x, y)$ has no pole at (a_1, a_2). □

Corollary 2.2. *Let K be a field extension of k, and $i_1, \dots, i_l \in I_0$, $a_1, \dots, a_l \in K^{\times} = K \setminus \{0\}$.*
(1) *Assume that $R^{\mathrm{nor}}_{i_\nu, i_\mu}(x, y)$ does not have a pole at $(x, y) = (a_\nu, a_\mu)$ for $1 \le \nu < \mu \le l$. Then $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_l})_{a_l}$ is generated by $u_{i_1} \otimes \cdots \otimes u_{i_l}$ as a $U_q'(\mathfrak{g})_K$-module.*
(2) *Assume that $R^{\mathrm{nor}}_{i_\nu, i_\mu}(x, y)$ does not have a pole at $(x, y) = (a_\nu, a_\mu)$ for $1 \le \mu < \nu \le l$. Then any non-zero $U_q'(\mathfrak{g})_K$-submodule of $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_l})_{a_l}$ contains $u_{i_1} \otimes \cdots \otimes u_{i_l}$.*

Proof. We may assume that K is generated by $a_1, \dots, a_l$ over k. Since $\bar{k}$ is an algebraically closed field with infinite transcendental dimension over k, there exists an embedding $K \hookrightarrow \bar{k}$. Hence we may assume $K = \bar{k}$.

Since the proof of (2) is similar, we shall only prove (1). We prove (1) by induction on the number of pairs (ν, μ) with $\nu < \mu$ and $a_\nu \not\leq a_\mu$, which we denote by n. If $n = 0$, the assertion follows immediately from Conjecture 1. If $n > 0$, take ν such that $a_\nu \not\leq a_{\nu+1}$. Hence $a_{\nu+1} \leq a_\nu$. Then Corollary 2.1 implies that $R^{\mathrm{nor}}_{i_{\nu+1},i_\nu}(x,y)$ does not have a pole at $(x,y) = (a_{\nu+1}, a_\nu)$. Since $R^{\mathrm{nor}}_{i_\nu,i_{\nu+1}}(x,y)$ does not have a pole at $(x,y) = (a_\nu, a_{\nu+1})$ by the assumption,

$$R^{\mathrm{nor}}_{i_\nu,i_{\nu+1}}(a_\nu, a_{\nu+1}) : V(\varpi_{i_\nu})_{a_\nu} \otimes V(\varpi_{i_{\nu+1}})_{a_{\nu+1}} \to V(\varpi_{i_{\nu+1}})_{a_{\nu+1}} \otimes V(\varpi_{i_\nu})_{a_\nu}$$

and

$$R^{\mathrm{nor}}_{i_{\nu+1},i_\nu}(a_{\nu+1}, a_\nu) : V(\varpi_{i_{\nu+1}})_{a_{\nu+1}} \otimes V(\varpi_{i_\nu})_{a_\nu} \to V(\varpi_{i_\nu})_{a_\nu} \otimes V(\varpi_{i_{\nu+1}})_{a_{\nu+1}}$$

are inverse of each other. Hence we can reduce the original case to the case where ν and $\nu+1$ are exchanged, in which n is smaller than the original one by 1. Hence the induction proceeds. □

Assume the condition (1) in the preceding Corollary 2.2. Let R be the intertwiner

$$R : V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_l})_{a_l} \to V(\varpi_{i_l})_{a_l} \otimes \cdots \otimes V(\varpi_{i_1})_{a_1}$$

sending $u_{i_1} \otimes \cdots \otimes u_{i_l}$ to $u_{i_l} \otimes \cdots \otimes u_{i_1}$, obtained as the product of $R^{\mathrm{nor}}_{i_\nu,i_\mu}(a_\nu, a_\mu)$ with $1 \leq \nu < \mu \leq l$.

Corollary 2.3. *Under the condition* (1) *in Corollary 2.2,* $\mathrm{Im}(R)$ *is irreducible.*

Note that the condition (1) is satisfied if $K = \bar{k}$ and $a_1 \leq \cdots \leq a_l$, and hence we can apply the corollary.

Proof. By Corollary 2.2 (1), $\mathrm{Im}(R)$ is generated by the dominant extremal vector $u_{i_l} \otimes \cdots \otimes u_{i_1}$. Since any submodule of $\mathrm{Im}(R)$ contains the same vector by Corollary 2.2 (2), $\mathrm{Im}(R)$ is irreducible. □

In fact, $\mathrm{Im}(R)$ is absolutely irreducible. Let us recall that, for a (not necessarily algebraically closed) field K containing k, a $U'_q(\mathfrak{g})_K$-module M finite-dimensional over K is called absolutely irreducible if the following equivalent conditions are satisfied.

(1) For some algebraically closed field K' containing K, $K' \otimes_K M$ is an irreducible $U'_q(\mathfrak{g})_{K'}$-module.

(2) For any algebraically closed field K' containing K, $K' \otimes_K M$ is an irreducible $U'_q(\mathfrak{g})_{K'}$-module.

(3) M is irreducible and $\mathrm{End}_{U'_q(\mathfrak{g})_K}(M) \cong K$. We denote by $\mathfrak{m}$ the maximal ideal $\bigcup_{n>0} q^{1/n}\mathbb{C}[[q^{1/n}]]$ of $\bar{A}$.

Corollary 2.4. *For $a \in \bar{k}^\times$, $y/x = a$ is a pole of $R^{\rm nor}_{ij}(x,y)$ if and only if $a \in \mathfrak{m}$ and $V(\varpi_i) \otimes V(\varpi_j)_a$ is reducible.*

Proof. By Corollary 2.1, if $y/x = a$ is a pole of $R^{\rm nor}_{ij}(x,y)$, then $a \in \mathfrak{m}$. By a similar argument to Corollary 2.1, the irreducibility of $V(\varpi_i)\otimes V(\varpi_j)_a$ implies that $y/x = a$ is not a pole of $R^{\rm nor}_{ij}(x,y)$. Now assume that $y/x = a \in \mathfrak{m}$ is not a pole of $R^{\rm nor}_{ij}(x,y)$. Since $R^{\rm nor}_{ji}(a,1)$ is well defined, $R^{\rm nor}_{ij}(1,a)$ is invertible. Hence $V(\varpi_i) \otimes V(\varpi_j)_a$ is irreducible by Corollary 2.3. □

Corollary 2.5. *Let K be an algebraically closed field containing k. If M and M' are irreducible finite-dimensional integrable $U'_q(\mathfrak{g})_K$-modules, then $M \otimes M'_z$ is an irreducible $U'_q(\mathfrak{g})_K$-module except finitely many $z \in K$.*

Proof. Let M (resp. M') be the irreducible subquotient of $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_m})_{a_m}$ (resp. $V(\varpi_{i'_1})_{a'_1} \otimes \cdots \otimes V(\varpi_{i'_{m'}})_{a_{m'}}$) such that $R^{\rm nor}_{i_\nu,i_\mu}(x,y)$ (resp. $R^{\rm nor}_{i'_\nu,i'_\mu}(x,y)$) does not have a pole at $(x,y) = (a_\nu,a_\mu)$ (resp. $(x,y) = (a'_\nu,a'_\mu)$) for $1 \le \nu < \mu \le m$ (resp. $1 \le \nu < \mu \le m'$). Then Corollary 2.3 implies that M is isomorphic to the image of the R-matrix

$$R : V \longrightarrow W,$$

where $V = V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_m})_{a_m}$ and $W = V(\varpi_{i_m})_{a_m} \otimes \cdots \otimes V(\varpi_{i_1})_{a_1}$. Similarly M' is isomorphic to the image of

$$R' : V' \longrightarrow W' ,$$

where $V' = V(\varpi_{i'_1})_{a'_1} \otimes\cdots\otimes V(\varpi_{i'_{m'}})_{a'_{m'}}$ and $W' = V(\varpi_{i'_{m'}})_{a'_{m'}} \otimes\cdots\otimes V(\varpi_{i'_1})_{a'_1}$. If z is generic , $R^{\rm nor}_{i_\nu,i'_{\nu'}}(x,y)$ does not have a pole at $(x,y) = (a_\nu, za'_{\nu'})$ and $R^{\rm nor}_{i'_{\nu'},i_\nu}(x,y)$ does not have a pole at $(x,y) = (za'_{\nu'}, a_\nu)$. Hence the R-matrix $W \otimes W'_z \to W'_z \otimes W$ is an isomorphism. Hence the image of the composition

$$V \otimes V'_z \xrightarrow{R\otimes R'_z} W \otimes W'_z \xrightarrow{\ \sim\ } W'_z \otimes W$$

is isomorphic to $M \otimes M'_z$ and it is irreducible by Corollary 2.3. □

Hence the intertwiner $M \otimes M'_z \to M'_z \otimes M$ is unique up to constant.

We give a conjecture on the poles of the R-matrices.

Conjecture 2. For $i,j \in I_0$, the pole of the normalized R-matrix $R^{\rm nor}_{ij}(x,y)$ has the form $y/x = \pm q^n$ for $n \in \gamma^{-1}Z$ with $0 < n \le (\delta,\rho)$ except $D^{(3)}_4$ (where γ is defined in (1.2)). In the $D^{(3)}_4$ case the third root of unity appears in the coefficients.

As seen in the appendix, this is true for $A^{(1)}_n$ and $C^{(1)}_n$.

We can also ask if the following statements are true.

(2.2) $R^{\rm nor}_{ij}(x,y)$ has only a simple pole.

(2.3) If $(x, y) = (a, b)$ is a pole of $R^{\mathrm{nor}}_{ij}(x, y)$, then the kernel of $R^{\mathrm{nor}}_{ji}(b, a) : V(\varpi_j)_b \otimes V(\varpi_i)_a \to V(\varpi_i)_a \otimes V(\varpi_j)_b$ is irreducible.

