THE FLAG MANIFOLD OF KAC-MOODY LIE ALGEBRA

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0. Introduction. In this paper, we shall construct the flag variety of a Kac-Moody Lie algebra as an infinite-dimensional scheme. There are several constructions by Kac-Peterson ([K-P]), Kazhdan-Lusztig ([K-L]), S. Kumar ([Ku]), O. Mathieu ([M]), P. Slodowy ([S]), J. Tits ([T]), but there the flag variety is understood as a union of finite-dimensional varieties.

We give here two methods of construction of the flag variety. For a Kac-Moody Lie algebra \( g \), let \( \hat{g} \) be the completion of \( g \). The first construction is to realize the flag variety as a subscheme of \( \text{Grass}(\hat{g}) \), the Grassmann variety of \( \hat{g} \). More precisely, taking the Borel subalgebra \( b_- \subset \hat{g} \) and regarding this as a point of \( \text{Grass}(\hat{g}) \), we define the flag variety as its orbit by the infinitesimal action of \( \hat{g} \) in \( \text{Grass}(\hat{g}) \).

The other construction is to realize the flag variety as \( G/B_- \). Of course, in the Kac-Moody Lie algebra case, we cannot expect that there is a group scheme whose Lie algebra is \( g \). But we can construct a scheme \( G \) on which \( g \) acts infinitesimally from the left and the right. Then we define the flag variety \( G/B_- \), where \( B_- \) is the Borel subgroup. More precisely, we consider the ring of regular functions as in [K-P]. Then its spectrum admits an infinitesimal action of \( g \). But its action is not locally free. Roughly speaking, \( G \) is the open subscheme where \( g \) acts locally freely (Proposition 6.3.1).

The flag variety of a Kac-Moody algebra shares the similar properties to the finite-dimensional ones, such as Bruhat decompositions.

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1. Scheme of countable type.

1.1. In this paper, we treat infinite-dimensional schemes such as $A^\infty$, $P^\infty$, etc. We shall discuss their local properties briefly.
Let $k$ be a commutative ring.

Definition 1.1.1. A $k$-algebra $A$ is called of countable type over $k$, if $A$ is generated by $k$ and countable numbers of elements.

The following is easily proven just as in EGA.

Lemma 1.1.2. Let $X$ be a scheme over $k$. Assume that there is an open affine covering $X = \bigcup U_j$ of $X$ such that $\Gamma(U_j; \mathcal{O}_X)$ is of countable type. Then, for any open affine subset $U$ of $X$, $\Gamma(U; \mathcal{O}_X)$ is of countable type.

Definition 1.1.3. A scheme $X$ over $k$ is called of countable type if for any open affine subset $U$ of $X$, $\Gamma(U; \mathcal{O}_X)$ is a $k$-algebra of countable type.

Lemma 1.1.4. Let $k$ be a noetherian ring. Then any ideal of a $k$-algebra $A$ of countable type is generated by countable elements.

Proof. Assume $A$ is generated by $x_i$, $(i = 1, 2, \ldots)$. Then for any ideal $I$ of $A$, $I \cap k[x_1, \ldots, x_n]$ is generated by finitely many elements.

Lemma 1.1.5. Let $k$ be an algebraically closed field such that $k$ is not a countable set, and let $X$ be a $k$-scheme of countable type. If $X$ has no $k$-valued point, then $X$ is empty.

Proof. We may assume $X = \text{Spec}(A)$ and $A \cong k[T_n; n \in \mathbb{N}] / I$, where $T_n$ are indeterminates. Then $I$ is generated by countably many elements $f_j$. Let $k^\prime$ be the subring of $k$ generated by the coefficients of the $f_j$. Set $A^\prime = k^\prime[T_n; n \in \mathbb{Z}] / I^\prime$ where $I^\prime$ is the ideal generated by $f_j$. Then $A \cong k \otimes_k A^\prime$. If $A \neq 0$, there is a homomorphism $A^\prime \to K^\prime$ from $A^\prime$ to a field $K^\prime$. We may assume $K^\prime$ is generated by the image of $A^\prime$ as a field. Then $K^\prime$ has at most countable transcendental dimension over the prime field. Hence $k^\prime \to k$ splits $k^\prime \to K^\prime \varphi \to k$ for some $\varphi$. Therefore $X$ has a $k$-valued point.

Proposition 1.1.6. Let $k$ be a noetherian ring, and $A \cong \lim_{n} A_n$, where $\{A_n\}_{n \in \mathbb{N}}$ is an inductive system of $k$-algebra of finite type and $A_n \to A_{n+1}$ is flat. Then $\mathcal{O}_{\text{Spec}(A)}$ is a coherent ring.

Proof. Any homomorphism $\varphi : A^{\otimes m} \to A$ comes from some $\varphi^\prime$:
$A^\otimes m \to A_n$. Then $\text{Ker } \varphi'$ is finitely generated over $A_n$ and hence $\text{Ker } \varphi \cong A \otimes_{A_n} \text{Ker } \varphi'$ is also finitely generated over $A$.

Let us give an example.

Example 1.1.7. Infinite-dimensional affine space: $A^\infty = \text{Spec } k[X_i; i \in \mathbb{N}]$. The set of $k$-valued points of $A^\infty$ is $\{(x_i)_{i \in \mathbb{N}}; x_i \in k\}$. The structure ring is coherent by Proposition 1.1.6, since $k[X_i; i \in \mathbb{N}] = \bigcup_{m,k} k[X_1, \ldots, X_m]$.

2. Grassmann variety.

2.1. Let $k$ be a base field.

Definition 2.1.1. An l.c. $k$-vector space $V$ is a $k$-vector space with a topology satisfying

(i) The addition map $V \times V \to V$ is continuous.
(ii) $V$ is Hausdorff and complete.
(iii) The open $k$-vector subspaces form a neighborhood system of 0.

Let $V_1$ and $V_2$ be two l.c. vector spaces. We set

\[(2.1.1) \quad V_1 \otimes V_2 = \lim_{U_1, U_2} (V_1/U_1) \otimes (V_2/U_2)\]

where $U_j$ ranges over open linear subspaces of 0 in $V_j$ ($j = 1, 2$). We endow $V_1 \otimes V_2$ with the structure of l.c. vector space such that $\text{Ker } (V_1 \otimes V_2 \to (V_1/U_1) \otimes (V_2/U_2))$ form a neighborhood system of 0.

Definition 2.1.2. An l.c. $k$-vector space $V$ is called a c.l.c. $k$-vector space if $V$ is an l.c. $k$-vector space and it satisfies furthermore

(iv) There is a decreasing sequence $\{W_n\}_{n \in \mathbb{Z}}$ of open vector subspaces forming a neighborhood system of 0 such that $V = \bigcup_{n \in \mathbb{Z}} W_n$ and $\dim W_n/W_m < \infty$ for $n \leq m$.

Remark that in this case the family $\mathcal{F}(V)$ of open vector subspace $W$ of $V$ which is contained by some $W_n$ is independent from the choice of $\{W_n\}$. In fact, $\mathcal{F}(V)$ is the family of open vector subspaces $W$ of $V$ such that $\dim(W/W') < \infty$ for any open subspace $W' \subset W$. 
2.2. For a c.l.c. vector space $V$, define the Grassmann variety as follows.

For a $k$-scheme $S$, set $\mathcal{O}_S \otimes V = \lim_{W \in \mathcal{F}(V)} \mathcal{O}_S \otimes (V/W)$ and consider the functor

\[(2.2.1) \quad \text{Grass}(V) : S \mapsto \{ \mathcal{F} ; \mathcal{F} \text{ is a sub-$\mathcal{O}_S$-module of } \mathcal{O}_S \otimes V \text{ such that locally in the Zariski topology there exists a } W \in \mathcal{F}(V) \text{ such that } \mathcal{F} \to \mathcal{O}_S \otimes (V/W) \text{ is an isomorphism} \} \]

For $W \in \mathcal{F}(V)$, we set

\[(2.2.2) \quad \text{Grass}_W(V) : S \mapsto \{ \mathcal{F} ; \mathcal{F} \text{ is a sub-$\mathcal{O}_S$-module of } \mathcal{O}_S \otimes V \text{ such that } \mathcal{F} \to \mathcal{O}_S \otimes (V/W) \text{ is an isomorphism} \} \]

Hence $\text{Grass}(V) = \bigcup_W \text{Grass}_W(V)$ in the Zariski topology.

**Proposition 2.2.1.** Grass($V$) is represented by a separated scheme.

**Proof.** This proposition follows from the following two statements

\[(2.2.3) \quad \text{Grass}_W(V) \text{ is represented by an affine scheme of countable type.} \]

\[(2.2.4) \quad \text{For } W, W' \in \mathcal{F}(V), \text{ there exists } f \in \Gamma(\text{Grass}_W(V) ; \mathcal{O}) \text{ and } f' \in \Gamma(\text{Grass}_{W'}(V) ; \mathcal{O}) \text{ such that } \text{Grass}_W(V) \cap \text{Grass}_{W'}(V) \text{ is represented by the open subscheme defined by } f \neq 0 \text{ of } \text{Grass}_W(V) \text{ and that we have } ff' = 1 \text{ on } \text{Grass}_W(V) \cap \text{Grass}_{W'}(V) . \]

We shall prove first (2.2.3). Let us take $\{ e_i \}_{i \in I}$ in $V$ such that $\{ e_i \}$ forms a base of $V/W$. Take $\{ u_j \}_{j \in J}$ in $W$ such that $u_j$ tends to 0 and any element of $W$ is uniquely written as $\Sigma a_j u_j$ ($a_j \in k$). Then for a scheme $S$ and $\mathcal{F} \in \text{Grass}_W(V)(S)$, there exist $a_{ij} \in \mathcal{O}(S)$ such that $\mathcal{F}$ is generated by $e_i + \Sigma_j a_{ij} u_j$. Hence Grass$_W(V)$ is represented by Spec$(k[T_{ij} ; i \in I, j \in J])$.

Now, we shall prove (2.2.4).

