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SECOND-MICROLOCALIZATION AND ASYMPTOTIC EXPANSIONS

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Abstract

The asymptotic expansions of (holonomic) microfunctions is neatly dealt with by the second micro-localization.

§1. Introduction.

Several years ago, Jeanquartier [5] proved the following:

For a real-valued real analytic function $f(x)$ defined on a real analytic manifold of dimensions n, $\delta(t-f(x))$ has an asymptotic expansion of the form

$$(1.1)\qquad \sum_{\mu=0}^{n-1}\sum_{\nu=1}^{N}\sum_{j=0}^{\infty} a_{\nu,\mu,j}(x)t^{\lambda_\nu+j}(\log t)^{\mu}.$$

for distributions $a_{\nu,\mu,j}(x)$, as t tends to zero. Here the meaning of the asymptotic expansion is given through the pairing with a C^∞-function $\varphi(x)$ with compact support.

Recently, by using a group-theoretic technique, Barbasch-Vogan [1] proved that any eigendistribution on a real semi-simple group G has an asymtpotic expansion and they studied some properties of the initial term of the expansion. Here we note that an eigendistribution satisfies, by definition, the following system of linear differential equations :

$$(1.2)\quad \begin{cases} Pu = \chi(P)u & \text{for each } P \text{ in } \mathfrak{z} \\ L_A u = 0 & \text{for each } A \text{ in the Lie algebra of } G, \end{cases}$$

where $\mathfrak{z}$ is the ring of G-biinvariant differential operators on G, χ is the character of $\mathfrak{z}$ and L_A is the vector field on G corresponding to the inner automorphism of G on G.

We can show that $\delta(t-f(x))$ satisfies a holonomic system with R.S.(*) and that (1.2) is also a holonomic system with R.S. As we show later (§4.2) each microfunction solution of a holonomic system with R.S. has an asymptotic expansion, whose meaning will be clarified in subsequent sections. Hence the result of Jeanquartier and that of Barbasch-Vogan are explained in a neat and unified manner by employing the theory of holonomic systems with R.S.

We now introduce the following space $M^{(r)}$ so that we may consider such asymptotic expansions of distributions mentioned above in a more general situation .

<u>Definition 1.1</u>. A distribution u on $\mathbb{R}^n$ belongs to $M^{(r)}$ $(r\in\mathbb{R})$ if and only if there exist positive constants ε, m and C such that

$$(1.3)\quad \left|\int u(tx)\varphi(x)\,dx\right| \leqq Ct^{-\frac{n}{2}-r}\sum_{|\alpha|\leqq m}\sup|D^{\alpha}\varphi|$$

holds for every φ in $C_0^{\infty}(\{x\in\mathbb{R}^n; |x|<\varepsilon\})$.

These spaces $M^{(r)}$ can be used as a scale in considering the asymptotic expansion of a distribution u as follows:

We say that u has an asymptotic expansion $\sum_{j=0}^{\infty} u_j$ and we write $u \sim \sum_{j=0}^{\infty} u_j$ if there exist a sequence $\{u_j\}_{j\geqq 0}$ of distributions which satisfy the following condition:

(1.4) For any $r\in\mathbb{R}$, there exists $N_0\in\mathbb{Z}$ such that $u-\sum_{j=0}^{N_0} u_j \in M^{(r)}$.

In particular, this condition implies the following:

(1.5) For any $r\in\mathbb{R}$, there exists $N_0\in\mathbb{Z}$ such that $u_j\in M^{(r)}$ for any $j \geqq N_0$.

(*) See Kashiwara-Kawai [8], [9] for the theory of holonomic systems with regular singularities.

In what follows we exclusively consider the case where u_j is homogeneous of degree $\lambda+j$ for some $\lambda \in \mathbb{C}$, or, a little more generally, the case where $(\sum_{\nu=1}^{n} x_\nu D_\nu - \lambda - j)^p u_j = 0$ holds for some $p \in \mathbb{N}$. In the latter case, u_j belongs to $M^{(-\mathrm{Re}\lambda - j - \frac{n}{2} + \varepsilon)}$ for any $\varepsilon > 0$.

In order to exemplify the notion introduced here, we give the following example, where the j-th terms in the right hand side are homogeneous of degree $2\lambda+j$ and $3\lambda+\frac{1}{2}+j$, respectively.

$$(1.6)\quad (y^2-x^3)_+^\lambda \sim \sum_{j=0}^{\infty} \frac{\Gamma(-\lambda+\frac{1}{2})}{4^j j!\,\Gamma(-\lambda+\frac{1}{2}+j)} x^{3j} D_y^{2j} |y|^{2\lambda}$$

$$+ \sum_{j=0}^{\infty} \frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{4^j j!\,\Gamma(\lambda+j+\frac{3}{2})\cos\pi\lambda} x^{3j} D_y^{2j} (-x_+^{3\lambda+\frac{3}{2}} + \sin\pi\lambda\, x_-^{3\lambda+\frac{3}{2}})\,\delta(y).$$

The asymptotic expansion done by using $M^{(r)}$ as the scale can be regarded as an asymptotic expansion with respect to the origin $\{x=0\}$. Hence it is natural to try to micro-localize the notion as follows:

Let Λ be the conormal bundle of the origin, i.e., $\sqrt{-1}\,T^*_{\{0\}}\mathbb{R}^n = \{(x,\sqrt{-1}\xi) \in \mathbb{R}^n \times \sqrt{-1}\,\mathbb{R}^n;\ x=0\}$. Let $p = (0,\sqrt{-1}\xi_0)$ be a point in Λ. In this situation, we introduce the following spaces $M^{(r)}_{\Lambda,p}$ $(r \in \mathbb{R})$.

<u>Definition 1.2.</u> A distribution u belongs to $M^{(r)}_{\Lambda,p}$ if and only if the following condition is satisfied:

There exist $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ with $\chi(0) \neq 0$, an open neighborhood U of ξ_0 and constants C and m such that

$$(1.7)\quad \left|\iint \exp(-\sqrt{-1}\langle x,\tau\xi\rangle)(\chi u)(x)\,dx\,\psi(\xi)\,d\xi\right| \leqq C\tau^{r-\frac{n}{2}} \sum_{|\alpha|\leqq m} \sup|D_\xi^\alpha \psi(\xi)|$$

holds for $\tau \geqq 1$ for every $\psi \in C_0^\infty(U)$.

The following lemma easily follows from the definition.

<u>Lemma 1.3.</u> (i) If (1.7) holds for some $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(0) \neq 0$, then (1.7) holds for any $\tilde{\chi} \in C^\infty(\mathbb{R}^n)$ such that $\tilde{\chi}(0) \neq 0$ and that supp $\tilde{\chi}$ is contained in the domain of definition of u.

(ii) If p is not in $WF(u)$,(*) then u belongs to $M^{(r)}_{\Lambda,p}$ for every r.

(*) See Hörmander [3] for the definition of the wave front set.

(iii) If u belongs to $M^{(r)}_{\Lambda,p}$ and $a(x)$ is a C^∞-function defined in a neighborhood of the origin, then au belongs to $M^{(r)}_{\Lambda,p}$. If we further suppose that $a(0) = 0$, then au belongs to $M^{(r-1)}_{\Lambda,p}$.

(iv) Let u be a tempered distribution and denote by v its Fourier transform. If there exist an open neighborhood U of ξ_0 and constants C, m such that

$$(1.8) \quad \left|\int v(\tau\xi)\psi(\xi)d\xi\right| \leqq C\tau^{r-\frac{n}{2}} \sum_{|\alpha|\leqq m} \sup|D^\alpha\psi|$$

holds for $\tau \geqq 1$ and for every $\psi \in C_0^\infty(U)$, then u belongs to $M^{(r)}_{\Lambda,p}$.

(v) If u belongs to $M^{(r)}_{\Lambda,p}$, then $\partial u/\partial x_k$ belongs to $M^{(r+1)}_{\Lambda,p}$.

(vi) Let P be a classical pseudo-differential operator of order at most 0. If u belongs to $M^{(r)}_{\Lambda,p}$, then Pu belongs to $M^{(r)}_{\Lambda,p}$.

(vii) The following three conditions are equivalent:

(a) $u \in M^{(r)}$

(b) $u \in M^{(r)}_{\Lambda,p}$ for any $p \in \Lambda$

(c) $u \in M^{(r)}_{\Lambda,p}$ for $p = (0;0)$

(viii) For each distribution u there exists r such that u belongs to $M^{(r)}_{\Lambda,p}$.

Remark. The converse of (ii) is not true, namely, the fact that u belongs to $M^{(r)}_{\Lambda,p}$ for every r does not imply that $WF(u)$ does not contain p; For example, let n be 1 and consider the C^∞-function $f(x) \underset{\text{def}}{=} \exp(-1/x^2)$ and a continuous function $g(x)$ which is not C^∞ at x_j, where $\{x_j\}$ is a sequence tending to 0. Then (iii) and (viii) combined implies that $u(x) \underset{\text{def}}{=} f(x)g(x)$ is in $M^{(r)}_{\Lambda,p}$ for any p and r. However $WF(u)$ contains some p in Λ, because x_j is contained in sing supp u.

We can also prove the following result.

Theorem 1.4. Let $a(x,\theta)$ be a C^∞-function defined in a neighborhood of $(x,\theta) = (0,\theta_0) \in \mathbb{R}^n \times \mathbb{R}^N$ $(N \geqq n)$. Suppose that $a(x,\theta)$ satisfies the following conditions:

$$(1.9) \quad a(0,\theta) = 0$$

$$(1.10) \quad \left(\frac{\partial^2 a(x,\theta)}{\partial x_i \partial \xi_j}\right)_{i,j} \text{ has rank } n \text{ at } (0,\theta_0).$$

Denote by p the point $(0, \sqrt{-1}d_x a(0,\theta_0)) \in \sqrt{-1}T^*_{\{0\}}\mathbb{R}^n$. Let u be a compactly supported distribution which belongs to $M^{(r)}_{\Lambda,p}$. Define a function $w(\theta)$ by

$$\int \exp(-\sqrt{-1}a(x,\theta))u(x)dx - \int \exp(-\sqrt{-1}\langle \mathrm{grad}_x a(0,\theta),x\rangle)u(x)dx.$$

Then there exist constants C, m and $\varepsilon(>0)$ such that

$$(1.11)\quad \left|\int w(t\theta)\varphi(\theta)d\theta\right| \leqq C\tau^{r-\frac{n}{2}-2} \sum_{|\alpha|\leqq m} \sup|D^\alpha_\theta \varphi|$$

holds for $\tau \geqq 1$ and for every φ in $C^\infty_0(\{\theta; |\theta-\theta_0| < \varepsilon\})$.

It immediately follows from this result that a Fourier integral operator(*) A of order at most 0 such that the associated canonical transformation Φ preserves Λ sends $M^{(r)}_{\Lambda,p}$ to $M^{(r)}_{\Lambda,\Phi(p)}$. In view of this fact we can define $M^{(r)}_{\Lambda,p}$ for every Lagrangian submanifold Λ of $\sqrt{-1}T^*\mathbb{R}^n$ by using a suitable phase function $a(x,\theta)$, even though we exclusively consider the case where $\Lambda = \sqrt{-1}T^*_{\{0\}}\mathbb{R}^n$ in this section. For example, if $\Lambda = \{(x',x''; \sqrt{-1}(\xi',\xi'')\in\mathbb{R}^\ell\times\mathbb{R}^{n-\ell}\times\sqrt{-1}(\mathbb{R}^\ell\times\mathbb{R}^{n-\ell}); x'=0, \xi''=0\}$, then a distribution u belongs to $M^{(r)}_{\Lambda,p}$ $(p=(0;\sqrt{-1}(\xi'_0,0))$ if and only if there exist $\chi\in C^\infty_0(\mathbb{R}^n)$ with $\chi(0)\neq 0$ and constants C, m and $\varepsilon > 0$ such that

$$(1.12)\quad \left|\iint\left(\int \exp(-\sqrt{-1}\langle\tau\xi',x'\rangle)(\chi u)(x',x'')dx'\right)\psi(\xi',x'')d\xi'dx''\right| \leqq C\tau^{r-\frac{\ell}{2}} \sum_{|\alpha|+|\beta|\leqq m} \sup|D^\alpha_{\xi'}D^\beta_{x''}\psi|$$

holds for $\tau\geqq 1$ and for every $\psi(\xi',x'')$ in $C^\infty_0(\{(\xi',x'')\in\mathbb{R}^\ell\times\mathbb{R}^{n-\ell}; |\xi'-\xi'_0|+|x''| < \varepsilon\}$.

Now we want to investigate how these scaling spaces are related to micro-differential equations with regular singularities. Let us denote by $\Lambda^{\mathbb{C}}$ the complexification of $\Lambda = \sqrt{-1}T^*_{\{0\}}\mathbb{R}^n$, i.e., $\Lambda^{\mathbb{C}} = T^*_{\{0\}}\mathbb{C}^n$. Then $\mathcal{E}_{\Lambda^{\mathbb{C}}}(m)$ is, by definition, $\{P = \sum_j p_j(x,D)\in\mathcal{E}; p_j(x,\xi)$ has the zero of degree $(j-m)$ along $\Lambda^{\mathbb{C}}\}$. We abbreviate $\mathcal{E}_{\Lambda^{\mathbb{C}}}(0)$ to $\mathcal{E}_{\Lambda^{\mathbb{C}}}$. It then follows from Lemma 1.3 (iii) that

(*) See Hörmander [3] for the definition.

(1.13) $\quad \mathcal{E}_{\Lambda^{\mathbb{C}}}(-1) M_{\Lambda,p}^{(r)} \subset M_{\Lambda,p}^{(r-1)}.$

Note also that, if a micro-differential operator P belongs to $\mathcal{E}_{\Lambda^{\mathbb{C}}}$, then it has the form

(1.14) $\quad \sum_{0 \leqq |\alpha| \leqq m} A_\alpha(D) x^\alpha + Q,$

where $A_\alpha(D)$ is homogeneous of degree $|\alpha|$ and Q belongs to $\mathcal{E}_{\Lambda^{\mathbb{C}}}(-1)$. In order to see how $\mathcal{E}_{\Lambda^{\mathbb{C}}}(m)$ is related to the asymptotic expansions, we consider a simple case where a distribution solution $u(x)$ of the equation $Pu = 0$ has an asymptotic expansion of the form $\sum_{j=0}^{\infty} u_{\lambda+j}(x)$ where $u_{\lambda+j}$ is homogeneous of degree $\lambda+j$. It is clear that $u_{\lambda+j}$ belongs to $M^{(-\mathrm{Re}\lambda - j - \frac{n}{2})}$. In this case, we find the following relation:

(1.15) $\quad Pu \equiv \sum_{0 \leqq |\alpha| \leqq m} A_\alpha(D) x^\alpha u_\lambda(x) \quad \mathrm{mod} \quad M^{(-\mathrm{Re}\lambda - 1 - \frac{n}{2})}.$

If we define a homogeneous differential operator $L(P)(\xi, D_\xi)$ by

$$\sum_{0 \leqq |\alpha| \leqq m} A_\alpha(\sqrt{-1}\xi)(\sqrt{-1}\,\frac{\partial}{\partial \xi})^\alpha,$$

then we obtain another equation

(1.16) $\quad L(P)\hat{u}_\lambda(\xi) = 0,$

where $\hat{u}_\lambda(\xi) = \int \exp(-\sqrt{-1}\langle x, \xi \rangle) u_\lambda(x) dx$. If we denote the image of L by $\mathcal{A}_{\Lambda^{\mathbb{C}}}$, $\mathcal{A}_{\Lambda^{\mathbb{C}}}$ is isomorphic to $\mathcal{E}_{\Lambda^{\mathbb{C}}} / \mathcal{E}_{\Lambda^{\mathbb{C}}}(-1)$. (Cf. [8], Chap. I, §5.) It follows from the definition that $\mathcal{A}_{\Lambda^{\mathbb{C}}}$ consists of homogeneous linear differential operators of degree 0 defined on $\Lambda^{\mathbb{C}}$. The following Theorem 1.5 shows the importance of the associated equation (1.16), especially because we will later prove that, if u is a solution of holonomic $\mathcal{E}$-Module with R.S., then its asymptotic expansion is determined by its initial term. (See §4.2, Theorem 4.2.12 for the precise statement.)

<u>Theorem 1.5</u>. Let X be a complex manifold and Λ a Lagrangian submanifold of T^*X. Let $\mathcal{J}$ be a left Ideal of $\mathcal{E}_X$ such that $\mathcal{M} \underset{\mathrm{def}}{=} \mathcal{E}_X / \mathcal{J}$ is a holonomic $\mathcal{E}_X$-Module with R.S. Then

$$\mathcal{L}: \quad L(P)v = 0 \qquad (P \in \mathcal{J} \cap \mathcal{E}_\Lambda)$$

is a holonomic system with R.S. on $T^*\Lambda$.

Proof. By a quantized contact transformation, we may assume the following:

(1.17) X is an open subset of $\mathbb{C}^{1+n} = \{(t,x);\ t \in \mathbb{C},\ x \in \mathbb{C}^n\}$.

(1.18) Λ is the conormal bundle of $Y \underset{\mathrm{def}}{=} \{(t,x) \in X;\ t=0\}$.

(1.19) Supp $\mathcal{M}$ is in a generic position (in the sense of [8] Chap. I, §6).

Then we know ([8] Chap. V, §1) that there exists a holonomic $\mathcal{D}_X$-Module $\mathcal{F}$ with R.S. and a section $\tilde{u}$ of $\mathcal{F}$ such that

(1.20) $\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \mathcal{F} \simeq \mathcal{M}, \quad \mathcal{H}^0_Y(\mathcal{F}) = 0$

and

(1.21) $1 \otimes \tilde{u}$ corresponds to the section $u = (1 \bmod \mathcal{J})$ of $\mathcal{M}$ by the above isomorphism.

We also know ([6]) the following:

(1.22) $\mathcal{D}[s](t^s u)$ is a coherent subholonomic $\mathcal{D}_X$-Module.

Here and in what follows, $\mathcal{D}[s]$ denotes $\mathbb{C}[s] \underset{\mathbb{C}}{\otimes} \mathcal{D}_X$. By using the same reasoning as in [6] we can also show that

(1.23) $\mathcal{D}[s](t^s u)/\mathcal{D}[s](t^{s+1}u)$ is a holonomic $\mathcal{D}_Y$-Module with R.S.

Let us now denote by $\mathcal{D}'$ the sub-Algebra of $\mathcal{D}_X$ generated by $\mathcal{O}_X$, D_{x_j} and tD_t. Then, for any P in $\mathcal{D}'$, t^sPt^{-s} belongs to $\mathcal{D}[s]$. Moreover, any section of $\mathcal{D}_X$ can be written in the form $\sum_{j \geq 0} D_t^j t^s P_j t^{-s}$ with $P_j \in \mathcal{D}'$. Here we note the following two lemmas.

Lemma 1.6. Let $P(s)$ be a section of $\mathcal{D}[s]$ of the form

$\sum_{j\geq 0} D_t^j t^s P_j t^{-s}$. If $P(s)t^s u = 0$ holds, then $P_j u = 0$ holds for any j.

Proof. It follows from the assumption that $\sum_{j\geq 0} D_t^j t^s P_j u = 0$ holds. It suffices to show that $P_j u = 0$ $(j>r)$ entails $P_r u = 0$ in this case for any integer r. If $P_j u = 0$ $(j>r)$, then we have

$$0 = \sum_{j=0}^{r} D_t^j t^s P_j u = \sum_{0\leq k\leq j\leq r} s(s-1)\cdots(s-k+1)t^{s-k} D_t^{j-k} P_j u.$$

Looking at the coefficients of s^r in the right hand side, we find $t^{s-r} P_r u = 0$. Hence we have $P_r u = 0$. Q.E.D.