3. Reduction of the conjecture

In this section we shall prove that Conjecture 1 follows from Conjecture 3 below. Let $\mathfrak{m} = \bigcup_{n>0} q^{1/n}\mathbb{C}[[q^{1/n}]]$ be the maximal ideal of $\bar{A}$.

Conjecture 3. For every $i \in I_0$, there exist $N \in \mathbb{N}$, $b_1, \cdots, b_N$, $c_1, \dots, c_N \in \mathfrak{m}\setminus\{0\}$, $s_1, \dots, s_N$, $t_1, \dots, t_N \in I_0$, an irreducible finite-dimensional $U'_q(\mathfrak{g})_{\bar{k}}$-module W_μ and a $U'_q(\mathfrak{g})_{\bar{k}}$-linear map $\varphi_\mu : V(\varpi_i) \otimes V(\varpi_{s_\mu})_{b_\mu} \longrightarrow V(\varpi_{t_\mu})_{c_\mu} \otimes W_\mu$ for μ with $1 \le \mu \le N$, satisfying the following conditions. Define $F_0 = \bigoplus_{\xi \neq -\varpi_{i^*}} V(\varpi_i)_\xi$ (recall that $-\varpi_{i^*}$ is the lowest weight vector of $V(\varpi_i)$) and $F_\mu = \{v \in F_{\mu-1} | \varphi_\mu(v \otimes u_{s_\mu}) = 0\}$ for $0 < \mu \le N$.
(1) $F_N = \bar{k}u_i$.
(2) $\varphi_\mu(F_{\mu-1} \otimes u_{s_\mu}) \subset V(\varpi_{t_\mu})_{c_\mu} \otimes w_\mu$.
(3) $V(\varpi_{s_\mu})_{b_\mu}$ is not isomorphic to $V(\varpi_{t_\mu})_{c_\mu}$.
(4) $V(\varpi_{s_\mu})_{b_\mu}$ is not a component of W_μ (see Definition 1.3).
Here u_{s_μ} and w_μ are dominant extremal vectors of $V(\varpi_{s_\mu})_{b_\mu}$ and W_μ, respectively.

Let us show that Conjecture 3 implies Conjecture 1 (2).
For $a_1, \dots, a_p \in \bar{k}^*$, let $P(a_1, \dots, a_p)$ denote the following statement.

$P(a_1, \dots, a_p)$: For indeterminates $x_1, \dots, x_l$, any dominant extremal vector of the $U'_q(\mathfrak{g})_{\bar{k}(x_1,\dots,x_l)}$ -module $V(\varpi_{j_1})_{x_1} \otimes \cdots \otimes V(\varpi_{j_l})_{x_l} \otimes V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_p})_{a_p}$ is a constant multiple of $u_{j_1} \otimes \cdots \otimes u_{j_l} \otimes u_{i_1} \otimes \cdots \otimes u_{i_p}$.

Assuming Conjecture 3, we shall prove the following lemma.

Lemma 3.1. *If $a_1, \dots, a_p \in \bar{k}^*$ satisfy $a_1 \ge \cdots \ge a_p$, then $P(a_1, \dots, a_p)$ holds.*

Since any non-zero finite-dimensional module contains a dominant extremal vector, this lemma implies Conjecture 1 (2). We shall prove this lemma by induction on p. First assume $p \ge 1$. Then $P(a_1, \dots, a_{p-1})$ holds by the hypothesis of induction. Set $K = \bar{k}(x_1, \dots, x_l)$. Let x be another indeterminate. By the existence of R-matrix, $V(\varpi_{j_1})_{x_1} \otimes \cdots \otimes V(\varpi_{j_l})_{x_l} \otimes V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_{p-1}})_{a_{p-1}} \otimes V(\varpi_{i_p})_x$ is isomorphic to $V(\varpi_{j_1})_{x_1} \otimes \cdots \otimes V(\varpi_{j_l})_{x_l} \otimes V(\varpi_{i_p})_x \otimes V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_{p-1}})_{a_{p-1}}$. Hence $P(a_1, \dots, a_{p-1})$ implies that a dominant extremal vector of the $U'_q(\mathfrak{g})_{K(x)}$-module $V(\varpi_{j_1})_{x_1} \otimes \cdots \otimes V(\varpi_{j_l})_{x_l} \otimes V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_{p-1}})_{a_{p-1}} \otimes V(\varpi_{i_p})_x$ is a constant multiple of $u_{j_1} \otimes \cdots \otimes u_{j_l} \otimes u_{i_1} \otimes \cdots \otimes u_{i_p}$. Then it follows that a dominant extremal vector of $U'_q(\mathfrak{g})_K$-module $V(\varpi_{j_1})_{x_1} \otimes \cdots \otimes V(\varpi_{j_l})_{x_l} \otimes V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_{p-1}})_{a_{p-1}} \otimes V(\varpi_{i_p})_z$ is a constant multiple of $u_{j_1} \otimes \cdots \otimes u_{j_l} \otimes u_{i_1} \otimes \cdots \otimes u_{i_p}$ except for finitely many $z \in \bar{k}$. This means that $P(a_1, \dots, a_{p-1}, z)$ holds except finitely many $z \in \bar{k}$. Arguing by induction on the order of the zero of a_p, we may assume from the beginning

$$P(a_1, \dots, a_{p-1}, z) \text{ holds for any } z \in \mathfrak{m}a_p \setminus \{0\}. \tag{3.1}$$

Let v be a dominant extremal vector of $U_q'(\mathfrak{g})_K$-module $V(\varpi_{j_1})_{x_1} \otimes \cdots \otimes V(\varpi_{j_l})_{x_l} \otimes V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_p})_{a_p}$. We shall prove that v is a constant multiple of $u_{j_1} \otimes \cdots \otimes u_{j_l} \otimes u_{i_1} \otimes \cdots \otimes u_{i_p}$.

We have $\varphi_0 : V(\varpi_{i_p}) \otimes V(\varpi_{i_p^*})_y \to \bar{k}$ with $y = (-1)^{(\delta,\rho^\vee)} q^{(\delta,\rho)}$ by (1.6). Set $V' = V(\varpi_{j_1})_{x_1} \otimes \cdots \otimes V(\varpi_{j_l})_{x_l} \otimes V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_{p-1}})_{a_{p-1}}$. Then we have a morphism

$$\mathrm{id}_{V'} \otimes (\varphi_0)_{a_p} : V' \otimes V(\varpi_{i_p})_{a_p} \otimes V(\varpi_{i_p^*})_{a_p y} \to V'.$$

Lemma 3.2. *We have* $(\mathrm{id}_{V'} \otimes (\varphi_0)_{a_p})(v \otimes u_{i_p^*}) = 0$.

Proof. Assume that $w = (\mathrm{id}_{V'} \otimes (\varphi_0)_{a_p})(v \otimes u_{i_p^*}) \neq 0$. Then w is a dominant extremal vector of V'. Hence w is equal to $u_{j_1} \otimes \cdots \otimes u_{j_l} \otimes u_{i_1} \otimes \cdots \otimes u_{i_{p-1}}$ up to a constant multiple by $P(a_1, \dots, a_{p-1})$. Therefore Theorem 1.2 implies that $V(\varpi_{i_p^*})_{a_p y}$ is isomorphic to one of $V(\varpi_{j_1})_{x_1}, \dots, V(\varpi_{j_l})_{x_l}$, $V(\varpi_{i_1})_{a_1}, \cdots, V(\varpi_{i_{p-1}})_{a_{p-1}}$. This is a contradiction since $y \in qA$. □

Since $F_0 = \{w \in V(\varpi_{i_p}); \varphi_0(w \otimes u_{i_p^*}) = 0\}$, we have $v \in V' \otimes (F_0)_{a_p}$. Now we shall show $v \in V' \otimes (F_\mu)_{a_p}$ by induction on μ. Applying Conjecture 3 with $i = i_p$, we have $U_q'(\mathfrak{g})$-linear maps $\varphi_\mu : V(\varpi_{i_p}) \otimes V(\varpi_{s_\mu})_{b_\mu} \longrightarrow V(\varpi_{t_\mu})_{c_\mu} \otimes W_\mu$ for $1 \le \mu \le N$ satisfying the conditions (1) – (4) in Conjecture 3. Then this induces a homomorphism

$$\mathrm{id}_{V'} \otimes (\varphi_\mu)_{a_p} : V' \otimes V(\varpi_{i_p})_{a_p} \otimes V(\varpi_{s_\mu})_{a_p b_\mu} \longrightarrow V' \otimes V(\varpi_{t_\mu})_{a_p c_\mu} \otimes (W_\mu)_{a_p}.$$

Suppose that $v \in V' \otimes (F_{\mu-1})_{a_p}$, which is the case when $\mu = 1$.

Lemma 3.3. *We have* $(\mathrm{id}_{V'} \otimes (\varphi_\mu)_{a_p})(v \otimes u_{s_\mu}) = 0$.

