RIMS Workshop

# Introduction to Idealistic Filtration Program

## An approach to resolution of singularities in positive characteristics

**Hiraku Kawanoue and Kenji Matsuki**

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# Lecture 3

## Can we lift

## the algorithm in $\mathrm{char} = 0$

## via $(\sigma, \widetilde{\mu}, s)$-method

## to the one in $\mathrm{char} = p > 0$ ?

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# 1 Question of translating the algorithm

$\boxed{\text{Question}}$ **Can we translate**

**the algorithm in char $= 0$ via $(\sigma, \widetilde{\mu}, s)$-method**

**into the one in char $= p > 0$ ?**

$\boxed{\text{Answer}}$ **YES !**

$\boxed{\text{Basic structure remains the same.}}$

**Inductive weaving of the strand**

**& construction of the modification**

- **Unit $(\sigma, \widetilde{\mu}, s)$ makes sense**

  **via the notion of LGS $\mathbb{H}$.**

- **Modification $(\mathbb{I}, E)$ makes sense**

  **via the construction of**

  **"Cpc" (at the analytic level) and "Bd"**

**Termination in the horizontal direction**

- **Main mechanism of induction on $(\sigma, t)$**

  **is valid.**

## Choice of the center

**$C_i = \mathrm{Supp}(\mathbb{I}_i^{m_i})$ nonsingular**

**Case:** $\cdots(\sigma_i^{m_i}, \infty, 0)$

**Use Nonsinularity Principle**

**Case:** $\cdots(\sigma_i^{m_i}, 0, 0, \Gamma)$ **MONOMIAL CASE**

**Use what ???**

**$C_i = \mathrm{Supp}(\mathbb{I}_i^{m_i})$ transversal to $E_i$**

**The same argument as before.**

## Termination in the vertical direction

**Crucial Claim** $\mathrm{inv}(P_i) \leq \mathrm{inv}(P_{i-1})$.

**Proof ???**

**Claim** $\mathrm{inv}(P_i) < \mathrm{inv}(P_{i-1})$.

**Assuming Crucial Claim,**

**the same proof works.**

**Claim** $\nexists$ **an infinite sequence**

$$\mathrm{inv}(P_0) > \mathrm{inv}(P_1) > \cdots > \mathrm{inv}(P_{i-1}) > \mathrm{inv}(P_i) > \cdots.$$

**The same proof works.**

# 2 Power Series Expansion w.r.t. LGS

**Given**

$\mathbb{I} = \mathfrak{D}(\mathbb{I})$; **a $\mathfrak{D}$-saturated idealisitic filtration**

$\mathbb{H}$; **an LGS of** $\mathbb{I}$

**Power Series Expansion**

**w.r.t.**

$$\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^{l} \text{ LGS}$$

$$h_\alpha = x_\alpha^{p^{e_\alpha}} \bmod \mathfrak{m}_P^{p^{e_\alpha}+1}$$

**and its associated reg. sys. of parameters**

$$(x_1, \cdots, x_l, x_{l+1}, \cdots, x_d).$$

$\forall f \in \widehat{\mathcal{O}_{W,P}}$

$$\exists! \ f = \sum c_B(f) H^B, \quad H^B = h_1^{b_1} \cdots h_l^{b_l}$$

**and**

$$\deg_{x_\alpha} c_B(f) \leq p^{e_\alpha} - 1 \text{ for } \alpha = 1, \cdots, l,$$

**i.e.,**

$$c_B(f) = \sum_{0 \leq n_\alpha \leq p^{e_\alpha}-1} c_{n_1 \cdots n_l} x_1^{n_1} \cdots x_l^{n_l}$$

**with** $c_{n_1 \cdots n_l} \in k[[x_{l+1}, \cdots, x_d]]$

## $\boxed{\text{Coefficient Lemma}}$

$$(f, a) \in \mathfrak{D}(\widehat{\mathbb{I}}) = \widehat{\mathfrak{D}(\mathbb{I})}$$

$\Longrightarrow$

$$(c_B(f), a - |[B]|) \in \mathfrak{D}(\widehat{\mathbb{I}})$$

where $|[B]| = b_1 p^{e_1} + \cdots + b_l p^{e_l}$.