Lemma 1.7. We have an injective $\mathcal{D}_Y$-linear homomorphism

$$\mathcal{D}'u/t\,\mathcal{D}'u \longrightarrow \mathcal{D}[s](t^s u)/\mathcal{D}[s](t^{s+1}u)$$

by assigning $t^s Pu$ to Pu $(P \in \mathcal{D}')$.

Proof. Let P be an element of $\mathcal{D}'$. If $t^s Pu$ belongs to $\mathcal{D}[s](t^{s+1}u)$, then we can find $P(s)=\sum_j D_t^j t^{-s} P_j t^s$ $(P_j \in \mathcal{D}')$ such that

$$t^s Pu = P(s)(t^{s+1}u) = \sum_j D_t^j t^s P_j tu.$$

Hence it follows from the preceding lemma that

$$Pu = P_0(tu) \in \mathcal{D}'(tu) = t\,\mathcal{D}'u.$$

Q.E.D.

Now we resume the proof of Theorem 1.5. Let $\mathcal{J}'$ denote the Ideal of $\mathcal{D}_Y$ which annihilates $t^s u \mod \mathcal{D}[s](t^{s+1}u)$. Then it follows from (1.23) that $\mathcal{D}_Y/\mathcal{J}'$ is a holonomic $\mathcal{D}_Y$-Module with R.S. On the other hand, Lemma 1.7 implies

$$\mathcal{J}'u \subset t\,\mathcal{D}'u.$$

Hence for any P in $\mathcal{J}'$, we can find Q in $t\,\mathcal{D}'$ such that $(P+Q)u = 0$. Since $\mathcal{D}' \subset \mathcal{E}_\Lambda$ and $L(P+Q) = L(P)$, the system $\mathcal{L}$ in Theorem 1.5 is stronger than the system $\mathcal{D}_Y/\mathcal{J}'$. This means that $\mathcal{L}$ is a holonomic $\mathcal{D}_Y$-Module with R.S. Q.E.D.

So far we have developed the theory of asymptotic expansions by the aid of scaling spaces $M^{(r)}$ (or its micro-localization $M_{\Lambda}^{(r)}$). However, it would be much more desirable if we could find a suitable sheaf on which another suitable sheaf of rings of operators acts and in which the asymptotic expansion of (a class of) microfunctions can be considered. In the example (1.6), we can rewrite the right hand side

$$(1.24)\qquad \left(\sum_{j=0}^{\infty} c_j x^{3j} D_y^{2j}\right) |y|^{2\lambda} + \left(\sum_{j=0}^{\infty} c_j' x^{3j} D_y^{2j}\right) \times \left\{ \frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{3}{2})\cos\pi\lambda} (-x_+^{3\lambda+\frac{3}{2}} + \sin\pi\lambda\ x_-^{3\lambda+\frac{3}{2}}) \right\} \delta(y),$$

where $c_j=\Gamma(-\lambda+\frac{1}{2})/4^j j!\,\Gamma(-\lambda+\frac{1}{2}+j)$ and $c_j' = \Gamma(\lambda+\frac{3}{2})/4^j j!\,\Gamma(\lambda+\frac{3}{2}+j)$.

In view of the growth order of c_j and c_j', it is easy to see that the infinite series of operators appearing in (1.24) do not preserve the local character but that it has "propagation velocity" of order $|x|^{\frac{3}{2}}$. (Cf. Kashiwara-Kawai [7]). Having this in mind, we will introduce the sheaves $\tilde{\mathcal{D}}_{\Lambda}^{\infty}$ and $\tilde{\mathcal{E}}_{\Lambda}^{\infty}$ in the next section. We note that it has turned out that $\tilde{\mathcal{E}}_{\Lambda}^{\infty}$ coincides with the sheaf $\mathcal{E}_{\Lambda}^{2\infty}$ introduced by Laurent [14], [15] in different context.

§2. The second-microlocalization of operators.

As we mentioned at the end of the introduction, we want to find a sheaf of operators which is suitable for the manipulation of the asymptotic expansion of microfunctions. For this purpose we introduce the sheaves $\tilde{\mathcal{D}}_{\Lambda}^{\infty}$ and $\tilde{\mathcal{E}}_{\Lambda}^{\infty}$ starting from the sheaf $\mathcal{M}$ of simple holonomic $\mathcal{E}_X$-Module supported by a complex Lagrangian submanifold Λ of T^*X. Our procedure for finding the desired sheaves is similar to the way of constructing the sheaves of hyperfunctions and (micro-) differential operators starting from the sheaf of holomorphic functions. For example, the sheaf $\mathcal{D}_{\mathbb{C}^n}^{\infty}$ of linear differential operators (of infinite order) on $\mathbb{C}^n$ is, by definition $\mathcal{H}_{\Delta}^n(\mathcal{O}_{\mathbb{C}^n} \hat{\otimes}\, \Omega_{\mathbb{C}^n}^n)$, where $\Delta = \{(x,y)\in\mathbb{C}^n\times\mathbb{C}^n;\ x=y\}$ and $\Omega_{\mathbb{C}^n}^n$ denotes the sheaf of holomorphic n-forms. Note that $\Omega^n_{\mathbb{C}^n}$ is a dual sheaf of $\mathcal{O}_{\mathbb{C}^n}$ and locally isomorphic to $\mathcal{O}_{\mathbb{C}^n}$. We will follow this procedure in defining the sheaf $\tilde{\mathcal{D}}^\infty_\Lambda$ (Definition 2.2). We first prepare several notations.

Let X be a complex manifold of dimension n and Λ a non-singular Lagrangian submanifold of T^*X. We denote by γ the projection from $T^*X - T^*_XX$ to P^*X. In what follows, we will consider the problem locally in Λ, and hence we may assume that $\Lambda = \gamma^{-1}\tilde{\Lambda}$ with $\tilde{\Lambda} = \gamma(\Lambda)$.

Let $\mathcal{M}$ be a simple holonomic $\mathcal{E}_X$-Module with support Λ and let $\mathcal{M}^*$ denote $\mathcal{E}xt^n_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X)$. Let $\mathcal{N}$ be $\mathcal{M} \hat{\otimes} \mathcal{M}^*$ and $\Delta^a \underset{\text{def}}{=} T^*_\Delta(X\times X) = \{(x,y;\xi,\eta) \in T^*(X\times X);\ x=y,\ \xi+\eta=0\}$. Here we note the following

Lemma 2.1. The system $\mathcal{N}$ is independent of the choice of $\mathcal{M}$.

Proof. Let $\mathcal{M}'$ be another simple holonomic system with support Λ. Then there exists a constant λ such that $\mathcal{M}'$ is isomorphic to $\mathcal{M}'' \underset{\text{def}}{=} \mathcal{E}_{(\lambda)} \otimes_{\mathcal{E}} \mathcal{M}$, where $\mathcal{E}_{(\lambda)}$ denotes the sheaf of micro-differential operators of fractional order $\lambda+j$ $(j\in\mathbb{Z})$. Furthermore the isomorphism φ from $\mathcal{M}'$ to $\mathcal{M}''$ is unique up to constant factor. On the other hand, $\mathcal{M}''^* = \mathcal{E}_{(-\lambda)} \otimes_{\mathcal{E}} \mathcal{M}^*$ and hence $\mathcal{M}'' \hat{\otimes} \mathcal{M}''^* = (\mathcal{E}_{(\lambda)} \otimes_{\mathcal{E}} \mathcal{M}) \hat{\otimes} (\mathcal{E}_{(-\lambda)} \otimes_{\mathcal{E}} \mathcal{M}^*) = \mathcal{M} \hat{\otimes} \mathcal{M}^* = \mathcal{N}$. Therefore φ and $\varphi^{*-1}: \mathcal{M}''^* \to \mathcal{M}'^*$ give rise to an isomorphism ψ from $\mathcal{M}' \hat{\otimes} \mathcal{M}'^*$ onto $\mathcal{M}'' \hat{\otimes} \mathcal{M}''^* = \mathcal{M} \hat{\otimes} \mathcal{M}^*$. Since $((c\varphi)^*)^{-1} = c^{-1}\varphi^{*-1}$ for $c \in \mathbb{C}^\times$, ψ does not depend on φ. Q.E.D.

After this observation we introduce the following

Definition 2.2. $\tilde{\mathcal{D}}^\infty_\Lambda \underset{\text{def}}{=} \mathcal{H}^n_{\Delta^a}(\mathcal{E}^\infty_{X\times X} \otimes_{\mathcal{E}_{X\times X}} \mathcal{N})$.

We can also consider the micro-localization of $\tilde{\mathcal{D}}^\infty_\Lambda$ by the same procedure used in S-K-K [17] Chap. II to define the sheaf of micro (=pseudo)-differential operators starting from $\mathcal{O}^{(0,n)}_{X\times X}$. In what follows, $\widetilde{{}_{\Delta^a}\Lambda\times\Lambda^{a*}}$ denotes the comonoidal transform of $\Lambda\times\Lambda^a$ with center Δ^a, i.e., $(\Lambda\times\Lambda^a - \Delta^a) \sqcup S^*_{\Delta^a}(\Lambda\times\Lambda^a)$. The projection from $\widetilde{{}_{\Delta^a}\Lambda\times\Lambda^{a*}}$ to $S^*_{\Delta^a}(\Lambda\times\Lambda^a)$ and the projection from $S^*_{\Delta^a}(\Lambda\times\Lambda^a)$ to $P^*_{\Delta^a}(\Lambda\times\Lambda^a)$ are denoted by π and γ respectively.

Definition 2.3.

(i) $\tilde{\mathcal{E}}^{\mathbb{R}}_{\Lambda} \underset{\text{def}}{=} \mathcal{H}^{n}_{S^*_{\Lambda^a}(\Lambda\times\Lambda^a)}(\pi^{-1}\mathcal{N}^{\infty})$.

(ii) $\tilde{\mathcal{E}}^{\infty}_{\Lambda} \underset{\text{def}}{=} \gamma^{-1}\gamma_* \tilde{\mathcal{E}}^{\mathbb{R}}_{\Lambda}$.

Here and in what follows $\mathcal{N}^{\infty}$ denotes $\mathcal{E}^{\infty} \otimes_{\mathcal{E}} \mathcal{N}$. Since $\mathcal{N}$ is independent of the choice of $\mathcal{M}$, the definitions introduced above are independent of the choice of $\mathcal{M}$. Hence we usually choose $\mathcal{C}_{Y|X}$ as $\mathcal{M}$ when Λ has the form $T^*_Y X$ for a non-singular submanifold of X.

We now begin to explain how to obtain the concrete expression of the germ of $\tilde{\mathcal{D}}^{\infty}_{\Lambda}$. The corresponding result for $\tilde{\mathcal{E}}^{\infty}_{\Lambda}$ is obtained by combining the results given below and the reasoning of S-K-K [17] Chap. II, §1.4, so we will state the result (Theorem 2.12), leaving the detailed arguments to the reader. See also Laurent [14].

We begin our discussion with proving several vanishing theorems needed for the concrete calculation of the relative cohomology groups introduced above.

Proposition 2.4. Let Y be a non-singular hypersurface of $X = \mathbb{C}^n$ and let Λ denote $T^*_Y X$. Let U be an open subset of Λ such that each fiber of $U \to \gamma(U)$ is contractible and that $\gamma(U)$ is a Stein manifold. Then we have

(2.1) $$H^j(U; \mathcal{C}^{\infty}_{Y|X}) = 0 \quad (j\neq 0)$$

Proof. Since $V \underset{\text{def}}{=} \gamma(U)$ is Stein, there exists a fundamental system of Stein open neighborhoods $W_k \subset X$ of V. Then $W_k \cap (X-Y)$ is also Stein. Denote by j the inclusion map from X-Y into X and let $\mathcal{P}$ denote the sheaf $j_* j^{-1}\mathcal{O}_X$. Denote $\mathcal{P}/\mathcal{O}_X$ by $\mathcal{F}$. Then it follows from the definition that

(2.2) $$\mathcal{C}^{\infty}_{Y|X} \cong \gamma^{-1}\mathcal{F} \oplus \gamma^{-1}\mathcal{O}_X.$$

On the other hand, we have

(2.3) $$H^j(W_k; \mathcal{P}) = H^j(W_k - Y; \mathcal{O}_X) = 0$$

for $j \geqq 1$. Hence we have

$$(2.4) \quad H^j(W_k; \mathcal{F}) = 0 \quad (j \geqq 1).$$

Therefore we find

$$(2.5) \quad H^j(U; \mathcal{C}^{\infty}_{Y|X}) = \varinjlim_{k} H^j(W_k; \mathcal{F} \oplus \mathcal{O}_X) = 0$$

for $j \geqq 1$. Q.E.D.

By a quantized contact transformation we can easily deduce the following corollary from Proposition 2.4.

Corollary 2.5. Let Λ be a Lagrangian submanifold of T^*X and $\mathcal{M}$ a simple holonomic $\mathcal{E}_X$-Module with support Λ. Then for any p in $\Lambda - T^*_X X$, we can find a neighborhood W of p which satisfies the following:

For each open subset U of W satisfying the conditions

(2.6) $\gamma(U)$ is a Stein manifold

and

(2.7) each fiber of $U \to \gamma(U)$ is contractible,

we find

$$H^j(U; \mathcal{M}^{\infty}) = 0 \quad \text{for } j \neq 0.$$

Next we deduce the following vanishing theorem (Theorem 2.6) from Corollary 2.5. In what follows, X denotes $\mathbb{C}^n$, V denotes $T^*_{\{0\}}\mathbb{C}^n \times (T^*_{\{0\}}\mathbb{C}^n)^a \simeq T^*_{\{0\}}\mathbb{C}^{2n} \simeq \mathbb{C}^{2n}$ and $\mathcal{N}$ denotes $\mathcal{C}_{\{0\}|\mathbb{C}^n} \hat{\otimes} \mathcal{C}_{\{0\}|\mathbb{C}^n} \simeq \mathcal{C}_{\{0\}|\mathbb{C}^{2n}}$. We denote a point in $\mathbb{C}^{2n}$ by (x,y) and a point in $T^*_{\{0\}}\mathbb{C}^{2n}$ by $(0,0;\xi,\eta)$. Let us take a point p in $\Delta^a \cap V - T^*_{X\times X}(X\times X) = \{(0,0;\ \xi,\eta);\ \xi=-\eta\neq 0\}$. By a linear transformation we may assume that p is $(0,0;\ \xi^0,-\xi^0)$ with $\xi^0 = (0,\cdots,0,1)$. Define V_ℓ by $\{(0,0;\ \xi,\eta)\in V;\ \xi_\ell+\eta_\ell \neq 0\}$ for $\ell=1,\ldots,n$ and let U be a sufficiently small neighborhood of p which satisfies conditions (2.6) and (2.7). Then it follows from Corollary 2.5 that

$$H^j(U\cap V_{\nu_1}\cap\cdots\cap V_{\nu_k};\mathcal{N}^\infty)=0$$

holds for $j\neq 0$ and $1\leqq\nu_1<\cdots<\nu_k\leqq n$ $(0\leqq k\leqq n)$. Since $\{U\cap V_\ell\}$ is an open covering of $U-(\Delta^a\cap V)$, we obtain by a theorem of Leray

$$H^j_{\Delta^a}(U;\mathcal{N}^\infty)=0 \quad \text{for } j>n \tag{2.8}$$

and

$$H^n_{\Delta^a}(U;\mathcal{N}^\infty)=\mathcal{N}^\infty(U\cap\bigcap_{\ell=1}^n V_\ell)/(\sum_{j=1}^n\mathcal{N}^\infty(U\cap(\bigcap_{\ell\neq j}V_\ell)). \tag{2.9}$$

On the other hand, Theorem 1.2.2 of [8] Chap. I, §2 asserts that

$$\mathcal{E}xt^j_{\mathcal{E}_{X\times X},\Delta^a\cap V}(\mathcal{E}_{X\times X},\mathcal{N}^\infty)=0$$

holds for $j<\operatorname{codim}_{T^*(X\times X)}(\Delta^a\cap V)-\operatorname{proj}\dim\mathcal{N}$. It is easy to see that

$$\operatorname{codim}_{T^*(X\times X)}(\Delta^a\cap V)=3n,\ \operatorname{proj}\dim\mathcal{N}=2n$$

and

$$\mathcal{E}xt^j_{\mathcal{E}_{X\times X},\Delta^a\cap V}(\mathcal{E}_{X\times X},\mathcal{N}^\infty)=\mathcal{H}^j_{\Delta^a}(\mathcal{N}^\infty).$$

Thus we obtain the following

<u>Theorem 2.6</u>. $\mathcal{H}^j_{\Delta^a}(\mathcal{N}^\infty)=0 \quad (j\neq n)$.

We next try to find a symbol sequence corresponding to an element in $\mathcal{D}^\infty_{\Lambda.p}$ $(p\in\Lambda)$.

First we note that

$$\mathcal{H}^n_{\Delta^a}(\mathcal{N}^\infty)_p=\varinjlim_{U\ni p}\mathcal{N}^\infty(U\cap\bigcap_{\ell=1}^n V_\ell)/(\sum_{j=1}^n\mathcal{N}^\infty(U\cap(\bigcap_{\ell\neq j}V_\ell)))$$

holds by (2.9). Let P be in $\mathcal{D}^\infty_{\Lambda,p}$. Let $(\sum_j p_j(D_x,D_y))\delta(x,y)$ be a corresponding element in $\mathcal{N}^\infty(U\cap\bigcap_{\ell=1}^n V_\ell)$ for sufficiently small U which satisfies conditions (2.6) and (2.7). Here and in what follows we denote by $\delta(x,y)$ the generator of $\mathcal{N}$. It follows from the definition that $p_j(\xi,\eta)$ is homogeneous of degree j and satisfies the following conditions (2.10) and (2.11):

(2.10) For each $\varepsilon > 0$ and each compact subset K of $U \cap (\bigcap_{\ell=1}^{n} V_\ell)$ there exists a constant $C_{\varepsilon,K}$ such that

$$\sup_K |p_j| \leqq C_{\varepsilon,K}\varepsilon^j/j! \qquad (j \geqq 0)$$

holds.

(2.11) For each compact subset K of $U \cap (\bigcap_{\ell=1}^{n} V_\ell)$, there exists a constant R_K such that

$$\sup_K |p_j| \leqq R_K^{-j}(-j)! \qquad (j < 0)$$

holds.

