RIMS WorkShop

Introduction to Idealistic Filtration Program

An approach to resolution of singularities in positive characteristics

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Lecture 1

Overview of
Idealistic Filtration Program

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Contents

1 Introduction

$k = \overline{k}$: **alg. closed field,** $\mathrm{char}\, k = p \geq 0$

Problem **For algebraic variety V/k,**

construct resolution of singularities of V

- $\mathrm{char}\, k = 0$ **with** $\forall \dim V$

  $\Rightarrow$ **solved by Hironaka**

- $\mathrm{char}\, k = p > 0$ **with** $\dim V \leq 3$

  $\Rightarrow$ **solved by Abhyankar**

  (partially Cossart-Piltant)

- $\mathrm{char}\, k = p > 0$ **with** $\dim V \geq 4$

  $\Rightarrow$ **open!**

We attack this problem along

Idealistic Filtration Program (IFP)

which is designed to "extend"

constructive algorithm in $\operatorname{char} k = 0$

into arbitrary characteristic case.

Plan of our lectures We have NO complete proof yet. We present

- philosophy and framework of IFP (today)
- candidate of algorithm for "formal uniformization" (2,3,4-th day)
- algebraization problem (5-th day)

Today: we only consider the case with NO exceptional divisors

2 Review for $\mathrm{char}\, k = 0$ approach

Brief review for "classical" approach

(mixture of known algorithms in $\mathrm{char}\, k = 0$).

$\boxed{\text{Strategy}}$ **Construct embedded resolution**

of $V \underset{\text{closed}}{\subset} M$ (M: nonsingular variety/k)!

- **attach inv_P for closed point $P \in M$**

  (inv_P: minimal $\Rightarrow$ V is resolved at P)

- **Blowup the max. locus ($\longleftarrow$nonsingular)**

  of inv_P, and check the decrease of inv_P.

$\boxed{\text{Recipe of } \mathrm{inv}_P}$ **Invariant is of the shape**

$$\mathrm{inv}_P = (\mu_0, \mu_1, \ldots, \mu_t, \infty) \quad (\mu_i \in \mathbb{Q}_{\geq 0}),$$

(in the case with NO exc. divisors!)

and μ_i's are defined as follows:

1. Initial Step

Put $b_0 := 1$,

$R_0 := \mathcal{O}_{M,P}$ **: local ring at** $P \in M$

$\cup$

$I_0 := \mathcal{I}_{V,P}$ **: defining ideal of** V

$\longrightarrow (I_0 \subset R_0, b_0)$ **: initial data**

Put $\mu_0 := \dfrac{\mathrm{ord}_P(I_0)}{b_0}$ $\longleftarrow$ **order of** I_0 **at** P

Example **Put** $P := \mathbf{0} \in M := \mathbb{A}^3_k$,

$f := x^2 - y^3$, **and** $V := \mathrm{V}(f) \subset M$.

Then,

$$R_0 = k[x,y,z]_{\mathfrak{m}_\mathbf{0}},$$
$$I_0 = (x^2 - y^3)R_0,$$

$$\boxed{((x^2 - y^3) \subset k[x,y,z]_{\mathfrak{m}_\mathbf{0}}, 1) \ \textbf{: initial data}}$$

$$\mathrm{ord}_P(I_0) = \mathrm{ord}_\mathbf{0}(x^2 - y^3) = 2$$
$$\mu_0 = 2/1 = 2$$

2. “Restriction” Step

Take

$$D^{\mathrm{ord}_P(I_0)-1}(I_0) \ni \phi_0 \text{ with } \mathrm{ord}_P(\phi_0) = 1$$

$(DJ = J + (\partial g \mid g \in J,\ \partial$: derivation of $R))$

Put $b_1 := \mathrm{ord}_P(I_0)!$,

$$R_1 := R_0/(\phi_0) \quad : \text{local ring at } P \in \mathrm{V}(f)$$

$$\cup$$

$$I_1 := \sum_{j=1}^{\mathrm{ord}_P(I_0)} (D^{\mathrm{ord}_P(I_0)-j} I_0)^{b_1/j}$$