Proof. The proof is similar to the one of the preceding lemma. Suppose that $w = (\mathrm{id}_{V'} \otimes (\varphi_\mu)_{a_p})(v \otimes u_{s_\mu})$ is not zero. Write w as $v'' \otimes w_\mu$ in virtue of the condition (2) in Conjecture 3, where v'' is a non-zero vector of $V' \otimes V(\varpi_{t_\mu})_{a_p c_\mu}$. Since $v \otimes u_{s_\mu}$ is extremal, so is $v'' \otimes w_\mu$. Hence v'' is a dominant extremal vector by Lemma 1.18. Since $a_p c_\mu \in \mathfrak{m}a_p$, the property $P(a_1, \dots, a_{p-1}, a_p c_\mu)$ holds by (3.1), and hence v'' is a nonzero scalar multiple of $u_{j_1} \otimes \cdots \otimes u_{j_l} \otimes u_{i_1} \otimes \cdots \otimes u_{i_p} \otimes u_{t_\mu}$. Then Theorem 1.2 implies that $V(\varpi_{s_\mu})_{a_p b_\mu}$ is isomorphic to one of $V(\varpi_{j_1})_{x_1}, \dots, V(\varpi_{j_l})_{x_l}$, $V(\varpi_{i_1})_{a_1}, \cdots, V(\varpi_{i_{p-1}})_{a_{p-1}}$, $V(\varpi_{t_\mu})_{a_p c_\mu}$ or to a component of $(W_\mu)_{a_p}$. It, however, is not the case because of the conditions (3), (4) in Conjecture 3 and $a_p b_\mu \in \mathfrak{m}a_p$. □

By this we have $v \in V' \otimes (F_\mu)_{a_p}$. Applying this process successively, we obtain $v \in V' \otimes (F_N)_{a_p}$. Hence we have $v \in V' \otimes u_{i_p}$ by the condition (1) in Conjecture 3. Write v as $v' \otimes u_{i_p}$, where v' is a nonzero vector of V'. Lemma

1.18 implies that v' is dominant and extremal. Therefore v' is a nonzero scalar multiple of $u_{j_1} \otimes \cdots \otimes u_{j_l} \otimes u_{i_1} \otimes \cdots \otimes u_{i_{p-1}}$ by the induction hypothesis on p. We have deduced the p case from the $p-1$ case.

It remains to prove $p = 0$ case, which follows from the following lemma.

Lemma 3.4. *Any dominant extremal vector of the $U'_q(\mathfrak{g})_{\bar{k}(x_1,\dots,x_l)}$-module $V(\varpi_{i_1})_{x_1} \otimes \cdots \otimes V(\varpi_{i_l})_{x_l}$ is a constant multiple of $u_{i_1} \otimes \cdots \otimes u_{i_l}$. Here $x_1, \dots, x_l$ are indeterminates.*

Proof. It is enough to prove the assertion with $x_1 = \cdots = x_l = 1$. Let V denote $V(\varpi_{i_1}) \otimes \cdots \otimes V(\varpi_{i_l})$. By Corollary 1.13, V is irreducible. Suppose now that V has a dominant extremal vector v that is not a constant multiple of $u_{i_1} \otimes \cdots \otimes u_{i_l}$. Then $U'_q(\mathfrak{g})v$ does not contain $u_{i_1} \otimes \cdots \otimes u_{i_l}$ since $\mathrm{wt}(U'_q(\mathfrak{g})v) \subset \mathrm{wt}(v) + \sum_{i \in I_0} \mathbb{Z}_{\le 0} \mathrm{cl}(\alpha_i)$, which is a contradiction. □

Thus we have proved

Proposition 3.5. *Conjecture 3 implies Conjecture 1.*

4. Proof of Conjecture 3 for $A_n^{(1)}$ and $C_n^{(1)}$

In this section, we shall prove the following theorem.

Theorem 4.1. *Conjecture 3 holds if $\mathfrak{g}$ is $A_n^{(1)}$ or $C_n^{(1)}$.*

4.1. $A_{n-1}^{(1)}$ **Case.** For the fundamental representations of $U'_q(\widehat{\mathfrak{sl}}_n)$, see Appendix B.1. We identify crystal bases of the fundamental representations with the corresponding global bases.

Let us prove Conjecture 3.

Since the $i = n-1$ case can be reduced to the case $i = 1$ by the Dynkin diagram automorphism, we assume $1 \le i < n-1$. Set $N = i$. For $1 \le \mu \le N = i$, take $s_\mu = \mu$, $t_\mu = i+1$, $b_\mu = (-q)^{i-\mu+2}$, $c_\mu = -q$, $W_\mu = V(\varpi_{\mu-1})_{(-q)^{i-\mu+1}}$ and define $\varphi_{i,\mu} : V(\varpi_i) \otimes V(\varpi_\mu)_{(-q)^{i-\mu+2}} \longrightarrow V(\varpi_{i+1})_{-q} \otimes W_\mu$ as the composition (see Lemma B.1):

(4.1)

$$\begin{array}{ccc} V(\varpi_i) \otimes V(\varpi_\mu)_{(-q)^{i-\mu+2}} & \xrightarrow{V(\varpi_i)\otimes(i_{1,\mu-1})_{(-q)^{i-\mu+2}}} & V(\varpi_i) \otimes V(\varpi_1)_{(-q)^{i+1}} \otimes W_\mu \\ & & \Big\downarrow {\scriptstyle (p_{i,1})_{-q} \otimes W_\mu} \\ & & V(\varpi_{i+1})_{-q} \otimes W_\mu . \end{array}$$

Then it is easy to check that Conjecture 3 holds with

$$F_\mu = \bigoplus_{\mu < a_{\mu+1} < \cdots a_i \le n} k(1, \dots, \mu,\ a_{\mu+1}, \dots, a_i)\,.$$

4.2. $C_n^{(1)}$ **Case.** For the fundamental representations of $U'_q(C_n^{(1)})$, see Appendix C.1.

For $1 \le i < n$, let $p_i : V(\varpi_i) \otimes V(\varpi_1)_{(-q_s)^{i+1}} \to V(\varpi_{i+1})_{-q_s}$ be $(p_{i,1})_{-q_s}$. Let $p_n : V(\varpi_n) \otimes V(\varpi_1)_{(-q_s)^{n+3}} \to V(\varpi_{n-1})_{-q_s}$ be the composition

$$\begin{array}{c} V(\varpi_n) \otimes V(\varpi_1)_{(-q_s)^{n+3}} \\ \Big\downarrow {\scriptstyle i_{n-1,1} \otimes V(\varpi_1)_{(-q_s)^{n+3}}} \\ V(\varpi_{n-1})_{-q_s} \otimes V(\varpi_1)_{(-q_s)^{1-n}} \otimes V(\varpi_1)_{(-q_s)^{n+3}} \xrightarrow{V(\varpi_{n-1})_{-q_s} \otimes \mathrm{tr}} V(\varpi_{n-1})_{-q_s}. \end{array}$$

Here tr is given in (C.1).

For $1 \le i \le n-1$, set $N = i$. For $1 \le \mu \le N = i$, we set $s_\mu = \mu$, $t_\mu = i+1$, $b_\mu = (-q_s)^{i-\mu+2}$, $c_\mu = -q_s$ and $W_\mu = V(\varpi_{\mu-1})_{(-q_s)^{i-\mu+1}}$. We define $\varphi_{i,\mu} : V(\varpi_i) \otimes V(\varpi_\mu)_{(-q_s)^{i-\mu+2}} \longrightarrow V(\varpi_{i+1})_{-q_s} \otimes W_\mu$ as the composition:

(4.2)
$$\begin{array}{ccc} V(\varpi_i) \otimes V(\varpi_\mu)_{(-q_s)^{i-\mu+2}} & \xrightarrow{V(\varpi_i) \otimes (i_{1,\mu-1})_{(-q_s)^{i-\mu+2}}} & V(\varpi_i) \otimes V(\varpi_1)_{(-q_s)^{i+1}} \otimes W_\mu \\ & & \Big\downarrow {\scriptstyle p_i \otimes W_\mu} \\ & & V(\varpi_{i+1})_{-q_s} \otimes W_\mu. \end{array}$$

Note that $b_\mu,\ c_\mu \in q_s A$.

For $i = n$, set $N = n$. For $1 \le \mu \le n$, we set $s_\mu = \mu$, $t_\mu = n-1$, $b_\mu = (-q_s)^{n-\mu+4}$, $c_\mu = -q_s$ and $W_\mu = V(\varpi_{\mu-1})_{(-q_s)^{n-\mu+3}}$. We define $\varphi_{n,\mu}$: $V(\varpi_n) \otimes V(\varpi_\mu)_{(-q_s)^{n-\mu+4}} \longrightarrow V(\varpi_{i+1})_{-q_s} \otimes W_\mu$ as the composition ;

(4.3)
$$\begin{array}{ccc} V(\varpi_n) \otimes V(\varpi_\mu)_{(-q_s)^{n-\mu+4}} & \xrightarrow{V(\varpi_n) \otimes (i_{1,\mu-1})_{(-q_s)^{n-\mu+4}}} & V(\varpi_n) \otimes V(\varpi_1)_{(-q_s)^{n+3}} \otimes W_\mu \\ & & \Big\downarrow {\scriptstyle p_n \otimes W_\mu} \\ & & V(\varpi_{n-1})_{-q_s} \otimes W_\mu. \end{array}$$

Note that $b_\mu,\ c_\mu \in q_s A$.

Then we have

$$F_\mu = \{v \in V(\varpi_i); p_i(v \otimes G(j)) = 0 \quad \text{for } 1 \le j \le \mu\}.$$

Then Conjecture 3 easily follows from the following lemma.