In particular,

$$(c_{\mathbb{O}}(f), a) \in \mathfrak{D}(\widehat{\mathbb{I}}).$$

**Example of the use of Power Series Expansion & $\boxed{\text{Coefficient Lemma}}$ in translating the algorithm**

**Case:** $\mathrm{inv}^{\leq j-1}(P_i) < \mathrm{inv}^{\leq j-1}(P_{i-1})$

○ **Description of**

$$\begin{aligned}\widetilde{\mu}_i^j &= \mu_{\mathbb{H}}(\mathfrak{D}(\mathbb{I}_i^{j-1}) \\ &= \inf\left\{\mathrm{ord}\left(\boxed{c_{\mathbb{O}}(f)}\right)/a; \right. \\ &\qquad \left.(f,a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}\right\}\end{aligned}$$

**makes sense thanks to Power Series Expansion.**

○ **Decsription of**

$$\mathrm{NaiveCpc}(\mathbb{I}_i^{j-1})$$

$$= G\left(\mathfrak{D}(\mathbb{I}_i^{j-1}) \cup \left\{\begin{array}{l}(\boxed{c_{\mathbb{O}}(f)}, \widetilde{\mu}_i^j \cdot a); \\ (f,a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}\end{array}\right\}\right)$$

**and**

$$\mathrm{Cpc}(\mathbb{I}_i^{j-1}) = G\left[IL\left\{\mathfrak{D}\left(\mathrm{NaiveCpc}(\mathbb{I}_i^{j-1})\right)\right\}\right]$$

**makes sense thanks to Power Series Expansion**

**at the analytic level.**

**(at the algebraic level ? $\xleftarrow{\text{partial}}$ Lecture 5 by Kawanoue)**

**∘ Independence of "Cpc" from the choice of LGS $\mathbb{H}$ and $(x_1, \cdots, x_l, x_{l+1}, \cdots, x_d)$ holds thanks to $\mathfrak{D}$-saturation and $\boxed{\text{Coefficient Lemma}}$**

**Case:** $\mathrm{inv}^{\leq j-1}(P_i) < \mathrm{inv}^{\leq j-1}(P_{i-1})$

**We use the "logarithmic versions" of**

**Power series Expansion &**

**Coefficient Lemma**

# 3 Question of validity of the translated algorithm

$\boxed{\text{Question}}$ **Will the translated algorithm via $(\sigma, \widetilde{\mu}, s)$-method in $\mathrm{char} = p > 0$ work ?**

$\boxed{\text{Answer}}$ **NO !**

**We have trouble handling**

**MONOMIAL CASE**

$\boxed{\text{Crucial Claim}}$ $\mathrm{inv}(P_i) \leq \mathrm{inv}(P_{i-1})$.

**We present some easy BAD EXAMPLES.**

# 4 Bad examples

**Example 1**

○ **Invariant $\widetilde{\mu}$ strictly increases after "permissible" blowup.**; **Trouble with** $\boxed{\text{Crucial Claim}}$

$\boxed{\mathrm{char}(k) = 2}$

$$\mathbb{I}_i^{j-1} \qquad (x^2 + f^{11}y^4, 2)$$

$$(f^{18}z^4, 2) \qquad /f^{18} \quad (z^4, 2)$$

$$\mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1}) \quad f\frac{\partial}{\partial f}(x^2 + f^{11}y^4, 2)$$

$$= (f^{11}y^4, 1) \qquad /f^9 \quad (f^2y^4, 1)$$

$$\begin{cases} E_{i,\mathrm{young}}^{j-1} & = \{F\}, \ F = \{f = 0\} \\ \mathbb{H}_i & = \{(x^2 + f^{11}y^4, 2)\} = \{(h, 2)\} \end{cases}$$

