Toward the exact WKB analysis for instanton-type solutions of Painlevé hierarchies

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This is a joint work with T. Kawai, T. Aoki, T. Koike, and partly with N. Honda, Y. Nishikawa, S. Sasaki.

At the Ramis Conference held at Toulouse in September in 2003, as a generalization of the exact WKB analysis for traditional Painlevé transcendents, Kawai proposed a program to analyze (P_J) (J = I, II-1, II-2 or IV) hierarchies of higher order Painlevé equations. After the venue of the Conference we named this program "the Toulouse Project". The purpose of this talk is to discuss to what extent the Toulouse Project is carried out.

Recently the so-called instanton-type solutions of higher order Painlevé equations have been constructed for the (P_{I}) hierarchy. (Cf. [T3]. Note that Koike's recent results reported in this symposium entail that we can construct the instanton-type solutions also for the $(P_{\mathrm{II-2}})$ and (P_{IV}) hierarchies.) The construction of instanton-type solutions is one of the most important steps in the Toulouse Project; the instanton-type solutions are expected to be indispensable for the description of Stokes phenomena, as is exemplified by the explicit connection formula for the traditional (P_{I}) equation given in [T1]. Roughly speaking, instanton-type solutions for higher order Painlevé equations play the role of WKB solutions for linear ordinary differential equations with a large parameter.

Our exact WKB analysis for instanton-type solutions of Painlevé hierarchies, however, is not a straightforward generalization of the exact WKB analysis for linear differential equations. In a sense the exact WKB analysis for Painlevé hierarchies is a generalization of the asymptotic analysis for integral representations of solutions of linear equations; we make full use of the underlying Lax pair as a substitute of integral representations. (The existence of the Lax pair is an expression of the "integrability" of higher order Painlevé equations.) In the talk we review the definition of the Stokes geometry of higher order Painlevé equations ([KKNT]) from this viewpoint, explain several results obtained so far ([KT1, KT2], [T2], [N], [S], [AH]), and list up some important open problems to be discussed in the future.

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