By the Laurent expansion, we have

(2.12) $$p_j(\xi,\eta) = \sum_{\alpha\in\mathbb{Z}^n} p_{j,\alpha}(\xi) \prod_{\ell=1}^{n} (\xi_\ell+\eta_\ell)^{-\alpha_\ell-1}.$$

It is then easy to see that, if we define $q_j^{(\ell)}(\xi,\eta)$ by

$$\sum_{\substack{\alpha_1,\ldots,\alpha_{\ell-1}\geqq 0\\ \alpha_\ell<0}} p_{j,\alpha}(\xi) \prod_{\ell=1}^{n} (\xi_\ell+\eta_\ell)^{-\alpha_\ell-1},$$

then $\sum_j q_j^{(\ell)}(D)\delta$ belongs to $\mathcal{N}^\infty(U \cap (\bigcap_{k\neq\ell} V_k))$. Hence we may assume from the first that $p_j(\xi,\eta)$ has the form

(2.13) $$\sum_{\alpha\in\mathbb{Z}_+^n} p_{j,\alpha}(\xi) \prod_{\ell=1}^{n} (\xi_\ell+\eta_\ell)^{-\alpha_\ell-1}.$$

Here we note that $p_{j,\alpha}(\xi)$ is homogeneous of degree $j+|\alpha|+n$ and defined on a neighborhood of ξ^0. Furthermore Cauchy's integral formula combined with (2.10) and (2.11) entails the following:

There exists a neighborhood W of $\xi^0 = (0,\ldots,0,1)$ on which p_j's are defined and satisfy the following conditions:

(2.14) For each $\varepsilon > 0$ and $\kappa > 0$ there exists a constant $C_{\varepsilon,\kappa}$ such that

$$\sup_W |p_{j,\alpha}| \leqq C_{\varepsilon,\kappa}\varepsilon^j\kappa^{|\alpha|}/j! \qquad (j \geqq 0)$$

(2.15) For each $\kappa > 0$ there exists a constant R_κ such that

$$\sup_W |p_{j,\alpha}| \leqq (-j)!\kappa^{|\alpha|}R_\kappa^{-j} \qquad (j < 0).$$

On the other hand, we have the following formula:

$$(2.16)\qquad \prod_{\ell=1}^n (D_{x_\ell}+D_{y_\ell})^{-1-\alpha_\ell}\delta(x,y) = \frac{x^\alpha}{\alpha!}\prod_{\ell=1}^n (D_{x_\ell}+D_{y_\ell})^{-1}\delta(x,y).$$

Hence, as a section of $\mathcal{N}^\infty(U \cap (\bigcap_{\ell=1}^n V_\ell))$,

$$\sum_{\substack{j\in\mathbb{Z}\\ \alpha\in\mathbb{Z}_+^n}} p_{j,\alpha}(D_x)\prod_{\ell=1}^n (D_{x_\ell}+D_{y_\ell})^{-\alpha_\ell-1}\delta(x,y)$$

is equal to

$$\sum_{\substack{j\in\mathbb{Z}\\ \alpha\in\mathbb{Z}_+^n}} \frac{1}{\alpha!}p_{j,\alpha}(D_x)x^\alpha\prod_{\ell=1}^n (D_{x_\ell}+D_{y_\ell})^{-1}\delta(x,y).$$

Furthermore we have

$$(2.17)\qquad (D_{x_\ell}+D_{y_\ell})\prod_{k=1}^n (D_{x_k}+D_{y_k})^{-1}\delta(x,y) = \prod_{k\neq\ell}(D_{x_k}+D_{y_k})^{-1}\delta(x,y)\in\mathcal{N}^\infty(U\cap(\bigcap_{k\neq\ell}V_k))$$

and we find

$$(2.18)\qquad (x_\ell-y_\ell)\prod_{k=1}^n (D_{x_k}+D_{y_k})^{-1}\delta(x,y) = 0.$$

Hence, as an element of $H^0(U;\widetilde{\mathscr{E}}_\Lambda^\infty)$,

$$w \underset{\text{def}}{=} \prod_{k=1}^n (D_{x_k}+D_{y_k})^{-1}\delta(x,y)$$

satisfies the system of differential equations

$$\begin{cases} (D_{x_\ell}+D_{y_\ell})w = 0 & (\ell=1,\dots,n)\\ (x_\ell-y_\ell)w = 0 & (\ell=1,\dots,n). \end{cases}$$

Hence w can be identified with $\delta(x-y)$. Therefore w can be identified with the identity operator, and hence, $\sum_j p_j(D_x,D_y)\delta(x,y)$

can be identified with $\left(\sum_{\substack{j\in\mathbb{Z}\\ \alpha\in\mathbb{Z}_+^n}} \frac{1}{\alpha!} p_{j,\alpha}(D_x)x^\alpha\right)$.

Now, if we define $q_{j,i}(\xi,x)$ by $\sum_{|\alpha|=i} \frac{1}{\alpha!} p_{j-i-n,\alpha}(\xi)x^\alpha$, then $q_{j,i}(\xi,x)$ is a holomorphic function defined on a neighborhood Ω of ξ^0 and a polynomial in x, where Ω is independent of i and j. It is homogeneous of degree i in x and of degree j in ξ. Further, they satisfy the following growth conditions:

(2.19) For each $\varepsilon > 0$ and each compact set K of $\Omega \times \mathbb{C}_x^n$, there exists a constant $C_{\varepsilon,K}$ such that

$$\sup_K |q_{j,i}| \leqq \frac{C_{\varepsilon,K}\varepsilon^{i+j}}{i!(j-i)!}$$

holds if $j \geqq i$.

(2.20) For each $\varepsilon > 0$ and each compact set K in $\Omega \times \mathbb{C}_x^n$, there exists a constant $R_{\varepsilon,K}$ such that

$$\sup_K |q_{j,i}| \leqq \frac{(i-j)!}{i!} \varepsilon^i R_{\varepsilon,K}^{i-j}$$

holds if $i > j$.

Now we introduce the following

<u>Definition 2.7</u>. The symbol sequence of $P \in \tilde{\mathcal{D}}_\Lambda^\infty(\Omega)$ $(\Omega \subset \Lambda = T^*_{\{0\}}\mathbb{C}^n)$ is, by definition, the doubly indexed sequence $\{q_{j,i}(\xi,x)\}_{\substack{i\in\mathbb{Z}_+\\ j\in\mathbb{Z}}}$ which satisfies the following conditions:

(2.21) $q_{j,i}(\xi,x)$ is a holomorphic function defined on $\Omega \times \mathbb{C}_x^n$ which is homogeneous of degree j in ξ and a polynomial of degree i in x and which satisfies the growth conditions (2.19) and (2.20).

Using this definition, we can summalize our result in the following form:

Theorem 2.8. An operator P in $\widetilde{\mathcal{D}}^{\infty}_{\Lambda}(\Omega)$ determines a symbol sequence $\{q_{j,i}(\xi,x)\}_{\substack{i\in\mathbb{Z}_+\\ j\in\mathbb{Z}}}$ which satisfies conditions (2.19) and (2.20) in a unique manner and each symbol sequence $\{q_{j,i}(\xi,x)\}_{\substack{i\in\mathbb{Z}_+\\ j\in\mathbb{Z}}}$ satisfying the the above conditions determine an operator P belonging to $\widetilde{\mathcal{D}}^{\infty}_{\Lambda}(\Omega)$ by defining it by $\sum_{j,i} q_{j,i}(D_x,x)$. Here $q_{j,i}(D_x,x)$ means that the multiplication by x should be applied first and next the action by D_x is considered.

Remark 2.9. We often express the symbol sequence of $P\in\widetilde{\mathcal{D}}^{\infty}_{\Lambda}(\Omega)$ as $\{p_{i,j}(x,\xi)\}_{\substack{i\in\mathbb{Z}_+\\ j\in\mathbb{Z}}}$, where $p_{i,j}$ is homogeneous of degree j in ξ and a homogeneous polynomial of degree i in x. When the symbol sequence is given in this form, we assign $P\in\widetilde{\mathcal{D}}^{\infty}_{\Lambda}(\Omega)$ to it by setting $P = \sum_{i,j} p_{i,j}(x,D_x)$, namely, the (micro-)differentiation is applied first and next comes the multiplicaiton by x. In this case $\{p_{i,j}(x,\xi)\}$ is related to $\{q_{i,j}(\xi,x)\}$ in Theorem 2.8 by

$$p_{i,j}(x,\xi) = \sum_{\alpha} \frac{1}{\alpha!} D^{\alpha}_{\xi} D^{\alpha}_{x} q_{j+|\alpha|,i+|\alpha|}(\xi,x)$$

and

$$q_{j,i}(\xi,x) = \sum_{\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} D^{\alpha}_{\xi} D^{\alpha}_{x} p_{i+|\alpha|,j+|\alpha|}(x,\xi).$$

Furthermore $\{p_{i,j}(x,\xi)\}$ satisfies the same growth conditions that $\{q_{j,i}(\xi.x)\}$ satisfies. Hence $\widetilde{\mathcal{D}}^{\infty}_{\Lambda}$ is closed under the operation of taking the formal adjoint.

Remark 2.10. Let $P = \sum_j p_j(x,D)$ be in $\mathcal{E}^{\infty}_{X}|_{\Lambda}$. Then we can assign $\sum_{i,j} p_{i,j}(x,D)$ in $\widetilde{\mathcal{D}}^{\infty}_{\Lambda}$ to P, by defining $p_{i,j}(x,\xi)$ to be the part of $p_j(x,\xi)$ that is homogeneous of degree i in x. This assignment is consistent with the embedding $\mathcal{E}^{\infty}_{X}|_{\Lambda} \rightarrowtail \widetilde{\mathcal{D}}^{\infty}_{\Lambda}$. Note also that the Ring structure of $\widetilde{\mathcal{D}}^{\infty}_{\Lambda}$ coincides with that of $\mathcal{E}^{\infty}_{X}|_{\Lambda}$ when it is restricted to this subsheaf, namely, the composition $\sum_{m,n} r_{m,n}(x,D_x)$

of two operators $\sum_{i,j} p_{i,j}(x,D_x)$ and $\sum_{k,\ell} q_{k,\ell}(x,D_x)$ in $\widetilde{\mathcal{D}}^\infty_\Lambda$ is given by the following:

$$(2.22)\quad r_{m,n}(x,\xi) = \sum_{\substack{m=i+k-|\alpha| \\ n=j+\ell-|\alpha|}} \frac{1}{\alpha!} D^\alpha_\xi p_{i,j}(x,\xi) D^\alpha_x q_{k,\ell}(x,\xi)$$

Remark 2.11. Theorem 2.8 establishes the correspondence between an operator in $\widetilde{\mathcal{D}}^\infty_\Lambda$ and a suitable symbol sequence, when $\Lambda = T^*_{\{0\}}\mathbb{C}^n$. An analogous result can be obtained in the case where $\Lambda = T^*_Y\mathbb{C}^n$ with an affine variety Y of $\mathbb{C}^n$ in the following manner:

Let Y be given by $\{x \in \mathbb{C}^n;\ x' \underset{\text{def}}{=} (x_1,\dots,x_d) = 0\}$ and $\Lambda = T^*_Y\mathbb{C}^n$. Denote $(\xi_{d+1},\dots,\xi_n)$ by ξ''. Then there exists a one-to-one correspondence between $\widetilde{\mathcal{D}}^\infty_\Lambda(\Omega)$ $(\Omega \subset \Lambda)$ and the set of doubly-indexed sequences $\{q_{i,j}(x,\xi)\}_{\substack{i\in\mathbb{Z}_+ \\ j\in\mathbb{Z}}}$ which satisfy the following condition (2.23) and growth conditions (2.19) and (2.20).

(2.23) $q_{i,j}(x,\xi)$ is a holomorphic function defined on $\Omega \times \mathbb{C}^n_{(x',\xi'')}$ which is homogeneous of degree j in ξ and a homogeneous polynomial of degree i in (x',ξ'').

In a similar way we can find the characterization of operators in $\widetilde{\mathcal{E}}^\infty_\Lambda$. (Cf. S-K-K [17] Chap. II, §1.4.)

Theorem 2.12. Let Λ be $T^*_{\{0\}}\mathbb{C}^n$ and let P be an operator in $\widetilde{\mathcal{E}}^\infty_\Lambda(U)$, where U is an open subset of $T^*\Lambda \cong \mathbb{C}^n_x \times \mathbb{C}^n_\xi$. Then there exists a symbol sequence $\{p_{i,j}(x,\xi)\}_{i,j\in\mathbb{Z}}$ which satisfies the following conditions (2.24) $\sim$ (2.28) so that P can be expressed as $\sum_{i,j} p_{i,j}(x,D_x)$. Conversely, if $\{p_{i,j}(x,\xi)\}_{i,j\in\mathbb{Z}}$ satisfies the conditions (2.24) $\sim$ (2.28), then $\sum_{i,j} p_{i,j}(x,D_x)$ defines an operator in $\widetilde{\mathcal{E}}^\infty_\Lambda(\Omega)$.

(2.24) $p_{i,j}(x,\xi)$ is a holomorphic function defined on U and it is homogeneous of degree i in x and of degree j in ξ.

(2.25) For each $\varepsilon > 0$ and each compact subset K of U, there exists a constant $C_{\varepsilon,K}$ such that

$$\sup_K |p_{i,j}| \leqq \frac{C_{\varepsilon,K}}{j!} \varepsilon^j$$

holds for $j \geqq i \geqq 0$.

(2.26) For each $\varepsilon > 0$ and each compact subset K of U, there exists a constant $R_{\varepsilon,K}$ such that

$$\sup_K |p_{i,j}| \leqq \frac{(i-j)!}{i!} R_{\varepsilon,K}^{i-j} \varepsilon^i$$

holds for $i \geqq 0$ and $j < i$.

(2.27) For each compact subset K of U, there exists a constant R_K such that

$$\sup_K |p_{i,j}| \leqq (-j)! R_K^{-j}$$

holds for $j < i < 0$.

(2.28) For each compact subset K of U, there exists a constant R_K such that for each $\varepsilon > 0$ we can find another constant $C_{\varepsilon,K}$ so that

$$\sup_K |p_{i,j}| \leqq \frac{(-i)!}{(j-i)!} C_{\varepsilon,K} \varepsilon^{j-i} R_K^{-i}$$

holds for $i < 0$ and $j \geqq i$.

Remark 2.13. If Λ is $T_Y^*\mathbb{C}^n$ with $Y = \{x \in \mathbb{C}^n;\ x_1 = \cdots = x_d = 0\}$, then by replacing the condition (2.24) with the following condition (2.29), we obtain the same result.

(2.29) $p_{i,j}(x,\xi)$ is a holomorphic function defined on U and it is homogeneous of degree i in $(x_1,\ldots,x_d,\xi_{d+1},\ldots,\xi_n)$ and of degree j in ξ.

§3. Construction of the sheaves on which $\tilde{\mathcal{D}}_\Lambda^\infty$ and $\tilde{\mathcal{E}}_\Lambda^\infty$ act.

In this section we construct sheaves $\tilde{\mathcal{B}}_\Lambda$ and $\tilde{\mathcal{C}}_\Lambda$ on which $\tilde{\mathcal{D}}_\Lambda^\infty$ and $\tilde{\mathcal{E}}_\Lambda^\infty$ act respectively. The first subsection is devoted to the proof of two theorems which relate the cohomology groups on tangential sphere bundles and those on cotangential sphere bundles. The results obtained there are effectively used in the second subsection to study basic properties of $\tilde{\mathcal{B}}_\Lambda$ and $\tilde{\mathcal{C}}_\Lambda$.

§3.1. Correspondence between sheaves on sphere bundles and co-sphere bundles.

Let M be a topological space and V a real vector bundle over M with fiber dimension d. Let V^* be the dual bundle of V. Denote by S (resp., S^*) the sphere bundle associated with V (resp., V^*), namely, $S = (V-M)/\mathbb{R}^+$ and $S^* = (V^*-M)/\mathbb{R}^+$. Denote by γ (resp., γ^*) the projection from $V-M$ (resp., V^*-M) to S (resp., S^*). Let D denote $\{(x,\bar{v}_x,\bar{\eta}_x) = (x,\bar{v}_x;x,\bar{\eta}_x) \in S \underset{M}{\times} S^* ;\ \langle v_x,\eta_x\rangle \geqq 0\}$, where $(\bar{v}_x,\bar{\eta}_x)$ is the equivalence class of $(x,v_x,\eta_x) \in (V-M) \underset{M}{\times} (V^*-M)$. We denote by τ (resp., π) the projection from S onto M (resp., from S^* onto M). The projection from D onto S^* (resp., onto S) is also denoted by τ (resp., π).

<u>Theorem 3.1.1</u>. Let $\mathcal{F}$ be a complex of sheaves on S and let $\mathcal{G}$ denote the complex of sheaves $\mathbb{R}\tau_*\pi^{-1}\mathcal{F}$. Let K be a closed and properly convex$^{(*)}$ set of S^*. Let $K°$ be the polar of K. Then we have

$$(3.1.1)\quad \mathbb{R}\Gamma_K(S^*;\mathcal{G}) = \mathbb{R}\Gamma_{\tau^{-1}\pi(K)}(K°;\mathcal{F})[1-d].$$

<u>Proof</u>. Set $M' = \pi(K)$. Then we have

$$\mathbb{R}\Gamma_K(S^*;\mathcal{G}) = \mathbb{R}\Gamma_K(S^*;\mathbb{R}\Gamma_{\pi^{-1}(M')}(\mathcal{G}))$$

(*) Convexity means that each fiber $K_x \underset{\text{def}}{=} K \cap \pi^{-1}(x)$ is convex, i.e., $\gamma^{*-1}(K_x)$ is convex. Proper convexity means that $\gamma^{*-1}(K_x)$ does not contain a line.

and

$$\mathbb{R}\Gamma_{\tau^{-1}(M')}(K^\circ; \mathcal{F}) = \mathbb{R}\Gamma(K^\circ; \mathbb{R}\Gamma_{\tau^{-1}(M')}(\mathcal{F})).$$

Since $\mathbb{R}\Gamma_{\pi^{-1}(M')}(\mathcal{G}) = \mathbb{R}\tau_*\pi^{-1}\mathbb{R}\Gamma_{\tau^{-1}(M')}(\mathcal{F})$ holds, by replacing M with M', we may assume from the first that

(3.1.2) $\pi(K) = M.$

Furthermore, by considering a flabby resolution of $\mathcal{F}$, we can reduce the problem to the case where $\mathcal{F}$ is a flabby sheaf.

Let f denote $\pi|_{\tau^{-1}(S^*-K)}$. Then we have

(3.1.3) $\mathbb{R}\Gamma(S^*-K; \mathcal{G})$

$$= \mathbb{R}\Gamma(S^*-K; \mathbb{R}\tau_*\pi^{-1}\mathcal{F})$$

$$= \mathbb{R}\Gamma(\tau^{-1}(S^*-K); \pi^{-1}\mathcal{F})$$

$$= \mathbb{R}\Gamma(S; \mathbb{R}f_*f^{-1}\mathcal{F}).$$

On the other hand, we also have

(3.1.4) $\mathbb{R}\Gamma(S^*; \mathcal{G})$

$$= \mathbb{R}\Gamma(\tau^{-1}S^*; \pi^{-1}\mathcal{F})$$

$$= \mathbb{R}\Gamma(\pi^{-1}S; \pi^{-1}\mathcal{F})$$

$$= \mathbb{R}\Gamma(S; \mathcal{F}).$$

Hence, by combining (3.1.3) and (3.1.4) with the long exact sequence of relative cohomology groups, we obtain the following triangle:

(3.1.5)
$$\begin{array}{ccc} & \mathbb{R}\Gamma_K(S^*; \mathcal{G}) & \\ \swarrow & & \nwarrow {\scriptstyle +1} \\ \mathbb{R}\Gamma(S; \mathcal{F}) & \longrightarrow & \mathbb{R}\Gamma(S; \mathbb{R}f_*f^{-1}\mathcal{F}). \end{array}$$

On the other hand, the following triangle follows from the definition of $\mathbb{R}\mathcal{D}ist_f(\mathcal{F})$ (cf. S-K-K [17] p.270 Remark):

$$\begin{array}{ccc} & \mathbb{R}\Gamma(S;\ \mathbb{R}\,\mathcal{D}ist_f(\mathcal{F})) & \\ \swarrow & & \nwarrow^{+1} \\ \mathbb{R}\Gamma(S;\mathcal{F}) & \longrightarrow & \mathbb{R}\Gamma(S;\ \mathbb{R}f_*f^{-1}\mathcal{F}). \end{array} \tag{3.1.6}$$

Therefore we find

$$\mathbb{R}\Gamma_K(S^*;\mathcal{G}) = \mathbb{R}\Gamma(S;\ \mathbb{R}\,\mathcal{D}ist_f(\mathcal{F})). \tag{3.1.7}$$

Thus the problem is reduced to the calculation of $\mathbb{R}\,\mathcal{D}ist_f(\mathcal{F})$. In order to calculate it, we appeal to a limitting procedure. For this purpose we choose a decreasing family of open and properly convex subset U_n of S^* so that $K = \bigcap_{n\geq 0} U_n$ holds. Let f_n denote $\pi|_{\tau^{-1}(S^*-U_n)}$. Note that

$$f_n^{-1}(x,v_x) = \{(x,v_x,\eta_x) \in S \underset{M}{\times} S^*\ ;\ \ (x,\eta_x) \in U_n^c \text{ and } \langle v_x,\eta_x\rangle \geq 0\}, \tag{3.1.8}$$

where $U_n^c = S^*-U_n$.