$\mu_F = 9$ **i.e., divisible mod $\mathbb{H}_i$ by $f^9$ per level**

$$\boxed{\widetilde{\mu}_i^j = 2}$$

$$\boxed{\text{Blowup with center } (x, f, z)}$$

## Description after blowup

$$\textbf{w.r.t. } (x' = x/f, y, z' = z/f, f)$$

$$\pi^\sharp(\mathbb{I}_i^{j-1}) \qquad \begin{aligned} &\pi^\sharp(x^2 + f^{11}y^4, 2) \\ &= (x'^2 + f^9 y^4, 2) \\ &\pi^\sharp(f^{18}z^4, 2) \\ &= (f^{20}z'^4, 2) && /f^{18} \ (f^2 z'^4, 2) \\ &\pi^\sharp(f^{11}y^4, 1) \\ &= (f^{10}y^4, 1) && /f^9 \ \ (f y^4, 1) \end{aligned}$$

$$\mathfrak{D}_{E^{j-1}_{i+1,\text{young}}}(\mathbb{I}^{j-1}_{i+1}) =$$

$$\mathfrak{D}_{E^{j-1}_{i+1,\text{young}}}(\pi^\sharp(\mathbb{I}^{j-1}_i))$$

$$\begin{aligned} &f\frac{\partial}{\partial f}(x'^2 + f^9 y^4, 2) \\ &= (f^9 y^4, 1) && /f^9 \ \ (y^4, 1) \end{aligned}$$

$$\begin{cases} E^{j-1}_{i+1,\text{young}} & = \{F_{\text{new}}\}, \ F_{\text{new}} = \{f = 0\} \\ \mathbb{H}_{i+1} & = \{(x'^2 + f^9 y^4, 2)\} \end{cases}$$

$\mu_{F_{\text{new}}} = 9$ **i.e., divisible mod $\mathbb{H}_{i+1}$ by $f^9$ per level**

$$\boxed{\widetilde{\mu}^j_{i+1} = 3 > 2 = \widetilde{\mu}^j_i}$$

## Source of trouble

$$
\begin{array}{ccc}
(f^{11}y^4, 1) & \xleftarrow{f\frac{\partial}{\partial f}} & (x^2 + f^{11}y^4, 2) \\
\downarrow \pi^\sharp & & \downarrow \pi^\sharp \\
(f^{10}y^4, 1) \color{red}{\neq} (f^9y^4, 1) & \xleftarrow{f\frac{\partial}{\partial f}} & (x'^2 + f^9y^4, 2)
\end{array}
$$

**Conclusion: We will have trouble showing**

$\boxed{\text{Crucial Claim}}$ $\mathrm{inv}(P_i) < \mathrm{inv}(P_{i-1})$.

## Example 2

∘ We get thrown out of the monomial case after blowup.; **Trouble with MONOMIAL CASE.**

$\boxed{\mathrm{char}(k) = 2}$

$$
\begin{array}{lll}
\mathbb{I}_i^{m_i-1} & (x^2 + f^{10}yz, 2) & \\
 & (f^{20}, 2) & /f^{20} \ (1, 2) \\
\mathfrak{D}_{E_{i,\text{young}}^{m_i-1}}(\mathbb{I}_i^{m_i-1}) & \frac{\partial}{\partial y}(x^2 + f^{10}yz, 2) & \\
 & = (f^{10}z, 1) & /f^{10} \ (z, 1) \\
 & \frac{\partial}{\partial z}(x^2 + f^{10}yz, 2) & \\
 & = (f^{10}y, 1) & /f^{10} \ (y, 1)
\end{array}
$$

$$
\begin{cases}
E_{i,\text{young}}^{m_i-1} = \{F\}, \ F = \{f = 0\} \\
\mathbb{H}_i = \{(x^2 + f^{10}yz, 2)\} = \{(h, 2)\}
\end{cases}
$$

$\mu_F = 10$ **i.e., divisible mod $\mathbb{H}_i$ by $f^{10}$ per level**

$$\boxed{\widetilde{\mu}_i^{m_i} = 0}$$

**We are in MONOMIAL CASE.**

$$\boxed{\begin{array}{c}\textbf{Blowup with center } (x, f) \textbf{ defining} \\ \mathrm{Supp}\left(\mathfrak{D}_{E_{i,\text{young}}^{m_i-1}}(\mathbb{I}_i^{m_i-1})|_F\right)\end{array}}$$