Lemma 4.2. *Fix $1 \le i \le n$. Then*

$$\{v \in V(\varpi_i) \mid p_i(v \otimes G(j)) = 0 \quad \textit{for all } 1 \le j \le i\} = kG(1, \dots, i). \tag{4.4}$$

Proof. Let E be the left-hand-side of (4.4). Then E is invariant by e_k for any $k \in I_0$. Let us prove $E_\lambda = 0$ by induction on the weight $\lambda \ne \varpi_i$. We can easily check the assertion when $\lambda = \varpi_i - \alpha_i$, since $V(\varpi_i)_\lambda = kf_iu_i$. If a weight λ of $V(\varpi_i)$ is not $\varpi_i - \alpha_i$, then $\lambda + \alpha_k \neq \varpi_i$ for any $k \in I_0$. Therefore any $v \in E_\lambda$ satisfies $e_k v = 0$ for all $k \in I_0$ by the induction hypothesis. This implies $v = 0$. □

Appendix A. Universal R-matrix

In this appendix we shall calculate the normalized and universal R-matrices of $U_q'(\mathfrak{g})$ for the fundamental representations following a variant of the recipe of Frenkel-Reshetikhin [8] in the $A_{n-1}^{(1)}$ and $C_n^{(1)}$ cases.

Let us choose the following universal R-matrix. Let us take a base P_ν of $U_q^+(\mathfrak{g})$ and Q_ν of $U_q^-(\mathfrak{g})$ dual to each other with respect a suitable coupling between $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$. Then for $U_q'(\mathfrak{g})$-modules M and N define

$$R_{MN}^{\mathrm{univ}}(u \otimes v) = q^{(\mathrm{wt}(u),\mathrm{wt}(v))} \sum_\nu P_\nu v \otimes Q_\nu u\,, \tag{A.1}$$

so that R_{MN}^{univ} gives a $U_q'(\mathfrak{g})$-linear homomorphism from $M \otimes N$ to $N \otimes M$ provided the infinite sum has a meaning. If M and N are finite-dimensional integrable modules, then $R_{M,N_z}^{\mathrm{univ}}$ converges in the z-adic topology. The existence of the universal R-matrix for (M, N) is proved by [6] (see also [18]). For a scalar a, the composition

$$(R_{M,N}^{\mathrm{univ}})_a : M_a \otimes N_a \cong (M \otimes N)_a \to (N \otimes M)_a \cong M_a \otimes N_a$$

is equal to $R_{M_a,N_a}^{\mathrm{univ}}$, and we sometimes confuse them.

For irreducible $U_q'(\mathfrak{g})$-modules M and N, let us denote by $R_{MN}^{\mathrm{nor}}(z)$ the R matrix $M \otimes N_z \to N_z \otimes M$ normalized by $R_{MN}^{\mathrm{nor}}(z)(u \otimes v) = v \otimes u$ for dominant extremal vectors u (resp. v) of M (resp. N). Let $d_{MN}(z)$ be a denominator of $R_{MN}^{\mathrm{nor}}(z)$. Namely $c(z) \in k[z, z^{-1}]$ is divisible by $d_{MN}(z)$ if and only if $c(z)R_{MN}^{\mathrm{nor}}(z)$ has no poles. Then $d_{MN}(z)$ is uniquely determined modulo $k[z, z^{-1}]^\times$. Here $k[z, z^{-1}]^\times$ is the set of invertible elements of $k[z, z^{-1}]$. Hence

$$k[z, z^{-1}]^\times = \{cz^n; n \in \mathbb{Z},\ c \in k \setminus \{0\}\}. \tag{A.2}$$

Since the intertwiner from $M \otimes N_z$ to $N_z \otimes M$ is unique up to a constant multiple by Corollary 2.5, we can write

$$R_{MN}^{\mathrm{univ}}(z) = a_{MN}(z) R_{MN}^{\mathrm{nor}}(z). \tag{A.3}$$

If λ and μ are the dominant extremal weight of M and N respectively, we have

$$a_{MN}(z) \in q^{(\lambda,\mu)}(1 + zk[[z]]). \tag{A.4}$$

For $i, j \in I_0$, we denote $R_{ij}^{\mathrm{univ}}(z) = R_{V(\varpi_i)V(\varpi_j)}^{\mathrm{univ}}(z)$, $R_{ij}^{\mathrm{nor}}(z) = R_{V(\varpi_i)V(\varpi_j)}^{\mathrm{nor}}(z)$, $a_{ij}(z) = a_{V(\varpi_i)V(\varpi_j)}(z)$ and $d_{ij}(z) = d_{V(\varpi_i)V(\varpi_j)}(z)$.

For a finite-dimensional $U_q'(\mathfrak{g})$-module M, let M^* be the left dual of M and ${}^*M$ the right dual of M. Hence we have

$$M^* \otimes M \xrightarrow{\ \mathrm{tr}\ } k \qquad k \xrightarrow{\ \iota\ } M \otimes M^*$$

$$M \otimes {}^*M \xrightarrow{\ \mathrm{tr}\ } k \qquad k \xrightarrow{\ \iota\ } {}^*M \otimes M.$$

We have

$$M^{**} \cong M_{q^{-2(\delta,\rho)}} \quad \text{and} \quad {}^{**}M \cong M_{q^{2(\delta,\rho)}}. \tag{A.5}$$

We have

$$V(\varpi_i)^* \cong V(\varpi_{i^*})_{p^{*-1}} \quad \text{and} \quad {}^*V(\varpi_i) \cong V(\varpi_{i^*})_{p^*},$$

where $p^* = (-1)^{(c,\rho^\vee)} q^{(\delta,\rho)}$.

Let $a \mapsto \overline{a}$ be the ring automorphism of $U_q'(\mathfrak{g})$ given by $\overline{q} = q^{-1}$, $(e_i)^- = e_i$, $(f_i)^- = f_i$, $q(h)^- = q(-h)$. For a $U_q'(\mathfrak{g})$-module M, let M^- be the $U_q'(\mathfrak{g})$-module whose underlying vector space is M with the new action $U_q'(\mathfrak{g}) \xrightarrow{\ -\ } U_q'(\mathfrak{g}) \longrightarrow \mathrm{End}\,(M)$. Then $(M \otimes N)^- \cong N^- \otimes M^-$ and $V(\varpi_i)^- \cong V(\varpi_i)$. Hence we have

$$d_{ji}(z) \equiv d_{ij}(z^{-1})^- \mod k[z, z^{-1}]^\times. \tag{A.6}$$

The conjecture 2 implies

$$d_{ji}(z) \equiv d_{ij}(z) \mod k[z, z^{-1}]^\times. \tag{A.7}$$

Proposition A.1. *For irreducible finite-dimensional integrable $U_q'(\mathfrak{g})$-modules V and W, we have*

$$a_{V,W}(z) a_{{}^*V,W}(z) \equiv \frac{d_{VW}(z)}{d_{W,{}^*V}(z^{-1})} \mod k[z, z^{-1}]^\times. \tag{A.8}$$

Proof. For a $U_q'(\mathfrak{g})$-linear homomorphism $\phi : V \otimes W_z \to W_z \otimes V$, we shall define $\mathcal{T}r(\phi) : W_z \otimes {}^*V \to {}^*V \otimes W_z$ as the composition

$$W_z \otimes {}^*V \xrightarrow{\iota \otimes W_z \otimes {}^*V} {}^*V \otimes V \otimes W_z \otimes {}^*V \xrightarrow{{}^*V \otimes \phi \otimes {}^*V} {}^*V \otimes W_z \otimes V \otimes {}^*V \xrightarrow{{}^*V \otimes W_z \otimes \mathrm{tr}} {}^*V \otimes W_z.$$

The correspondence $\phi \mapsto \mathcal{T}r(\phi)$ gives an isomorphism

$$\mathrm{Hom}(V \otimes W_z, W_z \otimes V) \xrightarrow{\ \sim\ } \mathrm{Hom}(W_z \otimes {}^*V, {}^*V \otimes W_z). \tag{A.9}$$

If we consider them as modules over $k[z, z^{-1}]$, then $\mathrm{Hom}(V \otimes W_z, W_z \otimes V)$ is generated by $d_{VW}(z) R^{\mathrm{nor}}_{VW}(z)$, and $\mathrm{Hom}(W_z \otimes {}^*V, {}^*V \otimes W_z)$ is generated by $d_{W,{}^*V}(z^{-1}) R^{\mathrm{nor}}_{W,{}^*V}(z^{-1})$. Hence we have

$$\mathcal{T}r(d_{VW}(z) R^{\mathrm{nor}}_{VW}(z)) \equiv d_{W,{}^*V}(z^{-1}) R^{\mathrm{nor}}_{W,{}^*V}(z^{-1}) \qquad \mod k[z, z^{-1}]^\times.$$

Then the result follows from $R^{\mathrm{nor}}_{W,{}^*V}(z^{-1}) = (R^{\mathrm{nor}}_{{}^*V,W}(z))^{-1}$ and a well known result $\mathcal{T}r(R^{\mathrm{univ}}_{VW}(z)) = (R^{\mathrm{univ}}_{{}^*V,W}(z))^{-1}$ (see [8]). □

This proposition implies

$$a_{i,j}(z)a_{i^*,j}(p^{*-1}z) \equiv \frac{d_{i,j}(z)}{d_{j,i^*}(p^* z^{-1})} \mod k[z,z^{-1}]^\times. \tag{A.10}$$

Applying (A.8) with ${}^*V$ instead of V, we have

$$a_{{}^*V,W}(z)a_{{}^{**}V,W}(z) \equiv \frac{d_{{}^*V,W}(z)}{d_{W,{}^{**}V}(z^{-1})} \mod k[z,z^{-1}]^\times. \tag{A.11}$$

Using (A.5) we obtain the q-difference equation

$$\frac{a_{VW}(z)}{a_{VW}(q^{-2(\delta,\rho)}z)} \equiv \frac{d_{VW}(z)d_{WV}(q^{2(\delta,\rho)}z^{-1})}{d_{W,{}^*V}(z^{-1})d_{{}^*V,W}(z)} \mod k[z,z^{-1}]^\times. \tag{A.12}$$

Write

$$d_{ji}(z) = \prod_\nu (z - x_\nu) \quad \text{and} \quad d_{j,i^*}(z) = \prod_\nu (z - y_\nu).$$

Then by (A.6), we have

$$d_{ij}(z) = \prod_\nu (z - \overline{x_\nu}^{-1}) \quad \text{and} \quad d_{i^*,j}(z) = \prod_\nu (z - \overline{y_\nu}^{-1}).$$

Then using (A.4), we can solve the q-difference equation (A.12),

$$a_{ij}(z) = q^{(\varpi_i,\varpi_j)} \frac{\prod_\nu (p^* y_\nu z; p^{*2})_\infty (p^* \overline{y_\nu} z; p^{*2})_\infty}{\prod_\nu (x_\nu z; p^{*2})_\infty (p^{*2}\overline{x_\nu} z; p^{*2})_\infty}\ . \tag{A.13}$$

Here $p^* = (-1)^{(\delta,\rho^\vee)}q^{(\delta,\rho)}$ and $(z;q)_\infty = \prod_{n=0}^\infty (1-q^n z)$.

