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GOCHUMOKU ONEGAISHIMASU: KANGAROO POINTS ⊙

HERWIG HAUSER

In September 2008, Heisuke Hironaka gave a series of lectures at the Clay Mathematics Institute explaining his approach to the resolution of singularities of algebraic varieties in positive characteristic [Hi1]. This was complemented by more detailed lectures delivered during the workshop on Resolution of Singularities at the Research Institute for Mathematical Sciences (RIMS) at Kyoto in December 2008. In the course of the lectures, Hironaka relied on results of the author from an unpublished manuscript written in 2003 [Ha1]. These results investigate the main obstruction for resolution in positive characteristic – the occurence of *kangaroo points* at certain stages of the resolution process of a singular variety. The paper [Ha1] describes in detail their structure and proposes various approaches how one can try to profit from this knowledge for the resolution in positive characteristic. Kangaroo points make their reappearance in [Hi1] under the name of *metastatic points*.

The present note, which is based on the author's lecture at RIMS, shall provide a brief introduction to the theory of kangaroo points. More details can be found in the survey [Ha2]. For the proofs, we refer to [Ha1].

The outset: The key instance in nowadays resolution is the *monomialization* or *principalization* of ideals: Transform a given ideal sheaf $\mathcal{J}$ on a smooth scheme W by a sequence of blowups into an ideal $\mathcal{J}^*$ which is locally principal and generated by one monomial. The singular subscheme X of W defined by $\mathcal{J}$ is thus transformed into a normal crossings divisor. Several extra conditions can be imposed on the resolution (e.g. *equivariance*, *excision*, *effectiveness*, *explicitness*, see [EH]). This leads to the notion of a *strong resolution* of an ideal or scheme, requiring all these properties to be realized by the sequence of blowups.

All relevant notions of a resolution of a singular scheme (embedded, abstract, weak, ...) follow by general arguments from the monomialization of ideals (cf. the last section of [EH]). The proof for the existence of monomialization is usually built on the principle of *cartesian induction.* It combines a horizontal descent (mostly given by a decrease in the embedding dimension, and defined only locally at a given point a of W) by a vertical induction on a suitably defined resolution invariant. Denote by subscript "minus" the descent and by superscript "prime" the transformation under blowup. Both have to be specified explicitly, and vary from author to author. We write capital letters for the ideal stalks at given points. One obtains a diagram

$$\begin{array}{llllllll} W' \text{ at } a' & \rightsquigarrow & (W')_- & \qquad & J' & \rightsquigarrow & (J')_- \\ \downarrow & & & & \downarrow & & \\ W \text{ at } a & \rightsquigarrow & W_- & & J & \rightsquigarrow & J_- \end{array}$$

⊙ Jap.: "Have a look please!" MSC-2000: 14B05, 14E15, 12D10.

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where the vertical arrow denotes a blowup of W in a given center Z (which is closed and regular), J' is the total transform (pullback) of J in W', and where W_-, respectively $(W')_-$ denote some regular ambient schemes associated in a natural way to J and J' locally at a and a'. The ideals J_- and $(J')_-$ are known as *coefficient ideals* of J and J' in W_- and $(W')_-$. In characteristic zero, W_- and $(W')_-$ are chosen as local regular hypersurfaces of *maximal contact*.[□] In the recent approaches in positive characteristic (where maximal contact fails), they are either again local hypersurfaces (defined suitably in a new way), or equal to W and W' (in which case the missing decrease of dimension has to be replaced by a "dimensional" invariant associated to J_- and $(J')_-$ which is smaller than the respective invariant of J and J'). The descent to the "minus-setting" can be performed for instance by restriction (as in characteristic zero), by projection (via elimination as proposed by Bravo-Villamayor), or by enlargement (of the ideal J, as proposed by Kawanoue-Matsuki). Actually, the involved ideals are often replaced by more sophisticated objects carrying detailed information (idealistic filtration, Rees algebras, characteristic algebras, or mobiles). For simplicity, we stick to ideals.

The descent in dimension is only required at points a' in W' where the transform J' of J has not improved. By improvement we mostly understand the drop of local invariants such as the multiplicity or the order of the ideal at the considered point. Let us call such points a' *equiconstant points* for J. These are the points where the (vertical) induction on the local invariant fails. The idea then is to measure at these points the improvement of J' by a secondary invariant, typically the order of the ideal $(J')_-$ at a' in $(W')_-$. The argument only works if the order of $(J')_-$ does not exceed the order of J_- at a in W_-. In characteristic zero, this can be shown to happen because for a careful choice of the center of blowup Z (which will locally at a be contained in W_-), the ideal $(J')_-$ equals the *controlled* transform of the ideal J_- under the blowup $(W_-)'$ of W_- along Z (this is a transform in between the total and weak transform). After factoring from it exceptional components, we obtain a notion of order of J_- which does not increase under blowup. We call this secondary order the *shade* of J at a (the precise definition is given below). If it decreased, we are done by induction, if it remained constant, a further descent in dimension becomes necessary. By exhaustion one arrives at a stage where the order must decrease (this always happens at least in dimension 1). We may summarize this argument in the cartesian diagram