For $\mathcal{F} \in \text{Grass}(V)(S)$, let $\mathcal{G}$ be the cokernel of $\mathcal{F} \to \mathcal{O}_S \otimes V/(W \cap W')$, and consider the diagram
\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_S \otimes W/(W \cap W') \overset{\cong}{\longrightarrow} \mathcal{O}_S \otimes W/(W \cap W') \longrightarrow 0 \\
0 & \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_S \otimes V/(W \cap W') \longrightarrow \mathcal{G} \longrightarrow 0 \\
\mathcal{F} & \longrightarrow \mathcal{O}_S \otimes V/W
\end{align*}
\]

Hence if \( \mathcal{F} \in \text{Grass}_W(V)(\mathcal{S}) \), \( \mathcal{G} \) is isomorphic to \( \mathcal{O}_S \otimes W/(W \cap W') \). The similar diagram obtained by exchanging \( W \) and \( W' \) shows that \( \mathcal{O}_S \otimes W'/(W' \cap W') \rightarrow \mathcal{G} \) and \( \mathcal{F} \rightarrow \mathcal{O}_S \otimes V/W' \) has the same kernel and the cokernel. Hence if we denote by \( f \) the determinant of \( \psi : \mathcal{O}_S \otimes W'/(W \cap W') \rightarrow \mathcal{G} \cong \mathcal{O}_S \otimes W/(W \cap W') \), then \( \text{Grass}_W(V) \cap \text{Grass}_{W'}(V) \) is defined by \( f \neq 0 \). On \( \text{Grass}_W(V) \), we define \( f' \) as the determinant of \( \psi' : \mathcal{O}_S \otimes W/(W \cap W') \rightarrow \mathcal{G} \cong \mathcal{O}_S \otimes W'/(W \cap W') \). Then since \( \psi \) and \( \psi' \) are inverse to each other on \( \text{Grass}_W(V) \cap \text{Grass}_{W'}(V) \), we have \( ff' = 1 \) there.

**Corollary 2.2.2.** Grass\(_W(V)\) is open in Grass\((V)\) and isomorphic to \( A^\infty \) (if \( \dim V = \infty \)).

**Corollary 2.2.3.** (i) For \( W, W' \in \mathcal{F}(V) \), Grass\(_W(V) \cap \text{Grass}_{W'}(V) = \emptyset \) if \( \dim W/(W \cap W') \neq \dim W'/(W \cap W') \).

(ii) Fix \( W \in \mathcal{F}(V) \). Then

\[
\text{Grass}(V) = \bigcup_{d \in \mathbb{Z}} \text{Grass}^d(V) \quad \text{and} \quad \text{Grass}^d(V) = \bigcup_{W'} \text{Grass}_W(V)
\]

where \( W' \) ranges over \( \mathcal{F}(V) \) with \( \dim W/(W \cap W') = \dim W'/(W \cap W') = d \).

2.3. Let \( G \) be an affine group scheme over a field \( k \). We say that \( G \) acts on a \( k \)-vector space (or \( V \) is a \( G \)-module) if \( V \) is an \( \mathcal{O}(G) \)-comodule; i.e. there is a comultiplication \( \mu : V \to \mathcal{O}(G) \otimes V \) such that

\[
(2.3.1) \quad V \longrightarrow \mathcal{O}(G) \otimes V \quad \text{and} \quad V \longrightarrow \mathcal{O}(G) \otimes V \\
\downarrow \quad \quad \quad \quad \downarrow \mu_g \otimes V \\
k \otimes V \quad \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes V
\]

commutes, where \( \mathcal{O}(G) \to k \) is the evaluation map at the identity and \( \mu_G \).
\( \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \) is the comultiplication. As well-known, in this case, \( V \) is a union of finite-dimensional sub-\( G \)-modules.

Now, let \( V \) be an \( l.c. \) \( k \)-vector space. We endow \( \mathcal{O}(G) \) with the discrete topology. We say that \( V \) is a \( (l.c.) \) \( G \)-module if there is given a continuous comultiplication \( V \to \mathcal{O}(G) \hat{\otimes} V \) such that

\[
(2.3.2) \quad V \to \mathcal{O}(G) \hat{\otimes} V \quad \text{and} \quad V \to \mathcal{O}(G) \hat{\otimes} V
\]

commute. In this case, there exists a neighborhood system of \( 0 \) by linear subspaces \( U_i (i \in I) \) such that \( V/U_i \) is a \( G \)-module and \( V/U_i \to V/U'_i \) is a morphism of \( G \)-modules if \( U_i \subset U'_i \).

**Proposition 2.3.1.** If \( V \) is a \( c.l.c. \) \( G \)-module, then \( G \) acts on Grass(\( V \)).

**Proof.** It is enough to construct

\[
G(\mathcal{S}) \times \text{Grass}(V)(\mathcal{S}) \to \text{Grass}(V)(\mathcal{S})
\]

functorially in \( S \). An \( S \)-valued point of \( G \) gives \( \mathcal{O}(G) \to \mathcal{O}(S) \).

Then we obtain

\[
g : \mathcal{O}_S \hat{\otimes} V \overset{\mathcal{O}_S \hat{\otimes} G}{\to} \mathcal{O}_S \hat{\otimes} \mathcal{O}(G) \hat{\otimes} V \overset{a}{\to} \mathcal{O}_S \hat{\otimes} V.
\]

This is an isomorphism. Hence for \( F \subset \mathcal{O}_S \otimes V, \varphi(F) \subset \mathcal{O}_S \hat{\otimes} V \) and it gives the map Grass(\( V \))(\( S \)) \to Grass(\( V \))(\( S \)).

3. **Kac-Moody Lie algebra.**

3.1. Following Kac, Moody, Mathieu, we start by the following data: a free \( \mathbb{Z} \) module \( P \), at most countably generated, and \( a_i \in P \) and \( h_i \in \text{Hom}_\mathbb{Z}(P, \mathbb{Z}) \) indexed by an index set \( I \).

We set \( t^0 = C \otimes_\mathbb{Z} P, \ t = \text{Hom}_C(t^0, C) \equiv \text{Hom}_\mathbb{Z}(P, C) \) with the structure of \( l.c. \) vector space induced from the discrete topology of \( t^0 \). We assume the following conditions:
(3.1.1) \( \left\{ \langle \alpha_i, h_j \rangle \right\}_{i,j} \) is a generalized Cartan matrix, i.e. \( \langle \alpha_i, h_j \rangle \in \mathbb{Z} \), \( \langle \alpha_i, h_i \rangle = 2 \), \( \langle \alpha_i, h_j \rangle \leq 0 \) for \( i \neq j \) and \( \langle \alpha_i, h_j \rangle = 0 \) iff \( \langle \alpha_i, h_i \rangle = 0 \).

(3.1.2) For any \( i \), there is \( \lambda \in \mathcal{P} \) such that \( \langle \lambda, h_i \rangle > 0 \) and \( \langle \lambda, h_j \rangle = 0 \) for any \( j \neq i \).

(3.1.3) \( \{ \alpha_i \}_{i \in I} \) is linearly independent.

(3.1.4) For any \( \lambda \in \mathcal{P} \), \( \langle h_i, \lambda \rangle = 0 \) except finitely many \( i \).

Let \( \mathcal{G} \) be the Lie algebra generated by \( t \) and symbols \( e_i, f_i \) \( (i \in I) \) with the following recover relations:

(3.1.5) \( [h, e_i] = \alpha_i(h)e_i \) and \( [h, f_i] = -\alpha_i(h)f_i \) for \( h \in t \).

(3.1.6) \( [e_i, f_j] = \delta_{ij}h_i \).

(3.1.7) \( (a e_i)^{1-\alpha_i(h_i)} e_j = 0 \) and \( (a f_i)^{1-\alpha_i(h_i)} f_j = 0 \) for \( i \neq j \).

Let \( n \) (resp. \( n_- \)) be the Lie subalgebra generated by \( e_i \) (resp. \( f_i \)) \( i \in I \). Then we have (e.g. [K])

\[
\mathcal{G} = n \oplus t \oplus n_-. \tag{3.1.8}
\]

Set

\[
b = t \oplus n, \quad b_- = t \oplus n_- \tag{3.1.9}
\]

\[
\mathcal{G}_i = t \oplus Ce_i \oplus Cf_i, \quad p_i = \mathcal{G}_i + n, \quad p_i^- = \mathcal{G}_i + n^- \tag{3.1.10}
\]

Let \( \Delta \) be the set of roots of \( \mathcal{G} \) and \( \Delta_+ \) and \( \Delta_- \) the set of roots of \( n \) and \( n_- \), respectively, and let \( \mathcal{G}_\alpha \) be the root space with root \( \alpha \in \Delta \). We set

\[
n_i = \bigoplus_{\alpha \in \Delta_+} \mathcal{G}_\alpha, \quad n_i^- = \bigoplus_{\alpha \in \Delta_-} \mathcal{G}_\alpha \tag{3.1.11}
\]

Let \( W \) be the Weyl group, i.e. the subgroup of \( \text{GL}(t^0) \) generated by the simple reflections \( s_i \) \( (i \in I) \), where
(3.1.12) \[ s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i. \]

We also denote by \( W' \) the braid group generated by \( s_i \ (i \in I) \) with the fundamental relation

\[ s_i s_j = s_j s_i \quad \text{if} \quad \langle h_i, \alpha_j \rangle = 0 \]

\[ s_i s_j s_i = s_j s_i s_j \quad \text{if} \quad \langle h_i, \alpha_j \rangle = \langle h_j, \alpha_i \rangle = -1 \]

(3.1.13)

\[ (s_i s_j)^2 = (s_j s_i)^2 \quad \text{if} \quad \langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle = 2 \]

\[ (s_i s_j)^3 = (s_j s_i)^3 \quad \text{if} \quad \langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle = 3 \]

Then as is well-known, \( W \) is isomorphic to the quotient of \( W' \) by the subgroup generated by \( w s_i^2 w^{-1} \ (i \in I) \).

For \( w \in W \), we denote by \( l(w) \) the length of \( w \), i.e. the smallest number \( l \) such that \( w \) is the product of a sequence of length \( l \) in \( \{s_i\} \). Recall that

(3.1.14) \[ l(w) = \#(\Delta_+ \cap w\Delta_-). \]

Also recall that \( l(s_i w) < l(w) \) if and only if \( w^{-1} \alpha_i \in \Delta_- \). Note also there exists a unique injection \( \iota : W \to W' \) such that

(3.1.15) \[ \iota(1) = 1, \quad \iota(s_i) = s'_i \quad \text{and} \quad \iota(ww') = \iota(w)\iota(w') \]

\[ \text{if} \quad l(ww') = l(w) + l(w'). \]

By this, we sometimes embed \( W \) into \( W' \).