# Description after blowup

**w.r.t.** $(x' = x/f, y, z, f)$

$$\pi^\sharp(\mathbb{I}_i^{m_i-1}) \quad \pi^\sharp(x^2 + f^{10}yz, 2)$$

$$= (x'^2 + f^8 yz, 2)$$

$$\pi^\sharp(f^{20}, 2) = (f^{18}, 2) \qquad / f^{16} \qquad (f^2, 2)$$

$$\pi^\sharp(f^{10}z, 1) = (f^9 z, 1) \qquad / f^8 \qquad (fz, 1)$$

$$\pi^\sharp(f^{10}y, 1) = (f^9 y, 1) \qquad / f^8 \qquad (fy, 1)$$

$$\mathfrak{D}_{E_{i+1,\mathrm{young}}^{m_i-1}}(\mathbb{I}_{i+1}^{m_i-1}) = \mathfrak{D}_{E_{i+1,\mathrm{young}}^{m_i-1}}(\pi^\sharp(\mathbb{I}_i^{m_i-1}))$$

$$\frac{\partial}{\partial y}(x'^2 + f^8 yz, 2)$$

$$= (f^8 z, 1) \qquad / f^8 \qquad (z, 1)$$

$$\frac{\partial}{\partial z}(x'^2 + f^8 yz, 2)$$

$$= (f^8 y, 1) \qquad / f^8 \qquad (y, 1)$$

$$\begin{cases} E_{i+1,\mathrm{young}}^{m_i-1} & = \{F_{\mathrm{new}}\}, \ F_{\mathrm{new}} = \{f = 0\} \\ \mathbb{H}_{i+1} & = \{(x'^2 + f^8 yz, 2)\} \end{cases}$$

$\mu_{F_{\mathrm{new}}} = 8$ **i.e., divisible mod** $\mathbb{H}_{i+1}$ **by** $f^8$ **per level**

$$\boxed{\widetilde{\mu}_{i+1}^{m_i} = 1 > 0 = \widetilde{\mu}_i^j}$$

## Source of trouble

$$\begin{array}{ccc}
(f^{10}z, 1) & \xleftarrow{\frac{\partial}{\partial y}} & (x^2 + f^{10}yz, 2) \\
\downarrow \pi^\sharp & & \downarrow \pi^\sharp \\
(f^9 z, 1) \neq (f^8 z, 1) & \xleftarrow{f\frac{\partial}{\partial f}} & (x'^2 + f^8 yz, 2)
\end{array}$$

**Conclusion: We can NOT stay**

**in the MONOMIAL CASE.**

## Example 3

∘ **Failure to choose the nice center**

**(as naively expected) in MONOMIAL CASE**

$\boxed{\mathrm{char}(k) = 5}$

$$\mathbb{I}_i^{m_i-1} \quad (x^5 + f^4(y^2 - z^3), 5)$$

$$(f^4, 4) \qquad /f^4 \ (1, 4)$$

$$\mathfrak{D}_{E_{i,\mathrm{young}}^{m_i-1}} \quad (\mathbb{I}_i^{m_i-1})$$

$$f\tfrac{\partial}{\partial f}(x^5 + f^4(y^2 - z^3), 5)$$

$$= (4f^4(y^2 - z^3), 4) \qquad /f^4 \ (4(y^2 - z^3), 4)$$

$$\tfrac{\partial}{\partial y}(x^5 + f^4(y^2 - z^3), 5)$$

$$= (2yf^4, 4) \qquad /f^4 \ (2y, 4)$$

$$\tfrac{\partial}{\partial z}(x^5 + f^4(y^2 - z^3), 5)$$

$$= (-3z^2f^4, 4) \qquad /f^4 \ (-3z^2, 4)$$

$$\begin{cases} E_{i,\mathrm{young}}^{m_i-1} & = \{F\}, \ F = \{f = 0\} \\ \mathbb{H}_i & = \{(x^5 + f^4(y^2 - z^3), 5)\} = \{(h, 5)\} \end{cases}$$

$\mu_F = 4$ **i.e., divisible mod $\mathbb{H}_i$ by $f^4$ per level**

$$\boxed{\widetilde{\mu}_i^{m_i} = 0}$$

**We are in MONOMIAL CASE.**

**Naive choice of the center $(x, f)$ defining**

$$\mathrm{Supp}\left(\mathfrak{D}_{E^{m_i-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{m_i-1})|_F\right)$$

**following the analogy to the classical method**

## HOWEVER

$$\mathrm{Supp}\left(\mathfrak{D}_{E^{m_i-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{m_i-1})\right) \not\supset \mathrm{Supp}\left(\mathfrak{D}_{E^{m_i-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{m_i-1})|_F\right)$$

**defined by** $\underbrace{(x, f, y^2 - z^3)}$ **NOT nonsingular**

**defined by** $(x, f)$

**Source of trouble**

$$(4(y^2 - z^3)f^4, 4) \xleftarrow{f\frac{\partial}{\partial f}} (x^5 + f^4\underbrace{(y^2 - z^3)}, 5)$$

$$(2yf^4, 4) \xleftarrow{\frac{\partial}{\partial y}}$$

$$(-3z^2f^4, 4) \xleftarrow{\frac{\partial}{\partial z}}$$

**invisible**

**after setting $f = 0$**

**Conclusion: We can NOT choose**

**a nice nonsingular center**

**in MONOMIAL CASE.**

## 5 Analysis of trouble

**An element in LGS is of the form**

$$h = \underbrace{x^{p^e}}_{\text{Principal part}} + \boxed{\text{TAIL}}.$$

**It is $\boxed{\text{TAIL}}$ that is causing all the TROUBLE.**

$\boxed{\text{Conclusion}}$ **We should incorporate the information on $\boxed{\text{TAIL}}$ into our algorithm.**

$\boxed{\text{How ?}}$

$$h = \underbrace{x^{p^e}}_{\text{Principal part}} + \boxed{\text{TAIL}}$$

**$\delta$: a diff. operator (of degree $0 < \deg \delta < p^e$)**

$$\delta h = \underbrace{\delta x^{p^e}}_{\substack{\| \\ 0}} + \delta \boxed{\text{TAIL}}$$

**Study of the derivatives of $\delta h = \delta \boxed{\text{TAIL}}$**

$\longrightarrow$

**Information on $\boxed{\text{TAIL}}$**

# 6 Introduction of invariant $\widetilde{\nu}$

**SPIRIT: Mimic the construction of invariant $\widetilde{\mu}$**

**$\widetilde{\mu}$: Weak order**

**(order after divided as much as possible by**

**the defining equations of the exceptional divisors)**

**in regard to all the elements in $\mathfrak{D}_{E_{\text{young}}}(\mathbb{I})$**

**$\widetilde{\mu}$: Weak order**

**(order after divided as much as possible by**

**the defining equations of the exceptional divisors)**

**in regard to all the derivatives of LGS**

## Definition of $\widetilde{\nu}_i^j$

$\mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1}) := \mathbb{J}$

$\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^{l}$ an LGS

$\{p^{e_\alpha}\}_{\alpha=1}^{l} = \{p^{e_1} < \cdots p^{e_m}\} = \{p^{e_\beta}\}_{\beta=1}^{m}$

$$\boxed{\begin{aligned} \widetilde{\nu}_i^j &= \nu_{\mathbb{H}, E_{i,\mathrm{young}}^{j-1}}(\mathbb{J}) \\ &:= \nu_{\mathbb{H}}(\mathbb{J}) - \textstyle\sum_{F_\lambda \subset E_{i,\mathrm{young}}^{j-1}} \nu_\lambda \end{aligned}}$$

where

$$\begin{cases} D^t_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{J}_{p^{e_\beta}}) \text{ with } t \in \mathbb{Z}_{>0}, p^{e_\beta} - t > 0 \\ \qquad := \{(f, p^{e_\beta} - t) = (\delta(g), p^{e_\beta} - t); \\ \qquad\quad \delta \in \mathrm{Diff}^t_{E_{i,\mathrm{young}}}, g \in \mathbb{J}_{p^{e_\beta}}\} \\ \nu_{\mathbb{H}}(\mathbb{J}) \quad := \inf\{\mathrm{ord}(c_{\mathbb{O}}(f))/(p^{e_\beta} - t); \\ \qquad (f, p^{e_\beta} - t) \in D^t_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{J}_{p^{e_\beta}}), \\ \qquad t \in \mathbb{Z}_{>0}, p^{e_\beta} - t > 0, f = \sum c_B(f) H^B\} \\ \nu_\lambda \quad := \inf\{n/(p^{e_\beta} - t); \\ \qquad c_{\mathbb{O}}(f) \text{ is divisible by } f_\lambda^n, \\ \qquad (f, p^{e_\beta} - t) \in D^t_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{J}_{p^{e_\beta}}), \\ \qquad t \in \mathbb{Z}_{>0}, p^{e_\beta} - t > 0, f = \sum c_B(f) H^B\} \end{cases}$$

$\boxed{\textbf{Lemma}}$

$\boldsymbol{\nu}_{\mathbb{H},\boldsymbol{E}^{j-1}_{i,\text{young}}}(\mathbb{J})$ **is independent of the choice of** $\mathbb{H}$ **(or** $\mathbb{H}_V$**). Therefore,** $\widetilde{\boldsymbol{\nu}}^j_i$ **is well-defined.**

# 7 Observe how invariant $\widetilde{\nu}$ overcomes trouble

**Example 1**

$\boxed{\text{Center } (x,f,z)}$

$(\sigma, \widetilde{\mu}, s)$-permissible NOT $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-permissible

$$\widetilde{\mu}_i^j = 2 \quad < \quad \widetilde{\mu}_{i+1}^j = 3$$

**Compute**

$\widetilde{\nu}_i^j = 4$

$$f\frac{\partial}{\partial f}(x^2 + f^{11}y^4, 2)$$

$$= (f^{11}y^4, 1)$$

$$\nu_F = 11$$

divisible mod $\mathbb{H}$ by $f^{11}$per level

$$/f^{11} \ (y^4, 1)$$

$\boxed{\text{New Center } (x,f,y,z)}$

$(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-permissible

$$\widetilde{\mu}_i^j = 2 \quad \geq \quad \widetilde{\mu}_{i+1}^j = 2$$

**Example 2 (Say $E^{j-1}_{i,\mathrm{aged}}$)**

$\boxed{(\sigma, \widetilde{\mu}, s)\text{-method}}$

$$\begin{array}{ccc} (\sigma^j_i, & \widetilde{\mu}^j_i, & s^j_i) \\ & \| & \| \\ & 0 & 0 \end{array}$$

$\longrightarrow$ **End of weaving. Try to determine the center as in the classical monomial case.**

**Center** $(x, f)$

$$\widetilde{\mu}^j_i = 0 < \widetilde{\mu}^j_{i+1} = 1.$$

$\boxed{(\sigma, \widetilde{\mu}, \nu, s)\text{-method}}$

$$\begin{array}{cccc} (\sigma^j_i, & \widetilde{\mu}^j_i, & \nu^j_i, & s^j_i) \\ & \| & \| & \| \\ & 0 & 1 & 0 \end{array}$$

$\longrightarrow$ **Weaving continues. Center determined by the later units.**

**Center** $((x, f, y, z)$: $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$**-permissible**

$$\widetilde{\mu}^j_i = 0 \ = \ \widetilde{\mu}^j_{i+1} = 0.$$

## Computation of $\widetilde{\nu}_i^j = 1$.

$$\begin{cases} \frac{\partial}{\partial y}(x^2 + f^{10}yz, 2) = (f^{10}z, 1) \;\; / f^{10} \;\; (z, 1) \\ \frac{\partial}{\partial z}(x^2 + f^{10}yz, 2) = (f^{10}y, 1) \;\; / f^{10} \;\; (y, 1) \\ \nu_F = 10 \end{cases}$$

**We have to add $y$ and $z$ to make the center**

**$(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-permissible**

**Center $(x, y)$: NOT $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-permissible**

**Center $(x, f, y, z)$: $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-permissible**

**Example 3 (Say $E^{j-1}_{i,\mathrm{aged}}$)**

$\boxed{(\sigma, \widetilde{\mu}, s)\text{-method}}$

$$\begin{array}{ccc} (\sigma^j_i, & \widetilde{\mu}^j_i, & s^j_i) \\ & \| & \| \\ & 0 & 0 \end{array}$$

$\longrightarrow$ **End of weaving. Try to determine the center as in the classical monomial case.**

**We have TROUBLE choosing a good center.**

$\boxed{(\sigma, \widetilde{\mu}, \nu, s)\text{-method}}$

$$\begin{array}{cccc} (\sigma^j_i, & \widetilde{\mu}^j_i, & \nu^j_i, & s^j_i) \\ & \| & \| & \| \\ & 0 & 1/4 & 0 \end{array}$$

$\longrightarrow$ **Weaving continues. Center determined by the later units.**

**Center $(x, f, y, z)$: nonsingular, $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-permissible**