$$\begin{array}{ccc} J' & \rightsquigarrow & (J')_- = (J_-)' \\ \downarrow & & \downarrow \\ J & \rightsquigarrow & J_- \end{array}$$

where $(J_-)'$ denotes the controlled transform of J_-. We may now write J'_- for $(J_-)' = (J')_-$. The diagram commutes at all points a' of W' where the order of J' has remained constant. In particular, the descent $(W')_-$ of W' at a' coincides with the strict transform $(W_-)'$ of W_- under the blowup of W along Z. As the center Z is assumed to be contained in W_- (locally at a) $(W_-)'$ equals the blowup of W_- along Z.

$$\begin{array}{ccc} W' & \rightsquigarrow & (W')_- = (W_-)' \\ \downarrow & & \downarrow \\ W & \rightsquigarrow & W_- \end{array}$$

[□] This means that their transforms under blowup contain all points where the order of J has remained constant.

We may now write W'_- for $(W')_- = (W_-)'$. In this argument it is crucial to show that the invariant in lower dimension does not depend on the choice of the hypersurface W_-. Indeed, again in characteristic zero, the order of J_- with respect to a hypersurface W_- of maximal contact is the *maximal* value of the orders over all choices of W_-. It is therefore intrinsic.$^{\otimes}$ By the *persistence* of maximal contact at equiconstant points, W'_- has again maximal contact with J' at a'. Hence the order of J'_- and the shade of J' are well defined at a'.

All this works fine in characteristic zero – up to some "minor" technicalities. One of these complications is the necessary factorization of the controlled transform into an exceptional monomial and a remaining ideal. More explicitly, write $J_- = M_- \cdot I_-$ with exceptional monomial M_- (stemming from earlier blowups) and some ideal I_-, and similarly for J'_-. Then I'_- is the weak transform of I_- and thus $\mathrm{ord}_{a'} I'_- \leq \mathrm{ord}_a I_-$, by general properties of blowups. We set $\mathrm{shade}_a J = \mathrm{ord}_a I_-$ and get

$$(\mathrm{ord}_{a'} J', \mathrm{shade}_{a'} J') \leq_{lex} (\mathrm{ord}_a J, \mathrm{shade}_a J).$$

We now come to positive characteristic and the main difficulty there. It relies on the observation that the local hypersurface $(W_-)'$ obtained as blowup of W_- along Z need no longer have maximal contact with J' at an equiconstant point a' of W'. As a consequence, the controlled transform $(J_-)'$ of J_- in $(W_-)'$ may have an order which is not maximal over all possible choices of hypersurfaces at a'. In this case it is necessary to choose a new hypersurface $U' = (W')_-$ in W' at a' so as to maximize the order of the associated ideal $(J')_-$. In particular, the ideals $(J_-)'$ and $(J')_-$ need no longer coincide. As Abhyankar, Cossart and Moh (and probably others) observed, also their orders may be different: $\mathrm{ord}_{a'}(J')_-$ may be larger than $\mathrm{ord}_{a'}(J_-)'$ (see [Co, Mo]). This destroys our required inequality $\mathrm{ord}_{a'}(I')_- \leq \mathrm{ord}_a I_-$. The descent in dimension has become obsolete.

In the present note, we propose to look more closely at the equiconstant points where $\mathrm{ord}_{a'}(I')_- > \mathrm{ord}_a I_-$. These are the *kangaroo points*. A good understanding of their occurrence is certainly helpful for advancing in characteristic p. Moreover, they serve as a testing ground for proposed resolution invariants.$^{\blacksquare}$

Example: This is the simplest example of a kangaroo point in a resolution process. Consider the following sequence of three point blowups in characteristic 2,

$$f^0 = x^2 + 1 \cdot (y^7 + yz^4) \text{ (oasis point } a^0), \qquad (x,y,z) \to (xy, y, zy),$$

$$f^1 = x^2 + y^3 \cdot (y^2 + z^4), \qquad (x,y,z) \to (xz, yz, z),$$

$$f^2 = x^2 + y^3 z^3 \cdot (y^2 + z^2) \text{ (antelope point } a^2), \qquad (x,y,z) \to (xz, yz + z, z),$$

$$f^3 = x^2 + z^6 \cdot (y+1)^3((y+1)^2 + 1),$$

$$= x^2 + z^6 \cdot (y^5 + y^4 + y^3 + y^2) \text{ (kangaroo point } a^3).$$

The first two blowups are monomial (the reference point in the exceptional divisor is an origin of an affine chart) and yield a point a^2 at the intersection of the two exceptional components.

$\otimes$ One can also show the independence of the order by Hironaka's technique of auxiliary blowups on cylinders.