First suppose that (x,v_x) is not contained in U_n°, the polar set of U_n. Then $f_n^{-1}(x,v_x)$ is a non-empty contractible set. In fact, it follows from the assumption that there exists (x,η^0) in U_n such that $\langle v_x,\eta^0\rangle < 0$. Hence $f_n^{-1}(x,v_x)$ contains $(x,v_x,-\eta^0)$. Then, for any η in $f_n^{-1}(x,v_x)$ and any t such that $0 \leq t \leq 1$,

$$t\eta - (1-t)\eta^0 \in f_n^{-1}(x,v_x).$$

In fact, if $(x,\ t\eta-(1-t)\eta^0)$ were in U_n, then $(x,t\eta)$ $=(x,t\eta-(1-t)\eta^0+(1-t)\eta^0)$ should be in U_n because of its convexity.

Thus we have seen that $f_n^{-1}(x,v_x)$ is contractible to the point $(x,v_x,-\eta^0)$ in this case. Hence we have

$$\mathbb{R}\,\mathcal{D}ist_{f_n}(\mathcal{F})_{(x,v_x)} = 0, \quad \text{if } (x,v_x) \notin U_n^\circ. \tag{3.1.9}$$

Next we consider the case where (x,v_x) is contained in U_n°. In this case, U_n is contained in $\{(x,v_x)\}^\circ$. On the other hand, it follows from the definition that (x,v_x,η_x) is contained in $f_n^{-1}(x,v_x)$ if and only if (x,η_x) is contained in $\{(x,v_x)\}^\circ \cap U_n^c$. Since U_n is open, $f_n^{-1}(x,v_x)$ is a closed annulus, namely, homotopic to S^{d-2}. Therefore we have

$$(3.1.10)\quad \mathbb{R}\mathcal{D}ist_{f_n}(\mathcal{F})_{(x,v_x)} = \mathbb{R}\Gamma(S^{d-2} \to \{\text{point}\}; \mathcal{F}_{(x,v_x)}) = \mathcal{F}_{(x,v_x)}[1-d], \quad \text{if } (x,v_x) \text{ in } U_n^\circ.$$

Thus we find

$$(3.1.11)\quad \mathbb{R}\mathcal{D}ist_{f_n}(\mathcal{F}) = \mathcal{F}[1-d]\big|_{U_n^\circ}$$

and hence

$$(3.1.12)\quad \mathbb{R}\Gamma(S; \mathbb{R}\mathcal{D}ist_{f_n}(\mathcal{F})) = \mathbb{R}\Gamma(U_n^\circ; \mathcal{F})[1-d].$$

Since $\{U_n\}_{n\geqq 0}$ is a decreasing family of proper convex sets whose limit set K is a closed and proper convex set each of whose fiber is not void, $\{U_n^\circ\}_{n\geqq 0}$ is an increasing family of properly convex sets whose limit set is K°. On the other hand, we have assumed that $\mathcal{F}$ is flably. Therefore $\{H^j(U_n^\circ;\mathcal{F})\}_{n\geqq 0}$ satisfies the (ML)-condition. Hence we have

$$(3.1.13)\quad \varprojlim_n H^{j+1-d}(U_n^\circ;\mathcal{F}) = H^{j+1-d}(K^\circ;\mathcal{F}).$$

Combining (3.1.7), (3.1.12) and (3.1.13), we finally obtain

$$\mathbb{R}\Gamma_K(S^*;\mathcal{G}) = \mathbb{R}\Gamma(S; \mathbb{R}\mathcal{D}ist_f(\mathcal{F})) = \mathbb{R}\Gamma(K^\circ;\mathcal{F})[1-d]. \qquad \text{Q.E.D.}$$

In application, we usually use Theorem 3.1.1 when $\mathcal{F}$ has a special structure. It enables us to obtain more concrete expression of the cohomology groups in question, as we show below. In what follows, we consider the problem in the following geometric situation:

Let N be a submanifold of codimension d of a real analytic manifold M. Denote by $\widetilde{{}^{N}M}$ the (real) monoidal transform of M with center N, i.e., $\widetilde{{}^{N}M} = (M-N) \sqcup S_N M$. We denote by τ the projection from $\widetilde{{}^{N}M}$ to M. In what follows, the triplet $(S_N M, S_N^* M, N)$ corresponds to (S, S^*, M) in Theorem 3.1.1.

<u>Theorem 3.1.2</u>. Let $\mathcal{K}$ be a sheaf on M and define $\mathcal{F}$ by $\mathbb{R}\Gamma_{S_N M}(\tau^{-1}\mathcal{K})$. Let $\mathcal{G}$ be $\mathbb{R}\tau_* \pi^{-1}\mathcal{F}$ and let K be a closed and properly convex subset of $S^* = S_N^* M$. Then we have

$$(3.1.14) \qquad H_K^j(S^*; \mathcal{G}) = \varinjlim_{Z,U} H_{Z \cap U}^{j+1-d}(U; \mathcal{K}),$$

where U ranges over a system of open neighborhood of $\pi(K)$ and Z ranges over the set of closed subset of U such that $Z \cap N \subset \pi(K)$ and $C_N(Z) \cap K^\circ = \phi$. Here $C_N(Z)$ denotes the normal cone of Z along N.

<u>Proof</u>. Theorem 3.1.1 asserts

$$H_K^j(S^*; \mathcal{G}) = H^{j+1-d}_{\tau^{-1}\pi(K)}(K^\circ; \mathcal{F}).$$

Hence it suffices to show that

$$H^j_{\tau^{-1}\pi(K)}(K^\circ; \mathcal{F}) = \varinjlim_{Z,U} H_Z^j(U; \mathcal{K}).$$

On the other hand, it follows from the definition of $\mathcal{F}$ that

$$\mathbb{R}\Gamma_{\tau^{-1}\pi(K)}(K^\circ; \mathcal{F}) = \mathbb{R}\Gamma_{\tau^{-1}\pi(K) \cap K^\circ}(W; \tau^{-1}\mathcal{K}),$$

where W is an open subset of $\widetilde{{}^{N}M}$ such that $W \cap S_N M = K^\circ$. Hence we obtain the following triangle:

$$(3.1.15) \qquad \begin{array}{ccc} & \mathbb{R}\Gamma_{\tau^{-1}\pi(K)}(K^\circ; \mathcal{F}) & \\ \swarrow & & \nwarrow {\scriptstyle +1} \\ \varinjlim_{W} \mathbb{R}\Gamma(W; \tau^{-1}\mathcal{K}) & \longrightarrow & \varinjlim_{W} \mathbb{R}\Gamma(W-(K^\circ \cap \tau^{-1}\pi(K)); \tau^{-1}\mathcal{K}) \end{array},$$

where W ranges over the set of neighborhoods of K° in $\widetilde{{}^{N}M}$ such that $W \cap S_N M = K^\circ$. Then we have

$$(3.1.16) \qquad \varinjlim_{W} \mathbb{R}\Gamma(W;\ \tau^{-1}\mathcal{K}) = \mathbb{R}\Gamma(K^\circ;\ \tau^{-1}\mathcal{K}).$$

Now we note that a fiber of K° over a point x in N is contractible or $(S_N M)_x$ according as x is in $\pi(K)$ or not. Hence we obtain the following triangle:

$$(3.1.17) \qquad \begin{array}{ccc} & \mathbb{R}\Gamma(N;\mathcal{K}) & \\ \swarrow & & \nwarrow^{+1} \\ \mathbb{R}\Gamma(K^\circ;\tau^{-1}\mathcal{K}) & \longrightarrow & \mathbb{R}\Gamma(N-\pi(K);\mathcal{K})[1-d]. \end{array}$$

Next let us define a subset $\tilde{W}$ of $M-N$ by $\tau(W-S_N M)$. Then we have $W-(K^\circ \cap \tau^{-1}\pi(K)) = \tau^{-1}(\tilde{W} \cup (N-\pi(K)))$, because $K^\circ - \tau^{-1}\pi(K) = \tau^{-1}(N-\pi(K))$. On the other hand, we easily see

$$\mathbb{R}\mathcal{D}ist_{\tau}(\mathcal{K}) = \mathcal{K}|_N[-d].$$

Therefore we obtain the following triangle:

$$(3.1.18) \qquad \begin{array}{ccc} & \mathbb{R}\Gamma(\tilde{W} \cup (N-\pi(K));\mathcal{K}) & \\ \swarrow & & \nwarrow^{+1} \\ \mathbb{R}\Gamma(W-(K^\circ \cap \tau^{-1}\pi(K));\mathcal{K}) & \longrightarrow & \mathbb{R}\Gamma(N-\pi(K);\mathcal{K})[1-d]. \end{array}$$

Comparing (3.1.15), (3.1.16), (3.1.17) and (3.1.18), we find the following triangle:

$$\begin{array}{ccc} & \mathbb{R}\Gamma_{\tau^{-1}\pi(K)}(K^\circ;\mathcal{F}) & \\ \swarrow & & \nwarrow^{+1} \\ \mathbb{R}\Gamma(N;\mathcal{K}) & \longrightarrow & \varinjlim_{W} \mathbb{R}\Gamma((W-S_N M) \cup (N-\pi(K));\mathcal{K}). \end{array}$$

Here we note that $\mathbb{R}\Gamma(N;\mathcal{K}) = \varinjlim_{U} \mathbb{R}\Gamma(U;\mathcal{K})$ holds when U ranges over the set of neighborhoods of N. Hence, by setting $W' = (W-S_N M) \cup (N-\pi(K))$, we obtain

$$\begin{array}{ccc} & \varinjlim_{U,W} \mathbb{R}\Gamma_{U-W'}(U;\mathcal{K}) & \\ \swarrow & & \nwarrow^{+1} \\ \varinjlim_{U} \mathbb{R}\Gamma(U;\mathcal{K}) & \longrightarrow & \varinjlim_{U,W} \mathbb{R}\Gamma(U \cap W';\mathcal{K}) \end{array}$$

Thus we finally obtain

(3.1.19) $\mathbb{R}\Gamma_{\tau^{-1}\pi(K)}(K^\circ;\mathcal{F}) = \varinjlim_{U,W} \mathbb{R}\Gamma_{U-W'}(U;\mathcal{K})$.

Since $U-W'$ ranges over the set of closed subset Z of U such that $Z\cap N = K$ and that $C_N(Z) \subset K^\circ$, (3.1.19) is the required result.

Q.E.D.

§3.2. Definition and basic properties of sheaves $\tilde{\mathcal{B}}_\Lambda$ and $\tilde{\mathcal{C}}_\Lambda$.

In §2 we introduced sheaves $\tilde{\mathcal{D}}^\infty_\Lambda$ and $\tilde{\mathcal{E}}^\infty_\Lambda$ for a Lagrangian submanifold Λ of T^*X. As we observed there, $\tilde{\mathcal{E}}^\infty_\Lambda$ deserves the name of second-microlocalization of operators. Hence it is natural to try to find sheaves $\tilde{\mathcal{B}}_\Lambda$ and $\tilde{\mathcal{C}}_\Lambda$ on which $\tilde{\mathcal{D}}^\infty_\Lambda$ and $\tilde{\mathcal{E}}^\infty_\Lambda$ act respectively, as in the case of (micro-)differential operators. In order to do this job, we hereafter assume that X is a complexification of a real analytic manifold M of dimension n and that the purely imaginary locus $\Lambda^{\mathbb{R}}$ of Λ, i.e., $\Lambda \cap \sqrt{-1}T^*M$, is also a (purely imaginary) Lagrangian submanifold of $\sqrt{-1}T^*M$. Throughout this subsection we choose and fix a simple holonomic $\mathcal{E}_X$-Module $\mathcal{M}$ with support Λ. We denote $\mathcal{E}^{\mathbb{R}}_X \otimes_{\mathcal{E}_X} \mathcal{M}$ by $\mathcal{M}^{\mathbb{R}}$. In parallel with the discussion in §2, we will introduce sheaves $\tilde{\mathcal{B}}_\Lambda$ and $\tilde{\mathcal{C}}_\Lambda$ by using the same procedure employed in defining the sheaf $\mathcal{B}_M$ of hyperfunctions and the sheaf $\mathcal{C}_M$ of microfunctions. In defining new sheaves $\tilde{\mathcal{B}}_\Lambda$ and $\tilde{\mathcal{C}}_\Lambda$, we start with $\mathcal{M}^{\mathbb{R}}$ instead of $\mathcal{O}_X$. In what follows, we denote by π the projection from $\widetilde{\Lambda^{\mathbb{R}}\Lambda^*}$ to Λ.

<u>Definition 3.2.1.</u> (i) $\tilde{\mathcal{B}}_\Lambda \underset{\text{def}}{=} \mathcal{H}^n_{\Lambda^{\mathbb{R}}}(\mathcal{M}^{\mathbb{R}})$

$$\text{(ii)}\quad \widetilde{\mathcal{C}}_\Lambda \underset{\text{def}}{=} \mathcal{H}^n_{T^*_{\Lambda^{\mathbb{R}}}\Lambda}(\pi^{-1}\mathcal{M}^{\mathbb{R}})^a.$$

Since we can easily prove that $\mathcal{M}^{\mathbb{R}}$ is a $\widetilde{\mathcal{D}}^\infty_\Lambda$-Module, $\widetilde{\mathcal{B}}_\Lambda$ is also a $\widetilde{\mathcal{D}}^\infty_\Lambda$-Module. Using the same reasoning as in S-K-K [17] Chap. III, §1, we can also verify that $\widetilde{\mathcal{C}}_\Lambda$ is an $\widetilde{\mathcal{E}}^\infty_\Lambda$-Module.

In order to study basic properties of $\widetilde{\mathcal{B}}_\Lambda$ and $\widetilde{\mathcal{C}}_\Lambda$, we need several cohomology vanishing theorems.

We first show that $\Lambda^{\mathbb{R}}$ is purely n-codimensional with respect to $\mathcal{M}^{\mathbb{R}}$, namely, we prove the following

Theorem 3.2.2. $\mathcal{H}^j_{\Lambda^{\mathbb{R}}}(\mathcal{M}^{\mathbb{R}}) = 0$ for $j \neq n$.

Proof. Let p be a point in $\Lambda^{\mathbb{R}}$. Since the problem is local in T^*X, we can find a real contact transformation which brings $(\Lambda^{\mathbb{R}}, p)$ to $(\sqrt{-1}T^*_N M, p^0)$ for a non-singular hypersurface N of M. We may choose a local coordinate system $z = (z_1, \ldots, z_n) = (x_1+\sqrt{-1}y_1, \ldots, x_n+\sqrt{-1}y_n)$ around $\pi(p^0) \in X$ so that $M = \{y_1 = \cdots = y_n = 0\}$, $N = \{x_1 = y_1 = \cdots = y_n = 0\}$ and $p^0 = (0; \sqrt{-1}dx_1)$ hold. We denote by Y the complex hypersurface $\{z_1 = 0\}$. By a quantized contact transformation, we can bring $\mathcal{M}$ to a simple holonomic system whose support is $T^*_Y X$. Hence we may assume without loss of generality that $\mathcal{M}^{\mathbb{R}}$ is $\mathcal{C}^{\mathbb{R}}_{Y|X}$. (S-K-K [17] Chap. II, Definition 1.1.4 and Theorem 4.2.5.) Then Theorem 3.1.2 implies the following:

$$\text{(3.2.1.)}\quad \mathcal{H}^j_{\sqrt{-1}T^*_N M}(\mathcal{C}^{\mathbb{R}}_{Y|X})_{p^0} = \varinjlim_{U,Z} H^j_{U\cap Z}(U; \mathcal{O}_X),$$

where U ranges over the set of open neighborhoods of 0 in $\mathbb{C}^n$ and Z ranges over the set of closed subsets of $\mathbb{C}^n$ which satisfy the following two conditions:

$$\text{(3.2.2)}\quad Z \cap Y \subset N$$

$$\text{(3.2.3)}\quad C_Y(Z) \cap \{z = x+\sqrt{-1}y \in \mathbb{C}^n;\ y_1 > 0\} = \phi.$$

Here we have identified $T_Y X$ with X by using the linear coordinate system on X. Furthermore, by shrinking U and replacing Z with $\overline{U \cap Z}$ if necessary, we may assume without loss of generality

(3.2.4) Z is compact

(3.2.5) $\quad Z \subset \{z \in \mathbb{C}^n;\ y_1 \leqq |x_1|\}.$

In what follows, for compact subset Z of $\mathbb{C}^n$, $\hat{Z}$ denotes $\{z \in \mathbb{C}^n;\ |f(z)| \leq \max_Z |f|$ for each entire function $f\}$. We say that a compact set Z is holomorphically convex, if $\hat{Z} = Z$ holds.

We now recall the following result ([10]).

<u>Proposition 3.2.3.</u> Let Z_1 and Z_2 be compact subsets of $\mathbb{C}^n$. Assume that both Z_1 and Z_2 are holomorphically convex. Then we have

$$H^j_{Z_1 - Z_2}(\mathbb{C}^n - Z_2;\ \mathcal{O}_{\mathbb{C}^n}) = 0 \qquad \text{for} \quad j \neq n.$$

We will prove the vanishing of (3.2.1) by using this result.

Let $\mathfrak{Z}$ denote the family of compact subsets Z of $\mathbb{C}^n$ satisfying conditions (3.2.2), (3.2.3), (3.2.4) and (3.2.5). We shall show that $\hat{Z}$ belongs to $\mathfrak{Z}$ if Z belongs to $\mathfrak{Z}$. First we show that $\hat{Z}$ satisfies the condition (3.2.2). Note that the condition (3.2.2) is equivalent to

(3.2.6) $\quad (Z-M) \cap Y = \phi.$

This condition implies the following:

For each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

(3.2.7) $\quad Z \cap \{z \in \mathbb{C}^n;\ |y'| \geqq \varepsilon,\ |z_1| \leqq \delta_\varepsilon\} = \phi.$

Here and in what follows, z', x' and y' denote $(z_2, \ldots, z_n)$, $(x_2, \ldots, x_n)$ and $(y_2, \ldots, y_n)$, respectively.

Let z^0 be a point in $Y-M$. Define an open subset R_c $(c > 0)$ of $\mathbb{C}$ by $\mathbb{C} - \{w = u + \sqrt{-1}v \in \mathbb{C};\ v \geq c\}$. We will show that there exists an entire function $f(z)$ $(z \in \mathbb{C}^n)$ which satisfies

(3.2.8) $\quad f(z^0) \notin R_c$

and

(3.2.9) $\quad f(Z) \subset R_c.$

Since R_c is a simply connected open subset of $\mathbb{C}$, R_c is a Runge domain. Hence there exists an entire function $\varphi(w)$ $(w \in \mathbb{C})$ which satisfies

$$(3.2.10) \qquad \operatorname{Re} \varphi(f(z^0)) \geqq 0$$

and

$$(3.2.11) \qquad \operatorname{Re} \varphi(f(z)) < 0 \quad \text{if } z \in Z.$$

Therefore the existence of such $f(z)$ will entail that $\hat{Z}$ satisfies the condition (3.2.2).