$\longrightarrow (I_1 \subset R_1, b_1)$: **new data**

Put $\mu_1 = \dfrac{\mathrm{ord}_P(I_1)}{b_1}$

Example

Recall $\mathrm{ord}_P(I_0) = 2.\qquad b_1 = 2! = 2.$

$$D^{2-1}(x^2 - y^3) = (x^2 - y^3, 2x, 3y^2)$$

Take $\phi_0 = x.$

Then,
$$\begin{aligned} R_1 &= k[x,y,z]_{\mathfrak{m}_0}/(x) \cong k[y,z]_{\mathfrak{m}_0} \\ I_1 &= ((2x, 3y^2)^2 + (x^2 - y^3))R_1 \\ &= y^3 R_1 \end{aligned}$$

$\boxed{((y^3) \subset k[y,z]_{\mathfrak{m}_0}, 1) \text{ : new data}}$

$$\mathrm{ord}_P(I_1) = \mathrm{ord}_{\mathbf{0}}(y^3) = 3$$

$$\mu_1 = 3/b_1 = 3/2$$

3. Last Step

Replace "initial data" by "new data," go back to Step 2, and repeat same procedure! Continue it until $\mu_{t+1} = \infty$.

Remark

- $H = \mathrm{V}(\phi_0)$: **maximal contact of I_0 at P.**

$$\{Q \in M \mid \mathrm{ord}_Q(I_0) \geq \mathrm{ord}_P(I_0)\} \underset{\text{near } P}{\subset} H$$

By construction, we have more:

- $$\begin{array}{c} \{Q \in M \mid \mathrm{ord}_Q(I_0) \geq \mathrm{ord}_P(I_0)\} \\ \| \quad \text{near } P \\ \{Q \in H \mid \mathrm{ord}_Q(I_1) \geq b_1\} \end{array}$$

Example

$\mathrm{ord}_P(I_1) = 3. \qquad b_2 = 3! = 6.$

$D^{3-1}(y^3) = (y).$ **Take** $\phi_1 = y.$

$$R_2 = k[y,z]_{\mathfrak{m}_0}/(y) \cong k[z]_{\mathfrak{m}_0}$$
$$I_2 = y^6 R_2 = (0), \quad \mathrm{ord}_P(I_2) = \infty$$

$\therefore \quad \mu_2 = \infty. \qquad \boxed{\mathrm{inv}_0 = (2, 3/2, \infty)}$

$H = \mathrm{V}(x)$: **max. cont. of** $x^2 = y^3$ **at 0.**

$$\{Q = (x,y,x) \mid \mathrm{ord}_Q(x^2 - y^3) \geq 2\}$$
$$\|$$
$$\{Q = (0,y,z) \mid \mathrm{ord}_Q(y^3) \geq 2\}$$

Summary Classical case

Invariant is defined in the following scheme:

initial data: pair $(I_0, b_0) = (I_V, 1)$ **on** M

obj.	(I_0, b_0)	(I_1, b_1)	$\cdots$	(I_t, b_t)	$(0, b_{t+1})$
amb.	M	$\supset H_1$	$\cdots$	$\supset H_t$	$\supset H_{t+1}$
order	μ_0	μ_1	$\cdots$	μ_t	∞

$$\Downarrow$$

$$\mathrm{inv}_P = (\mu_0, \mu_1, \ldots, \mu_t, \infty)$$

3 Framework of IFP

Try to apply the "classical" argument to char $k > 0$ case $\Rightarrow$ Fails! since

In positive characteristic, maximal contact does not exist in general

To overcome this hurdle, we introduce

- $\mathbb{I}$: **idealistic filtration (I.F.)**

  (refinement of idealistic exponent, $\cdots$)

  $\downarrow$ **analyzing algebraic structure**

- $\mathbb{H}$: **Leading Generator System (LGS) of $\mathbb{I}$**

  (collective substitute of maximal contacts, with possibly singular elements)

By using LGS as substitute of max. cont, we define inv_P as in previous section.