We are going to determine $d_{ij}(z)$ and $a_{ij}(z)$ in the $A^{(1)}_{n-1}$ and $C^{(1)}_n$ cases.

Remark . We can see easily

$$d_{V^*,W^*}(z) \equiv d_{{}^*V,{}^*W}(z) \equiv d_{V,W}(z). \tag{A.14}$$

Hence

$$a_{V^*,W^*}(z) = a_{{}^*V,{}^*W}(z) = a_{V,W}(z), \tag{A.15}$$

and

$$d_{i^*,j^*}(z) \equiv d_{i,j}(z). \tag{A.16}$$

Appendix B. $A^{(1)}_{n-1}$ case

We shall review the fundamental representations and R-matrices for $A^{(1)}_{n-1}$.

B.1. **Fundamental representations.** The root data of $\mathfrak{g} = A_{n-1}^{(1)}$ are as follows.

$$
\begin{aligned}
&I = \{0, 1, \dots, n-1\} \\
&(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j \\ -\delta(i \equiv j+1 \bmod n) - \delta(i \equiv j-1 \bmod n) & \text{otherwise} \end{cases} \\
&\delta = \alpha_0 + \cdots + \alpha_{n-1}\,, \\
&c = h_0 + \cdots + h_{n-1}\,, \\
&(\delta, \rho) = (\delta, \rho^\vee) = n.
\end{aligned}
$$

Here for the statement P, we define $\delta(P) = 1$ or -1 according that P is true or false.

Hence by (1.6) the duality morphisms are given by

$$
k \xrightarrow{\iota} V(\varpi_{n-i})_{(-q)^n} \otimes V(\varpi_i) \quad \text{and} \quad V(\varpi_i) \otimes V(\varpi_{n-i})_{(-q)^n} \xrightarrow{\mathrm{tr}} k.
$$

By [16], the vectors of the crystal base B_k of the fundamental representation $V(\varpi_k)$ $(1 \le k \le n-1)$ are labeled by the subsets of $\mathbb{Z}/n\mathbb{Z} = \{1, \dots, n\}$ with exactly k elements. For $0 \le i \le n-1$ and $K \subset \mathbb{Z}/n\mathbb{Z}$, we have

$$
\begin{aligned}
\tilde{e}_i(K) &= \begin{cases} (K \setminus \{i+1\}) \cup \{i\} & \text{if } i+1 \in K \text{ and } i \notin K, \\ 0 & \text{otherwise,} \end{cases} \\
\tilde{f}_i(K) &= \begin{cases} (K \setminus \{i\}) \cup \{i+1\} & \text{if } i \in K \text{ and } i+1 \notin K, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

In the case of the fundamental representations of $U_q'(\widehat{\mathfrak{sl}}_n)$, all the weights are extremal. Therefore we have $e_i G(b) = G(\tilde{e}_i b)$ and $f_i G(b) = G(\tilde{f}_i b)$ for every b in the crystal base. Here $G(b)$ is the corresponding global base. Hence we can and do identify its crystal bases with the corresponding global bases.

We have

$$
t_i K = q^{\delta(i \in K) - \delta(i+1 \in K)} K\,.
$$

We present a lemma that is easily verified by calculation.

Lemma B.1. *For $j, k \ge 0$ such that $j + k \le n$, there exist following non-zero $U_q'(\widehat{\mathfrak{sl}}_n)$-linear homomorphisms.*

(1) $i_{j,k} : V(\varpi_{j+k}) \longrightarrow V(\varpi_j)_{(-q)^k} \otimes V(\varpi_k)_{(-q)^{-j}}$ *given by*

$$
i_{j,k}(M) = \sum_{\substack{\sharp J = j, \sharp K = k \\ M = J \cup K,\ J \cap K = \emptyset}} (-q)^{\psi(J,K)} J \otimes K\,.
$$

Here $\psi(J, K) = \sharp\{(\nu, \mu) \in J \times K\,;\, \nu > \mu\}$.

(2) $p_{j,k} : V(\varpi_j)_{(-q)^{-k}} \otimes V(\varpi_k)_{(-q)^j} \longrightarrow V(\varpi_{j+k})$
given by

$$p_{j,k}(J \otimes K) = \begin{cases} (-q)^{\psi(J,K)}(J \cup K) & \text{if } J \cap K = \emptyset \\ 0 & \text{if } J \cap K \neq \emptyset. \end{cases}$$

Here $V(\varpi_0)$ and $V(\varpi_n)$ are understood to be the trivial representation.

B.2. **R-matrices.** We shall recall the result of Date-Okado [5].

Proposition B.2 ([5])**.** *For $k, l \in I_0$*

$$d_{kl}(z) = \prod_{\nu=1}^{\min(k,l,n-k,n-l)} (z - (-q)^{2\nu+|k-l|}). \tag{B.1}$$

The universal R-matrices can be easily obtained by (A.13) and (B.1).

Proposition B.3 ([5])**.** *For $k, l \in I_0 = \{1, \dots, n-1\}$, we have*

$$a_{kl}(z) = q^{\min(k,l)-kl/n} \frac{((-q)^{|k-l|}z; q^{2n})_\infty((-q)^{2n-|k-l|}z; q^{2n})_\infty}{((-q)^{k+l}z; q^{2n})_\infty((-q)^{2n-k-l}z; q^{2n})_\infty}.$$

Appendix C. $C_n^{(1)}$ case

C.1. **Fundamental representations.** The Dynkin diagram of $C_n^{(1)}$ is

$$\underset{-2\varepsilon_1}{\overset{0}{\bigcirc}} \Longrightarrow \underset{\varepsilon_1-\varepsilon_2}{\overset{1}{\bigcirc}} — \underset{\varepsilon_2-\varepsilon_3}{\overset{2}{\bigcirc}} —\cdots\cdots— \underset{\varepsilon_{n-1}-\varepsilon_n}{\overset{n-1}{\bigcirc}} \Longleftarrow \underset{2\varepsilon_n}{\overset{n}{\bigcirc}}$$

Here $(\varepsilon_i)_{i=1,\dots,n}$ is an orthogonal basis of $\mathfrak{t}_{\mathrm{cl}}^{*0}$ such that $(\varepsilon_i, \varepsilon_i) = 1/2$. We have

$$\begin{aligned} q_i &= \begin{cases} q & \text{if } i = 0 \text{ or } n \\ q^{1/2} & \text{if } 1 \le i < n, \end{cases} \\ \delta &= \alpha_0 + 2(\alpha_1 + \cdots \alpha_{n-1}) + \alpha_n, \\ c &= h_0 + h_1 + \cdots + h_n, \\ (\delta, \rho) &= n + 1, \\ \langle \rho^\vee, \delta \rangle &= 2n, \\ \varpi_i &= \Lambda_i - \Lambda_0 = \varepsilon_1 + \cdots + \varepsilon_i. \end{aligned}$$

We set $q_s = q^{1/2}$. Hence by (1.6) the duality morphisms are given by

$$k \xrightarrow{\iota} V(\varpi_i)_{q_s^{2(n+1)}} \otimes V(\varpi_i) \quad \text{and} \quad V(\varpi_i) \otimes V(\varpi_i)_{q_s^{2(n+1)}} \xrightarrow{\mathrm{tr}} k. \tag{C.1}$$

We review the crystal base $(L_k,\ B_k)$ of the fundamental representation $V(\varpi_k)$ $(1 \le k \le n)$ of $U_q'(C_n^{(1)})$. Recall that $V(\varpi_k)$ is as a $U_q(C_n)$-module isomorphic to the k-th fundamental representation of $U_q(C_n)$. Hence by [17], B_k is labeled by

$$\{(m_i)_{i=1}^k \mid m_1 \prec \cdots \prec m_k,\, m_i \in \{1, \dots, n,\ \bar{n}, \dots, \bar{1}\},$$
$$i + (k - j + 1) \le m_i \quad \text{if} \quad m_i = \bar{m_j}\ (i < j)\},$$

where the ordering on $\{1, \dots, n,\ \overline{n}, \dots, \overline{1}\}$ is defined by

$$1 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{1}. \tag{C.2}$$

On B_k the actions of $\tilde{f}_i$ and $\tilde{e}_i$ with $0 \le i \le n$ are defined as follows. As for $i \neq 0$, write i and $\overline{i+1}$ as $+$, $i+1$ and $\bar{i}$ as $-$, and others as 0. Then first ignore 0 and next ignore $+-$. Then $\tilde{f}_i b$ is obtained by replacing the leftmost $+$ with $-$ and $\tilde{e}_i b$ is obtained by replacing the rightmost $-$ with $+$.