An element \( h \) of \( i \) is called regular if \( \langle h, \alpha \rangle \neq 0 \) for any \( \alpha \in \Delta \). Such an element always exists. We set

(3.1.16) \[ P_+ = \{ \lambda \in P; \langle \lambda, h_i \rangle \geq 0 \ \text{for any} \ i \}. \]

For any finite set \( J \) of \( I \), we set

(3.1.17) \[ P_J^+ = \{ \lambda \in P_+; \langle \lambda, h_i \rangle = 0 \ \text{for} \ i \in I \setminus J \}. \]

If we set \( P_0 = \{ \lambda \in P; \langle \lambda, h_i \rangle = 0 \ \text{for} \ i \in I \} \) then \( P_0 \) is a free \( \mathbb{Z} \)-module and \( P_J^+/P_0 \) is a finitely generated semigroup.
3.2. Now, we shall define a completion of $\mathcal{G}$. For a subset $\mathcal{S}$ of $\Delta_+$, we set

$$n_\mathcal{S} = \bigoplus_{\alpha \in \mathcal{S}} \mathcal{G}_\alpha.$$  

We set

$$(3.2.2) \quad \hat{\mathcal{G}} = \lim_{\mathcal{T}} \mathcal{G}/n_\mathcal{S} = b_+ \bigoplus_{\alpha \in \Delta_+} \mathcal{G}_\alpha$$

where $\mathcal{S}$ ranges over the subsets of $\Delta_+$ such that $\Delta_+ \setminus \mathcal{S}$ is finite. We define the subalgebras $\hat{\mathcal{P}}, \hat{\mathcal{N}}, \hat{\mathcal{B}}, \hat{\mathcal{N}}$ of $\hat{\mathcal{G}}$, similarly. We set also

$$(3.2.3) \quad \hat{\mathcal{U}}(\mathcal{G}) = \bigcup_l \hat{\mathcal{U}}_l(\mathcal{G})$$

Then $\hat{\mathcal{U}}(\mathcal{G})$ is an algebra containing $\mathcal{U}(\mathcal{G})$ as a subalgebra.

3.3. In general, let $\mathcal{G}$ be a Lie algebra. A vector $v$ of a $\mathcal{G}$-module $V$ is called $\mathcal{G}$-finite if $v$ is contained in a finite-dimensional sub-$\mathcal{G}$-module of $V$. We call a $\mathcal{G}$-module $V$ is locally finite if any element of $V$ is $\mathcal{G}$-finite.

Let us define a ring homomorphism

$$(3.3.1) \quad \delta : \mathcal{U}(\mathcal{G}) \to \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})$$

by $\delta(A) = A \otimes 1 + 1 \otimes A$ for $A \in \mathcal{G}$, and an anti-ring automorphism

$$(3.3.2) \quad a : \mathcal{U}(\mathcal{G}) \to \mathcal{U}(\mathcal{G})$$

by $A^* = -A$ for $A \in \mathcal{G}$. Then $\delta$ defines $\mathcal{U}(\mathcal{G})^* \otimes \mathcal{U}(\mathcal{G})^* \to (\mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}))^*$

$\to \mathcal{U}(\mathcal{G})^*$ and this gives a commutative ring structure on $\mathcal{U}(\mathcal{G})^*$.

The right and left multiplication of $\mathcal{G}$ on $\mathcal{U}(\mathcal{G})$ induces the two $\mathcal{G}$-module structures on $\mathcal{U}(\mathcal{G})^*$:

$$(3.3.3) \quad (R(A)f)(P) = f(PA), \quad (L(A)f)(P) = f(a(A)P)$$
for $A \in U(\mathfrak{G})$, $f \in U(\mathfrak{G})^*$ and $P \in U(\mathfrak{G})$. Then $R(A)$ and $L(A)$ are derivations of the ring $U(\mathfrak{G})^*$ for any $A \in \mathfrak{G}$.

Now let $\mathfrak{G}$ be an abelian Lie algebra acting on the Lie algebra $\mathfrak{G}$ semi-simply, $t$ an abelian subalgebra of $\mathfrak{G}$ stable by $\mathfrak{G}$, and $P \subset t^*$ a sub-$\mathbb{Z}$-module stable by $\mathfrak{G}$. We assume that $t$ acts semi-simply on $\mathfrak{G}$ by the adjoint action and its weights belong to $P$.

Then, we set

$$
(3.3.5) \quad A(\mathfrak{G}, t, P, \mathfrak{G}) = \bigoplus_{\lambda \in P} \{ f \in U(\mathfrak{G})^*; f \text{ satisfies the following conditions (3.3.6), (3.3.7) and (3.3.8)} \}.
$$

(3.3.6) $f$ is $\mathfrak{G}$-finite with respect to $L$ and $R$.

(3.3.7) $f$ is a weight vector with weight $\lambda$ with respect to the left action of $t$.

(3.3.8) $f$ is $\mathfrak{G}$-finite.

Then $f \in U(\mathfrak{G})^*$ belongs to $A(\mathfrak{G}, t, P, \mathfrak{G})$ if and only if there exists a two-sided ideal $I$ of $U(\mathfrak{G})$ such that

(3.3.9) $f(U(\mathfrak{G})/I) = 0$,

(3.3.10) $\dim U(\mathfrak{G})/I < \infty$,

(3.3.11) $I$ is $\mathfrak{G}$-invariant,

(3.3.12) $t$ acts semi-simply on $U(\mathfrak{G})/I$ by the left multiplication and its weights belong to $P$.

Then one can see easily that $A(\mathfrak{G}, t, P, \mathfrak{G})$ is a subring of $U(\mathfrak{G})^*$ and the multiplication map $\mu : U(\mathfrak{G}) \otimes U(\mathfrak{G}) \to U(\mathfrak{G})$ induces the homomorphism

$$
(3.3.13) \quad U(\mathfrak{G})^* \bigcap U(\mathfrak{G})^* \quad \longrightarrow \quad (U(\mathfrak{G}) \otimes U(\mathfrak{G}))^*
$$

With this, $\text{Spec}(A(\mathfrak{G}, t, P, \mathfrak{G}))$ becomes an affine group scheme (see [M]).
We write

\[(3.3.14) \quad G(\mathcal{G}, t, P, \alpha) = \text{Spec}(A(\mathcal{G}, t, P, \alpha)).\]

Remark that \(g \mapsto g^{-1}\) is given by \(a : U(\mathcal{G}) \to U(\mathcal{G})\).

When \(\alpha = 0\), we write \(G(\mathcal{G}, t, P)\) for \(G(\mathcal{G}, t, P, \alpha)\) for short.

\[3.4.\quad \text{Coming back to the situation in Section 3.1, we define the affine group schemes } B, B_-, T, U, U_-, G_i, U_i, U_i^-, P_i, P_i^- \text{ as follows. This construction is due to Mathieu [M].}\]

\[
\begin{align*}
B &= G(b, t, P), \\
B_- &= G(b_-, t, P), \\
T &= G(t, t, P), \\
U &= G(n, 0, 0, t), \\
U_- &= G(n_-, 0, 0, t), \\
G_i &= G(\mathcal{G}_i, t, P), \\
U_i &= G(n_i, 0, 0, t), \\
U_i^- &= G(n_i^-, 0, 0, t), \\
P_i &= G(p_i, t, P), \\
P_i^- &= G(p_i^-, t, P), \\
G_i^+ &= G(t \oplus C e_i, t, P), \\
G_i^- &= G(t \oplus C f_i, t, P).
\end{align*}
\]

Then we have ([M])

\[B = T \bowtie U = G_i^+ \times U_i,\]
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\[ B_- = T \times U_- = G_i^- \times U_i^- , \]

\[ P_i = G_i \times U_i \supset B \supset T , \]

\[ P_i^- = G_i^{-} \times U_i^{-} \supset B_- \supset T , \]

\[ T = \text{Spec } \mathbb{C}[P] , \]

\[ U \cong \text{Spec } S( \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^{\alpha^*} ) , \]

\[ U_- \cong \text{Spec } S( \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}^{\alpha^*} ) , \]

\[ G_i^+ = G_i \cap B , \quad G_i^- = G_i \cap G_- . \]

More generally, for a subset \( S \) of \( \Delta^+ \) such that \( (S + S) \cap \Delta^+ \subset S \), we set \( n_S = \bigoplus \mathfrak{g}_{\alpha} \) and \( U_S = G(n_S, 0, 0, t) \).

Then for \( S \supset S' \) such that \( S \setminus S' \) is a finite set and that \( (S + S') \cap \Delta^+ \subset S' \), \( n_S/n_{S'} \) is a finite-dimensional nilpotent Lie algebra and if we denote by \( \exp(n_S/n_{S'}) \) the associated unipotent group, we have

\[ U_S \cong \lim_{S \to S'} \exp(n_S/n_{S'}). \]

3.5. The group \( P_i \) acts on the c.l.c. space \( \mathcal{G} \) by the adjoint action. In fact, \( \text{ad} : p_i \to \text{End } (\mathcal{G}) \) extends to \( \text{ad} : U(p_i) \to \text{End } (\mathcal{G}) \). Moreover, for any ideal \( \mathcal{A} \) of \( p_i \) with \( \text{codim } p_i/\mathcal{A} < \infty \), \( \mathcal{G}/\mathcal{A} \) is locally \( p_i \)-finite. Hence, for any \( A \in \mathcal{G} \), there is a two-sided ideal \( I \) of \( U(p_i) \) with \( \dim \ U(p_i)/I < \infty \) and \( \text{ad}(I)A \subset \mathcal{A} \). Hence the morphism \( P \to \text{ad}(P)A \) from \( U(p_i) \) to \( \mathcal{G}/\mathcal{A} \) splits as \( U(p_i)/I \to \mathcal{G}/\mathcal{A} \). Hence this gives an element of \( (U(p_i)/I)^* \otimes \mathcal{G}/\mathcal{A} \subset U(p_i)^* \otimes (\mathcal{G}/\mathcal{A}) \). This element clearly belongs to \( A(p_i, t, P) \otimes (\mathcal{G}/\mathcal{A})^* \). Thus we obtained \( \mathcal{G}/\mathcal{A} \to A(p_i, t, P) \otimes (\mathcal{G}/\mathcal{A})^* \). Since \( \lim_{i \to I} \mathcal{G}/\mathcal{A} = \hat{\mathcal{G}} \), we obtain \( \hat{\mathcal{G}} \to \mathcal{O}(P_i) \otimes \hat{\mathcal{G}} \). This gives an action of \( P_i \) on \( \hat{\mathcal{G}} \).

Clearly the action of \( B \) on \( \hat{\mathcal{G}} \) obtained from the action of \( P_i \) does not depend on \( i \in I \).

Especially, \( P_i \) acts on the Grassmann variety \( \text{Grass}(\hat{\mathcal{G}}) \) by Proposition 2.3.1.
4. The first construction of the flag variety.

4.1. In this section, for a Kac-Moody Lie algebra $\mathfrak{g}$, we construct its flag variety as a subscheme of $\text{Grass}(\mathfrak{g})$. We keep the notations in Section 3.

4.2. Since $\mathfrak{g}$ is a c.l.c. vector space, $\text{Grass}(\mathfrak{g})$ is a separated scheme. Since $\mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{n}_+$, $\mathfrak{b}_-$ gives a $\mathbb{C}$-valued point of $\text{Grass}(\mathfrak{g})$. We denote this point by $x_0$. By Section 3.5, $P_i$ and $B$ act on $\text{Grass}(\mathfrak{g})$.

4.3. Set $s_i' = \exp(-e_i)\exp(f_i)\exp(-e_i) \in G_i \subset P_i$. Then $s_i'^4 = 1$ and $s_i'$ acts on $\mathfrak{g}$. This extends to the group homomorphism:

\[(4.3.1) \quad W' \to \text{Aut}(\mathfrak{g}).\]

In order to see this, it is enough to prove the braid relation (3.1.13) when the Lie algebra generated by $e_i, e_j, f_i, f_j$ is finite-dimensional. Then the braid condition holds in the corresponding simply connected semi-simple group.