We also emphasize 2 points:

1. In Classical case, ambient space changed by restricting to max. cont. in each step

$$M \supset H_1 \supset H_2 \supset \cdots$$

In IFP, we stay in the same ambient M, but enlarging I.F. in each step

$$I_0 \subset I_1 \subset I_2 \subset \cdots$$

2. In Classical case, invariant is of the shape

$$\mathrm{inv}_P = (\mu_0, \mu_1, \ldots, \mu_t, \infty)$$

In IFP, invariant is of the shape

$$\mathrm{inv}_P = ((\sigma_0, \mu_0^{\sim}), \ldots, (\sigma_t, \mu_t^{\sim}), (\sigma_{t+1}, \infty))$$

The pair $(\sigma, \mu^{\sim})$ is called paired invariant

Summary IFP case

Invariant is defined in the following scheme:

initial data: I.F. $\mathbb{I}_0 = \mathrm{G}(I_V \times \{1\})$ **on** M

obj.	$\mathbb{I}_0$	$\subset \mathbb{I}_1$	$\cdots$	$\subset \mathbb{I}_t$	$\subset \mathbb{I}_{t+1}$
amb.	M	M	$\cdots$	M	M
order	$(\sigma_0, \mu_0^{\sim})$	$\cdots$	$\cdots$	$(\sigma_t, \mu_t^{\sim})$	(σ_{t+1}, ∞)

$$\Downarrow$$

$$\mathrm{inv}_P = ((\sigma_0, \mu_0^{\sim}), \ldots, (\sigma_t, \mu_t^{\sim}), (\sigma_{t+1}, \infty))$$

4 Idealistic Filtration

R: regular k-algebra, $\mathbb{I} \subset R \times \mathbb{R}$: subset

(We denote $\mathbb{I}_a = \{f \in R \mid (f, a) \in \mathbb{I}\}$)

Definition 1 $\mathbb{I}$ is called idealistic filtration (I.F.) on R if the following condition holds:

1. $\mathbb{I}_0 = R$
2. $\mathbb{I}_a$: ideal of R $(a \in \mathbb{R})$
3. $\mathbb{I}_a \mathbb{I}_b \subset \mathbb{I}_{a+b}$ $(a, b \in \mathbb{R})$
4. $\mathbb{I}_a \supset \mathbb{I}_b$ $(a \leq b)$

Definition 2 $\mathbb{T} \subset R \times \mathbb{R}$: subset

The minimal I.F. containing $\mathbb{T}$ is called the I.F. generated by $\mathbb{T}$ and denoted as $\mathrm{G}(\mathbb{T})$.

Example

If $\mathbb{I} = \mathrm{G}(I \times \{b\})$ **(**I**:ideal,** $b \in \mathbb{R}_{>0}$**)**

$$\mathbb{I}_a = \begin{cases} R & : a \le 0 \\ I & : 0 < a \le b \\ I^2 & : b < a \le 2b \\ I^n & : (n-1)b < a \le nb \end{cases}$$

(I, b)**: pair** $\longleftrightarrow$ $\mathrm{G}(I \times \{b\})$**: I.F.**

Definition 3 Denote $U = \mathrm{maxSpec}\, R$.

We define the support $\mathrm{Supp}(\mathbb{I}) \subset U$ of $\mathbb{I}$ as

$$\mathrm{Supp}(\mathbb{I}) = \{Q \in U \mid \mathrm{ord}_Q(\mathbb{I}_a) \geq a \ (\forall a \in \mathbb{R})\}$$

Saturate I.F. to visualize more information!

Definition 4 $\mathbb{I}$: I.F. on R is called

$\mathfrak{D}$-saturated if the following condition holds:

$\forall \partial \in \mathrm{Diff}^t(R/k)$ (diff. operators of deg $\leq t$),

$$\partial \mathbb{I}_a \subset \mathbb{I}_{a-t} \quad (\forall a \in \mathbb{R})$$

The minimum $\mathfrak{D}$-saturated I.F. containing $\mathbb{I}$ is called $\mathfrak{D}$-saturation of $\mathbb{I}$ denoted as $\mathfrak{D}(\mathbb{I})$.

Example $\mathbb{I} = G((x^2 - y^3) \times \{2\}) \Rightarrow$

$$\mathfrak{D}(\mathbb{I}) = G(\{(x^2 - y^3, 2), (2x, 1), (3y^2, 1)\})$$

5 Leading Generator System

$\mathbb{I}$: $\mathfrak{D}$-saturated I.F. on R

Assumptions (Always assumed)

- $R = (R, \mathfrak{m}) = \mathcal{O}_{M,P}$: local ring at closed point P in non-singular variety M
- $\mu(\mathbb{I}) \geq 1 \quad \left(\mu(\mathbb{I}) := \inf_{a>0} \dfrac{\mathrm{ord}_P(\mathbb{I}_a)}{a}\right)$
  (corresp. to the condition $P \in \mathrm{Supp}(\mathbb{I})$)

Definition 5 $\quad \pi_n : \mathfrak{m}^n \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$: proj.