Now we try to find such $f(z)$. We may assume without loss of generality that

$$\operatorname{Im} z^0 = (0, b, 0, \ldots, 0) \quad (b > 0).$$

Define $f(z)$ by $z_1 - \sqrt{-1}a(z_2 - x_2^0)^2$, where a is a positive constant which we shall specify later. Choose δ_ε as in (3.2.7) for $\varepsilon = b/2$. First let us consider the case where $|y'| \leqq \varepsilon = b/2$. Let g and h denote $\operatorname{Re}(-\sqrt{-1}a(z_2 - x_2^0)^2)$ and $\operatorname{Im}(-\sqrt{-1}a(z_2 - x_2^0)^2)$, respectively. Then we have

$$g^2 = 4a^2(x_2 - x_2^0)^2 y_2^2 \leqq 4a^2\varepsilon^2(x_2 - x_2^0)^2$$

and

$$h = a(y_2^2 - (x_2 - x_2^0)^2) \leqq a\varepsilon^2 - a(x_2 - x_2^0)^2.$$

It then follows that

$$(3.2.12) \qquad h + \frac{g^2}{4a\varepsilon^2} \leqq a\varepsilon^2.$$

If there were z in Z such that $f(z) \notin R_c$, then we should have

$$(3.2.13) \qquad \begin{cases} x_1 + g = 0 \\ y_1 + h = t \end{cases}$$

with $t \geqq c$. In view of (3.2.5), we should then conclude

$$(3.2.14)\qquad |g| + h \geqq c.$$

Combining (3.2.12) and (3.2.14), we should obtain

$$|g| - \frac{g^2}{4a\varepsilon^2} \geqq c - a\varepsilon^2,$$

and hence

$$(3.2.15)\qquad g^2-4a\varepsilon^2|g|+4ac\varepsilon^2-4a^2\varepsilon^4 = (|g|-2a\varepsilon^2)^2+4a\varepsilon^2(c-2a\varepsilon^2) \leqq 0.$$

If we choose $c = ab^2$, $f(z^0) \notin R_c$. On the other hand, we have

$$c-2a\varepsilon^2 = ab^2-2a(\tfrac{b}{2})^2 = \frac{ab^2}{2} > 0 .$$

This implies that (3.2.15) is not satisfied. Hence, for this choice of c, $f(Z\cap\{|y'| < \varepsilon\})$ is contained in R_c. Note that we need not have fixed a so far. Next let us consider the case where $|y'| \geqq \varepsilon$ $(= b/2)$. Since Z is compact, there exists a constant κ such that

$$Z \subset \{z \in \mathbb{C}^n; |z| \leqq \kappa\}.$$

In this case it follows from (3.2.5) and (3.2.7) that

$$(3.2.16)\qquad |\frac{z_1}{a} - \sqrt{-1}t| \geqq \frac{\delta_\varepsilon}{\sqrt{2}a}$$

holds for every positive t. Hence if we choose a (>0) so that

$$-\frac{\delta_\varepsilon}{\sqrt{2}a} + (\kappa+|x_2^0|)^2 < b^2$$

and

$$\frac{\delta_\varepsilon}{\sqrt{2}a} - (\kappa+|x_2^0|)^2 > 0$$

hold, then $f(z)/a$ is contained in R_{b^2}. Then it is obvious that for sufficiently small $a > 0$, $f(z)/a$ is contained in R_{b^2}. Hence $f(z)$ is contained in R_c. Thus we have verified that $\hat{Z}$ satisfies the condition (3.2.2).

Next we show that $\hat{Z}$ satisfies the condition (3.2.3). The condition (3.2.3) is equivalent to the following:

For any $\varepsilon > 0$, there exists δ_ε such that

(3.2.17) $Z \subset \{z \in X;\ y_1 \leqq \varepsilon|x_1|\} \cup \{z \in X;\ |z_1| \geqq \delta_\varepsilon\}$.

Then it is easy to see that $\hat{Z}$ satisfies the condition (3.2.17), and hence (3.2.3). It is also clear that $\hat{Z}$ satisfies conditions (3.2.4) and (3.2.5), if Z satisfies them. Therefore we may replace $\mathfrak{Z}$ with $\mathfrak{Z}_0 \underset{\text{def}}{=} \{Z \in \mathfrak{Z};\ \hat{Z} = Z\}$ when we consider the inductive limit in (3.2.1). Next we shall show that, for each $\varepsilon > 0$ and each Z_1 in $\mathfrak{Z}_0$, we can find a compact and holomorphically convex set Z_2 such that

(3.2.18) $0 \notin Z_2$

and

(3.2.19) $Z_1 - Z_2 \subset \{z \in \mathbb{C}^n;\ |z| < 4\varepsilon\} \underset{\text{def}}{=} U_\varepsilon$.

If this is the case, we have

(3.2.20) $$\varinjlim_{Z_1, U} H^j_{Z_1 \cap U}(U;\ \mathcal{O}_{\mathbb{C}^n})$$

$$= \varinjlim_{Z_1, U} H^j_{Z_1 \cap U \cap (Z_2^c)}(U \cap Z_2^c;\ \mathcal{O}_{\mathbb{C}^n}).$$

$$= \varinjlim_{Z_1, U} H^j_{Z_1 - Z_2}(U \cap Z_2^c;\ \mathcal{O}_{\mathbb{C}^n}).$$

Here Z_2^c denotes $\mathbb{C}^n - Z_2$. Then it follows from Proposition 3.2.3 and the excision theorem that

$$H^j_{Z_1 - Z_2}(U \cap Z_2^c;\ \mathcal{O}_{\mathbb{C}^n}) = H^j_{Z_1 - Z_2}(Z_2^c;\ \mathcal{O}_{\mathbb{C}^n}) = 0$$

holds for $j \neq n$.

Now we embark on the construction of required Z_2. Let K denote the set $\{w = u + \sqrt{-1}v \in \mathbb{C};\ v \leq |u| - \beta\}$ and let $\varphi(z)$ denote $z_1 - \sqrt{-1}\alpha(z_2^2 + \cdots + z_n^2)$. Here α and β are sufficiently small positive constants which we will specify later under the constraint

(3.2.21) $\beta < 2\alpha\varepsilon^2$.

We now define Z_2 by $\varphi^{-1}(K)\cap Z_1$. Since Z_1 is compact and since we may assume that Z_1 is contained in $\{z = x+\sqrt{-1}y \in \mathbb{C}^n;\ 2y_1 \leqq |x_1|\}$, $Z_1\cap\{|z_1| \geqq \varepsilon\} - U_\varepsilon$ is contained in Z_2 if we choose α and β sufficiently small. In view of (3.2.7) we can also conclude that $Z_1\cap\{|y'|\geqq\varepsilon\}-U_\varepsilon$ is contained in Z_2. Hence it suffices to show that $Z_1' \underset{\mathrm{def}}{=} Z_1\cap\{|x'|\geqq 3\varepsilon,\ |y'|\leqq\varepsilon,\ |z_1|\leqq\varepsilon\} - U_\varepsilon$ is contained in Z_2. In order to see this, we have only to verify that $\mathrm{Im}\,\varphi(z) \leqq |\mathrm{Re}\,\varphi(z)|-\beta$ holds on Z_1', namely,

$$y_1+\alpha(|y'|^2-|x'|^2) \leqq |x_1+2\alpha\langle x',y'\rangle|-\beta \tag{3.2.22}$$

holds on Z_2'. Since $y_1 \leqq |x_2|/2$ holds on Z_1, (3.2.22) is satisfied if

$$\alpha(|y'|^2-|x'|^2+2|\langle x',y'\rangle|) \leqq -\beta \tag{3.2.23}$$

holds. Since $|x'| \geqq 3\varepsilon$ and $|y'| \leqq \varepsilon$ hold on Z_1, (3.2.23) follows from the constraint (3.2.21). Q.E.D.

Corollary 3.2.4. The sheaf $\widetilde{\mathcal{B}}_\Lambda$ is flabby.

Proof. Since the flabby dimension of $\mathcal{M}^{\mathbb{R}}$ is equal to or at most n, this immediately follows from the theorem.

By a reasoning similar to the proof of Theorem 3.2.2, we can prove

Theorem 3.2.5. There exists an injective $\mathcal{E}_X^\infty$-homomorphism from $\mathcal{C}_M|_\Lambda$ to $\widetilde{\mathcal{B}}_\Lambda$. More precisely, if we choose an $\mathcal{E}_X$-linear injective homomorphism φ from $\mathcal{M}$ into $\mathcal{C}_M$, then there exists ψ such that the composition $\psi\circ\varphi\colon \mathcal{M}\to\widetilde{\mathcal{B}}_\Lambda$ coincides with the composition of the canonical homomorphism from $\mathcal{M}$ to $\mathcal{M}^{\mathbb{R}}$ and that from $\mathcal{M}^{\mathbb{R}}$ to $\widetilde{\mathcal{B}}_\Lambda$.

Probably the same argument will work for the proof of the following conjecture. However, so far, we have been unable to prove the conjecture completely. (The point which we cannot prove at the moment is the vanishing of $\mathcal{H}^j_{T^*_{\Lambda^{\mathbb{R}}}\Lambda}(\pi^{-1}\mathcal{M}^{\mathbb{R}})_p$ $(j \neq n)$ at p where $\omega = \sum\xi_j dx_j$ vanishes.)

Conjecture 3.2.6. (i) $\mathcal{H}^{j}_{T^*_{\Lambda}\mathbb{R}^{\Lambda}}(\pi^{-1}\mathcal{M}^{\mathbb{R}}) = 0 \quad (j\neq n)$

$$\text{(ii)}\quad 0 \longrightarrow \mathcal{M}^{\mathbb{R}} \longrightarrow \widetilde{\mathcal{B}}_{\Lambda} \xrightarrow{\widetilde{\mathrm{sp}}} \pi_* \widetilde{\mathcal{C}}_{\Lambda} \longrightarrow 0$$

is exact.

In what follows, for a section f of the sheaf $\widetilde{\mathcal{B}}_{\Lambda}$, we denote supp $\widetilde{\mathrm{sp}}(f)$ by $\widetilde{\mathrm{SS}}_{\Lambda}f$.

§4. Asymptotic expansion of solutions of holonomic systems with regular singularities.

In this section we clarify the meaning of the asymptotic expansion of a microfunction solution of a holonomic system with R.S.(*) by the aid of sheaves $\tilde{\mathcal{D}}^{\infty}_{\Lambda}$ and $\tilde{\mathcal{B}}_{\Lambda}$. Our procedure is as follows:

First, for a (microfunction) solution of a holonomic system with R.S., we find a micro-differential equation of a special type which it satisfies. (Theorem 4.1.1 and Proposition 4.1.6 in §4.1.) Next we employ $\tilde{\mathcal{D}}^{\infty}_{\Lambda}$ to bring each equation of that type to the simple and canonical equation whose $\tilde{\mathcal{B}}_{\Lambda}$-solution is easy to describe. Using this result we clarify the meaning of the asymptotic expansion of a microfunction solution of the equation in question with respect to a Lagrangian manifold Λ, namely, we grasp the notion of the asymptotic expansion of a microfunction by regarding it as a section of the sheaf $\tilde{\mathcal{B}}_{\Lambda}$.

§4.1. Holonomic systems with R.S. and indicial polynomials.

The purpose of this subsection is to prove Theorem 4.1.1 stated below. The structure of the equation $(b(\vartheta)-P)u = 0$ appearing there will be investigated in the next sybsection by the aid of the sheaf $\tilde{\mathcal{D}}^{\infty}_{\Lambda}$. Before stating the theorem, we prepare some notations.

Let Λ be a Lagrangian submanifold of T^*X. Denote by I_{Λ} the defining Ideal of Λ and by $\mathcal{J}_{\Lambda}$ the Ideal $\{P \in \mathcal{E}_X(1); \sigma_1(P)|_{\Lambda} = 0\}$. Let $\mathcal{E}_{\Lambda}$ denote the sub-Ring of $\mathcal{E}_X$ generated by $\mathcal{J}_{\Lambda}$ and define $\mathcal{E}_{\Lambda}(m)$ by $\mathcal{E}_{\Lambda}\mathcal{E}_X(m)$ $(= \mathcal{E}_X(m)\mathcal{E}_{\Lambda})$. As we noted in §1, $\mathcal{E}_{\Lambda}/\mathcal{E}_{\Lambda}(-1)$ is locally isomorphic to $\mathcal{A}_{\Lambda}$, which is a sub-Ring of the sheaf of linear differential operators on Λ.

Now let us consider a micro-differential operator $\vartheta = \vartheta_1 + \vartheta_0 + \cdots$ which satisfies the conditions below. Here ϑ_1 and ϑ_0 are respectively the first order and the zeroth order term of ϑ.

(4.1.1) $\quad \vartheta \in \mathcal{J}_{\Lambda}$

(4.1.2) $\quad d\,\vartheta_1 \equiv \omega_X \ (= \sum_j \xi_j dx_j) \mod I_{\Lambda}\Omega^1$

(*) R.S. is the abbreviation of "regular singularities". See Kashiwara-Kawai [8], [9] for the definition.

(4.1.3) $\vartheta_0 - \frac{1}{2}\sum_j \frac{\partial^2 \vartheta_1}{\partial x_j \partial \xi_j} = 0$ on Λ.

Here Ω^1 denotes the sheaf of holomorphic 1-forms on T^*X. It is then easy to verify that such ϑ always exists and it is unique modulo $\mathcal{J}_\Lambda^2 \mathcal{E}_X(-1)$. For example, if $\Lambda = T_Y^*\mathbb{C}^n$ with $Y = \{x \in \mathbb{C}^n; x_1 = \dots = x_d = 0\}$, then we may choose $\frac{1}{2}(\sum_{j=1}^{d}(x_jD_j + D_jx_j))$ as ϑ. In what follows, we always use the letter ϑ to denote such an operator. Now we have the following

Theorem 4.1.1. Let u be a section of a holonomic $\mathcal{E}_X$-Module with R.S. and let Λ be an arbitrary Lagrangian submanifold of T^*X. Then there exist a non-zero polynomial $b(s)$ and a micro-differential operator $P \in \mathcal{E}_\Lambda(-1)$ which satisfy the following conditions:

(4.1.4) $\deg b \geq \text{order } P$

(4.1.5) $(b(\vartheta) - P)u = 0$.

In order to prove Theorem 4.1.1, we prepare several lemmas. In what follows $\mathcal{E}(m)$ denotes the sheaf of micro-differential operators of order at most m. For an $\mathcal{E}(0)$-Module $\mathcal{L}$, $\mathcal{L}(m)$ denotes $\mathcal{E}(m)\mathcal{L}$. The symbols $\mathring{T}^*X$ and $\mathring{\pi}$ denote respectively $T^*X - T_X^*X$ and the projection from $\mathring{T}^*X$ onto X.

Lemma 4.1.2. Let X be $\mathbb{C}^{1+n} = \{(t,x); t \in \mathbb{C},\ x \in \mathbb{C}^n\}$, $V = \{(t,x;\tau,\xi) \in T^*X;\ t=0\}$ and $\Lambda = \{(t,x;\tau,\xi) \in T^*X;\ t=\xi=0,\ \tau \neq 0\}$. Let Θ denote tD_t. Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-Module with R.S. which is defined on a neighborhood of a point p of Λ, and let $\mathcal{L}$ be a coherent $\mathcal{E}(0)$-sub-Module of $\mathcal{M}$. If $\operatorname{Supp}\mathcal{M}$ is contained in V, then there exists a non-zero polynomial $b(s)$ such that $b(\Theta)\mathcal{L} \subset \mathcal{L}(-1)$.

Proof. First we note that it suffices to show the existence of a coherent $\mathcal{E}(0)$-sub-Module $\mathcal{L}'$ of $\mathcal{M}$ and a non-zero polynomial $b(\Theta)$ such that $\mathcal{E}\mathcal{L}' = \mathcal{M}$, $b(\Theta)\mathcal{L}' \subset \mathcal{L}'(-1)$ and $\Theta\mathcal{L}' \subset \mathcal{L}'$ hold. In fact, if it is the case, then there exists an integer r such that $\mathcal{L} \cap \mathcal{L}'(-r) \subset \mathcal{L}(-1)$ holds. On the other hand,

$$b(\Theta - j)\mathcal{L}'(-j) = b(\Theta - j)D_t^{-j}\mathcal{L}' = D_t^{-j}b(\Theta)\mathcal{L}' \subset \mathcal{L}'(-j-1)$$

holds. Hence, by setting $b(\Theta - r + 1)\cdots b(\Theta)$ to be $\tilde{b}(\Theta)$, we obtain

$\tilde{b}(\Theta)\mathcal{L}' \subset \mathcal{L}'(-r)$. Therefore $\tilde{b}(\Theta)\mathcal{L} \subset \mathcal{L}'(-r) \cap \mathcal{L} \subset \mathcal{L}(-1)$ holds.

Now we shall prove the existence of such $\mathcal{L}'$ and b. Since $\mathcal{M}$ is with R.S. and since Supp $\mathcal{M}$ is contained in V, $\mathcal{M}$ has regular singularities along $V \cap \mathring{T}^*X$. ([8] Chap.V, §1, Corollary 5.1.7.) Let M be the monodromy of $\mathcal{M}$ in the sense of [11]. Then, by the definition, M is an endomorphism of $\mathcal{M}$. Since End($\mathcal{M}$) is finite-dimensional, $\mathcal{M}$ is a direct sum of sub-Modules $\mathcal{M}_j$ such that $(M-\lambda_j)^{N_j}|_{\mathcal{M}_j} = 0$ for some $\lambda_j \in \mathbb{C}^\times$ and some integer N_j. Hence we may assume from the first that there exist $\lambda \in \mathbb{C}$ and $N \in \mathbb{Z}_+$ such that $(M-\exp(2\pi\sqrt{-1}\lambda))^N\mathcal{M} = 0$ holds. Since $\mathcal{M}$ has regular singularities along V, there exists a coherent $\mathcal{E}(0)$-sub-Module $\mathcal{L}$ such that $\mathcal{M} = \mathcal{E}\mathcal{L}$ and that $\Theta\mathcal{L} \subset \mathcal{L}$. Let $u_1, \ldots, u_m$ be a system of generators of $\mathcal{L}$ as an $\mathcal{E}(0)$-Module. Then there exist $P_{ij} \in \mathcal{E}(0)$ $(1 \leq i,j \leq m)$ such that $\Theta u_i = \Sigma P_{ij}u_j$. If we denote by u the column vector ${}^t(u_1, \ldots, u_m)$ and by P the matrix (P_{ij}), we have $\Theta u = Pu$. We know ([11]) that there exists $U \in GL(m; \mathcal{E}(0))$ and a matrix A(x) of functions in x such that $\Theta - P = U(\Theta - A(x))U^{-1}$. Hence, by replacing u with $U^{-1}u$, we may assume that $\Theta u = A(x)u$ holds. Then it follows from the definition of M that $Mu = \exp(2\pi\sqrt{-1}A(x))u$ holds. Since $(M-\exp(2\pi\sqrt{-1}\lambda))^r u = 0$ holds, we have $(\exp(2\pi\sqrt{-1}A(x)) - \exp(2\pi\sqrt{-1}\lambda))^r u = 0$. Considering the problem on a neighborhood of x_0, we can find an integer c so that all the eigenvalues of A(x) are contained in $\{t \in \mathbb{C};\ |t-\lambda| < c\}$ for any x in the neighborhood. Let us now define $b_0(s)$ and G(x) by $\prod_{\nu=-c}^{c}(s-\nu)$ and $\frac{1}{2\pi\sqrt{-1}}\int_{|t-\lambda|=c+\frac{1}{2}} \frac{b_0(t)(t-A(x))^{-1}}{(\exp(2\pi\sqrt{-1}t)-\exp(2\pi\sqrt{-1}\lambda)}dt$.