The morphism (4.3.1) induces

\[(4.3.2) \quad W' \to \text{Aut}(\text{Grass}(\mathfrak{g})).\]

We have also

\[(4.3.3) \quad \text{The image of Ker}(W' \to W) \text{ in Aut}(\mathfrak{g}) \text{ belongs to the image of } T \text{ in Aut}(\mathfrak{g}).\]

In fact, Ker$(W' \to W)$ is generated by the $w s_i'^2 w^{-1}$, which belongs to $T$.

Since $[t, b_-] \subset b_-$, we have

\[(4.3.4) \quad Tx_0 = x_0.\]

Hence for $w \in W$, $w' x_0$ does not depend on the choice of a representative $w'$ of $w$ in $W'$. We denote it by $w x_0$.

4.4. As in (2.2.2), we set

\[(4.4.1) \quad \text{Grass}_a(\mathfrak{g}) = \{ W \in \text{Grass}(\mathfrak{g}); W \oplus \mathfrak{n} \cong \mathfrak{g} \}.\]
This is an affine open subscheme of Grass(\(\hat{G}\)).

**Lemma 4.4.1.** The morphism \(U \to \text{Grass}(\hat{G})\) given by \(U \ni g \mapsto gx_0\) is an embedding.

**Proof.** First we shall show \(Ux_0 \subset \text{Grass}_\alpha(\hat{G})\). For this, it is enough to show, for any \(g \in U\),

\[
(4.4.2) \quad gb_+ \oplus \hat{n} = \hat{G}.
\]

But this is obvious because \(\hat{n}\) is stable by \(U\). Hence it is enough to show that \(U \to Y = \text{Grass}_\alpha(\hat{G})\) is a closed embedding. In order to see this, let us take a regular element \(h\) of \(t\) (i.e. \(\langle h, \alpha \rangle \neq 0\) for any \(\alpha \in \Delta\)). Then for any \(F \in \text{Grass}_\alpha(\hat{G})\), \(F \oplus \hat{n} = \hat{G}\), and hence there exists \(\psi(F) \in \hat{n}\) with \(h - \psi(F) \in F\). This defines a morphism

\[
\psi : Y \to \hat{n}.
\]

If we combine \(U \to Y \overset{\psi}{\to} \hat{n}\), this is given by

\[
U \ni g \mapsto h - g^{-1}h \in \hat{n}.
\]

Hence it is enough to show the following lemma.

**Lemma 4.4.2.** Let \(h\) be a regular element of \(t\). Then, the morphism \(U \to h + \hat{n}\) given by \(g \mapsto gh\) is an isomorphism.

**Proof.** Let \(S\) be a subset of \(\Delta_+\) such that \((S + \Delta_+) \cap \Delta_+ \subset S\) and \(\Delta_+ \setminus S\) is finite. Then \(U \to h + \hat{n}\) induces \(U/U_S \to (h + n)/n_S\), and it is enough to show that this is an isomorphism. Now \(U/U_S\) acts on \(b/n_S\). For \(A \in n/n_S\), the isotropy group at \(h + A\) is the identity. In fact this follows from

\[
(4.4.3) \quad \{E \in n; [h + A, E] \in n_S\} = n_S.
\]

Since \(\dim (h + n)/n_S = \dim U/U_S\), \((U/U_S)(h + A)\) is open in \((h + n)/n_S\). Thus \((U/U_S)(h + A)\) and \((U/U_S)h\) intersect. This shows \(U/U_S \cong (U/U_S)h = (h + n)/n_S\).
4.5. We have

\[(4.5.1) \quad Bx_0 = Ux_0\]

because \(Tx_0 = x_0\) and \(B = UT\). For \(w \in W\), let us denote

\[(4.5.2) \quad B \cap ^wB = A(t \oplus \bigoplus_{\alpha \in \Delta_+ \cap ^w\Delta_+} S_\alpha, t, P)\]

\[B \cap ^wB_- = A(t \oplus \bigcap_{\alpha \in \Delta_+ \cap ^w\Delta_-} S_\alpha, t, P).\]

They are subgroups of \(B\). Similarly, we define \(U \cap ^wU\) and \(U \cap ^wU_-\). Then we have

\[(4.5.3) \quad U = (U \cap ^wU) \times (U \cap ^wU_-) \approx (U \cap ^wU_-) \times (U \cap ^wU).\]

We have also

\[(4.5.4) \quad (B \cap ^wB_-)wx_0 = x_0.\]

**Lemma 4.5.1.** For \(w \in W\), \(Bs_iBwx_0 \subseteq Bwx_0 \cup Bs_iwx_0\).

**Proof.** We have \(Bs_iBwx_0 \subseteq P_iwx_0\). Since \(P_i = BG_i \subseteq B(G_i \cap ^wB_-) \cup Bs_i(G_i \cap ^wB_-), \)

\(P_iwx_0 \subseteq B(G_i \cap ^wB_-)wx_0 \cup Bs_i(G_i \cap ^wB_-)wx_0 \subseteq Bwx_0 \cup Bs_iwx_0\).

Note that for \(w_1, w_2 \in W\), \(w_1Bw_2x_0\) does not depend on the representatives in \(W\) of \(w_1, w_2 \in W\). Hence we denote \(w_1Bw_2x_0\) for it.

**Lemma 4.5.2.** Let \(w \in W\).

(i) If \(l(w) > l(s,w)\), \(Bs_iBwx_0 = Bs_iwx_0\).

(ii) If \(l(w) < l(s,w)\), \(Bs_iBwx_0 = P_iwx_0 \cup Bs_iwx_0\).

**Proof.** If \(l(s,w) < l(w)\), then \(w^{-1}\alpha_i \in \Delta_-\). Hence \(G_i^+ = G_i \cap B \subset ^wB_-\) and \(s_iB \subset s_iU_iG_i^+ \subset Bs_iG_i^+\). Hence we have \(Bs_iBwx_0 = Bs_iG_i^+wx_0 = Bs_iwx_0\).

If \(l(s,w) > l(w)\), then we have \(Bs_iBs_iwx_0 = Bwx_0\) since \(l(s Is, w) < l(s, w)\). Hence \(Bs_iBwx_0 = Bs_iBs_iBwx_0\). Since \(Bs_iBs_iB = P_i\), \(Bs_iBwx_0 = P_iwx_0\) and it contains \(wx_0\) and \(s_iwx_0\).
Lemma 4.5.3. \( wBx_0 \subset U_{w' \leq w} Bw' x_0 \), where \( \leq \) is the Bruhat order (the order generated by \( s_{i_1} \cdots s_{i_k-1} s_{i_k+1} \cdots s_{i_l} \leq s_{i_1} \) for a reduced expression \( s_{i_1} \cdots s_{i_l} \)).

Proof. We shall prove by the induction of \( l(w) \). If \( l(w) = 0 \), it is trivial. Otherwise, set \( w = s_i w' \) with \( l(w) = 1 + l(w') \). Then by the hypothesis of the induction, \( wBx_0 \subset \bigcup \mathcal{w} \subset w \), \( s_i Bw'' x_0 \subset \bigcup \mathcal{w} \subset w \), \( Bs_i w'' x_0 \subset \bigcup \mathcal{w} \subset w \). Hence \( wBx_0 \subset \bigcup \mathcal{w} \subset w \). 

Lemma 4.5.4.

(i) \( Bw x_0 \cap \text{Grass}_{\mathring{n}}(\mathfrak{G}) = \emptyset \) if \( w \neq 1 \).

(ii) \( wBx_0 \cap \text{Grass}_{\mathring{n}}(\mathfrak{G}) \subset Bx_0 \).

Proof. (i) Let \( g \in B \) and assume that \( gwB \in \mathfrak{G}/\mathring{n} \). Then \( wb \in \mathfrak{G}/\mathring{n} \). Hence \( w \Delta = \Delta \), which implies \( w = 1 \).

(ii) follows from (i) and the preceding lemma.

Corollary 4.5.5. \( X = \bigcup_{w \in W} wBx_0 \) is a subscheme of \( \text{Grass}(\mathfrak{G}) \) and \( wBx_0 \) is open in \( X \) for any \( w \in W \).

This easily follows from \( X \cap \text{Grass}_{\mathring{n}}(\mathfrak{G}) = Bx_0 \).

Definition 4.5.6. We call \( X \) the flag variety of \( \mathfrak{G} \).

Since \( \text{Grass}(\mathfrak{G}) \) is a separated scheme, \( X \) is also a separated scheme, and \( \{ wBx_0 \} \) is an open affine covering of \( X \). Note that \( X \) is not quasi-compact if \( W \) is an infinite group. I do not know whether \( X \) is a closed subscheme of \( \text{Grass}(\mathfrak{G}) \) or not.

Lemma 4.5.7. \( Bw x_0 \) is a closed subscheme of \( wBx_0 \) and we have a commutative diagram:

\[
\begin{array}{ccc}
Bw x_0 & \rightarrow & wBx_0 \\
\uparrow & & \uparrow \\
\mathring{n} \cap w^{-1} \mathring{n} & \rightarrow & \mathring{n}
\end{array}
\]

(4.5.5)

Proof. We have \( U = (U \cap w U) \times (U \cap w U) \). Since \( (U \cap w U) x_0 = x_0 \), we have \( Uw x_0 = (U \cap w U)wx_0 = w^{-1} U \cap U x_0 \). Then the lemma follows from Lemma 4.4.1.

Corollary 4.5.8. \( Bw x_0 \) is affine and codimension \( l(w) \) in \( X \).

Proposition 4.5.9. \( X(\mathbb{C}) = \bigsqcup_{w \in W} Bw x_0 \).
Proof. By Lemma 4.5.3, it is enough to show \( Bw_x_0 = Bw'x_0 \) implies \( w = w' \).

We have \( wx_0 \in Bw'x_0 \subset w'Bx_0 \). Hence \( w'^{-1}wx_0 \subset Bw'^{-1}wx_0 \cap Bx_0 \). Then Lemma 4.5.4 implies \( w' = w \).

**Lemma 4.5.10.** Let \( w_1, w_2 \in W \) and assume \( l(w_1s_iw_2) = l(w_1) + l(w_2) + 1 \). Then \( Bw_1s_iw_2x_0 \subset Bw_1w_2x_0 \).