The leading algebra $\mathrm{L}(\mathbb{I})$ of $\mathbb{I}$ is defined as

$$\mathrm{L}(\mathbb{I}) = \bigoplus_{n\geq 0} \pi_n(\mathbb{I}_n) \subset \mathrm{Gr}(R) \left(= \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \right)$$

($\mathbb{I}_n \subset \mathfrak{m}^n$ since $\mu(\mathbb{I}) \geq 1$)

Example $\quad \mathbb{I} = \mathrm{G}((x^2 - y^3) \times \{2\}) \Rightarrow$

$\mathrm{L}(\mathfrak{D}(\mathbb{I})) \cong \quad k[x] \ (p \neq 2); \ k[x^2] \ (p = 2)$

Observation R: regular, $R/\mathfrak{m} = k$

$$\begin{array}{ccccl} \mathrm{Gr}(R) & \cong & k[X] & : & \text{polynomial ring}/k \\ \cup & & \cup & & \\ \mathrm{L}(\mathbb{I}) & \cong & L & : & \text{graded } k\text{-subalg. of } k[X] \end{array}$$

$\mathbb{I}$: $\mathfrak{D}$-sat. $\Rightarrow L$: stable under differentiation

i.e. $\partial_{X^J} L \subset L \quad (\forall J:$ multi-index$)$

(∂_{X^J} is defined by $\partial_{X^J} X^K = \binom{K}{J} X^{K-J}$)

What can we say on such L?

$\boxed{\text{char}\, k = 0 \Rightarrow L \text{ is generated by } L_1,}$
(L_1: homogeneous part of degree 1 of L)

Example $f = x^2 + xy$,

$L' = k[f] \subset k[x, y, z]$

$\downarrow$ enlarge L' to be stable under diff.

$\partial_x f = 2x + y$, $\partial_y f = x$,

$L'[\partial_x f, \partial_y f] = k[x, y] \subset k[x, y, z]$

$\operatorname{char} k = 0$, $\mathbb{I} = \mathrm{G}(\mathcal{I}_V \times \{\operatorname{ord}_P(I_V)\}) \Rightarrow$

$\overline{h} \in \mathrm{L}(\mathfrak{D}(\mathbb{I}))_1 \leftrightarrow$ **max. cont.** $\mathrm{V}(h)$ **of** $\mathcal{I}_V$

Example $\mathbb{I} = \mathrm{G}((x^2 - y^3) \times \{2\})$

$\mathrm{L}(\mathfrak{D}(\mathbb{I})) = k[x]$, $\mathrm{L}(\mathfrak{D}(\mathbb{I}))_1 = k \cdot x$

From these observations, we can see

the substitute of max. cont. should correspond to generators of $\mathrm{L}(\mathbb{I})$.

How is L **generated in** $p > 0$**?**

Proposition 6 **(Hironaka-Oda)**

S: **polynomial ring**$/k$, $\operatorname{char} k = p > 0$

$L \subset S$: **graded** k**-subalgebra of** S,

stable under differentiation

$$\Rightarrow \begin{cases} \exists\, x_1, \ldots, x_\ell \in S_1 & : k\text{-lin. indep.} \\ \exists\, e_1, \ldots, e_\ell \in \mathbb{Z}_{\geq 0} & : \text{non-neg. integers} \end{cases}$$

s.t. $x_1^{p^{e_1}}, \ldots, x_\ell^{p^{e_\ell}}$ **generate** S **as** k**-algebra.**

Remark Also VALID for $\mathrm{char}\, k = 0$ if we set $p = \infty$. $\Longrightarrow$ $\forall e_i = 0$.
i.e. L is generated by L_1.

Definition 7 A representative $\mathbb{H} \subset \mathbb{I}$ of generators of $L(\mathbb{I})$ in the shape as above is called a leading generator system (LGS) of $\mathbb{I}$. By definition, $\mathbb{H}$ is not unique.

$$\mathbb{H} = \{(h_i, p^{e_i}) \mid 1 \leq i \leq \ell\}$$
$$h_i = x_i^{p^{e_i}} + (\text{higher}).$$

Example LGS $\mathbb{H}$ of $\mathrm{G}((x^2 - y^3) \times \{2\})$:

$$\mathbb{H} = \begin{cases} \{(x, 1)\} & \mathrm{char}\, k \neq 2 \\ \{(x^2 - y^3, 2)\} & \mathrm{char}\, k = 2 \end{cases}$$

Remark $\mathrm{V}(h_i)$ may be singular.