Then $G(x)(\exp(2\pi\sqrt{-1}A(x))-\exp(2\pi\sqrt{-1}\lambda)) = b_0(A(x))$ holds, and, moreover, G(x) and A(x) are commutative. Hence we obtain

$$b_0(\Theta)^r u = b_0(A(x))^r u = G(x)^r(\exp(2\pi\sqrt{-1}A(x))-\exp(2\pi\sqrt{-1}\lambda))^r u = 0.$$

This entails $b_0(\Theta)^r\mathcal{L} \subset \mathcal{L}(-1)$. Q.E.D.

Lemma 4.1.3. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-sub-Module of $\mathcal{M}$ such that $\mathcal{D}\mathcal{F} = \mathcal{M}$ holds. Let $\mathcal{L}$ be a coherent $\mathcal{E}(0)$-sub-Module of $\mathcal{E} \underset{\mathcal{D}}{\otimes} \mathcal{M}$ which is generated by $\mathcal{F}$. Then we have the following:

(i) $\mathring{\pi}_*(\mathcal{L}(k)/\mathcal{L}(k-1)) = \mathcal{D}_k\mathcal{F}/\mathcal{D}_{k-1}\mathcal{F}$ holds for $k >> 0$. Here and in what follows $\mathcal{D}_k$ denotes the sheaf of linear differential operators of order at most k.

(ii) $\mathcal{D}_k\mathcal{F} = \{u \in \mathcal{M};\ 1 \otimes u$ is contained in $\mathcal{L}(k)$ on $\mathring{T}^*X\}$ holds for $k >> 0$.

(iii) If a section u of $\mathcal{M}$ belongs to $\mathcal{L}(k)$ at a point p of $\mathring{T}^*X$, then there exists a linear differential operator P of order m such that $Pu \in \mathcal{D}_{m+k-1}$ and $\sigma_m(P)(p) \neq 0$.

<u>Proof</u>. There exist integers r_1, r_2, N_0, N_1, N_2 $(r_1 \leqq r_2)$ and an exact sequence

$$0 \leftarrow \mathcal{M} \overset{\varphi}{\leftarrow} \mathcal{D}^{N_0} \overset{P_0}{\leftarrow} \mathcal{D}^{N_1} \overset{P_1}{\leftarrow} \mathcal{D}^{N_2}$$

such that $\varphi(\mathcal{O}^{N_0}) = \mathcal{F}$, $P_0(\mathcal{D}_k^{N_1}) \subset (\mathcal{D}_{k-r_1})^{N_0}$, $P_1(\mathcal{D}_{k-r_2}^{N_2}) \subset \mathcal{D}_{k-r_1}^{N_1}$ for $k \geqq 0$ and that

$$(4.1.6) \quad 0 \longleftarrow \frac{\mathcal{D}_k \mathcal{F}}{\mathcal{D}_{k-1}\mathcal{F}} \longleftarrow \left(\frac{\mathcal{D}_k}{\mathcal{D}_{k-1}}\right)^{N_0} \longleftarrow \left(\frac{\mathcal{D}_{k-r_1}}{\mathcal{D}_{k-r_1-1}}\right)^{N_1} \longleftarrow \left(\frac{\mathcal{D}_{k-r_2}}{\mathcal{D}_{k-r_2-1}}\right)^{N_2}$$

is exact for $k \geqq r_2$. Then the following sequence

$$\mathcal{E}(0)^{N_0} \overset{P_0}{\longleftarrow} \mathcal{E}(-r_1)^{N_1} \overset{P_1}{\longleftarrow} \mathcal{E}(-r_2)^{N_2}$$

is also exact. Since $\mathcal{M}$ is the cokernel of P_0, we obtain the exact sequence

$$0 \leftarrow \mathcal{L}(k) \leftarrow \mathcal{E}(k)^{N_0} \leftarrow \mathcal{E}(k-r_1)^{N_0}$$

for each k. Hence its symbol sequence

$$0 \leftarrow \mathcal{L}(k)/\mathcal{L}(k-1) \leftarrow (\mathcal{E}(k)/\mathcal{E}(k-1))^{N_0} \leftarrow (\mathcal{E}(k-r_1)/\mathcal{E}(k-r_1-1))^{N_1}$$

is also exact. Here we recall that $\mathcal{E}(k)/\mathcal{E}(k-1) \cong \mathcal{O}(k)$, the sheaf of holomorphic functions on $\mathring{T}^*X$ which are homogeneous of degree k with respect to the fiber coordinate. Then the following sequence (4.1.7) is exact for $k \gg 0$ by Serre's theorem on the vanishing of cohomology groups of coherent sheaves on a projective space.

$$(4.1.7) \quad 0 \leftarrow \mathring{\pi}_*(\mathcal{L}(k)/\mathcal{L}(k-1)) \leftarrow \mathring{\pi}_*(\mathcal{O}(k)^{N_0}) \leftarrow \mathring{\pi}_*(\mathcal{O}(k-r_1)^{N_1}).$$

Since $\mathring{\pi}_*(\mathcal{O}(k)) \cong \mathcal{D}_k/\mathcal{D}_{k-1}$ holds for $k \geqq 0$, by comparing (4.1.6) and (4.1.7), we obtain

$$(4.1.8) \quad \mathring{\pi}_*(\mathcal{L}(k)/\mathcal{L}(k-1)) = \mathcal{D}_k \mathcal{F}/\mathcal{D}_{k-1}\mathcal{F}$$

for $k \geqq k_0$ with a sufficiently large k_0. This proves (i). Next we shall prove (ii) for $k \geqq k_0$. Let u be a section of $\mathcal{M}$ such that $1 \otimes u$ belongs to $\mathcal{L}(k)|_{\mathring{T}^*X}$ for some $k \geqq k_0$. Then there exists j such that $\mathcal{D}_j \mathcal{F}$ contains u. If j is larger than k, $1 \otimes u$ is zero in $\mathring{\pi}_*(\mathcal{L}(j)/\mathcal{L}(j-1))$. Hence (4.1.8) implies that $\mathcal{D}_{j-1}\mathcal{F}$ contains u. Thus the induction on j porves that $\mathcal{D}_k \mathcal{F}$ contains u. This proves (ii).

Finally let us prove (iii). Let S denote the set of points where $1 \otimes u$ does not belong to $\mathcal{L}(k)$. Then there exists a homogeneous polynomial a on T*X such that $a(p) \neq 0$ and $a|_S = 0$. Let r be the homogeneous degree of a and let P be a linear differential operator of order r whose principal symbol is a. Then it follows from Hilbert's Nullstellensatz that $P^\nu u$ belongs to $\mathcal{L}(k+r\nu-1)$ for $\nu >> 0$. Hence (ii) implies $P^\nu u \in \mathcal{D}_{k+r\nu-1}\mathcal{F}$. This proves (iii). Q.E.D.

Lemma 4.1.4. Let X, V and Λ be the same as in Lemma 4.1.2 and let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module with R.S. Assume that $\mathrm{Supp}(\mathcal{E} \underset{\mathcal{D}}{\otimes} \mathcal{M})$ is contained in V on a neighborhood of Λ. Let u be a section of $\mathcal{M}$. Then there exist a non-zero polynomial b(s) and a linear differential operator P which satisfy the following:

(4.1.9) $b(tD_t)u = tPu$,

where $\mathrm{order}\, P \leqq \deg b$ and $P \in \mathcal{E}_\Lambda$.

Proof. We may assume without loss of generality that $\mathcal{M} = \mathcal{D}u$. Let $\tilde{u}$ be the section $1 \otimes u$ of $\mathcal{E} \otimes \mathcal{M}|_{\mathring{T}^*X}$. Then Lemma 4.1.2 guarantees that there exists a non-zero polynomial b(s) such that $b(\Theta)\tilde{u} \in \mathcal{E}(-1)\tilde{u}$ holds on a neighborhood of $p \in \Lambda$, where $\Theta = tD_t$. Hence it follows from Lemma 4.1.3 that there exists a linear differential operator P of order m such that

(4.1.10) $\sigma_m(P)(p) \neq 0$

(4.1.11) $Pb(\Theta)u = Qu$ with $Q \in \mathcal{D}_{m-1}$.

By the condition (4.1.10) we may assume that P has the form

$$D_t^m - \sum_{j=0}^{m-1} A_j(t,x,D_x)D_t^j,$$

where A_j is of order at most m-j. Let Q be written in the form

$$\sum_{j=0}^{m-1} Q_j(t,x,D_x)D_t^j,$$

where Q_j is of order at most $m-1-j$. Then, setting $\tilde{b}(s) = b(s)s(s+1)\cdots(s+m-1)$, we obtain from (4.1.11)

$$\tilde{b}(\Theta)u = t^m D_t^m b(\Theta)u = t(\sum_{j=0}^{m-1}(A_j b(\Theta-m+1)+Q_j)t^{m-1}D_t^j)u.$$

This proves the required result. Q.E.D.

Lemma 4.1.5. Let X and Λ be the same as in Lemma 4.1.2. Let $\mathcal{M}$ be an arbitrary holonomic $\mathcal{D}_X$-Module with R.S. and u a section of $\mathcal{M}$. Then there exists a non-zero polynomial $b(s)$ and a linear differential operator P which satisfy the following:

(4.1.12) $b(tD_t)u = tPu$,

where $\operatorname{order} P \leqq \deg b$ and $P \in \mathcal{E}_\Lambda$.

Proof. For each positive integer N we denote by f_N the map from X to X defined by $f_N(t,x) = (t^N,x)$. Then the associated map f_{N*} is an isomorphism on $T^*X - \{t=0\}$, because $f_{N*}(t,x;\tau,\xi) = (t^N,x;\tau/Nt^{N-1},\xi)$. Let W be $\operatorname{Supp}(\mathcal{E}\otimes\mathcal{M})$. Then W is a Lagrangian subvariety of T^*X, and hence $C_\Lambda(W) \subset \{t=0\} \subset T_\Lambda(T^*X)$. ([12], Chap.X.) Therefore there exist an integer N and a constant C such that

$$t^{N-1}\tau^N \leqq C(|t\tau|+|\xi|)^N,$$

or equivalently,

$$(t\tau)^N \leqq Ct(|t\tau|+|\xi|)^N.$$

Therefore we have

(4.1.13) $(t\tau/N)^N \leqq Ct^N(|t\tau/N|+|\xi|)^N$

on $W' \underset{\text{def}}{=} (f_{N*})^{-1}(W \cap \{t\neq 0\})$.

The inequality (4.1.13) implies that

$$\overline{W'} \cap \Lambda = \phi.$$

In the sequel, we fix such an N and set $f = f_N$. Let $\mathcal{M}'$ be $f^*(\mathcal{M})$

and let $\tilde{u}$ be the section $1\otimes u$ of $\mathcal{M}'$. Then $\mathcal{M}'$ is also with R.S. Further $\mathrm{Supp}(\mathcal{E}\otimes\mathcal{M}')$ is contained in $V\cap W'$. Hence $\mathcal{M}'$ satisfies all the conditions in the preceding lemma. Therefore there exist a polynomial $b(s)$ and $P\in\mathcal{D}\cap\mathcal{E}_\Lambda$ such that $b(tD_t)\tilde{u} = tP(t,x,tD_t,D_x)\tilde{u}$ holds with $\deg b \geq \mathrm{order}\, P$. Hence, if we denote by ε the primitive N-th root of 1, we obtain

$$(4.1.14)\quad b(tD_t)\tilde{u} = \frac{1}{N}\sum_{i=0}^{N-1}\varepsilon^i tP(\varepsilon t,x,tD_t,D_x)\tilde{u}.$$

It is easy to see that $\frac{1}{N}\sum_{i=0}^{N-1}\varepsilon^i tP(\varepsilon t,x,tD_t,D_x)$ can be written in the form $t^N Q(t^N,x,tD_t,D_x)$, because it is invariant under the transformation $t\mapsto \varepsilon t$. Therefore (4.1.14) implies that

$$b(NtD_t)u = tQ(t,x,NtD_t,D_x)u$$

holds outside $\{t=0\}$. Here we note that $\mathrm{order}\, Q = \mathrm{order}\, P \leq \deg b$. Hence it follows from Hilbert's Nullstellensatz that there exists r such that

$$(4.1.15)\quad t^r b(N\Theta)u = t^{r+1}Q(t,x,N\Theta,D_x)u.$$

By multiplying the both sides of (4.1.15) by D_t^r, we finally obtain

$$\tilde{b}(\Theta)u = t(\Theta+2)\cdots(\Theta+r+1)Q(t,x,N\Theta,D_x)u$$

with $\tilde{b}(s) = (s+1)\cdots(s+r)b(Ns)$. This completes the proof. Q.E.D.

Now let us prove Theorem 4.1.1. By a suitable contact transformation, we can bring $\mathrm{Supp}\,\mathcal{M}$ to a generic position and Λ to $T^*_Y X$ for a hypersurface Y of X. ([8] Chap. I, §6, Corollary 1.6.4.) Then we may assume that $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-Module with R.S., because $\mathrm{Supp}\,\mathcal{M}$ is in a generic position. ([8] Chap. V, §1, Theorem 5.1.4.) Therefore Theorem 4.1.1 follows from Lemma 4.1.5. Q.E.D.

The following proposition easily follows from Theorem 4.1.1.

Proposition 4.1.6. Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-Module with R.S. and let Λ be an arbitrary Lagrangian submanifold of T^*X. Then there exist a system of generators $u_1,\dots,u_m$ of $\mathcal{M}$, a non-zero polynomial $b(s)$ and micro-differential operators $P_{ij}\in\mathcal{E}_\Lambda(-1)$ $(1\leq i,j\leq m)$ which satisfy the following three conditions:

(4.1.16) $b(\vartheta)u_i = \sum_j P_{ij}u_j$

(4.1.17) $\deg b \geqq \text{order } P_{ij}$

(4.1.18) The difference of any two different roots of $b(s) = 0$ is not an integer.

Proof. It follows from Theorem 4.1.1 that there exist a coherent $\mathcal{E}(0)$-sub-Module $\mathcal{L}$ of $\mathcal{M}$ and a non-zero polynomial $b(s)$ of degree, say, r which satisfy

(4.1.19) $b(\vartheta)\mathcal{L} \subset \mathcal{J}_{\Lambda}^{r+1}\mathcal{L}(-1)$

and

(4.1.20) $\mathcal{M} = \mathcal{E}\mathcal{L}$.

We now show that we can choose $\mathcal{L}$ and b so that they satisfy the additional condition (4.1.18). For this purpose, we prove the following:

If $b(s)$ has the form $\tilde{b}(s)(s-\lambda)$ and satisfies (4.1.19) and (4.1.20), then there exists a coherent $\mathcal{E}(0)$-sub-Module $\mathcal{L}'$ which satisfies

(4.1.21) $(\vartheta-\lambda)\tilde{b}(\vartheta-1)\mathcal{L}' \subset \mathcal{J}_{\Lambda}^{r+1}\mathcal{L}'(-1)$.

We will prove this fact by showing that

$$\mathcal{L}' = \mathcal{J}_{\Lambda}\mathcal{L} + (\vartheta-\lambda-1)(\mathcal{L}(1))$$

is a required one. This $\mathcal{L}'$ is clearly coherent $\mathcal{E}(0)$-Module. Furthermore we see by an easy computation that $b(\vartheta-1)(\mathcal{L}(1))$ is contained in $\mathcal{J}_{\Lambda}^{r+1}\mathcal{L}$. Hence we have

$$(\vartheta-\lambda)\mathcal{L} \subset \mathcal{L}'(-1).$$

Therefore we obtain

$$\tilde{b}(\vartheta-1)\mathcal{L}' \subset \tilde{b}(\vartheta-1)\mathcal{J}_{\Lambda}\mathcal{L} + b(\vartheta-1)(\mathcal{L}(1)) \subset \mathcal{J}_{\Lambda}^{r+1}\mathcal{L}.$$

This implies

$$(\vartheta-\lambda)\tilde{b}(\vartheta-1)\mathcal{L}'\subset(\vartheta-\lambda)\mathcal{J}_{\Lambda}^{r+1}\mathcal{L}\subset\mathcal{J}_{\Lambda}^{r+2}(-1)\mathcal{L}+\mathcal{J}_{\Lambda}^{r+1}(\vartheta-\lambda)\mathcal{L}\subset\mathcal{J}_{\Lambda}^{r+1}\mathcal{L}'(-1).$$

Thus we obtain the required result. Q.E.D.

§4.2. Transforming equations to the canonical one.

The purpose of this section is to find the canonical form of the micro-differential equations of the form (4.1.5) by using $\tilde{\mathscr{D}}_{\Lambda}^{\infty}$. Needless to say, $\tilde{\mathscr{D}}_{\Lambda}^{\infty}$ does not act as a sheaf homomorphism on the sheaf of microfunctions, and hence the nature of the reduction to the canonical form discussed in this subsection is different from that of the classical one (e.g. S-K-K [17] Chap. II, §5). Actually the reduction discussed here changes the characteristic variety of the equation in question. Still $\tilde{\mathscr{D}}_{\Lambda}^{\infty}$ acts on $\tilde{\mathscr{B}}_{\Lambda}$ as a sheaf homomorphism. Therefore the structure of $\tilde{\mathscr{B}}_{\Lambda}$-solutions can be investigated by our reduction and it enables us to clarify the meaning of the asymptotic expansion of a microfunction solution. Before stating our main theorem, we re-examine the example (1.6) discussed in §1 from the view point of the transformation of the equations.

We first consider the following system of equations which admits $(y^2-x^3)_+^s$ as its solution:

$$(4.2.1)\qquad \begin{cases} (\frac{1}{3}xD_x+\frac{1}{2}yD_y-s)u = 0 \\ (2yD_x+3x^2D_y)u = 0 \end{cases}$$

Choosing xD_x+yD_y+1 as ϑ, we obtain the following equation from (4.2.1).

$$(4.2.2)\qquad (\vartheta-3s-3/2)(\vartheta-2s-1)u = x^3D_y^2u/4.$$

For simplicity we assume that $s+1/2$ is not an integer. Then u admits an asymptotic expansion of the following form:

$$(4.2.3)\qquad u \sim \sum_{j=0}^{\infty}u_j+\sum_{j=0}^{\infty}v_j,$$

where u_j and v_j are homogeneous of degree $2s+j$ and $3s+1/2+j$, respectively. By (1.6) in §1, we find

$$(4.2.4)\qquad u_j = \frac{\Gamma(-s+1/2)}{j!\Gamma(-s+1/2+j)}\times(\tfrac{1}{4}x^3D_y^2)^j u_0$$

(4.2.5) $v_j = \frac{\Gamma(s+3/2)}{j!\,\Gamma(s+3/2+j)}\times(\frac{1}{4}x^3D_y^2)^j v_0,$

namely,

(4.2.6) $u \sim (x^{3/2}D_y/2)^{s+1/2}\Gamma(-s+1/2)I_{-s-1/2}(x^{3/2}D_y)u_0$

$+(x^{3/2}D_y/2)^{-s-1/2}\Gamma(s+3/2)I_{s+1/2}(x^{3/2}D_y)v_0,$

where $I_\nu(z)$ is the ν-th modified Bessel function. Further, if we define v by $(\vartheta-5s/2-5/4)u$, then we obtain

(4.2.7) $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

with $A = (z/2)^{s+1/2}\Gamma(-s+1/2)I_{-s-1/2}(z),$

$B = (z/2)^{-s-1/2}\Gamma(s+3/2)I_{s+1/2}(z),$

$C = (z/2)^{s+3/2}\Gamma(-s+1/2)I'_{-s-1/2}(z)$

and $D = (z/2)^{-s+1/2}\Gamma(s+3/2)I'_{s+1/2}(z),$

where $z = x^{3/2}D_y$. Then Lommel's formula for Bessel functions entails

(4.2.8) $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

Hence this transformation reduces the equation (4.2.2) to

(4.2.9) $\begin{cases} (\vartheta-2s-1)u_0 = 0 \\ (\vartheta-3s-3/2)v_0 = 0. \end{cases}$

This example shows that, essentially speaking, the asymptotic expansion in our sense is nothing but the transformation of the equation by $\mathscr{D}_\Lambda^\infty$. Hence we want to have a general result on the transformation of euqations. The following theorem gives us a satisfactory result in this direction.