Lemma C.1. *If b is of the form $(1,\ a_1, \dots, a_{k-1})$, then $\tilde{e}_0 b = (a_1, \dots, a_{k-1},\ \overline{1})$. Otherwise $\tilde{e}_0 b = 0$.*
If b is of the form $(a_1, \dots, a_{k-1},\ \overline{1})$, then $\tilde{f}_0 b = (1,\ a_1, \dots, a_{k-1})$. Otherwise $\tilde{f}_0 b = 0$.

Proof. It is easy to check that B_k is a regular crystal with this definition of $\tilde{e}_0$ and $\tilde{f}_0$. Set $J = \{1, 2, \dots, n-1\} \subset I$. Then B_k decomposes, as a crystal over $\mathfrak{g}_J \simeq A_{n-1}$, into irreducible components with multiplicity 1. Hence there is a unique way to draw 0-arrows on the crystal B_k over C_n. □

The following proposition can be checked by a direct calculation.

Proposition C.2. *For μ, ν with $\mu + \nu \le n$, there exist following non-zero $U'_q(C_n^{(1)})$-linear maps:*

$$i_{\mu\nu} : V(\varpi_{\mu+\nu}) \longrightarrow V(\varpi_\mu)_{(-q_s)^\nu} \otimes V(\varpi_\nu)_{(-q_s)^{-\mu}}\,,$$
$$p_{\mu\nu} : V(\varpi_\mu)_{(-q_s)^{-\nu}} \otimes V(\varpi_\nu)_{(-q_s)^{\mu}} \longrightarrow V(\varpi_{\mu+\nu}).$$

C.2. **Normalized R-matrices.** Let us calculate R-matrices between a fundamental representation and the vector representation of $U'_q(C_n^{(1)})$. First recall that we have the following decomposition as $U_q(C_n)$-modules;

$$V(\varpi_k) \otimes V(\varpi_1) = V(\varpi_k + \varpi_1) \oplus V(\varpi_{k+1}) \oplus V(\varpi_{k-1}). \tag{C.3}$$

Here $V(\varpi_0)$ is understood to be the trivial representation and $V(\varpi_{n+1})$ to be 0. Therefore the R-matrix $R_{k1}^{\text{nor}}(x,y) : V(\varpi_k)_x \otimes V(\varpi_1)_y \longrightarrow V(\varpi_1)_y \otimes V(\varpi_k)_x$ can be written as $R_{k1}^{\text{nor}}(x,y) = \mathrm{P}_{\varpi_{k+1}+\varpi_{l-1}} \oplus \gamma_1(y/x)\mathrm{P}_{\varpi_{k+1}} \oplus \gamma_2(y/x)\mathrm{P}_{\varpi_{k-1}}$, where P_ϖ is a $U_q(C_n)$-linear projection from $V(\varpi_k) \otimes V(\varpi_1)$ to $V(\varpi)$ in $V(\varpi_1) \otimes V(\varpi_k)$ with $\varpi = \varpi_k + \varpi_1$, ϖ_{k+1} or ϖ_{k-1}.

Let u_i and u'_i $(i = 0,\ 1,\ 2)$ be highest-weight vectors in the $U_q(C_n)$-modules $V(\varpi_k) \otimes V(\varpi_1)$ and $V(\varpi_1) \otimes V(\varpi_k)$ with highest weights $\varpi_k + \varpi_1$ $(i = 0)$,

ϖ_{k+1} $(i = 1)$, ϖ_{k-1} $(i = 2)$. Remark that if $k = n$ we ignore ϖ_{k+1}, u_1, u'_1 and $\gamma_1(y/x)$. We set $Q_1 = f_0 f_1 \cdots f_{n-1} f_n f_{n-1} \cdots f_{k+1}$ and $Q_2 = f_0 f_1 \cdots f_{k-1}$. Then $Q_i u_i$ is proportional to u_0 because its weight is $\varpi_k + \varpi_1$. Let us first determine γ_1, assuming that $k \neq n$.

The following lemma is by direct calculation and we leave it to the reader. In the statement $G^{(low)}$ means the lower global base (cf. [12, 13]).

Lemma C.3. *Let b be an element of $V(\varpi_k) \otimes V(\varpi_1)$ which is a tensor product of two lower global bases of $V(\varpi_k)$ and $V(\varpi_1)$ and has the weight ϖ_{k+1}. Then $Q_1 b \neq 0$ if and only if $b = b_1 := G^{(low)}(1, \ldots, k) \otimes G^{(low)}(k+1)$ or $b = b_2 := G^{(low)}(2, \ldots, k+1) \otimes G^{(low)}(1)$. Moreover $Q_1 b_1 = q^{-1} y^{-1} u_0$ and $Q_1 b_2 = x^{-1} u_0$, where we set $u_0 = G^{(low)}(1, \ldots, k) \otimes G^{(low)}(1)$.*

Lemma C.4. *If we write $u_1 = b_1 + \sum_{b \neq b_1} a_b b$, where b runs over the set of tensor products of two lower global bases, then $a_{b_2} = (-q_s)^k$.*

Proof. There are relations

$$\begin{aligned} &e_i(G^{(low)}(1, \ldots, \widehat{i+1}, \ldots, k+1) \otimes G^{(low)}(i+1) \\ &\quad - q_s G^{(low)}(1, \ldots, \hat{i}, \ldots, k+1) \otimes G^{(low)}(i)) = 0 \quad \text{for } 1 \leq i \leq k. \end{aligned}$$

It follows that $a_{b_2} = (-q_s)^k$. □

By these lemmas we have $Q_1 u_1 = (q_s^{-1} y^{-1} + (-q_s)^k x^{-1}) u_0$ in $V(\varpi_k) \otimes V(\varpi_1)$. Similarly we obtain the following two lemmas.

Lemma C.5. *Let b be an element of $V(\varpi_1) \otimes V(\varpi_k)$ which is a tensor product of two lower global bases of $V(\varpi_1)$ and $V(\varpi_k)$ and has the weight ϖ_{k+1}. Then $Q_1 b \neq 0$ if and only if $b = b'_1 := G^{(low)}(1) \otimes G^{(low)}(2, \ldots, k+1)$ or $b = b'_2 := G^{(low)}(k+1) \otimes G^{(low)}(1, \ldots, k)$. Moreover $Q_1 b'_1 = q_s^{-1} x^{-1} u'_0$ and $Q_1 b'_2 = y^{-1} u'_0$, where we set $u'_0 = G^{(low)}(1) \otimes G^{(low)}(1, \ldots, k)$.*

Lemma C.6. *If we write $u'_1 = b'_1 + \sum_{b \neq b'_1} a_b b$, where b runs over the set of tensor products of two lower global bases, then $a_{b'_2} = (-q_s)^k$.*

By these lemmas we have in $V(\varpi_1) \otimes V(\varpi_k)$

$$Q_1 u'_1 = (q_s^{-1} x^{-1} + (-q_s)^k y^{-1}) u'_0. \tag{C.4}$$

Therefore we have

$$\gamma_1 = \frac{x - (-q_s)^{k+1} y}{y - (-q_s)^{k+1} x}. \tag{C.5}$$

Next let us determine γ_2. For brevity, we assume that $k \neq 1$ in the following four lemmas, and $G^{(up)}$ means the lower global base (cf. [13]).

Lemma C.7. *Let b be an element of $V(\varpi_k) \otimes V(\varpi_1)$ which is a tensor product of two upper global bases of $V(\varpi_k)$ and $V(\varpi_1)$ and has the weight ϖ_{k-1}. Then $Q_2 b \neq 0$ if and only if $b = b_3 := G^{(up)}(1, \dots, k) \otimes G^{(up)}(\bar{k})$ or $b = b_4 := G^{(up)}(2, \dots, k, \bar{k}) \otimes G^{(up)}(1)$. Moreover $Q_2 b_3 = q_s^{-1} y^{-1} u_0$ and $Q_2 b_4 = q_s x^{-1}[2]_1 u_0$, where we set $u_0 = G^{(up)}(1, \dots, k) \otimes G^{(up)}(1)$.*

This lemma is by direct calculation and we leave it to the reader.

Lemma C.8. *If we write $u_2 = b_3 + \sum_{b \neq b_3} a_b b$, where b runs over the set of tensor products of two upper global bases, then $a_{b_4} = -(-q_s)^{2n-k+1}/[2]_{k-1}$.*

Proof. There are relations.

$$\begin{aligned}
&e_i(G^{(up)}(1, \dots, k-1,\ i) \otimes G^{(up)}(\bar{i}) \\
&\quad -q_s G^{(up)}(1, \dots, k-1,\ i+1) \otimes G^{(up)}(\overline{i+1})) = 0 \quad \text{for } i = k, \dots, n-1,
\end{aligned}$$

$$\begin{aligned}
&e_n(G^{(up)}(1, \dots, k-1,\ n) \otimes G^{(up)}(\overline{n}) \\
&\quad -q_s^2 G^{(up)}(1, \dots, k-1,\ n) \otimes G^{(up)}(\overline{n})) = 0,
\end{aligned}$$

$$\begin{aligned}
&e_i(G^{(up)}(1, \dots, k-1,\ \overline{i+1}) \otimes G^{(up)}(i+1) \\
&\quad -q_s G^{(up)}(1, \dots, k-1,\ \bar{i}) \otimes G^{(up)}(i)) = 0 \quad \text{for } i = k, \dots, n-1,
\end{aligned}$$

$$\begin{aligned}
&e_{k-1}([2]_{k-1} G^{(up)}(1, \dots, k-1,\ n) \otimes G^{(up)}(\overline{n}) \\
&\quad -q_s G^{(up)}(1, \dots, k-1,\ n) \otimes G^{(up)}(\overline{n})) = 0,
\end{aligned}$$

$$\begin{aligned}
&e_i(G^{(up)}(1, \dots, \widehat{i+1}, \dots, k,\ \overline{k}) \otimes G^{(up)}(i+1) \\
&\quad -q_s G^{(up)}(1, \dots, \widehat{i}, \dots, k,\ \overline{k}) \otimes G^{(up)}(i)) = 0 \quad \text{for } i = 1, \dots, k-2.
\end{aligned}$$

It follows that $a_{b_4} = -(-q_s)^{2n-k+1}/[2]_{k-1}$. □

By these lemmas we have in $V(\varpi_k) \otimes V(\varpi_1)$

$$Q_2 u_2 = (q_s^{-1} y^{-1} + (-q_s)^{2n-k+2} x^{-1}) u_0. \tag{C.6}$$

Similarly we obtain the following two lemmas for $V(\varpi_1) \otimes V(\varpi_k)$.