**Proof.** Since \( l(w_1s_i) > l(w_1) \), we have \( w_1 \alpha_i \in \Delta_+ \), and hence \( G_i \cap w_i^{-1}B \subset G_i \cap B \). Since \( l(s_iw_2) > l(w_2) \), \( w_2^{-1} \alpha_i \in \Delta_+ \) and hence \( G_i \cap w_2B_- \subset G_i \cap B_- \). Since \( (G_i \cap B)(G_i \cap B_-) \) is dense in \( G_i \), we obtain

\[
Bw_1s_iw_2x_0 \subset Bw_1G_iw_2x_0 \subset \overline{Bw_1(G_i \cap w_i^{-1}B)(G_i \cap w_2B_-)w_2x_0} = \overline{Bw_1w_2x_0}.
\]

**Proposition 4.5.11.** \( \overline{Bw_0} = \cup_{w' \preceq w} \overline{Bw'x_0} \).

**Proof.** We shall prove first \( \overline{Bw_0} \supset Bw'x_0 \) if \( w' \preceq w \) by the induction of \( l(w') \). If \( l(w') = 0 \), then \( w = w' = e \) and this is evident. If \( l(w') > 0 \), there is \( w_1, w_2 \in W \) and \( i \) such that \( w' = w_1s_iw_2 \), \( w_1w_2 \preceq w \) and \( l(w') = l(w_1) + l(w_2) + 1 \). Hence \( Bw'x_0 \subset Bw_1w_2x_0 \subset Bw_0x_0 \).

Now, we shall prove the converse inclusion.

In order to see this, we shall prove that \( Bw_0 \supset Bw'x_0 \) implies \( w \preceq w' \) by the induction of \( l(w') \). If \( l(w') = 0 \), \( w \neq 1 \) implies \( Bw_0 \cap Bx_0 = 0 \). Hence \( Bw_0 \cap Bx_0 = 0 \). Assume that \( l(w') > 0 \). Then there is \( i \) such that \( l(s_iw') < l(w') \). Thus we have \( Bw_0 \supset Bw_0 \supset Bw'x_0 = Bw'x_0 \) by Lemma 4.5.2.

If \( l(s_iw) < l(w) \), then by Lemma 4.5.2, \( Bw_0 \supset Bw_0 \supset Bw'x_0 \) and hence \( s_iw \geq s_iw' \), which implies \( w \geq w' \).

If \( l(s_iw) > l(w) \), then \( Bw_0 \supset Bw_0 \supset Bw'x_0 \) and hence \( w \geq s_iw \geq w' \).

**Proposition 4.5.12.** \( BwBx_0 = \cup_{w' \preceq w} \overline{Bw'x_0} \).

**Proof.** By Lemma 4.5.3, it is enough to show \( BwBx_0 \supset Bw'x_0 \) implies \( w \preceq w' \), or equivalently

\[
(4.5.8) \quad wBx_0 \cap Bw'x_0 \neq \emptyset \quad \text{implies} \quad w \preceq w'.
\]

We shall prove this by the induction on \( l(w) \). If \( l(w) = 0 \), this is
already proven. Assume \( l(w) > 0 \). Then there exists \( i \) such that \( w'' = s_i w \)
satisfies \( l(w'') < l(w) \). Then \( wb\mathcal{X}_0 \cap Bw'x_0 \neq \emptyset \) implies \( w''\mathcal{X}_0 \cap Bs_iBw'x_0 \neq \emptyset \).

If \( l(s_iw') < l(w') \), Lemma 4.5.2 implies \( w''\mathcal{X}_0 \cap Bs_iw'x_0 \neq \emptyset \).
Hence the hypothesis of the induction implies \( w'' \geq s_iw' \), which gives \( w \geq w' \). If \( l(s_iw') > l(w') \), then \( w''\mathcal{X}_0 \cap (Bs_iw'x_0 \cup Bw'x_0) \neq \emptyset \).
Hence \( w' \geq s_iw' \) or \( w'' \geq w' \). Hence in the both cases, we have \( w \geq w' \).

**Corollary 4.5.13.** \( Bw\mathcal{X}_0 = \bigcup_{w' \leq w} w'Bw\mathcal{X}_0 \).

**Proof.** If \( w' \leq w \), \( w'Bw\mathcal{X}_0 \subset \bigcup_{w' \leq w} Bw''x_0 \subset BwB\mathcal{X}_0 \). The inverse inclusion follows from \( w'Bw\mathcal{X}_0 \supset Bw'x_0 \) (Lemma 4.5.7).

**Remark 4.5.14.** For \( w, w' \in W \), we have

\[
\overline{Bwx_0} \cap w'Bw\mathcal{X}_0 \cong (U \cap \langle w' \rangle U) \times (\overline{Bwx_0} \cap w'(B \cap \langle w' \rangle B_-)x_0)
\]

because \( w'B\mathcal{X}_0 = (U \cap \langle w' \rangle U) \times w'(B \cap \langle w' \rangle B_-)x_0 \) and \( \overline{Bwx_0} \) is invariant by \( U \cap \langle w' \rangle U \). Then \( \overline{Bwx_0} \cap w'(B \cap \langle w' \rangle B_-)x_0 \) is a finite-dimensional variety. Thus, \( \overline{Bwx_0} \) is locally finite-dimensional or the product of a finite-dimensional variety and \( A^\infty \).

**Proposition 4.5.15.** \( X \) is irreducible.

**Proof.** Since \( X = \bigcup wB\mathcal{X}_0 \) is an open covering by irreducible subsets, it is enough to show \( wB\mathcal{X}_0 \cap w'B\mathcal{X}_0 \neq \emptyset \) for any \( w, w' \). This follows from \( Bw^{-1}wB\mathcal{X}_0 \supset B\mathcal{X}_0 \) (Proposition 4.5.12).

5. **The second construction of the flag variety.**

5.1. Following Kac-Peterson [K-P], we shall first define the ring of regular functions. Recall that \( U(\mathfrak{g})^* \) has the structure of two-sided \( \mathfrak{g} \)-modules (Section 3.3).

**Definition 5.1.1.** \( A(\mathfrak{g}, P) = \bigoplus_{\mu \in \mathcal{P}} \{ \varphi \in U(\mathfrak{g})^* ; \varphi \) satisfies the following conditions (5.1.1) and (5.1.2) \}.

(5.1.1) \( \varphi \) is finite with respect to the left action of \( p_i \) and the right action of \( p_i \) for all \( i \).

(5.1.2) \( \varphi \) is a weight vector of weight \( \mu \) with respect to the left action of \( t \).
**Lemma 5.1.2.**  \( A(\mathcal{G}, P) \) is a subring of \( U(\mathcal{G})^* \).

This easily follows from the fact that \( \delta : U(\mathcal{G}) \to U(\mathcal{G}) \otimes U(\mathcal{G}) \) is \( p_i \)-linear with respect to the left and right actions.

**Definition 5.1.3.** We define \( G_\infty \) as \( \text{Spec}(A(\mathcal{G}, P)) \).

**Lemma 5.1.4.** Let \( V \) be a \( p_i \)-module, and \( v \in V \).

(i) If \( v \) is \( b \)-finite, then \( f_i v \) is also \( b \)-finite.

(ii) If \( v \) is \( b \)-finite and \( f_i^N v = 0 \) for \( N \gg 0 \), then \( v \) is \( p_i \)-finite.

**Proof.** Since \( [b, f_i] \subset p_i = b + \mathbb{C}f_i \), we have

\[
(5.1.3) \quad U(b)f_i \subset U(b) + f_i U(b).
\]

This shows (i). If \( f_i^N v = 0 \), then \( U(p_i) v = \sum_{k<N} U(b) f_i^k v \), which shows (ii).

**Lemma 5.1.5.** Let \( V \) be a \( \mathcal{G} \)-module. Then, for any \( i \in I \), the set of \( p_i \)-finite vectors is a sub-\( \mathcal{G} \)-module.

**Proof.** It is enough to show that if \( v \) is a \( p_i \)-finite vector then \( f_j v \) is also \( p_i \)-finite vector for \( j \neq i \). By the preceding lemma, \( f_j v \) is \( b \)-finite. Hence it is enough to show \( f_i^N f_j v = 0 \) for \( N \gg 0 \). But this follows from (3.1.7) and \( f_i^N f_j v = \sum_k (a_{df_i}^k f_j) f_i^{N-k} v \).

**Lemma 5.1.6.** For any \( \lambda \in t^0 \), \( \lambda + N\alpha_i \) is not a weight of \( U(n_i) \) except finitely many \( N \in \mathbb{Z} \).

**Proof.** We may assume that \( \lambda \) is a weight of \( U(n_i) \) and \( I \) is finite. For \( \lambda = \sum \alpha_j \otimes \mathbb{Z} \alpha_j \), set \( |\lambda|' = \sum_{j \neq i} m_j \). Then if \( \alpha \) is a weight of \( n_i \), then \( |\alpha|' > 0 \). Now assume \( \lambda + N\alpha_i \) is a weight of \( U(n_i) \). Then

\[
\lambda + N\alpha_i = \sum_{\nu=1}^r \gamma_\nu
\]

where \( \gamma_\nu \) are weights of \( n_i \). Hence \( |\lambda|' = \sum_{\nu=1}^r |\gamma_\nu|' \). Hence \( r \leq |\lambda|' \) and \( |\gamma_\nu|' \leq |\lambda|' \). Since for any root \( \beta \), there is only finitely many roots of the form \( \beta + N\alpha_i \), there are only finitely many possibilities for \( \gamma_\nu \). Thus we obtain the result.

**Lemma 5.1.7.**

(i) \( [n_i, f_i] \subset n_i \).
(ii) \((\text{ad} f_i)\) acts locally nilpotently on \(U(n_i)\).
(iii) For any two-sided ideal \(I\) of \(U(n_i)\) such that \([t, I] \subseteq I\) and \(\dim(U(n_i)/I) < 0\), there exists \(N\) such that

\[(a) \ (\text{ad} f_i)^m U(n_i) \subseteq I \text{ for } m \geq N.\]
\[(b) \ f_i^{N+m} U(n_i) \subseteq IC[f_i] + U(n_i)C[f_i] f_i^m \text{ for } m \geq 0.\]

Proof.

(i) follows from \((\Delta_+ - \alpha_i) \cap \Delta \subseteq \Delta_+ \setminus \{\alpha_i\}\).
(ii) follows from the fact that weights of \(U(n_i)\) belong to \(\Sigma Z_{\geq 0} \alpha_j\).
(iii) In order to see (a), it is enough to show, for any weight \(\beta\) of \(U(n_i)\), \(\beta + N\alpha_i\) is not a weight of \(U(n_i)\) if \(N \gg 0\). This follows from Lemma 5.1.6. (b) follows from (a) and \(f_i^{N+m} U(n_i) \subseteq \Sigma ((\text{ad} f_i)^k U(n_i)) f_i^{N+m-k}\).

**Lemma 5.1.8.** If \(\phi \in U(\mathfrak{g})^*\) is left \(b\)-finite and right \(p^-\)-finite, then \(\phi\) is left \(p^-\)-finite.