Theorem 4.2.1. Let Λ be $T^*_Y\mathbb{C}^{1+n}$, where $Y=\{(t,x)=(t,x_1,\dots,x_n)\in\mathbb{C}^{1+n};\ t=0\}$. Assume that $A=(a_{\mu\nu}(x))_{1\leq\mu,\nu\leq N}$ and $Q=(Q_{\mu\nu}(x,D_x,D_t))_{1\leq\mu,\nu\leq N}$

satisfy the follwoing conditions (4.2.10) ~ (4.2.14) for some integers m_μ $(1 \leqq \mu \leqq N)$:

(4.2.10) $a_{\mu\nu}(x)$ is holomorphic on a neighborhood of the origin.

(4.2.11) $a_{\mu\nu} = 0$ if $m_\mu - m_\nu + 1 < 0$.

(4.2.12) If we denote by $\lambda_\mu(x)$ $(1 \leqq \mu \leqq N)$ the eigenvalues of the matrix A, then $\lambda_\mu(x)$ is holomorphic on a neighborhood of zero and $\lambda_\mu(0) - \lambda_\nu(0) \notin \mathbb{Z} - \{0\}$ $(1 \leqq \mu, \nu \leqq N)$.

(4.2.13) $Q_{\mu\nu} \in \mathcal{J}_\Lambda^{m_\mu - m_\nu + 2}(-1)$.

Then on a neighborhood of $\{(t,x;\tau,\xi);\ t=x=0,\ \xi=0\}$ there exists an invertible matrix $R(x,D_x,D_t)$ each of whose entry belongs to $\tilde{\mathscr{D}}_\Lambda^\infty$ and which satisfies

(4.2.14) $(tD_t - A(x) + Q(x,D_x,D_t))R(x,D_x,D_t) = R(x,D_x,D_t)(tD_t - A(x))$.

Furthermore $R(x,D_x,D_t)$ can be expressed in the form $\sum_p R^p(x,D_x)D_t^{-p}$, where $R^p(x,D_x) = 0$ for $p < 0$, $R^0 = I$, the identity matrix and order $R^p \leqq ap$ holds for $p > 0$ with $a = 2 + \max_{\mu,\nu}(m_\mu - m_\nu)$.

Corollary 4.2.2. Let $b(s)$ be a polynomial of degree m and Λ a Lagrangian submanifold of T^*X. Assume further that the difference of any two different roots of $b(s) = 0$ is not an integer. Let P be a micro-differential operator of order at most m which belongs to $\mathcal{E}_\Lambda(-1)$. Then we have

(4.2.15) $\tilde{\mathscr{D}}_\Lambda^\infty / \tilde{\mathscr{D}}_\Lambda^\infty (b(\vartheta) - P) \cong \tilde{\mathscr{D}}_\Lambda^\infty / \tilde{\mathscr{D}}_\Lambda^\infty b(\vartheta)$.

Proof of Corollary 4.2.2. By a suitable quantized contact transformation we may assume that $\Lambda = T_Y^*\mathbb{C}^{1+n}$ with $Y = \{(t,x) \in \mathbb{C}^{1+n};\ t=0\}$ and $\vartheta = tD_t + 1/2$. Furthermore, by the division theorem for micro-differential operators (S-K-K [17] Chap. II, §2, Theorem 2.2.2), we may assume without loss of generality that P is a polynomial of degree $(m-1)$ in t, i.e., $P = \sum_{j=0}^{m-1} P_j(x,D_x,D_t)\vartheta^j$. Then rewriting the equation $(b(\vartheta) - P)u = 0$ in the matrix form, we can apply Theorem 4.2.1 to obtain the required result. Q.E.D.

Proof of Theorem 4.2.1. We try to find an invertible matrix R in the following form:

$$(4.2.16)\quad R = \sum_{p=0}^{\infty} R^p(x,D_x)D_t^{-p},$$

where $R^p(x,D_x)$ is a matrix of linear differential operators of finite order and $R^0 = I$, the identity matrix. If we can construct R in this form, then its inverse can be constructed in a similar manner. We expand Q also in this form, namely,

$$(4.2.17)\quad Q = \sum_{j\geq 1} Q^j(x,D_x)D_t^{-j},$$

where $Q^j = (Q^j_{\mu\nu})$ is a matrix of linear differential operators. Note that (4.2.13) implies

$$(4.2.18)\quad \operatorname{order} Q^j_{\mu\nu} \leqq m_\mu - m_\nu + j + 1.$$

In order to satisfy (4.2.14) it suffices to define R^p $(p \geqq 1)$ successively by the following recursion formula:

$$(4.2.19)\quad pR^p - [R^p, A] = \sum_{\substack{j+k=p\\ j\geqq 1}} Q^j R^k \quad (p \geqq 1).$$

Here we note that $\operatorname{order} R^p \leqq ap$ $(p > 0)$ holds with $a = 2 + \max_{\mu,\nu}(m_\nu - m_\mu)$ in view of (4.2.18) and (4.2.19), if R^p exists.

Let us now define an $N \times N$ matrix $A^*(y)$ and a column vector $\tilde{R}^p$ of length N^2 by $(a_{\nu\mu}(y))_{1\leqq\nu,\mu\leqq N}$ and ${}^t(\tilde{r}^p_{11}(x,y,D_x),\ldots,\tilde{r}^p_{1N}(x,y,D_x), \tilde{r}^p_{21}(x,y,D_x),\ldots,\tilde{r}^p_{N1}(x,y,D_x),\ldots,\tilde{r}^p_{NN}(x,y,D_x))$, respectively. Then, instead of solving (4.2.19), we solve the following equation.

$$(4.2.20)\quad (p-(I_N \otimes A^*(y) - A(x)\otimes I_N))\tilde{R}^p(x,y,D_x) = \sum_{\substack{j+k=p\\ j\geqq 1}} (Q^j(x,D_x)\otimes I_N)\tilde{R}^p(x,y,D_x),$$

where $A(x)\otimes I_N$ etc. denote the Kronecker product of the matrices. In fact, rewriting $\tilde{r}^p_{\mu\nu}(x,y,D_x)\delta(x-y)$ in the form $r^p_{\mu\nu}(x,D_x)\delta(x-y)$, we can find the required $R^p = (r^p_{\mu\nu})$ which satisfies (4.2.19) from the solution $\tilde{R}^p$ of (4.2.20). If we denote $I_N \otimes A^*(y) - A(x)\otimes I_N$ by $\mathfrak{A}$, then the assumption (4.2.12) guarantees that $(p-\mathfrak{A})^{-1}$ exists for $p = 1,2,\cdots$. In order to facilitate the argument, we introduce a dummy variable x_{n+1} and define an $N^2 \times N^2$ matrix C by

$$I_N \otimes \begin{pmatrix} (D_{x_{n+1}})^{m_1} & & 0 \\ & \ddots & \\ 0 & & (D_{x_{n+1}})^{m_N} \end{pmatrix}.$$

Then C is invertible on $\Omega = \{(t,x,y,x_{n+1};\tau,\xi,\eta,\xi_{n+1}) \in T^*\mathbb{C}^{1+2n+1};\ \xi_{n+1} \neq 0\}$. Hence $\mathcal{B}(p) \underset{def}{=} C^{-1}(p-\mathcal{O}\!\mathcal{l})^{-1}C$ is well-defined there. Furthermore, by using the assumption (4.2.11), we can prove that $\mathcal{B}(p)$ has the form

(4.2.21) $\quad \sum_{\ell=0}^{b} C^{\ell}(p)$

where each entry $C^{\ell}_{\mu\nu}(p)$ of $C^{\ell}(p)$ is of order at most ℓ and it satisfies

$$N_{\ell}(C^{\ell}_{\mu\nu}(p);\delta) \leqq A/p^{\ell+1}.$$

Here N_{ℓ} denotes the formal norm introduced in Boutet de Monvel and Krée [2]. See Tahara [18] Lemma 1.2.8 for the proof of this assertion. Defining $\tilde{S}^p$ by $C^{-1}\tilde{R}^p$, we can rewrite (4.2.20) into the following form:

$$S^p = \mathcal{B}(p) \sum_{\substack{j+k=p \\ j\geqq 1}} (C^{-1}(Q_j \otimes I_N)C)S^k.$$

Thus we can write down S^p in terms of $\mathcal{B}(p)$ and $C^{-1}(Q_j \otimes I_N)C$ explicitly. Then, using the assumption (4.2.13) and the decomposition (4.2.21), we can verify that the i-th homogeneous part $s^p_i(x,y,\xi,\xi_{n+1})$ of the symbol sequence of S^p satisfies the following:

(4.2.22) For each $\varepsilon > 0$ and each compact subset K of Ω, there exist constants $C_{\varepsilon,K}$ and $M_{\varepsilon,K}$ such that

$$\sup_K |s^p_i| \leqq M_{\varepsilon,K} C^p_{\varepsilon,K} \varepsilon^i p!/i!$$

holds.

Note also that

(4.2.23) $\quad s^p_i = 0$, if $i \leqq (b+2)p$.

From the estimate (4.2.22), we easily find that $R = \sum_{p=0}^{\infty} R^p(x,D_x)D_t^{-p}$

belongs to $M_N(\tilde{\mathcal{D}}^\infty_\Lambda)$. Q.E.D.

Remark 4.2.3. Let Λ be the same as in Theorem 4.2.1. Then in view of (4.2.23), the operator R (and R^{-1} also) acts on $\hat{\mathcal{C}}_{Y|X}$ (S-K-K [17] Chap. II, §3), where $X = \mathbb{C}^{1+n}$. On the other hand, it is easy to verify that

$$\mathcal{E}xt^j(\mathcal{M}_0, \mathcal{C}_{Y|X}) \cong \mathcal{E}xt^j(\mathcal{M}_0, \hat{\mathcal{C}}_{Y|X}) \qquad (j = 0,1)$$

holds for $\mathcal{M}_0 = \mathcal{E}^N_X/\mathcal{E}^N_X(tD_t - A(x))$. Hence the comparison theorem of this type also holds for $\mathcal{M} = \mathcal{E}^N_X/\mathcal{E}^N_X(tD_t - A(x) + Q(x,D_x,D_t))$.

Remark 4.2.4. Because of the form (4.2.16) of the intertwining operator R constructed above, we find that R and its inverse S actually belong to a smaller class of operators introduced by Laurent [14], namely, R and S belong to $M_N(\mathcal{E}^{2\infty}_V|_\Lambda)$, where $V = \{(t,x;\tau,\xi) \in T^*\mathbb{C}^{1+n};\ \xi=0\}$.

In order to apply Theorem 4.2.1 to the study of microfunction solution of the equation in question, we prepare the following Lemma 4.2.5. In what follows, $\Lambda^{\mathbb{R}}$ denotes the purely imaginary locus of Λ, which is supposed to be Lagrangian.

Lemma 4.2.5. Let Λ be the same as in Theorem 4.2.1. Let u be a $\tilde{\mathcal{B}}_\Lambda$-solution of the following equation:

$$(4.2.24)\quad (tD_t - \lambda(x))u = 0$$

Then we have

$$u = \Gamma(-\lambda(x))(t+\sqrt{-1}0)^{\lambda(x)}\,\varphi(x)$$

$$= 2\pi\exp(-\pi\sqrt{-1}\lambda(x)/2)(D_t/\sqrt{-1})^{-\lambda(x)-1}\delta(t)\varphi(x) \quad \text{with } \varphi(x) \in \mathcal{B}_{\mathbb{R}^n}.$$

Proof. By applying $(D_t/\sqrt{-1})^{-\lambda(x)-1} = \exp(-(\lambda(x)+1)\log(D_t/\sqrt{-1}))$ to (4.2.24), we may replace (4.2.24) by

$$(4.2.25)\quad tu = 0$$

On the other hand, we have the following exact sequence:

$$(4.2.26)\quad 0 \to \pi^{-1}\mathcal{O}_Y \to \mathcal{C}^{\mathbb{R}}_{Y|X} \overset{t}{\to} \mathcal{C}^{\mathbb{R}}_{Y|X} \to 0,$$

where π is the projection from $T^*_Y X$ to Y. Since

$$\mathcal{H}^{j}_{\sqrt{-1}\,T^*_{\mathbb{R}^n}\mathbb{R}^{n+1}}(\pi^{-1}\mathcal{O}_{\mathbb{C}^n}) = \mathcal{H}^{j-1}_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n}),$$

(4.2.26) implies

$$0 \to \mathcal{B}_{\mathbb{R}^n} \to \tilde{\mathcal{B}}_\Lambda \overset{t}{\to} \tilde{\mathcal{B}}_\Lambda \to 0.$$

This means that $u = \delta(t)\varphi(x)$ holds with $\varphi(x) \in \mathcal{B}_{\mathbb{R}^n}$. Q.E.D.

The following theorem is an immediate consequence of Theorem 4.2.1 and Lemma 4.2.5.

<u>Theorem 4.2.6</u>. Assume the same conditions as in Theorem 4.2.1. For simplicity, assume further that $A(x)$ is a constant matrix A. Let U be a microfunction solution of $(tD_t - A + Q(x,D_x,D_t))U = 0$. Then U has the following expression as a section of $\tilde{\mathcal{B}}_\Lambda$:

$$U = R\begin{pmatrix} \Gamma(-\lambda_1)(t+\sqrt{-1}0)^{\lambda_1} L_1 & & \\ & \ddots & \\ & & \Gamma(-\lambda_d)(t+\sqrt{-1}0)^{\lambda_d} L_d \end{pmatrix}\begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_N(x) \end{pmatrix},$$

where $R \in M_N(\tilde{\mathcal{D}}^\infty_\Lambda)$, $\lambda_1,\dots,\lambda_d$ are the mutually different eigenvalues of A, L_j has the form

$$\begin{pmatrix} 1 & \log(t+\sqrt{-1}0) & (\log(t+\sqrt{-1}0))^2/2! & \dots & (\log(t+\sqrt{-1}0))^{m_j-1}/(m_j-1)! \\ & 1 & \log(t+\sqrt{-1}0) & \dots & (\log(t+\sqrt{-1}0))^{m_j-2}/(m_j-2)! \\ & & 1 & \ddots & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

with m_j being the multiplicity of λ_j and $\varphi_\ell(x)$ is a hyperfunction in x.

<u>Remark 4.2.7</u>. If $A(x)$ is not supposed to be a constant matrix, U has the form

$$R\Gamma(-A(x))(t+\sqrt{-1}0)^{A(x)}\Phi(x)$$

with $\Phi(x) \in \mathcal{B}^N_{\mathbb{R}^n}$.

Corollary 4.2.8. Assume the same conditions as in Corollary 4.2.2. Then for each microfunction solution u of the equation $(b(\vartheta)-P)u = 0$, there exist $R_\mu \in \widetilde{\mathcal{D}}^\infty_\Lambda$ $(\mu = 1,\ldots,N)$ and hyperfunctions ψ_μ $(\mu = 1,\ldots,N)$ which satisfy $b(\vartheta)\psi_\mu = 0$ so that

$$u = \sum_{\mu=1}^{N} R_\mu \psi_\mu$$

holds as section of $\widetilde{\mathcal{B}}_\Lambda$. In particular, if $\Lambda = T^*_Y\mathbb{C}^{1+n}$ with $Y = \{(t,x) \in \mathbb{C}^{1+n}; t=0\}$, then u has the form

$$\sum_{\nu=1}^{d} \sum_{j=0}^{j_\nu - 1} R_{\nu,j}(x,D_x,D_t)(t+\sqrt{-1}0)^{\lambda_\nu}(\log(t+\sqrt{-1}0))^j$$

where $R_{\nu,j} \in \widetilde{\mathcal{D}}^\infty_\Lambda$, $b(\lambda_\nu) = 0$ and j_ν is the multiplicity of the root λ_ν.

These results show that the sheaf $\widetilde{\mathcal{B}}_\Lambda$ is fitted for the study of the asymptotic expansion of a microfunction which satisfies a suitable micro-differential equation. Note that $R(x,D_x,D_t)(\varphi(x)(t+\sqrt{-1}0)^\lambda)$ can be formally rewritten in the form

$$\sum_{j=0}^{\infty} \varphi_j(x)(t+\sqrt{-1}0)^{\lambda+j},$$

if $\Lambda = T^*_Y\mathbb{C}^{1+n}$ and R is the operator constructed in Theorem 4.2.1.

In what follows, we call the expression $\sum_{\mu=1}^{N} R_\mu\psi_\mu (R_\mu \in \widetilde{\mathcal{D}}^\infty_\Lambda)$ given in Corollary 4.2.8 to be the asymptotic expansion with respect to Λ of the microfunction solution u of the equation $(b(\vartheta)-P)u = 0$. Note that Proposition 4.1.6 guarantees that such an equation always exists if u satisfies a holonomic system with R.S.

Now, as far as we start from a single equation $(b(\vartheta)-P)u = 0$, Corollary 4.2.8 is the best possible one of the sort. However, in practical problems equations of this type usually appear in connection with holonomic systems. In such circumstances, as we mentioned in §1, the top term of the asymptotic expansion is controlled again by holonomic systems on Λ. For example, in the case of the example (4.2.1) discussed at the beginning of this subsection, u_0 and v_0 satisfy the following equations beside (4.2.9):

$$(xD_x/3+yD_y/2-s)w = yD_xw = 0.$$

This situation is best formulated by regarding the original holonomic system as $\tilde{\mathcal{D}}^{\infty}_{\Lambda}$-Module. In connection with such a formulation we first prepare some notions related to the Levi condition of the system. (Cf. Monteiro-Fernandes [16].) In what follows, Λ is a Lagrangian submanifold of T^*X and $\mathcal{J}_\Lambda$ and $J_\Lambda(1)$ denote $\{P\in \mathcal{E}_X(1); \sigma_1(P)|_\Lambda = 0\}$ and $\{f\in \mathcal{O}_{T^*X}(1); f|_\Lambda = 0\}$, respectively.

Definition 4.2.9. (i) $A \underset{\text{def}}{=} \underset{k}{\oplus}(\mathcal{J}^k_\Lambda/(\mathcal{J}^{k-1}_\Lambda + \mathcal{J}^{k+1}_\Lambda(-1))$

$$(= \underset{k}{\oplus}(J_\Lambda(1)^k/J_\Lambda(1)^{k-1}\mathcal{O}_{T^*X}(1))).$$

(ii) Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-Module and $\mathcal{L}$ its coherent $\mathcal{E}(0)$-sub-Module such that $\mathcal{M} = \mathcal{E}_X\mathcal{L}$ holds. Then $\mathrm{Ch}_\Lambda(\mathcal{M})$ is, by definition, a subvariety of $T_\Lambda(T^*X)$, the normal bundle along Λ, which is given by

$$\mathrm{supp}(\mathcal{O}_{T_\Lambda(T^*X)} \underset{A}{\otimes} (\underset{k}{\oplus}(\mathcal{J}^k_\Lambda\mathcal{L}/\mathcal{J}^{k-1}_\Lambda\mathcal{L} + \mathcal{J}^{k+1}_\Lambda(-1)\mathcal{L})).$$

Remark 4.2.10. The variety $\mathrm{Ch}_\Lambda(\mathcal{M})$ is independent of the choice of $\mathcal{L}$. We conjecture that it is an involutory subvariety of $T^*\Lambda \cong T_\Lambda(T^*X)$.