Lemma C.9. *Let b be an element of $V(\varpi_1) \otimes V(\varpi_k)$ which is a tensor product of two upper global bases of $V(\varpi_1)$ and $V(\varpi_k)$ and has the weight ϖ_{k-1}. Then $Q_2 b \neq 0$ if and only if $b = b'_3 := G^{(up)}(1) \otimes G^{(up)}(2, \dots, k, \overline{k})$ or $b = b'_4 := G^{(up)}(\overline{k}) \otimes G^{(up)}(1, \dots, k)$. Moreover $Q_2 b'_3 = q_s^{-1} x^{-1}[2]_1 u'_0$ and $Q_2 b'_4 = y^{-1} u'_0$, where we set $u'_0 = G^{(up)}(1) \otimes G^{(up)}(1, \dots, k)$.*

Lemma C.10. *If we write $u'_2 = b'_3 + \sum_{b \neq b'_3} a_b b$, where b run over the set of tensor products of two upper global bases, then $a_{b'_4} = (-q_s)^{2n-k+2}[2]_{k-1}$.*

By these lemmas we have in $V(\varpi_1)\otimes V(\varpi_k)$

$$Q_2u_2' = (q_s^{-1}x^{-1} + (-q_s)^{2n-k+2}y^{-1})(q_s+q_s^{-1})u_0'. \tag{C.7}$$

Therefore up to a multiple of an element of k we have

$$\gamma_2 = \frac{x-(-q_s)^{2n-k+3}y}{y-(-q_s)^{2n-k+3}x}. \tag{C.8}$$

It is easy to check that this expression for γ_2 still holds even if $k=1$. So we obtain the following result.

Theorem C.11. *The normalized R-matrix is given by*

$$R^{\rm nor}_{k1}(z) = \begin{cases} \mathrm{P}_{\varpi_k+\varpi_1} + \dfrac{1-(-q_s)^{k+1}z}{z-(-q_s)^{k+1}}\mathrm{P}_{\varpi_{k+1}} + \dfrac{1-(-q_s)^{2n-k+3}z}{z-(-q_s)^{2n-k+3}}\mathrm{P}_{\varpi_{k-1}} & \\ & \text{if } 1\le k<n, \\ \mathrm{P}_{\varpi_k+\varpi_1} + \dfrac{1-(-q_s)^{n+3}z}{z-(-q_s)^{n+3}}\mathrm{P}_{\varpi_{k-1}} & \text{if } k=n. \end{cases}$$

Hence we have

$$d_{1,k}(z)=d_{k1}(z)=\begin{cases}(z-(-q_s)^{k+1})(z-(-q_s)^{2n+3-k}) & \text{if } 1\le k<n,\\ z-(-q_s)^{n+3} & \text{if } k=n.\end{cases} \tag{C.9}$$

We give the explicit form of R-matrix for the vector representation.

Proposition C.12. *For $b_1,b_2\in B(\varpi_1)$ we have*

$$R^{\rm nor}_{11}(z)(b_1\otimes b_2)=\begin{cases} b_1\otimes b_2 & \text{if } b_1=b_2,\\ \dfrac{(1-q_s^2)z^{\delta(b_2\prec b_1)}}{z-q_s^2}\,b_1\otimes b_2 + \dfrac{q_s(z-1)}{z-q_s^2}\,b_2\otimes b_1 & \text{if } b_1\neq b_2,\overline{b_2}.\end{cases}$$

For $1 \le a \le n$ we have

$$R^{\rm nor}_{11}(z)(a\otimes\overline{a}) = \frac{1-q_s^2}{z-q_s^2}\,a\otimes\overline{a} + \sum_{k=1}^{n}\frac{(-q_s)^{a+k}(1-q_s^2)(z-1)}{(z-q_s^2)(z-(-q_s)^{2n+2})}\,k\otimes\overline{k}$$
$$-\sum_{k>a}\frac{(-q_s)^{2n+a-k+2}(1-q_s^2)(z-1)}{(z-q_s^2)(z-(-q_s)^{2n+2})}\,\overline{k}\otimes k + \frac{q_s^2(z-1)(z-(-q_s)^{2n})}{(z-q_s^2)(z-(-q_s)^{2n+2})}\,\overline{a}\otimes a$$
$$-\sum_{k<a}\frac{(-q_s)^{a-k}(1-q_s^2)z(z-1)}{(z-q_s^2)(z-(-q_s)^{2n+2})}\,\overline{k}\otimes k\,,$$
$$R^{\rm nor}_{11}(z)(\overline{a}\otimes a) = -\sum_{k<a}\frac{(-q_s)^{2n-a+k+2}(1-q_s^2)(z-1)}{(z-q_s^2)(z-(-q_s)^{2n+2})}\,k\otimes\overline{k}$$
$$+\frac{q_s^2(z-1)(z-(-q_s)^{2n})}{(z-q_s^2)(z-(-q_s)^{2n+2})}\,a\otimes\overline{a} - \sum_{k>a}\frac{(-q_s)^{k-a}(1-q_s^2)z(z-1)}{(z-q_s^2)(z-(-q_s)^{2n+2})}\,k\otimes\overline{k}$$
$$+\sum_{k=1}^{n}\frac{(-q_s)^{2n-a-k+2}(1-q_s^2)z(z-1)}{(z-q_s^2)(z-(-q_s)^{2n+2})}\,\overline{k}\otimes k + \frac{(1-q_s^2)z}{z-q_s^2}\,\overline{a}\otimes a\,.$$

The general d_{ij} with $i,j \ne 1$ will be calculated at the end of this section with the aid of the universal R-matrices.

C.3. **Universal R-matrices.** We shall calculate the universal R-matrices. By (A.13) and (C.9), we have

$$\text{(C.10)}\quad a_{1k}(z) = a_{k1}(z) = q_s\frac{\{k-1\}\{2n+1-k\}\{2n+3+k\}\{4n+5-k\}}{\{k+1\}\{2n+3-k\}\{2n+1+k\}\{4n+3-k\}}\,.$$

Here we employed the notation

$$\text{(C.11)}\qquad \{m\} = ((-q_s)^m z; q_s^{4n+4})_\infty\,.$$

Now we shall calculate $a_{kl}(z)$ for $l \le k$. Consider the commutative diagram

(C.12)

$$\begin{array}{ccc}
V(\varpi_k)\otimes V(\varpi_{l-1})_{(-q_s)^{-1}z}\otimes V(\varpi_1)_{(-q_s)^{l-1}z} & \xrightarrow{\ \psi\ } & V(\varpi_k)\otimes V(\varpi_l)_z \\
\downarrow f & & \\
V(\varpi_{l-1})_{(-q_s)^{-1}z}\otimes V(\varpi_k)\otimes V(\varpi_1)_{(-q_s)^{l-1}z} & & \downarrow h \\
\downarrow g & & \\
V(\varpi_{l-1})_{(-q_s)^{-1}z}\otimes V(\varpi_1)_{(-q_s)^{l-1}z}\otimes V(\varpi_k) & \xrightarrow[\ \psi'\ ]{} & V(\varpi_l)_z\otimes V(\varpi_k)\,.
\end{array}$$

Here

$$\psi = V(\varpi_k)\otimes(p_{l-1,1})_z, \quad \psi' = (p_{l-1,1})_z\otimes V(\varpi_k),$$
$$f = R^{\mathrm{univ}}_{k,l-1}((-q_s)^{-1}z)\otimes V(\varpi_1)_{(-q_s)^{l-1}z},$$
$$g = V(\varpi_{l-1})_{(-q_s)^{-1}z}\otimes R^{\mathrm{univ}}_{k1}((-q_s)^{l-1}z) \text{ and } h = R^{\mathrm{univ}}_{kl}(z).$$

We have

$$\begin{aligned} p_{l-1,1}(G(1\dots,l-1)\otimes G(l)) &= G(1,\dots,l)\,, \\ R^{\mathrm{nor}}_{k,1}(z)(G(1,\dots,k)\otimes G(l)) &= G(l)\otimes G(1,\dots,k). \end{aligned}$$

Chasing the vector $G(1,\dots,k)\otimes G(1,\dots,l-1)\otimes G(l)$ of $V(\varpi_k)\otimes V(\varpi_{l-1})_{(-q_s)^{-1}z}\otimes V(\varpi_1)_{(-q_s)^{l-1}z}$ in the diagram C.12, we obtain the recurrence relation

$$a_{kl}(z) = a_{k,l-1}((-q_s)^{-1}z)a_{k,1}((-q_s)^{l-1}z).$$

Solving this, and noticing $a_{kl} = a_{lk}$, we obtain the following result.