**Proof.** By Lemma 5.1.4, it is enough to show

\[(5.1.5) \quad L(f_i)^N \phi = 0 \quad \text{for} \quad N \gg 0.\]

There exists a two-sided ideal \(I\) of \(U(b)\) such that \(\phi(IU(\mathfrak{g})) = 0\) and \(\dim U(b)/I < \infty\). Then by the preceding lemma, there exists \(N\) such that

\[f_i^{N+m} U(n_i) \subseteq IU(\mathfrak{g}) + U(n_i) f_i^m U(p^-) \quad \text{for} \quad m \geq 0.\]

Since \(U(\mathfrak{g}) = U(n_i)U(p^-)\), we have

\[
\phi(f_i^{N+m} U(\mathfrak{g})) \subseteq \phi(IU(\mathfrak{g}) + U(n_i) f_i^m U(p^-))
\]

\[\subseteq \{R(f_i)^m R(U(p^-))\phi\}(U(\mathfrak{g})) = 0\]

for \(m \gg 0\).

**Proposition 5.1.9.** \(\mathcal{O}(G_\infty)\) is a two-sided sub-\(\mathfrak{g}\)-module of \(U(\mathfrak{g})^*\). This follows immediately from Lemma 5.1.5.

Let \(e \in G_\infty\) be the point given by \(U(\mathfrak{g}) \rightarrow U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g} \cong \mathbb{C}\).

**Theorem 5.1.10.**

(i) \(P_i\) acts on \(G_\infty\) from the left and \(P^-\) acts on \(G_\infty\) from the right.
(ii) The action of $B$ on $G_\infty$ induced from the one of $P_i$ does not depend on $i$.

(iii) For $g \in G_i$, $ge = eg$.

Proof. The multiplication homomorphism $\mu_i : U(p_i) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ gives a $\varphi : U(\mathfrak{g})^* \to (U(p_i) \otimes U(\mathfrak{g}^*))^*$. We shall show that

$$
(5.1.6) \quad \varphi(\Theta(G_\infty)) \subset \Theta(P_i) \otimes \Theta(G_\infty).
$$

Then $\varphi$ is a ring homomorphism and defines $P_i \times G_\infty \to G_\infty$. It is easy to check this is an action of $P_i$. Similarly $U(\mathfrak{g}) \otimes U(p_i^\tau) \to U(\mathfrak{g})$ defines $G_\infty \times P_i^\tau \to G_\infty$ and it gives the right action of $P_i^\tau$ on $G_\infty$. The rest is easy to check. Now, we shall show (5.1.6).

Let $f \in \Theta(G_\infty)$. Then by the definition, there exists a two-sided ideal $I$ of $U(p_i)$ such that $f(IU(\mathfrak{g})) = 0$, $U(p_i)/I$ is finite-dimensional and that $t$ acts semisimply and the weights belong to $P$.

Hence $f \circ \mu_i : U(p_i) \otimes U(\mathfrak{g}) \to \mathbb{C}$ splits to $U(p_i) \otimes U(\mathfrak{g}) \to (U(p_i)/I) \otimes U(\mathfrak{g})$. Hence $f$ belongs to $(U(p_i)/I)^* \otimes U(\mathfrak{g})^* \subset \Theta(P_i) \otimes U(\mathfrak{g})^*$. Write $f = \sum \varphi_k \otimes \psi_k$ with $\varphi_k \in \Theta(P_i)$ and $\psi_k \in U(\mathfrak{g})^*$, such that $\{\varphi_k\}$ is linearly independent. Then there are $R_k \in U(p_i)$ such that $\varphi_k(R_k) = \delta_{kk'}$. Then $\psi_k(P) = f(R_kP)$ for any $P \in U(\mathfrak{g})$. Hence $\psi_k \in \Theta(G_\infty)$ by Proposition 5.1.9.

5.2. For $\Lambda \in t^0$, let us denote $K_\Lambda \in U(\mathfrak{g})^*$ given by

$$
(5.2.1) \quad K_\Lambda : U(\mathfrak{g}) \longleftrightarrow U(n) \otimes U(t) \otimes U(n_-) \longrightarrow U(t) \longrightarrow \mathbb{C}
$$

where the middle arrow is given by $U(n) \to U(n)/U(n)n \cong \mathbb{C}$ and $U(n_-) \to U(n_-)/U(n_-)n_- \cong \mathbb{C}$ and the last arrow is given by $h \mapsto -\Lambda(h)$. We have in the ring $U(\mathfrak{g})^*$

$$
(5.2.2) \quad K_{\Lambda_1} \cdot K_{\Lambda_2} = K_{\Lambda_1 + \Lambda_2} \quad \text{for} \quad \Lambda_1, \Lambda_2 \in t^0.
$$

(5.2.3) \quad L(h)K_\Lambda = \langle \Lambda, h \rangle K_\Lambda \quad \text{and} \quad R(h)K_\Lambda = -\langle \Lambda, h \rangle K_\Lambda

for $h \in t$, $\Lambda \in t^0$.

**Lemma 5.2.1.** Let $\varphi \in U(\mathfrak{g})^*$ be a left $b$-finite and right $b_-$-finite element, $a$, $b$ nonnegative integers. Assume that

$$
(5.2.4) \quad R(f_i)^{1+a} R(U(n_-)) \varphi = 0.
$$
(5.2.5) Either $R(e_i)^{1+b}(R(U(n_-)\varphi)|_{U(b)}) = 0$ or $L(e_i)^{1+b}L(U(n)\varphi) = 0$.

(5.2.6) Assume that $t$ acts, by $R$, semisimply on $(R(U(b_-))\varphi)|_{U(b)} \subset U(b)^*$ and its weight $\Lambda$ satisfies $\Lambda(h_i) \leq -a - b$ and $\Lambda(h_i) \in \mathbb{Z}$.

Then $\varphi$ is $p_i$-finite.

Proof. Let $N$ be an integer such that $N \geq 1 - \Lambda(h_i)$ for any weight $\Lambda$ of $R(U(b_-))\varphi|_{U(b)}$. By Lemma 5.1.4, it is enough to show

(5.2.7) $L(f_i)^{N+m}\varphi = 0$ if $m \gg 0$.

Let $I$ be the ideal of $U(b)$ given by $\{P \in U(b); L(P)\varphi = 0\}$. Then by Lemma 5.1.7 we have $f_i^{N+m}U(\mathfrak{g}) \subset U(n_i)f_i^N\mathcal{C}[e_i]U(b_-) + IU(\mathfrak{g})$. We have

(5.2.8) $f_i^Ne_i^k = \sum \frac{N!k!}{(N - \nu)!(k - \nu)!} e_i^{k-r}(-h_i - N - k + 2\nu; \nu)f_i^{N-r}$

where $(x; n) = x(x - 1) \cdots (x - n + 1)/n!$.

We obtain

(5.2.9) $\varphi(f_i^{N+m}U(\mathfrak{g})) \subset \sum_{0 \leq \nu \leq k, N} \varphi(U(n_i)e_i^{k-r}(-h_i - N - k + 2\nu; \nu)f_i^{N-r}U(b_-))$.

Hence it is enough to show

(5.2.10) $\varphi(U(n_i)e_i^{k-r}(-h_i - N - k + 2\nu; \nu)U(t)f_i^{N-r}U(n_-)) = 0$

for $0 \leq \nu \leq k, N$.

If $N - \nu \geq 1 + a$, (5.2.10) holds by (5.2.4). If $k - \nu \geq 1 + b$, (5.2.10) holds by (5.2.5). Hence we may assume $0 \leq N - \nu \leq a$ and $0 \leq k - \nu \leq b$. Then in this case, it is enough to show

(5.2.11) $(R((-h_i - N - k - 2\nu; \nu))R(U(b_-))\varphi)|_{U(b)} = 0$.

This is true, if for any weight $\Lambda$ of $R(U(b_-))\varphi|_{U(b)}$ satisfies

$0 \leq -\Lambda(h_i) - N - k + 2\nu \leq \nu - 1$.  

This is true if \( N \geq 1 - \Lambda(h_i), \) 0 \( \leq N - \nu \leq a \) and 0 \( \leq k - \nu \leq b. \)

**Corollary 5.2.2.** \( K_\Lambda \in \Theta(G_\infty) \) if \( \Lambda \in P_+. \)

In fact, we can apply the preceding lemma with \( a = b = 0. \)

### 5.3.

For a subset \( J \) of \( I, \) we set

\[
\Delta_J = \Delta \cap \left( \sum_{j \in J} \mathbf{Z} \alpha_j \right) \quad \text{and} \quad \Delta_J^\pm = \Delta^\pm \cap \Delta_J,
\]

\[
\mathcal{G}_J = t \oplus \bigoplus_{\alpha \in \Delta_J} \mathcal{G}_\alpha; \quad n_J^\pm = \bigoplus_{\alpha \in \Delta_+ \setminus \Delta_J} \mathcal{G}_\alpha.
\]

Then \( \mathcal{G} = n_J^+ \oplus \mathcal{G}_J \oplus n_J^- \) and \( U(\mathcal{G}) \cong U(n_J^+) \otimes U(\mathcal{G}_J) \otimes U(n_J^-). \)

We have

\[
[\mathcal{G}_J + n_J^+, n_J^+] \subset n_J^+.
\]

Since \( \mathcal{G}_J \) is also a Kac-Moody algebra, we set \( G_{J, \infty} \) the corresponding variety \( \text{Spec}(A(\mathcal{G}_J, P)). \) We also set \( U_J, U_J^+ \) the subgroups of \( U \) and \( U^- \) with the Lie algebra \( \hat{n}_J^+ \) and \( \hat{n}_J^- \). Set

\[
A_J = \bigoplus_{\mu \in P} \{ \varphi \in U(\mathcal{G})^*; \varphi \text{ is a weight vector of weight } \mu \text{ with respect to the left action of } t \text{ and } \varphi \text{ is left } p_j^- \text{-finite and right } p_j^- \text{-finite for any } j \in J \text{ and } \varphi \text{ is left } b^- \text{-finite and right } b^{-} \text{-finite} \}.
\]

Then we can easily show that

\[
A_J \text{ is a subring of } U(\mathcal{G})^* \text{ and a two-sided sub-} \mathcal{G} \text{-module of } U(\mathcal{G})^*.
\]

**Lemma 5.3.4.** \( A_J \cong \Theta(U_J) \otimes \Theta(G_J) \otimes \Theta(U_J). \)

**Proof.** We have

\[
\Theta(U_J) \otimes \Theta(G_J) \otimes \Theta(U_J)
\]

\[
\subset (U(n_J^+) \otimes U(\mathcal{G}_J) \otimes U(n_J^-))^* \cong (U(\mathcal{G}))^*.
\]