6 Enlargement

(I, b) **Classical**

$\downarrow$ **change level** $b \longrightarrow \mathrm{ord}_P(I) = b \cdot \mu$

$(I, \mathrm{ord}_P(I))$ **on low dim'l ambient** $H \subset M$

$\Downarrow$ **How to Translate?**

$\mathbb{I}$ **with LGS** $\mathbb{H}$ **IFP**

$\cap$ **"enlargement"**

$\mathbb{I}'$ **on original ambient** M

Rough idea

- **divide** $\mathbb{I}$ **into** $\mathbb{H}$ **and "remainder w.r.t.** $\mathbb{H}$**"**

$$\mathbb{I} \quad = \quad \text{“ } \mathbb{H} + (\textbf{Remainder})\text{”}$$

- **and change level of "remainder" part**

$$\mathbb{I}' \quad = \quad \text{“ } \mathbb{H} + (\textbf{level-adjusted Remainder}) \text{ ”}$$

Following example is the idealistic case.

Example **Translate as follows:**

$$\text{pair: } (x^2 - y^3) \subset k[x,y,z]_{\mathfrak{m}_0}/(x), 2)$$

$$\Downarrow$$

$$\text{I.F.: } \mathrm{G}((x,1),(x^2-y^3,2)) \text{ on } k[x,y,z]_{\mathfrak{m}_0}$$

and

$$\text{pair: } ((x^2 - y^3) \subset k[x,y,z]_{\mathfrak{m}_0}/(x), 3)$$

$$\Downarrow$$

$$\text{I.F.: } \mathrm{G}((x,1),(y^3,3)) \text{ on } k[x,y,z]_{\mathfrak{m}_0}$$

Is it always possible?

Yes, but in formal level. In $\widehat{R}$, we have

$$\boxed{\begin{aligned}\widehat{\mathbb{I}} &= \mathrm{G}(\mathbb{H} \cup \{(c_0(f), a) \mid (f,a) \in \mathbb{I}\}) \\ \widehat{\mathbb{I}}' &= \mathfrak{D}(\mathrm{G}(\mathbb{H} \cup \{(c_0(f), \mu^{\sim} a) \mid (f,a) \in \mathbb{I}\}))\end{aligned}}$$

where $c_0(f)$ is determined as follows:

Remainder in completion Take

$$\begin{cases} \mathrm{x} = \{x_1, \ldots, x_\ell\}, \mathrm{y} \subset R \\ \mathbb{H} = \{(h_i, p^{e_i}) \mid 1 \le i \le \ell\}\text{: LGS of } \mathbb{I} \end{cases}$$

such that

$$\begin{cases} h_i \in x_i^{p^{e_i}} + \mathfrak{m}^{p^{e_i}+1} \quad (1 \le i \le \ell) \\ \{\mathrm{x}, \mathrm{y}\}\text{: reg. sys. of par's(RSP) of } R \end{cases}$$

Proposition 8 Regard $\widehat{R} = k[[\mathrm{x}, \mathrm{y}]]$. Then,

$f \in \widehat{R} \Rightarrow \exists!\ c_0(f) \in k[[y]][x] \subset \widehat{R}$

s.t. $$\begin{cases} f - c_0(f) \in \sum_{i=1}^{r} h_i \widehat{R} \\ \deg_{x_i}(c_0(f)) < p^{e_i} \quad (1 \le i \le \ell) \end{cases}$$

$c_0(f)$: "the remainder of f w.r.t. $\mathbb{H}$".

Descent to Zariski local level Not finished.

Later we will investigate this subject again.