Proposition C.13. *For $k, l \in I_0 = \{1,\dots,n\}$, we have*

$$a_{kl}(z) = q_s^{\min(k,l)}\frac{\{|k-l|\}\{2n+2-k-l\}\{2n+2+k+l\}\{4n+4-|k-l|\}}{\{k+l\}\{2n+2-k+l\}\{2n+2+k-l\}\{4n+4-k-l\}}\,.$$

Here we used the notation $\{m\} = ((-q_s)^m z; q_s^{4n+4})_\infty$.

C.4. **Denominators of normalized R-matrices.** In this subsection we shall prove

Proposition C.14. *For $1 \le k, l \le n$, we have*

(C.13)
$$d_{kl}(z) = \prod_{i=1}^{\min(k,l,n-k,n-l)}(z-(-q_s)^{|k-l|+2i})\prod_{i=1}^{\min(k,l)}(z-(-q_s)^{2n+2-k-l+2i})\,.$$

This is already proved in the case $l = 1$. The case $k = l = n$ is proved in [16, Proposition 4.2.6]. We shall prove this proposition by reduction to those cases. Let $D_{kl}(z)$ be the right hand side of (C.13).

By (A.6), we may assume that $k \ge l$. First let us show that $d_{kl}(z)$ is a multiple of $D_{kl}(z)$. In order to see this, by using Corollary 2.4, it is enough to show that $V(\varpi_k)\otimes V(\varpi_l)_a$ is reducible for any root a of $D_{kl}(z)$. For $1 \le i \le$

$n-k, l$, we have

$$\begin{array}{c} V(\varpi_k)\otimes V(\varpi_l)_{(-q_s)^{k-l+2i}} \\ \Big\downarrow {\scriptstyle V(\varpi_k)\otimes(i_{i,l-i})_{(-q_s)^{k-l+2i}}} \\ V(\varpi_k)\otimes V(\varpi_i)_{(-q_s)^{k+i}}\otimes V(\varpi_{l-i})_{(-q_s)^{k-l+i}} \\ \Big\downarrow {\scriptstyle (p_{ki})_{(-q_s)^i}\otimes V(\varpi_{l-i})_{(-q_s)^{k-l+i}}} \\ V(\varpi_{k+i})_{(-q_s)^i}\otimes V(\varpi_{l-i})_{(-q_s)^{k-l+i}} \end{array}$$

Here $V(\varpi_0)$ is understood to be the trivial representation. Then one can easily see that the composition is not zero but $u_k\otimes u_l$ is sent to zero. Hence $V(\varpi_k)\otimes V(\varpi_l)_{(-q_s)^{k-l+2i}}$ is reducible. Similarly for $1\le i\le l$, let us consider

$$\begin{array}{c} V(\varpi_k)\otimes V(\varpi_l)_{(-q_s)^{2n+2-k-l+2i}} \\ \Big\downarrow {\scriptstyle i_{k-i,i}\otimes(i_{i,l-i})_{(-q_s)^{2n+2-k-l+2i}}} \\ V(\varpi_{k-i})_{(-q_s)^i}\otimes V(\varpi_i)_{(-q_s)^{i-k}}\otimes V(\varpi_i)_{(-q_s)^{2n+2-k+i}}\otimes V(\varpi_{l-i})_{(-q_s)^{2n+2-k-l+i}} \\ \Big\downarrow {\scriptstyle V(\varpi_{k-i})_{(-q_s)^i}\otimes \mathrm{tr}\otimes V(\varpi_{l-i})_{(-q_s)^{2n+2-k-l+i}}} \\ V(\varpi_{k-i})_{(-q_s)^i}\otimes V(\varpi_{l-i})_{(-q_s)^{2n+2-k-l+i}} \end{array}$$

In this case also, the composition is not zero but $u_k\otimes u_l$ is sent to zero. Hence $V(\varpi_k)\otimes V(\varpi_l)_{(-q_s)^{2n+2-k-l+2i}}$ is reducible.

By (A.10), we have

$$a_{kl}(z)\;a_{kl}((-q_s)^{-(2n+2)}z)\equiv\frac{d_{kl}(z)}{d_{kl}((-q_s)^{2n+2}z^{-1})}\quad \operatorname{mod} k[z,z^{-1}]^\times .$$

Hence we obtain

$$d_{kl}(z)=D_{kl}(z)\psi_{kl}(z) \tag{C.14}$$

for a polynomial $\psi_{kl}(z)$ satisfying

$$\psi_{kl}(z)\equiv\psi_{kl}((-q_s)^{2n+2}z^{-1})\quad \operatorname{mod} k[z,z^{-1}]^\times. \tag{C.15}$$

Now we shall use the following lemma.

Lemma C.15. *Let V', V'', V and W be irreducible $U_q'(\mathfrak{g})$-modules. Assume that there is a surjective morphism $V'\otimes V''\to V$. Then*

$$\frac{d_{W,V'}(z)d_{W,V''}(z)a_{W,V}(z)}{d_{W,V}(z)a_{W,V'}(z)a_{W,V''}(z)}\quad\text{and}\quad\frac{d_{V',W}(z)d_{V'',W}(z)a_{V,W}(z)}{d_{V,W}(z)a_{V',W}(z)a_{V'',W}(z)}$$

are in $k[z,z^{-1}]$.

Proof. In a commutative diagram

$$\begin{array}{ccc} W\otimes V'_z\otimes V''_z & \longrightarrow & W\otimes V_z \\ {\scriptstyle R'(z)\otimes V''_z}\downarrow & & \\ V'_z\otimes W\otimes V''_z & & {\scriptstyle R(z)}\downarrow \\ {\scriptstyle V'_z\otimes R''(z)}\downarrow & & \\ V'_z\otimes V''_z\otimes W & \longrightarrow & V_z\otimes W\,, \end{array}$$

if $R'(z)$ and $R''(z)$ do not have poles, then so is $R(z)$. To see the first assertion, it is enough to apply this to $R'(z) = d_{W,V'}(z)R^{\mathrm{nor}}_{W,V'}(z)$, $R''(z) = d_{W,V''}(z)R^{\mathrm{nor}}_{W,V''}(z)$ and

$$R(z) = \frac{d_{W,V'}(z)d_{W,V''}(z)a_{W,V}(z)}{d_{W,V}(z)a_{W,V'}(z)a_{W,V''}(z)}R^{\mathrm{nor}}_{W,V}(z)\,.$$

The second assertion can be proved similarly. □

We shall prove $\psi_{kl}(z) \equiv 1 \bmod k[z, z^{-1}]^{\times}$.

Case $k + l \le n$**.** We prove this by the induction on l. If $l = 1$, it is already proved. If $l > 1$ then applying the lemma above to $V(\varpi_{l-1})_{(-q_s)^{-1}} \otimes V(\varpi_1)_{(-q_s)^{l-1}} \to V(\varpi_l)$, we have

$$\frac{d_{k,l-1}((-q_s)^{-1}z)\ d_{k,1}((-q_s)^{l-1}z)\ a_{k,l}(z)}{d_{k,l}(z)\ a_{k,l-1}((-q_s)^{-1}z)\ a_{k,1}((-q_s)^{l-1}z)} \in k[z, z^{-1}].$$

Since

$$\frac{D_{k,l-1}((-q_s)^{-1}z)\ d_{k,1}((-q_s)^{l-1}z)\ a_{k,l}(z)}{D_{k,l}(z)\ a_{k,l-1}((-q_s)^{-1}z)\ a_{k,1}((-q_s)^{l-1}z)} \equiv 1,$$

$\psi_{k,l-1}(z) \equiv 1$ implies $\psi_{kl}(z) \equiv 1$.

Case $k + l > n$ We shall first reduce the assertion to the $k = n$ case. For $k < n$ consider a surjection

$$V(\varpi_{k+1})_{(-q_s)^{-1}} \otimes V(\varpi_1)_{(-q_s)^{2n+1-k}} \to V(\varpi_k)$$

given by the composition

$$V(\varpi_{k+1})_{(-q_s)^{-1}} \otimes V(\varpi_1)_{(-q_s)^{2n+1-k}} \to V(\varpi_k) \otimes V(\varpi_1)_{(-q_s)^{-1-k}} \otimes V(\varpi_1)_{(-q_s)^{2n+1-k}} \to V(\varpi_k)\,.$$

We have

$$\frac{D_{k+1,l}((-q_s)z)\ d_{1,l}((-q_s)^{k-2n-1}z)\ a_{kl}(z)}{D_{kl}(z)\ a_{k+1,l}((-q_s)z)\ a_{1,l}((-q_s)^{k-2n-1}z)} \equiv z - (-q_s)^{4n+4-k-l}.$$

Hence $\psi_{k+1,l}(z) \equiv 1$ implies that $\psi_{kl}(z)$ is a divisor of $z-(-q_s)^{4n+4-k-l}$. Then (C.15) implies that $\psi_{kl}(z) \equiv 1$. Hence, the descending induction on k reduces

the problem to the $k=n$ case.
We have

$$\frac{D_{k,l-1}((-q_s)^{-1}z)\ d_{k,1}((-q_s)^{l-1}z)\ a_{k,l}(z)}{D_{k,l}(z)\ a_{k,l-1}((-q_s)^{-1}z)\ a_{k,1}((-q_s)^{l-1}z)} \equiv z-(-q_s)^{2n+2-k+l}.$$

Hence by the similar argument to $k+l\leq n$ case, $\psi_{k,l-1}(z)\equiv 1$ implies that $\psi_{kl}(z)$ is a divisor of $z-(-q_s)^{2n+2-k+l}$. Hence if $l\neq k=n$ then we can reduce the l case to the $l-1$ case. This completes the proof of Proposition C.14.

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