FAMILY OF SOLUTIONS OF A GARNIER SYSTEM

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Let us consider a degenerate Garnier system of the form

$$
\text{(G)}\qquad
\begin{aligned}
&\frac{\partial q_i}{\partial s} = \frac{\partial H_1}{\partial p_i}, \quad \frac{\partial p_i}{\partial s} = -\frac{\partial H_1}{\partial q_i},\\
&\frac{\partial q_i}{\partial t} = \frac{\partial H_2}{\partial p_i}, \quad \frac{\partial p_i}{\partial t} = -\frac{\partial H_2}{\partial q_i} \qquad (i = 1, 2)
\end{aligned}
$$

with the Hamiltonian functions

$$
\begin{aligned}
3H_1 &= \Big(q_2^2 - q_1 - \frac{s}{3}\Big)p_1^2 + 2q_2p_1p_2 + p_2^2\\
&\qquad + 9\Big(q_1 + \frac{s}{3}\Big)q_2\Big(q_2^2 - 2q_1 + \frac{s}{3}\Big) - 3tq_1,\\
3H_2 &= q_2p_1^2 + 2p_1p_2 + 9\Big(q_2^4 - 3q_1q_2^2 + q_1^2 - \frac{s}{3}q_1 - \frac{t}{3}q_2\Big)
\end{aligned}
$$

for $(s, t) \in \mathbb{C}^2$. This system admits singular loci $s = \infty$ and $t = \infty$. For each $s_0 \in \mathbb{C}$, the restriction of (G) to the complex line $s = s_0$ is a fourth order differential equation belonging to PI-hierarchy. Recently, for the first Painlevé hierarchy of $2m$-th order with large parameter, Y. Takei constructed instanton-type formal solutions containing $2m$ integration constants.

In this talk, we give a family of asymptotic solutions of (G) near the singular locus $t = \infty$. By a suitable canonical transformation, the Hamiltonian system (G) is reduced to a system with the Hamiltonian functions

$$
\begin{aligned}
H_1 &= -(4\sqrt{5})^{-1} i\lambda^3 t^{1/2} Q_1P_1 + (4\sqrt{5})^{-1} i\overline{\lambda}^3 t^{1/2} Q_2P_2,\\
H_2 &= \big(-\lambda t^{1/6} - (8\sqrt{5})^{-1} i\lambda^3 s t^{-1/2}\big)Q_1P_1 + \big(-\overline{\lambda} t^{1/6} + (8\sqrt{5})^{-1} i\overline{\lambda}^3 s t^{-1/2}\big)Q_2P_2\\
&\qquad + t^{-1}\big(\kappa_{20}(Q_1P_1)^2 + \kappa_{11}Q_1P_1Q_2P_2 + \kappa_{02}(Q_2P_2)^2\big),
\end{aligned}
$$

where the constants λ, κ_{20}, κ_{11}, κ_{02} are given by

$$
\begin{gathered}
\lambda = 2^{3/4}15^{1/12}e^{-i(\omega-\pi/2)}, \quad \cos 2\omega = \sqrt{5/6}, \quad \sin 2\omega = \sqrt{1/6},\\
\kappa_{20} = (-7 + 2\sqrt{5}\, i)/24, \quad \kappa_{11} = 2\sqrt{30}/5, \quad \kappa_{02} = \overline{\kappa_{20}}\,.
\end{gathered}
$$

Substituting a solution of the new system, for example,

$$\begin{aligned}
Q_1 &= C_1 t^{2\kappa_{20}C_1C_2}\, \exp\bigl(-(6/7)\lambda t^{7/6} - (4\sqrt{5})^{-1} i\lambda^3 s t^{1/2}\bigr),\\
P_1 &= C_2 t^{-2\kappa_{20}C_1C_2}\, \exp\bigl((6/7)\lambda t^{7/6} + (4\sqrt{5})^{-1} i\lambda^3 s t^{1/2}\bigr),\\
Q_2 &= C_3 \exp\bigl(-(6/7)\overline{\lambda} t^{7/6} + (4\sqrt{5})^{-1} i\overline{\lambda}^3 s t^{1/2}\bigr),\\
P_2 &= 0
\end{aligned}$$

into the canonical transformation, we obtain a family of asymptotic solutions of (G) in a sector of the form

$$\bigl\{(s,t) \bigm| |s| < R_0,\, |t| > R_1,\, \theta_0 < \arg t < \theta_1\bigr\}$$

near $t = \infty$.

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