We shall show first \( A_J \subset \Theta(U_J) \otimes \Theta(G_J) \otimes \Theta(U_J). \) For \( f \in A_J, \) let \( \mathcal{G} \) be the annihilator in \( U(b) \) of \( L(U(b))f. \) Then \( f : U(\mathcal{G}) \rightarrow \mathbb{C} \) splits into \( U(\mathcal{G}) \cong U(n_J) \otimes U(\mathcal{G}_J) \otimes U(n_J^-) \rightarrow (U(n_J)/(\mathcal{G} \cap U(n_J))) \otimes U(\mathcal{G}_J) \otimes U(n_J^-). \) Hence \( f \) belongs to \( \Theta(U_J) \otimes (U(\mathcal{G}_J) \otimes U(n_J^-))^*. \) Similarly \( f \) belongs to
\[(U(n_J) \otimes U(G_J)) \otimes \Theta(U_J),\] and hence to the intersection \(\Theta(U_J) \otimes U(G_J)^* \otimes \Theta(U_J).\) Write \(f = \sum_{k=1}^N \varphi_k \otimes \psi_k \otimes \xi_k\) with \(\varphi_k \in \Theta(U_J), \psi_k \in U(G_J)^*, \xi_k \in \Theta(U_J).\) We take an expression such that \(N\) is minimal among them. Then there are \(S_k^* \in U(n_J)\) and \(R_k^* \in U(n_J)\) such that 

\[\varphi_k(S_k^*) \psi_k(R_k^*) = \delta_{kk}.\] Hence \(\psi_k(P) = f(S_k^*PR_k^*).\) Since \(A_J\) is a two-sided \(G\)-module, \(\psi_k\) belongs to \(\Theta(G_J).\)

We shall prove the converse inclusion \(A_J \supseteq \Theta(U_J) \otimes \Theta(G_J) \otimes \Theta(U_J).\) In order to see this, it is enough to show that any element in \(\Theta(U_J) \otimes \Theta(G_J) \subset (U(n_J \oplus G_J))^*\) is \(b\)-finite and \(p_j\)-finite for any \(j \in J.\) For any \(\varphi \in \Theta(U_J),\) there exists a two-sided ideal \(\mathcal{G}\) of \(U(n_J)\) such that \([b, \mathcal{G}] \subset \mathcal{G},\) \(\dim U(n_J)/\mathcal{G}\) and \(\varphi(\mathcal{G}) = 0.\) For any \(\psi \in \Theta(G_J),\) there exists an ideal \(k\) of \(U(G_J \cap b)\) such that \(\dim (U(G_J \cap b)/k) < \infty\) and \(\psi(k) = 0.\) Since \(bU(n_J)\) \(\subset U(n_J) + U(n_J)(b \cap G_J),\) \(U(n_J) \otimes k + \mathcal{G} \otimes U(G_J)\) is a left \(b\)-module. Since \(\varphi \otimes \psi\) decomposes into

\[U(n_J) \otimes U(G_J) \rightarrow U(n_J + G_J)/(U(n_J) \otimes kU(G_J) + \mathcal{G} \otimes U(G_J))\]

\[\equiv (U(n_J)/\mathcal{G}) \otimes (U(G_J)/kU(G_J)),\]

\(\varphi \otimes \psi\) is \(b\)-finite.

We have

\[\text{ad}_{f_i}^N U(n_J) \subset \mathcal{G} \quad \text{for} \quad N \gg 0 \quad \text{for} \quad i \in J.\]

In fact, this follows from the fact that for any \(\lambda \in \mathfrak{l}^0, \lambda + m\alpha_i\) is a weight of \(U(n_J)\) except finitely many integer \(m\) (Lemma 5.1.6). Hence \(\varphi \otimes \psi\) is \(f_i\)-finite. Thus, \(\varphi \otimes \psi\) is \(p_j\)-finite for any \(j \in J.\) Since \(\varphi \otimes \psi\) is \(b\)-finite, we obtain \(\varphi \otimes \psi \in A_J.\)

**Proposition 5.3.5. ([K-P]).** \(A_J = \Theta(G_\infty)[K^\Lambda_1^{-1}; \Lambda \in P_+, h_j(\Lambda) = 0\) for \(j \in J].\)

**Proof.** Since \(K_\Lambda\) is invertible in \(\Theta(G_{j_\infty})\) if \(h_j(\Lambda) = 0\) for \(j \in \Lambda,\) we have

\[A_J \supseteq \Theta(G_\infty)[K^\Lambda_1^{-1}; \Lambda \in P_+, h_j(\Lambda) = 0\) for \(j \in J].\]

Now, we shall show the converse inclusion.

Let \(\varphi \in A_J.\) Then there exists \(a > 0\) such that \(R(n_-)^{1+a}\varphi = L(n)^{1+a}\varphi = 0.\) Let \(S\) be the set of weights of \(R(U(b_-))\varphi\) with respect to the right
action of $t$. Taking a sufficiently large, we may assume that $\langle \lambda, h_i \rangle \leq a$ for any $i \in I$ and $\lambda \in S$. Moreover, there exists a finite set $K$ of $I$ such that $R(e_i) \varphi = L(e_i) \varphi = 0$, $\langle \lambda, h_i \rangle = 0$ for any $i \in I \setminus K$ and $\lambda \in S$.

Now, let $A \in P_+$ be such that $h_j(A) = 0$ for $j \in J$ and $h_j(A) \geq a$ for $j \in K \setminus J$. Then $\varphi \cdot K_A$ is $p_j$-finite for $j \in J$ and $p_j$-finite for $j \in I \setminus J$ by Lemma 5.2.1. Hence $\varphi K_A \in \mathcal{O}(G_{\infty})$.

5.4. By Proposition 5.3.5, for finite subsets $J$ and $J'$ with $J \subset J'$, Spec$(A_J)$ is an open subscheme of Spec$(A_{J'})$. We set $G_{\alpha f} = \bigcup_J U_J \times G_J \times U_J$ where $J$ ranges through finite subsets of $I$. Then $G_{\alpha f}$ is an irreducible separated scheme, and $U \times T \times U_-$ is an open subscheme of $G_{\alpha f}$. The groups $P_i$ and $P_i^-$ act on $G_{\alpha f}$ from the left and the right, respectively.

Definition 5.4.1. Let $G$ be the smallest open subset of $G_{\alpha f}$ containing $U \times T \times U_-$ closed by the left and right actions of $G_i$ ($i \in I$).

5.5. Hence $G$ is invariant by the left action of $P_i$, and the right action of $P_i^-$. Since $G_{\alpha f}$ is irreducible, $G$ is also irreducible. In Section 6, we shall study more precisely the structure of $G_{\alpha f}$ in the symmetrisable case.

5.6. Since $G_i$ acts on $G_{\infty}$, $G_{\alpha f}$ and $G$, $s_{i'} \in G_i$ acts on them. Then we have the braid condition (3.1.13). In fact, if $i, j \in I$ satisfies $\langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle \leq 3$, then the semisimple part of $G_{i, j}$ is a finite-dimensional group. Thus we can apply the braid condition for finite-dimensional Lie group and hence $s_{i'}$ and $s_{i'}^j$ satisfy the braid condition in $G_{i, j}$. Since we can check easily that $G_{i, j}$ acts on $G_{\alpha f}$, $G_{\alpha f}$ and $G$, we obtain (3.1.13). Thus the braid group $W'$ acts on $G$, $G_{\alpha f}$ and $G_{\infty}$.

Let us embed $W$ into $W'$ by $w \mapsto s_{i_1} \cdots s_{i_r}$ where $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression of $w$.

Lemma 5.6.1. $G = \bigcup_{w \in W} w(U \times T \times U_-)$

$= \bigcup_{w \in W} (U \times T \times U_-)w$.

In fact, we have $P_i^- = G_i \ U_- \text{ and } (U \times T \times U_-)P_i^- = U e \cdot P_i^- = U G_i e \cdot U_- = P_i e \cdot U_-$. Since $P_i \subset s_i B G_i \cup B G_i$, we have $P_i e \cdot U_- \subset s_i B e U_- \cup B e U_-$. Thus $\bigcup_{w \in W} w(U \times T \times U_-)$ is invariant by $P_i^-$. Hence if $A$ (resp. $A'$) is the smallest open subset containing $U \times T \times U_-$ and invariant by $P_i$ (resp. $P_i^-$) for any $i$, we have $A \supset \bigcup_{w \in W} w(U \times T \times U_-) \supset A'$. Similarly $A \subset A'$. Hence $A = A' = \bigcup_{w \in W} w(U \times T \times U_-)$. 
5.7. In general, let \( X \) be a scheme and \( G \) a group scheme acting on \( X \). We say that \( G \) acts locally freely on \( X \) if any point has a \( G \)-stable open neighborhood which is isomorphic to \( G \times U \) for some scheme \( U \). In this case, the quotient \( X/G \) in the Zariski topology is representable by a scheme. Note that \( X/G \) is not necessarily separated even if \( X \) is separated.

5.8. Now, \( B_- \) acts on \( G \) locally freely. Hence \( G/B_- \) is a scheme and covered by open affine subsets \( wU \times B_-/B_- \). Note that we have not yet shown that \( G/B_- \) is a separated scheme.

**Proposition 5.8.1.** \( X \equiv G/B_- \). Here \( X \) is the flag variety defined in Section 4.

**Proof.** We have \( G/B_- = \bigcup_{w \in W} wUB_-/B_- \) and \( X = \bigcup_{w \in W} wU x_0 \). We define for \( w \in W' \), the morphism

\[
\varphi_w : wUB_- \to wU x_0 \quad \text{by} \; \quad wgb_\rightarrow wg.
\]

We shall show

\[
(5.8.1) \quad \varphi_w = \varphi_{w'} \quad \text{on} \quad wUB_- \cap w'UB_-.
\]

This follows from the case where \( w' = 1 \). If \( w = 1 \), this is trivial. If \( w = s_i^\pm 1 \), then this is trivial because \( \varphi_w \) and \( \varphi_1 \) are the restrictions of \( P_i g U_i^- \to X \) given by \( g e g' \rightarrow g x_0 \; (g \in P_i, g' \in U_i^-) \).

Arguing by induction on the length of \( w \), we may assume \( w = s_i^\pm 1 w'' \) and

\[
\varphi_{w''} \big|_{w''U UB_- \cap U UB_-} = \varphi_1 \big|_{w''U UB_- \cap U UB_-}
\]

and hence

\[
\varphi_w \big|_{wU UB_- \cap s_i^\pm 1 U UB_-} = \varphi_{s_i^\pm 1} \big|_{wU UB_- \cap s_i^\pm 1 U UB_-}.
\]

Hence \( \varphi_w \) and \( \varphi_1 \) coincide on \( wU UB_- \cap s_i^\pm 1 U UB_- \cap U UB_- \). Since \( wU UB_- \cap s_i^\pm 1 U UB_- \cap U UB_- \) is open dense in \( wUB_- \cap w'UB_- \) and \( X \) is separated, we have (5.8.1).