7 Paired invariants

$\mathbb{I}$: $\mathfrak{D}$-saturated I.F. on R as before,

$\mathbb{H} = \{(h_i, p^{e_i}) \mid i\}$: LGS of $\mathbb{I}$

Definition 9 (σ: "dimension") Define

$$\sigma(\mathbb{I}) = (\sigma_0, \sigma_1, \ldots) \in \mathbb{Z}_{\geq 0}^{\infty}$$

where $\sigma_e = \dim R - \#\{i \mid e_i \leq e\}$

Example $\mathbb{I} = \mathfrak{D}(\mathrm{G}((x^2 - y^3) \times \{2\}))$:

$$\mathbb{H} = \begin{cases} \{(x, 1 = p^0)\} & (p \neq 2) \\ \{(x^2 - y^3, 2 = p^1)\} & (p = 2) \end{cases}$$

$\dim R = \dim k[x, y, z]_{\mathbf{0}} = 3$

$$\Rightarrow \sigma(\mathbb{I}) = \begin{cases} (2, 2, 2, \ldots) & p \neq 2 \\ (3, 2, 2, \ldots) & p \neq 2 \end{cases}$$

Remark $\operatorname{char} k = 0 \Rightarrow \sigma(\mathbb{I})$: const. seq.

Definition 10 ($\mu^{\sim}$: order mod. $\mathbb{H}$) Define

$$\mathrm{ord}_{\mathbb{H}}(J) = \sup\{n \in \mathbb{Z}_{\geq 0} \mid J \subset \mathfrak{m}^n + \sum_i Rh_i\}$$

and $$\mu^{\sim}(\mathbb{I}) = \inf_{a>0} \frac{\mathrm{ord}_{\mathbb{H}}(\mathbb{I}_a)}{a}$$

Example

$$\begin{aligned}\mathbb{I} &= \mathfrak{D}(\mathrm{G}((x^2 - y^3) \times \{2\})) \\ &= \mathrm{G}((x^2 - y^3, 2), (2x, 1), (3y^2, 1))\end{aligned}$$

$$\mathbb{H} = \begin{cases} \{(x, 1)\} & (p \neq 2) \\ \{(x^2 - y^3, 2)\} & (p = 2) \end{cases}$$

$p \neq 2 \Rightarrow$ modulo (x) $\Rightarrow \mu^{\sim}(\mathbb{I}) = \frac{3}{2}$

$p = 2 \Rightarrow$ modulo $(x^2 - y^3)$ $\Rightarrow \mu^{\sim}(\mathbb{I}) = 2$

Proposition 11 If $\mathbb{I}$ is $\mathfrak{D}$-saturated, $\sigma(\mathbb{I})$ and $\mu^{\sim}(\mathbb{I})$ are independent of the choice of LGS $\mathbb{H}$.

8 Basic Results

We have to arrange the situation to function without "nonsingularity of max. cont." These results are important in this context.

Theorem 12 (U.S.C. of paired inv.)

$$\begin{cases} \mathrm{Spec}\, R & : \text{nonsingular affine variety}/k \\ \mathbb{I} & : \mathfrak{D}\text{-saturated I.F. on } R \end{cases}$$

$\Rightarrow$ $(\sigma_P(\mathbb{I}), \widetilde{\mu_P}(\mathbb{I}))$ with lex. order is upper semi-continuous on $P \in \mathrm{maxSpec}\, R$.

We look at only $\mathrm{maxSpec}\, R$. It is enough since we define inv_P for only closed points P as in classical approach.

Theorem 13 (NonSingularity Principle)

Let $\mathbb{I}$ be $\mathfrak{D}$-saturated I.F. on $R = \mathcal{O}_{M,P}$ with $\mu(\mathbb{I}) \geq 1$. Assume $\mu^{\sim}(\mathbb{I}) = \infty$. Then,

1. $\mathbb{I}$ is generated by LGS $\mathbb{H}$ of $\mathbb{I}$.

2. $\begin{cases} \exists\{x_i \mid i\} \subset R & : \text{a part of RSP of } R \\ \exists\{e_i \mid i\} \subset \mathbb{Z}_{\geq 0} & : \text{non-neg. integers} \end{cases}$

   such that $\left\{(x_i^{p^{e_i}}, p^{e_i}) \mid i\right\}$ is an LGS of $\mathbb{I}$.

This theorem says:

In the final step of defining invariant, $\mu^{\sim}(\mathbb{I}) = \infty$. Then, Support of $\mathbb{I}$ is germ of nonsingular variety. i.e.

$$\mathrm{Supp}(\mathbb{I}) = \mathrm{V}(\{x_i \mid i\}) \quad \text{near } P$$