Remark 4.2.11. Denote by $\tilde{\sigma}_k$ the map from $\mathcal{J}^k_\Lambda$ to $J_\Lambda(1)^k/J_\Lambda(1)^{k-1}\mathcal{O}_{T^*X}(1)$, which is induced from the isomorphism between $\mathcal{J}^k_\Lambda/(\mathcal{J}^{k-1}_\Lambda + \mathcal{J}^{k+1}_\Lambda(-1))$ and $J_\Lambda(1)^k/J_\Lambda(1)^{k-1}\mathcal{O}_{T^*X}(1)$. Then, for a system $\mathcal{M}$ of the form $\mathcal{E}_X/\mathcal{J}$,

$$\mathrm{Ch}_\Lambda(\mathcal{M}) = \underset{k}{\oplus}A/(\tilde{\sigma}_k(\mathcal{J}^k_\Lambda\cap\mathcal{J})).$$

Now we begin the discussion on the structure of a holonomic system with R.S. regarded as a $\tilde{\mathcal{D}}^{\infty}_{\Lambda}$-Module. In order to simplify the description, we will consider the case where $X = \mathbb{C}^{1+n} = \{(t,x); t\in\mathbb{C}, x\in\mathbb{C}^n\}$ and $\Lambda = \{(t,x;\tau,\xi)\in T^*X;\ t=0,\ \xi=0,\ \tau\neq 0\}$. Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-Module with R.S. Take a coherent $\mathcal{E}_\Lambda$-sub-Module $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{E}_X\mathcal{N} = \mathcal{M}$. If we set $\mathcal{N}(-1) = \mathcal{E}(-1)\mathcal{N}$, then $\mathcal{N}/\mathcal{N}(-1)$ is a coherent $\mathcal{A}_\Lambda$-Module. Here $\mathcal{A}_\Lambda = \mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$. By identifying $\mathcal{A}_\Lambda$ with the sheaf of homogeneous linear differential operators on Λ, we can regard $\mathcal{N}/\mathcal{N}(-1)$ as a system of differential equations on Λ, i.e., x-space. If $\mathcal{N} = \mathcal{E}_\Lambda u$ for a section u of $\mathcal{M}$, then $\mathcal{N}/\mathcal{N}(-1)$ is nothing but the following equations:

$$L(P)\bar{u} = 0 \quad \text{for} \quad P\in\mathcal{E}_\Lambda \quad \text{such that} \quad Pu = 0.$$

As we have shown in §1 (Theorem 1.5), $\mathcal{N}/\mathcal{N}(-1)$ is a holonomic system with R.S.

Now it follows from Propositin 4.1.6 that there exist a coherent $\mathcal{E}(0)$-sub-Module $\mathcal{L}$ of $\mathcal{M}$ and a non-zero polynomial b(s) of degree, say, r which satisfy the following two conditions:

(4.2.27) $b(tD_t)\mathcal{L} \subset \mathcal{J}_\Lambda^{r+1}(-1)\mathcal{L}$

(4.2.28) The difference of any roots of $b(s)=0$ is not a non-zero integer.

Take a system of generators $u_1,\ldots,u_m$ of $\mathcal{L}$ and denote by u the column vector having $(tD_t)^{\alpha_0} D_{x_1}^{\alpha_1}\ldots D_{x_n}^{\alpha_n} u_j$ as components, where $\alpha_0,\ldots,\alpha_n$ range over all non-negative integers satisfying the constraint $\sum_{\nu=0}^{n}\alpha_\nu < r$ and $1 \leqq j \leqq m$. Let N be the length of the vector u. Then it is easy to see that u satisfies the equation

$$tD_t u = Au + Qu,$$

where A is a constant matrix and Q is a matrix of linear differential operators satisfying the conditions in Theorem 4.2.1. Here we note that the eigenvalues of A are the roots of $b(s) = 0$. Let us now consider the system $\tilde{\mathcal{M}} = \mathcal{E}^N/\mathcal{E}^N(tD_t-A-Q)$ and let $\tilde{u}$ be the corresponding vector whose components generate $\tilde{\mathcal{M}}$. Let f be the homomorphism from $\tilde{\mathcal{M}}$ onto $\mathcal{M}$ defined by assigning u to $\tilde{u}$. Let $\mathcal{M}'$ denote the kernel of f and let $\tilde{\mathcal{L}}$ be the $\mathcal{E}(0)$-Module generated by components of $\tilde{u}$. Then it is easy to see that $\tilde{b}(tD_t)\tilde{\mathcal{L}} \subset \mathcal{J}_\Lambda^{N+1}\tilde{\mathcal{L}}(-1)$ holds for the characteristic polynomial $\tilde{b}(s)$ of A. Note that deg $\tilde{b}$ is at most N. The Artin-Rees theorem can be modified to show that

$$(\mathcal{J}_\Lambda^{k}\tilde{\mathcal{L}})\cap\mathcal{M}' = \mathcal{J}_\Lambda^{k-k_0}(\mathcal{J}_\Lambda^{k_0}\tilde{\mathcal{L}}\cap\mathcal{M}')$$

holds for $k \geqq k_0$ if we take k_0 sufficiently large. We can easily prove that $\tilde{b}(tD_t)\mathcal{J}_\Lambda^{k}\tilde{\mathcal{L}} \subset \mathcal{J}_\Lambda^{k+N+1}\tilde{\mathcal{L}}(-1)$, and hence we obtain

$$\tilde{b}(tD_t)(\mathcal{J}_\Lambda^{k}\tilde{\mathcal{L}}\cap\mathcal{M}') \subset \mathcal{J}_\Lambda^{k+N+1}\tilde{\mathcal{L}}(-1)\cap\mathcal{M}' = \mathcal{J}_\Lambda^{N+1}(-1)(\mathcal{J}_\Lambda^{k}\tilde{\mathcal{L}}\cap\mathcal{M}').$$

Then, by the same reasoning used above, we can find a system of generators $v_1,\ldots,v_{\tilde{N}}$ of $\mathcal{J}_\Lambda^{k}\tilde{\mathcal{L}}\cap\mathcal{M}'$ such that the row vector $v = (v_1,\ldots,v_{\tilde{N}})$ satisfies

$$tD_t v = \tilde{A}v + \tilde{Q}v,$$

where $\tilde{A}$ is a constant matrix whose eigenvalues are the roots of $\tilde{b}(s) = 0$ (and hence those of $b(s) = 0$) and $\tilde{Q}$ satisfies the conditions in Theorem 4.2.1. Then, if we write $v = P(x,D_x,D_t)u$ with $P \in M(\tilde{N},N;\mathcal{E})$ which does not contain t, then we have $(tD_t-\tilde{A}-\tilde{Q})P = P'(tD_t-A-Q)$ with $P' \in M(\tilde{N},N;\mathcal{E})$. By comparing the coefficients of t, we can easily verify that $P' = P$. Hence we obtain

$$(tD_t-\tilde{A}-\tilde{Q})P = P(tD_t-A-Q)$$

It follows from Theorem 4.2.1 that there exist $U(x,D_x,D_t) \in GL(N;\tilde{\mathcal{D}}^\infty_\Lambda)$ and $\tilde{U}(x,D_x,D_t) \in GL(\tilde{N};\tilde{\mathcal{D}}^\infty_\Lambda)$ which satisfy

$$tD_t-A-Q = U(tD_t-A)U^{-1}$$

and

$$tD_t-\tilde{A}-\tilde{Q} = \tilde{U}(tD_t-\tilde{A})\tilde{U}^{-1}.$$

Hence we have

$$(tD_t-\tilde{A})(\tilde{U}^{-1}PU) = (\tilde{U}^{-1}PU)(tD_t-A).$$

Set $T = \tilde{U}^{-1}PU = \sum_j T^j(x,D_x)D_t^{-j}$. Then

(4.2.29) $jT^j = \tilde{A}T^j - T^jA$

holds. Note that the difference of any eigenvalues of A and that of $\tilde{A}$ are not a non-zero integer by (4.2.28). Hence (4.2.29) implies $T^j = 0$ for $j \neq 0$. Therefore T is a linear differential operator in x. On the other hand, U^{-1} and $\tilde{U}$ have respectively the form $\sum_{j=0}^{\infty} G^j(x,D_x)D_t^{-j}$ and $\sum_{j=0}^{\infty} H^j(x,D_x)D_t^{-j}$ with $G^0 = H^0 = I$ and $\operatorname{order} G^j$, $\operatorname{order} H^j \leq \alpha j$ for some α. Hence, if we write $P = \sum_{j=0}^{\infty} P^j(x,D_x)D_t^{-j}$, then we have

(4.2.30) $P^0 = T$ and $\operatorname{order} P^j \leq \alpha j$.

Since $\mathcal{M}$ is an $\mathcal{E}_X$-Module generated by u with the relation $Pu = (tD_t-A-Q)u = 0$, $\tilde{\mathcal{D}}^\infty_\Lambda \otimes_{\mathcal{E}_X} \mathcal{M}$ is generated by $1 \otimes u$ with the same relation. Hence, if we take another generator $w \underset{\text{def}}{=} U^{-1}(1 \otimes u)$ of $\tilde{\mathcal{D}}^\infty_\Lambda \otimes_{\mathcal{E}_X} \mathcal{M}$, we find

(4.2.31) $(tD_t - A)w = Tw = 0.$

Hence $\tilde{\mathcal{D}}^{\infty}_{\Lambda} \underset{\mathcal{E}_X}{\otimes} \mathcal{M}$ has the form $\tilde{\mathcal{D}}^{\infty}_{\Lambda} \underset{\mathcal{A}_\Lambda}{\otimes} \mathcal{F}$, where $\mathcal{F}$ is the $\mathcal{A}_\Lambda$-Module generated by a generator w with the relation (4.2.31). Here we identify $\mathcal{A}_\Lambda$ with a sub-Module of $\tilde{\mathcal{D}}^{\infty}_{\Lambda}$. Let us take as $\mathcal{N}$ the $\mathcal{E}_\Lambda$-Module generated by u. Then by using the fact that $\mathcal{E}_\Lambda v \cap \mathcal{M}'$ $= \mathcal{E}^{\tilde{N}}_{\Lambda} Pu$ and (4.2.30) we can easily verify that $\mathcal{N}/\mathcal{N}(-1) = \mathcal{F}$ holds. Furthermore, by using Remark 4.2.11 and an invertibility theorem for operators in $\tilde{\mathcal{E}}^{\infty}_{\Lambda}$ (Laurent [14]), we can easily show the characteristic variety of $\mathcal{F}$ is contained in $Ch_\Lambda(\mathcal{M})$. Thus we obtain the following

Theorem 4.2.12. Let $\mathcal{M}$ be a holonomic $\mathcal{E}$-Module with R.S. and let $\mathcal{N}$ be a coherent $\mathcal{E}(0)$-sub-Module of $\mathcal{M}$ such that there exists a non-zero polynomial b(s) which satisfies the condition (4.2.28) and $b(\vartheta)\mathcal{N} \subset \mathcal{N}(-1)$. Then we have

(i) $\mathcal{N}/\mathcal{N}(-1)$ is a holonomic $\mathcal{A}_\Lambda$-Module with R.S. whose characteristic variety is contained in $Ch_\Lambda(\mathcal{M})$.

(ii) $\tilde{\mathcal{D}}^{\infty}_{\Lambda} \underset{\mathcal{E}_\Lambda}{\otimes} \mathcal{M} = \tilde{\mathcal{D}}^{\infty}_{\Lambda} \underset{\mathcal{A}_\Lambda}{\otimes} (\mathcal{N}/\mathcal{N}(-1))$

This theorem implies that operators in $\tilde{\mathcal{D}}^{\infty}_{\Lambda}$ enable us to transform a holonomic system of micro-differential equations with R.S. to a differential equation in (t,x)-variables. It also asserts that any microfunction solution of such a system is determined by its initial terms in its asymptotic expansions.

§5. Discussion on the analyticity of the S-matrix.

It is now commonly accepted that the analyticity properties of the S-matrix is most neatly expressed in terms of microfunctions and/or essential support theory. However, in order to study the structure of the S-matrix, we sometimes need more precise information than the microanalyticity of the S-matrix. For example, Iagolnitzer-Stapp [4] makes essential use of "no sprout assumption" in the study of pole-factorization theorem. This condition cannot be described in terms of the micro-analyticity.

The purpose of this section is to discuss how such a delicate analyticity problem is related to the theory expounded so far, and how it is related to the holonomic character of the *S*-matrix. (See Kawai-Stapp [13] for the discussion on the holonomic character of the *S*-matrix.) The theory of double-microlocalization with respect to an involutory manifold, which Laurent [14],[15] is now developing, is also very useful for this purpose.

Now let Λ be $T^*_Y\mathbb{C}^n$ with $Y = \{(z_1,z')\in \mathbb{C}^n;\ z_1=0\}$ and let V be $\{(z,\zeta)\in T^*\mathbb{C}^n;\ \zeta_2=\dots=\zeta_n=0\}$. Let L and W denote their purely imaginary loci. For a microfunction f, $SS^2_W f$ (Kashiwara-Laurent [10]) and $\widetilde{SS}_L f$ are, by definition, subsets of $T_W(\sqrt{-1}T^*\mathbb{R}^n)$ and $T_L(\sqrt{-1}T^*\mathbb{R}^n)$, respectively. (Cf. the end of §3.2. Note that $\sqrt{-1}T^*_L\mathbb{R}^n$ and $T_L(\sqrt{-1}T^*\mathbb{R}^n)$ can be identified by the Hamiltonian map H, because L is Lagrangian. Actually $H(\sqrt{-1}dx_j) = -\sqrt{-1}\partial/\partial\xi_j$ and $H(\sqrt{-1}d\xi_j) = \sqrt{-1}\partial/\partial x_j$ hold.) Using the coordinate system we denote a point in $T_W(\sqrt{-1}T^*\mathbb{R}^n)$ and a point in $T_L(\sqrt{-1}T^*\mathbb{R}^n)$ respectively by $(x, \sqrt{-1}\xi_1;\ \sqrt{-1}\sum_{j=2}^{n} c_j\partial/\partial\xi_j)$ and $(x',\sqrt{-1}\xi_1;\sqrt{-1}(c_1\partial/\partial x_1+\sum_{j=2}^{n} c_j\partial/\partial\xi_j))$, where x,ξ_1 and c_j are real. Note that the subset of $T_L(\sqrt{-1}T^*\mathbb{R}^n)$ defined by $\omega = 0$ is identified with $T_W(\sqrt{-1}T^*\mathbb{R}^n)|_L$.

Let us now consider a microfunction f supported by $L^+ \overset{=}{\text{def}} \{(x,\sqrt{-1}\xi)\in\sqrt{-1}(T^*\mathbb{R}^n - T^*_{\mathbb{R}^n}\mathbb{R}^n);\ x_1=0,\ \xi_1>0,\ \xi' = (\xi_2,\dots,\xi_n) = 0\}$. Then

$$(5.1)\quad SS^2_W(f)\subset\{(x,\sqrt{-1}\xi_1;\ c')\in T_W(\sqrt{-1}T^*\mathbb{R}^n);\ c'=0\}$$

is equivant to the assertion that the defining function $F(z)$ of the microfunction f is holomorphic on $\{z\in\mathbb{C}^n;\ |z|<\varepsilon,\ \operatorname{Im} z_1>0\}$, namely, f satisfies the no sprout assumption. (In the condition (5.1) $c' = (c_2,\dots,c_n)$ is identified with $\Sigma c_j\partial/\partial\xi_j$.) Next consider the following condition

$$(5.2)\quad \widetilde{SS}_L(f)\subset\{(x',\sqrt{-1}\xi_1;\ c)\in T_L(\sqrt{-1}T^*\mathbb{R}^n);\ c=0\}.$$

Then, on the supposition that Conjecture 3.2.6 (ii) in §3.2 is valid, (5.2) is equivalent to the assertion that f belongs to $\mathcal{M}^{\mathbb{R}}(=\mathcal{E}^{\mathbb{R}}\otimes\mathcal{M})$ for a simple holonomic system $\mathcal{M}$ supported by Λ. In particular, the defining function $F(z)$ of f can be analytically continued to the universal covering space of $\Omega - Y$, where Ω is a neighborhood of $0\in\mathbb{C}^n$. Hence the condition (5.2) is close to the holonomic character of f. However, if f is holonomic, we find another important property of f, namely, the finite determination property. This property is not implied by (5.2). The simplest example of f that satisfies condition (5.2) but that is not holonomic is given by $(x_1+\sqrt{-1}0)^{x_2}$. Thus the sheaf $\tilde{\mathcal{C}}_\Lambda$ supplies us with a link between the no sprout assumption and the holonomic character of the function in question. Note also that

(5.3) $SS^2_W(f) \cong \widetilde{SS}_L(f)$

holds for a microfunction f which satisfies an equation dealt with in Corollary 4.2.2. (See Remark 4.2.4.) It is also noteworthy that (5.3) is valid for an arbitrary holonomic microfunction f (i.e., not necessarily supported by L^+). In fact, we know that, for each holonomic $\mathcal{E}_X$-Module $\mathcal{M}$, we can find a holonomic $\mathcal{E}_X$-Module $\mathcal{M}_{reg}$ with R.S. so that

$$\mathcal{E}^\infty \otimes \mathcal{M} \cong \mathcal{E}^\infty \otimes \mathcal{M}_{reg}$$

holds. ([8] Chap. V, §2, Theorem 5.2.1. See also our report [9] in this colloquium.) Since $\mathcal{E}^\infty_X|_\Lambda$ is a subsheaf of $\tilde{\mathcal{D}}^\infty_\Lambda$ and $\mathcal{E}^{2\infty}_V|_\Lambda$, this fact implies (5.3).

We end this report by mentioning a fact which is a generalization of the no sprout assumption and will probably turn out to be useful in application.

Let f be a hyperfunction defined on $\mathbb{R}^n$ whose singularity spectrum is contained in a properly convex cone $C=\{\sqrt{-1}\xi\in\sqrt{-1}\,\mathbb{R}^n;\ \varphi(\xi)>0\}$ with φ being a real-calued real analytic homogeneous function such that $d\varphi$ never vanishes on $V \underset{\text{def}}{=} \{\xi\in\mathbb{R}^n-\{0\};\varphi(\xi)=0\}$. Let Γ be the dual cone of C, i.e., $\{y\in\mathbb{R}^n;\ \langle y,\xi\rangle>0\ \ (\xi\in C)\}$. Suppose that $SS^2_V f\subset$

$\{(x,\sqrt{-1}\xi;\ c)\in T_V(\sqrt{-1}T^*\mathbb{R}^n);\ \varphi(\xi)=0,\ \sum_{j=1}^{n} c_j\,\partial\varphi/\partial\xi_j \geqq 0\}$. Then the defining

function $F(z)$ of f is holomorphic on $\{z\in\mathbb{C}^n;\ |\mathrm{Im}\, z|<\varepsilon(\mathrm{Re}\, z),$ $\mathrm{Im}\, z\in\Gamma\}$, where ε is a positive-valued function on $\mathbb{R}^n$.

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