Thus, we can construct \( \varphi : G \to X \) such that \( \varphi \big|_{wU UB_-} = \varphi_w \). Taking the quotient, we obtain \( \bar{\varphi} : G/B_+ \to X \).
By the definition, \( \tilde{\varphi} \) is \( W' \)-equivariant. Also, \( \tilde{\varphi} \) is \( B \)-equivariant. This is because \( \varphi_{|_{BeB_-}} \) is \( B \)-equivariant and \( BeB_- \) is open dense in \( G \).

Since \( \tilde{\varphi} \) is clearly a local isomorphism and surjective, it is enough to show that \( \tilde{\varphi} \) is injective. In order to see this, we shall prove that, for two \( C \)-valued points \( g, g' \) of \( G/B_- \), \( \varphi(g) = \varphi(g') \) implies \( g = g' \). Since \( \varphi \) is \( W' \)-equivariant, we may assume \( g \in BeB_-/B_- \). Since \( \varphi \) is \( B \)-equivariant, we may assume \( g = e \mod B_- \). Assume \( g' \in wUeb_-/B_- \) for \( w \in W \). Write \( g' = wuB_-/B_- \) for \( u \in U \). Then \( \varphi(g) = \varphi(g') \) implies \( x_0 = wux_0 \). Hence Proposition 4.5.9 implies \( w = 1 \) and Lemma 4.4.1 implies \( u = 1 \). Hence \( g = g' \).


6.1. In Section 6, we shall assume that the set \( I \) of simple roots is finite and the Kac-Moody Lie algebra is symmetrisable. Then by Gabber-Kac [G-K], any integrable \( U(\mathfrak{g}) \)-module generated by a highest weight vector is semisimple. For \( \Lambda \in P_+ \), let \( L_\Lambda \) be the irreducible \( \mathfrak{g} \)-module with highest weight \( \Lambda \). Then we have

**Lemma 6.1.1.** \(([K-P]). \ A(\mathfrak{g}, P) = \mathcal{O}(G_\infty) \cong \bigoplus_{\Lambda \in P_+} L_\Lambda \otimes L_\Lambda^*.

6.2. We shall assume further that any irreducible finite-dimensional representation of \( \mathfrak{g} \) is one-dimensional. This is equivalent to saying that any connected component of the Dynkin diagram of \( \mathfrak{g} \) is not finite-dimensional. In this case, letting \( P_0 = \{ \Lambda \in P; \langle \Lambda, h_j \rangle = 0 \text{ for any } j \} \), any irreducible finite-dimensional representation is \( C \) with weight \( \Lambda \in P_0 \).

**Lemma 6.2.1.** \( \bigoplus_{\Lambda \in P_+ \setminus P_0} (L_\Lambda \otimes L_\Lambda^*) \) is an ideal of \( A(\mathfrak{g}, P) \).

**Proof.** For \( \Lambda_1, \Lambda_2 \in P_+ \setminus P_0 \),

\[
(L_{\Lambda_1} \otimes L_{\Lambda_1}^*) \cdot (L_{\Lambda_2} \otimes L_{\Lambda_2}^*) \subset \sum_{\Lambda} L_\Lambda \otimes L_\Lambda^*
\]

where \( \Lambda \) ranges over the set \( \Lambda \) with \( L_\Lambda \subset L_{\Lambda_1} \otimes L_{\Lambda_2} \). If \( \Lambda \in P_0 \) and \( L_\Lambda \subset L_{\Lambda_1} \otimes L_{\Lambda_2} \), then we have a homomorphism \( L_{\Lambda_1}^* \otimes L_\Lambda \to L_{\Lambda_2} \). Therefore \( L_{\Lambda_2} \) has a lowest weight vector, which implies \( L_{\Lambda_2} \) is finite-dimensional. Hence \( \Lambda_2 \in P_0 \), which is a contradiction.

**Definition 6.2.2.** Let us define \( \infty \in G_\infty \) by
\[ A(\mathcal{G}, P) \to A(\mathcal{G}, P)/(\sum_{\Lambda \in P_+ \setminus P_0} L_\Lambda \otimes L_\Lambda^*) \cong \bigoplus_{\Lambda \in P_0} CK_\Gamma \to C \]

where the last arrow is given by \( K_\Lambda \mapsto 1 \).

Note that

\[(6.2.1) \quad T \cdot \infty \equiv \text{Spec}(\mathcal{C}[K_\Lambda; \Lambda \in P_0]) \]

\[(6.2.2) \quad P_i \infty = \infty P_i^- = T \cdot \infty \quad \text{for any } i. \]

6.3. Proposition 6.3.1.

\[ G_\infty \setminus T \cdot \infty = \bigcup_{w \in W'} \bigcup_{J \neq I} \langle w(U_J \times G_J \times U_J^-) \rangle = \bigcup_{w \in W'} \bigcup_{J \neq I} \langle (U_J \times G_J \times U_J^-)w \rangle. \]

Proof. The last identity can be proven as in the proof of Lemma 5.6.1. For \( v \in L_\Lambda, w \in L_\Lambda^* \), let us denote by \( \langle v, gw \rangle \) the corresponding function on \( g \in G_\infty \). Now, let \( g \) be an element of \( G_\infty \setminus T \cdot \infty \). Let us denote by \( G_f \) the subgroup of \( \text{Aut}(L_+) \) generated by the \( G_\). By the assumption, there is \( \Lambda \in P_+ \setminus P_0 \) and \( v \in L_\Lambda, w \in L_\Lambda^* \) such that \( \langle v, gw \rangle \neq 0 \). Then \( \langle v', L_\Lambda, \langle G_f v', gw \rangle = 0 \rangle \) is a \( \mathcal{G} \)-module. Hence, it is zero. Therefore, if we denote by \( v_\Lambda \) the highest weight vector of \( L_\Lambda \), then \( \langle G_f v_\Lambda, gw \rangle \neq 0 \). Hence there exists \( g_0 \in G_f \) such that \( \langle v_\Lambda, g_0^{-1} gw \rangle \neq 0 \). Since \( \bigcup w(U_J \times G_J \times U_J^-) \) is invariant by \( G_f \), we may assume from the beginning \( \langle v_\Lambda, gw \rangle \neq 0 \).

Similarly, \( \{w'; \langle v_\Lambda, gG_f w' \rangle = 0\} \) is \( \mathcal{G} \)-invariant and hence it is zero. Therefore if \( v^- \Lambda \) is the lowest weight vector of \( L_\Lambda^* \) such that \( \langle v_\Lambda, v^- \Lambda \rangle = 1 \), then \( \langle v^- \Lambda, gG_f v^- \Lambda \rangle \neq 0 \). Hence replacing \( g \) with an element in \( gG_f \), we may assume \( \langle v_\Lambda, g v^- \Lambda \rangle \neq 0 \). Since \( K_\Lambda(g) = \langle v_\Lambda, v^- \Lambda \rangle \neq 0, g \) belongs to \( U_{I \setminus \{j\}} \times G_{I \setminus \{j\}} \times U_{I \setminus \{j\}}^- \) for \( j \in I \) with \( \langle h_j, \Lambda \rangle \neq 0 \), by Proposition 5.3.5.

7. Example.

7.1. We shall give here one example \( A^{(1)} \). Let \( I \) be \( Z, P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i, \alpha_i = 2\Lambda_i - \Lambda_{i+1} - \Lambda_{i-1} \) and \( h_i \in t \) is given by \( \langle h_i, \Lambda_i \rangle = \delta_{ij} \).

Let \( V' = C^Z = \Pi_{i \in I} C_{V_i}, V_{\leq q} = \Pi_{i \leq q} C_{V_i} \subset V' \) for \( q \in \mathbb{Z} \) and \( V = \bigcup V_{\leq q} \). Let us define \( g \to \text{End}(V) \) by

\[ t \ni h: \sum a_i v_i \ni \sum (\Lambda_i(h) - \Lambda_{i-1}(h)) a_i v_i \]
\[ e_i : \sum a_j v_j \mapsto a_{i+1} v_i \]
\[ f_i : \sum a_j v_j \mapsto a_i v_{i+1} . \]

For \( p \leq q \), let \( \text{GL}_{p,q}(\infty) \) be the subgroup of \( \text{GL}(V) \) given by

\[ \{ g \in \text{End}(V) ; g \mid_{V_{\leq k}} \subset V_{\leq k} \text{ for } k < p \text{ or } k \geq q \text{ and } g \mid_{V_{\leq k}/V_{\leq k-1}} \text{ is invertible for } k < p \text{ or } k > q \text{ and } g \mid_{V_{\leq q}/V_{\leq p-1}} \text{ is invertible} \} . \]

This is an affine group scheme. With matrix expression, \( \text{GL}_{p,q}(\infty) = \{ (g_{ij}) ; g_{ij} = 0 \text{ for } j < i \text{ and } j < p, j < i \text{ and } i \geq q, g_{ii} \text{ invertible for } i < p \text{ or } i > q \text{ and } \det((g_{ij})_{p \leq i, j \leq q}) \text{ is invertible} \} . \) We define the affine group scheme \( \tilde{\text{GL}}_{p,q}(\infty) \) by

\[ \tilde{\text{GL}}_{p,q}(\infty) = \text{GL}_{p,q}(\infty) \times \mathbb{C}^* . \]

We define for \( p' \leq p \leq q \leq q' \) \( \tilde{\text{GL}}_{p,q}(\infty) \rightarrow \tilde{\text{GL}}_{p',q'}(\infty) \) by

\[ (g, c) \mapsto (g, c \det(g \mid_{V_{\leq q}/V_{\leq q}})). \]

Then for \( p'' \leq p' \leq p \leq q \leq q' \leq q'' \),

\[ \tilde{\text{GL}}_{p,q}(\infty) \rightarrow \tilde{\text{GL}}_{p',q'}(\infty) \rightarrow \tilde{\text{GL}}_{p'',q''}(\infty) \]

commutes. We set

\[ \tilde{\text{GL}}(\infty) = \lim_{\to} \tilde{\text{GL}}_{p,q}(\infty), \quad \text{GL}(\infty) = \lim_{\to} \text{GL}_{p,q}(\infty). \]

Then \( \tilde{\text{GL}}(\infty) \) and \( \text{GL}(\infty) \) are ind-objects in the category of schemes with group structure. The group \( \tilde{\text{GL}}_{p,q}(\infty) \) coincides with \( U_J \times G_J \) where \( J = \{ i \in \mathbb{Z} ; p \leq i \leq q \} . \) Note that we have an exact sequence

\[ 1 \rightarrow \mathbb{C}^* \rightarrow \tilde{\text{GL}}(\infty) \rightarrow \text{GL}(\infty) \rightarrow 1, \]

which does not split.
In this case, the flag variety is, under the notation in Corollary 2.2.3, \( \{(W_i)_{i \in \mathbb{Z}}; W_i \in \text{Grass}^i(V), W_i \subset W_{i+1}\} \).

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REFERENCES


