FAMILY OF SOLUTIONS OF A GARNIER SYSTEM

SHUN SHIMOMURA

Let us consider a degenerate Garnier system of the form

\[
\begin{align*}
\frac{\partial q_i}{\partial s} &= \frac{\partial H_1}{\partial p_i}, \quad \frac{\partial p_i}{\partial s} = -\frac{\partial H_1}{\partial q_i}, \\
\frac{\partial q_i}{\partial t} &= \frac{\partial H_2}{\partial p_i}, \quad \frac{\partial p_i}{\partial t} = -\frac{\partial H_2}{\partial q_i}
\end{align*}
\]

(G) for \((s, t) \in \mathbb{C}^2\). This system admits singular loci \(s = \infty\) and \(t = \infty\). For each \(s_0 \in \mathbb{C}\), the restriction of (G) to the complex line \(s = s_0\) is a fourth order differential equation belonging to PI-hierarchy. Recently, for the first Painlevé hierarchy of \(2m\)-th order with large parameter, Y. Takei constructed instanton-type formal solutions containing \(2m\) integration constants.

In this talk, we give a family of asymptotic solutions of (G) near the singular locus \(t = \infty\). By a suitable canonical transformation, the Hamiltonian system (G) is reduced to a system with the Hamiltonian functions

\[
3H_1 = \left(q_2^2 - q_1 - \frac{s}{3}\right)p_1^2 + 2q_2p_1p_2 + p_2^2 + 9\left(q_1 + \frac{s}{3}\right)q_2\left(q_2^2 - 2q_1 + \frac{s}{3}\right) - 3tq_1,
\]

\[
3H_2 = q_2p_1^2 + 2p_1p_2 + 9\left(q_2^4 - 3q_1q_2^2 + q_1^2 - \frac{s}{3}q_1 - \frac{t}{3}q_2\right)
\]

for \((s, t) \in \mathbb{C}^2\). The constants \(\lambda, \kappa_{20}, \kappa_{11}, \kappa_{02}\) are given by

\[
\lambda = 2^{3/4}15^{1/12}e^{-i(\omega - \pi/2)}, \quad \cos 2\omega = \sqrt{5/6}, \quad \sin 2\omega = \sqrt{1/6},
\]

\[
\kappa_{20} = (-7 + 2\sqrt{5}i)/24, \quad \kappa_{11} = 2\sqrt{30}/5, \quad \kappa_{02} = \kappa_{20}.
\]
Substituting a solution of the new system, for example,
\[ Q_1 = C_1 t^{2\pi/20} C_1 C_2 \exp\left( -(6/7)\lambda t^{7/6} - (4\sqrt{5})^{-1}i\lambda^3 st^{1/2} \right), \]
\[ P_1 = C_2 t^{-2\pi/20} C_1 C_2 \exp\left( (6/7)\lambda t^{7/6} + (4\sqrt{5})^{-1}i\lambda^3 st^{1/2} \right), \]
\[ Q_2 = C_3 \exp\left( -(6/7)\lambda t^{7/6} + (4\sqrt{5})^{-1}i\lambda^3 st^{1/2} \right), \]
\[ P_2 = 0 \]
into the canonical transformation, we obtain a family of asymptotic solutions of (G) in a sector of the form
\[ \{(s, t) \mid |s| < R_0, |t| > R_1, \theta_0 < \arg t < \theta_1 \} \]
near \( t = \infty \).

**Department of Mathematics, Keio University,**
3-14-1, Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan