RIMS WorkShop

Introduction to Idealistic Filtration Program

An approach to resolution of singularities in positive characteristics

Hiraku Kawanoue and Kenji Matsuki

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Lecture 5

Two topics in IFP:

Algebraization

and

Nonsingularity principle

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Contents

1 Plan of today's talk

Today I want to talk on 2 small topics:

- **One Prop. concerning algebraization problem**

- **Outline of the proof of nonsingularity principle**

We consider again in the case with NO exceptional divisors

2 Algebraization problem revisited

Setting for today

M: nonsing. var./k, $P \in M$: closed point,

$R = (R, \mathfrak{m}) = \mathcal{O}_{M,P}$: local ring at $P \in M$,

$\mathbb{I}$: $\mathfrak{D}$-sat. I.F. on R, $\mu(\mathbb{I}) \geq 1$, $\mathbb{H}$: LGS of $\mathbb{I}$.

Review

- We use enlargement to define invariant

$$\begin{array}{ccccccc} \mathbb{I}^0 & \subset & \mathbb{I}^1 \cdots & \subset & \mathbb{I}^t & \subset & \mathbb{I}^{t+1} \\ (\sigma_0, \mu_0^{\sim}) & \cdots & \cdots & & (\sigma_t, \mu_t^{\sim}) & & (\sigma_{t+1}, \infty) \end{array}$$

- Idea to define enlargement $\mathbb{I}'$ of $\mathbb{I}$

$$\begin{array}{rcl} \mathbb{I} & = & \text{“}\, \mathbb{H} + (\text{Remainder})\text{”} \\ & & \quad\cap \\ \mathbb{I}' & = & \text{“}\, \mathbb{H} + \begin{pmatrix} \text{Remainder} \\ \text{with } \alpha \times (\text{level}) \end{pmatrix} \text{”} \end{array}$$

where $\alpha := \mu^{\sim}(\mathbb{I}) \geq 1$.

- In completion level, enlargement exists

$$\widehat{\mathbb{I}} = G(\mathbb{H} \cup \{(c_0(f), a) \mid (f, a) \in \mathbb{I}\})$$
$$\widehat{\mathbb{I}}' = \mathfrak{D}(G(\mathbb{H} \cup \{(c_0(f), \alpha a) \mid (f, a) \in \mathbb{I}\}))$$

- $c_0(f)$: "the remainder of f w.r.t. $\mathbb{H}$".

$$\begin{cases} \widehat{R} = k[[\mathrm{x}, \mathrm{y}]] \\ \mathbb{H} = \{(h_i, p^{e_i}) \mid 1 \leq i \leq l\}\text{: LGS of } \mathbb{I} \\ h_i \in x_i^{p^{e_i}} + \text{(higher)} \quad (1 \leq i \leq l) \end{cases}$$

$$\Downarrow$$

$c_0(f)$ is element of $k[[y]][x] \subset \widehat{R}$ with

$$\begin{cases} f - c_0(f) \in \sum_{i=1}^{r} h_i \widehat{R} \\ \deg_{x_i}(c_0(f)) < p^{e_i} \quad (1 \leq i \leq l) \end{cases}$$

How to construct enlargement $\mathbb{I}'$ in Zariski local level?

Problem Let $f \in R$. Find suitable approximation of $c_0(f)$ in R!

(algebraization problem)

Abstract approach

- By Artin's approximation theorem, construct $c_0(f)$ in henselization R^*.
- Then, apply descent theory.

$$\left(\begin{array}{c}\text{Seems to work nicely, but today we skip it} \\ \text{by our ignorance of “well-known method”}\end{array}\right)$$

Constructive approach

What is the "suitable" approximation?

Fix $t > 0$. By definition of $\widehat{\mathbb{I}}'$,

we require $\boxed{\mathbb{I}_{\alpha t} \subset \mathbb{I}'_t.}$

Therefore, we can define $\mathbb{I}'_t$ if there exists

$$\phi\colon \boldsymbol{R} \longrightarrow \boldsymbol{R} \quad \text{(approximation of } \boldsymbol{c}_0\text{)}$$
$$\text{s.t. } (\boldsymbol{c}_0 - \phi)(\mathbb{I}_t) \in \mathbb{I}_{\alpha t}\widehat{\boldsymbol{R}}$$

We show the following:

Lemma 1

Let $l \in \mathbb{Z}_{\geq 0}$. If $\#\mathbb{H} = 1$, we can construct

$\Phi_l\colon \boldsymbol{R} \longrightarrow \boldsymbol{R}$ s.t. $(\boldsymbol{c}_0 - \Phi_l)(\boldsymbol{R}) \in \mathbb{I}_l\widehat{\boldsymbol{R}}$.

Remark $\mu(\mathbb{I}) \geq 1 \Rightarrow \mathbb{I}_l \subset \mathfrak{m}^l$. **Thus**

$\Phi_l(f)$ is a special $\mathfrak{m}$-adic approxim. of $\boldsymbol{c}_0(f)$.

3 Construction of Φ_l when $\#\mathbb{H} = 1$

Assume: $\mathbb{H} = \{(h, q)\}$ ($q = p^e$, $e \in \mathbb{Z}_{\geq 0}$)

How to construct Φ_l**: approxim. of** c_0 **?**

Idea for construction of Φ_l

1. Construct ψ_l**: approxim. of "**∂_h**"!**

i.e., $\psi_l : R \to R$ **with**

$$\begin{cases} \psi_l(x^u) = 0 & (0 \leq u < q) \\ \psi_l(h\beta) \in \beta + \mathbb{I}_l \widehat{R} & (\beta \in \widehat{R}) \end{cases}$$

2. Put $\boxed{\Phi_l = 1 - h\psi_l}$

3. Done! since

$$(1 - h\psi_l) : \begin{cases} R & \to & R \\ h\beta & \to & \mathbb{I}_l \\ c_0(f) & \mapsto & c_0(f) \end{cases}$$

Notation

- X: RSP of R

- $\left\{\partial_{X^I} : R \to R \mid I : \text{multi-index}\right\}$

  : partial diff. operators w.r.t. X

  $$\text{i.e.} \quad \partial_{X^I} X^J = \binom{J}{I} X^{J-I}$$

- Since $h =$(linear)^q+(higher),

  $$\exists x \in X \text{ such that } \partial_{x^q} h \in R^\times.$$

$$\partial_t := \begin{cases} \partial_{x^t} & : t \in \mathbb{Z}_{\geq 0} \\ 0 & : t \in \mathbb{Z}_{<0} \end{cases}$$

- $$h_t := \partial_t h \in R$$

Since $\partial_{x^{q-t}} h \in \partial_{x^{q-t}} \mathbb{I}_q \subset \mathbb{I}_t$,

$$\boxed{h_{q-t} \in \mathbb{I}_t}$$

- $M_l := (h_{q-i+j})_{0 \le i,j \le l} \in \mathrm{Mat}(l+1, R)$

i.e.

$$M_l = \begin{bmatrix} h_q & h_{q+1} & \cdots & h_{q+l} \\ h_{q-1} & h_q & \cdots & h_{q+l-1} \\ \vdots & & \ddots & \vdots \\ h_{q-l} & h_{q-l+1} & \cdots & h_q \end{bmatrix}$$

Since $h_q \in R^\times$ **and** $h_i \in \mathfrak{m}$ $(i < q)$,

$$\boxed{M_l \in \mathrm{GL}(R)}$$

- $c_t := \left(M_l^{-1}\right)_{t,0} \in R$

By definition,

$c_t = (-1)^t (\det M_l)^{-1} \times$ **(minor)**

where (minor) is calculated as follows:

$$M_l = \begin{bmatrix} h_q & \cdots & h_{q+t} & \cdots & h_{q+l} \\ h_{q-1} & \cdots & h_{q+t-1} & \cdots & h_{q+l-1} \\ \vdots & & \vdots & & \vdots \\ h_{q-l} & \cdots & h_{q+t-l} & \cdots & h_q \end{bmatrix}$$

Ignore colored entries and take determinant.

Monomials in (minor) are of the shape

$$\pm h_{s_1} h_{s_2} \cdots h_{s_l} \quad \textbf{with}$$

$$\sum_{u=1}^{l} s_u = (l+1)q - (q+t) = lq - t$$

$$\Rightarrow h_{s_1} h_{s_2} \cdots h_{s_l} \in \mathbb{I}_t \quad \textbf{i.e.} \quad \boxed{c_t \in \mathbb{I}_t}$$

Construction of ψ_l

Define

$$\boxed{\psi_l := \sum_{t=0}^{l} c_t \partial_{q+t} : R \to R}$$

Clearly, $\psi_l(x^u) = 0 \quad (0 \leq u < q)$

Calculation of $\psi_l(h\beta)$:

$$\begin{aligned}
\psi_l(h\beta) &= \sum_{t=0}^{l} c_t \partial_{q+t}(h\beta) \\
&= \sum_{t=0}^{l} c_t \sum_{s \geq 0} \partial_{q+t-s} h \cdot \partial_s \beta \\
&= \sum_{s \geq 0} \partial_s \beta \sum_{t=0}^{l} h_{q-s+t} c_t
\end{aligned}$$

For $0 \leq s \leq l$**,**

$$\sum_{s \geq 0} h_{q-s+t}\, c_t = \sum_{t=0}^{l} (M_l)_{s,t} (M_l^{-1})_{t,0}$$
$$= (M_l M_l^{-1})_{s,0} = \delta_{0,s}$$

For $l < s$**,**

$$h_{q-s+t}\, c_t \in \mathbb{I}_{s-t}\mathbb{I}_t \subset \mathbb{I}_s \subset \mathbb{I}_l$$

Therefore

$$\psi_l(h\beta) = \sum_{s \geq 0} \partial_s \beta \sum_{t=0}^{l} h_{q-s+t}\, c_t$$
$$\in \sum_{s=0}^{l} \delta_{0,s} \partial\beta + \mathbb{I}_l \widehat{R} = \beta + \mathbb{I}_l \widehat{R}.$$

i.e. $\boxed{\phi_l(h\beta) \in \beta + \mathbb{I}_\alpha.}$

Thus this ψ_l **is what we need.**

4 Nonsingularity principle revisited

Theorem 2 (Nonsingularity principle)

Assume $\mu^{\sim}(\mathbb{I}) = \infty$. Then,

1. $\mathbb{I}$ is generated by any LGS $\mathbb{H}$ of $\mathbb{I}$.

2. $\begin{cases} \exists\{x_i \mid i\} \subset R & : \text{a part of RSP of } R \\ \exists\{e_i \mid i\} \subset \mathbb{Z}_{\geq 0} & : \text{non-neg. integers} \end{cases}$

   such that $\left\{(x_i^{p^{e_i}}, p^{e_i}) \mid i\right\}$ is an LGS of $\mathbb{I}$.

This theorem guarantees the nonsingularity

of max. locus of inv_P,

which coincides $\mathrm{Supp}(\mathbb{I})$ in the last stage

of defining inv_P (when $\mu^{\sim} = \infty$).

Notation

$\mathbb{H} = \{(h_i, q_i) \mid 1 \leq i \leq \ell\}$: **LGS of** $\mathbb{I}$

X: **RSP of** R

$\{x_i, \ldots, x_\ell\} \subset X$

s.t. $h_i = x_i^{q_i}$+**(higher)** $(1 \leq i \leq \ell)$

$B = (b_1, b_2, \ldots, b_\ell)$: **multi-index**

$H^B = \prod_{i=1}^{\ell} h_i^{b_i}, \quad |B| = \sum_{i=1}^{\ell} b_i$

$[B] = (q_1 b_1, q_2 b_2, \ldots, q_\ell b_\ell)$

5 Sketch of proof: the former part

$\mu^{\sim}(\mathbb{I}) = \infty$ is equivalent to

$$\mathbb{I}_t \subset \sum_{i=0}^{\ell} R h_i \quad (\forall t > 0). \qquad \textbf{(1)}$$

$\mathbb{I} = \mathrm{G}(\mathbb{H})$ is equivalent to

$$\mathbb{I}_t \subset \sum_{|[B]| \geq t} R H^B \quad (\forall t > 0). \qquad \textbf{(2)}$$

Regarding RHS as “expansion w.r.t. $\mathbb{H}$”,

(1): constant part $= 0$.

(2): coeff. of $H^B = 0$ for $|[B]| <$(level).

We want to derive (2) from (1).

key point: $\boxed{\mathbb{I} \text{ is } \mathfrak{D}\text{-saturated}}$.

Example

$R = k[x,y]_{(x,y)}$, $\mathbb{H} = \{(x,1)\}$.

$\mathbb{I}_{2.5} \ni f = \sum_{i=0}^{4} c_i(y) x^i$

$\mu^{\sim}(\mathbb{I}) = \infty \Rightarrow \mathbb{I}_t \subset xR$ for $t > 0$.

$$
\begin{array}{lll}
\mathbb{I}_{2.5} \ni f = \sum_{i=0}^{4} c_i x^i & \Rightarrow & c_0 = 0. \\
\mathbb{I}_{1.5} \ni \partial_x f = \sum_{i=1}^{4} i c_i x^{i-1} & \Rightarrow & c_1 = 0. \\
\mathbb{I}_{0.5} \ni \partial_{x^2} f = \sum_{i=2}^{5} \binom{i}{2} c_i x^{i-1} & \Rightarrow & c_2 = 0. \\
\mathbb{I}_{-0.5} \ni \partial_{x^3} f \quad (\text{level} < 0) & \Rightarrow & \text{STOP!}
\end{array}
$$

Thus

$$f = \sum_{i=3}^{4} c_i x^i \in \sum_{i > 2.5} R x^i$$

Strategy

By differentiation, bring coeff's of H^B

into constant part and kill it!

General case: more complicated, i.e.

- **many variables**
- **levels of LGS > 1**

$\longrightarrow$ **Use of higher diff. operators:**

$\longrightarrow$ **Leibneiz rule is not usual.**

$$\partial_{X^K}(fg) = \sum_{I+J=K} \partial_{X^I} f \cdot \partial_{X^J} g$$

But, we can still find out the

leading part of coeff. of specified H^B:

$$\Uparrow$$

key point: $\boxed{\begin{array}{c}\text{If } \ i \neq j \ \text{ or } \ u \neq 0, \ q_i, \\ \text{then } \operatorname{ord} \partial_{x_j^u} h_i > q_i - u\end{array}}$.

We have only to repeat this procedure

and use Krull's intersection theorem!

6 Sketch of proof: the latter part

As LGS $\mathbb{H}$ is representative of $\mathrm{L}(\mathbb{I})$, usually we have "tail" i.e.

$$h_i = x_i^{q_i} + (\text{higher, not } q_i\text{-th power})$$

$$\text{with } q_i = p^{e_i} > 1.$$

The latter part of NSP asserts that LGS without tails exists provided $\mu^{\sim}(\mathbb{I}) = \infty$.

Let $\mathrm{F}^q \colon R \ni r \mapsto r^q \in R$ (Frobenius).

$\boxed{\text{Strategy of whole proof}}$ Fix LGS $\mathbb{H}$.

- Change $\mathbb{H}$ into tail-free from lower level.
- induction on r, where

$$r := r(\mathbb{H}) = \min\{q \in \mathbb{Z}_{>0} \mid \mathbb{H}_q \not\subset \mathrm{F}^q(\mathfrak{m})\}$$

Setup

As $\mathbb{I} = \mathrm{G}(\mathbb{H})$, we can express

$$\mathbb{I}_r = \sum_{|[B]| \geq r} RH^B = \sum_{q_i \geq r} Rh_i + J,$$

where J is factors from lower level, i.e.

$$J = (H^B \mid |[B]| \geq r,\ b_i = 0 \text{ if } q_i \geq r).$$

Since $h_i \in \delta_{r,q_i} x^{q_i} + \mathfrak{m}^{r+1}$ for $q_i \geq r$,

$$\mathbb{I}_r \subset \mathrm{F}^r(\mathfrak{m}) + \mathfrak{m}^{r+1} + J$$

Goal

Our goal is to prove

$$\mathbb{I}_r = \mathrm{F}^r(\mathfrak{m}) \cap \mathbb{I}_r + \mathbb{I}_r \cap \mathfrak{m}^{r+1} + J.$$

lives elt's of LGS $\uparrow$ $\uparrow$

(big order) (from lower)

To prove above equation, we show

$$\mathbb{I}_r \subset K_n \quad (\forall n \geq 0) \quad \text{where}$$

$$K_n := \mathrm{F}^r(\mathfrak{m}) + \mathbb{I}_r \cap \mathfrak{m}^{r+1} + J + \mathfrak{m}^n.$$

Step: $n \longrightarrow (n+1)$

For $f \in \mathbb{I}_n$, we show $f \in K_{n+1}$ as follows:

1. $\exists \phi \in \mathfrak{m}^n$ s.t. $f - \phi \in K_{n+1}$

2. Find suitable partial diff. operator ∂ of degree $< r$ to extract the information of $\mathrm{in}(\phi)$ (leading term)

3. By $\partial\, \mathrm{F}^r(\mathfrak{m}) = 0$, show that $\mathrm{in}(\partial\phi) \in \mathrm{in}(\mathbb{I}_{r-\deg\partial})$.

4. By $\mathbb{I}_{r-\deg\partial} \subset \mathbb{I}_1 = \sum_i Rh_i$, show $\mathrm{in}(\partial\phi) \in (\mathrm{in}(h_i) \mid i)$, and deduce $\phi \in K^{n+1}$.

Conclusion Regard K_n as R-mod. by F^r, and use Krull's intersection theorem to get the equation in "Goal".