$\blacksquare$ The characterization of kangaroo points stems from the manuscript [Ha1]. It is very well possible that various aspects were already known earlier (but possibly never made explicit) by people working in the field. The main issue is to exploit the precise information on the structure of kangaroo points in order to establish a subsequent resolution argument.

they give rise to the monomial factor y^3z^3 in front of y^2+z^2. The point immediately prior to a kangaroo point is called *antelope point*. The kangaroo point is a uniquely specified point a^3 of the exceptional divisor of the third blowup. It lies off the transforms of the exceptional components produced by the first two blowups (see Figure 1). The coordinate change $x \to x + yz^3$ at a^3 eliminates y^2z^6 and produces

$$f^3 = x^2 + z^6 \cdot (y^5 + y^4 + y^3).$$

The order of f has remained constant equal to 2 throughout. But the shade of f has increased between a^2 and a^3. Namely, in $y^3z^3 \cdot (y^2+z^2)$ the monomial y^3z^3 is exceptional and the remaining factor y^2+z^2 has order 2, whereas in $z^6 \cdot (y^5+y^4+y^3)$ the exceptional factor is z^6 and the remaining factor $y^5+y^4+y^3$ has order 3.

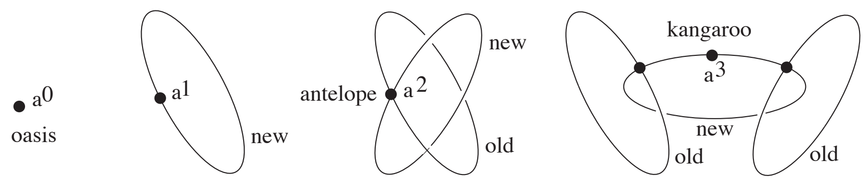

Figure 1: The configuration of kangaroo, antelope and oasis points.

The invariant. We define the first two components of the local resolution invariant at closed points a.• Moreover, we restrict to hypersurfaces. This suffices to describe the phenomena we are interested in. For convenience, we take a to be the origin of $W = \mathbb{A}^{1+m}$. Let f be a polynomial of order $o > 0$ at 0, generating the ideal J in W. Let $V \subset W$ be a regular hypersurface through 0 in an étale neighborhood of 0 (i.e., V is defined by an element in the completion $\widehat{\mathcal{O}}_{W,0}$ of the local ring at 0). Let $x, y_m, \ldots, y_1$ be local coordinates in W at 0 (= regular parameter system of $\widehat{\mathcal{O}}_{W,0}$) so that V is defined by $x = 0$ and $\widehat{\mathcal{O}}_{W,0} \cong K[[x,y]]$. Consider the Taylor expansion of f with respect to x, say

$$f(x,y) = \textstyle\sum_{i=0}^{\infty} a_i(y)x^i,$$

with $a_i \in \widehat{\mathcal{O}}_{V,0} \cong K[[y]]$. The ideal of $\widehat{\mathcal{O}}_{V,0}$ generated by the powers $a_i^{o!/(o-i)}$ of the coefficients a_i of f with $0 \le i < o$ is called the *coefficient ideal* $J_- = \mathrm{coeff}_V(f)$ of f with respect to V. We say that V has *weak maximal contact* with f at 0 if V realizes the maximal value of the order of J_- over all choices of hypersurfaces. If the order is unbounded, f equals a power of a local coordinate, a case which is simple and will be omitted here.

After a sequence of blowups, the coefficient ideal accumulates exceptional monomial factors, giving rise to a factorization $J_- = M_- \cdot I_-$. Here, M_- is a locally principal ideal supported by the exceptional components. Its exponents are prescribed by the preceding resolution process (or, in the language of mobiles of [EH], by the combinatorial handicap D). The ideal I_- represents the portion of J_- which is not monomialized yet. In this way, for V a hypersurface of weak maximal contact, the order of I_- at a is well defined and independent of any choices. We call it the shade of f (or J) at a, denoted by $\mathrm{shade}_a f$. The local resolution invariant for f (more precisely, its first two components) is then the lexicographic pair

$$(\mathrm{ord}_a f, \mathrm{shade}_a f).$$

• In general, the invariant is a whole vector of integers whose components are orders of various ideals.

This is just one candidate invariant, for alternatives in the case of surfaces see [Hi2], [Ha3] and [HWZ]. In the case of purely inseparable equations $f = x^{p^e} + y^r \cdot g(y)$ of order p^e at 0 with variables $y = (y_m, \dots, y_1)$ and exponent vector $r \in \mathbb{N}^m$ prescribing the exceptional multiplicities at $a = 0$, the shade of f is simply the order of g at 0 (up to the constant factor $(p^e - 1)!$ which is usually omitted). Instead of choosing a hypersurface V of weak maximal contact for f it then suffices to treat $y^r g(y)$ modulo p^e-th powers, i.e., as an equivalence class in the quotient $K[[y]]/K[[y^{p^e}]]$.[↺] Most authors consider at this stage only purely inseparable equations, whereas the paper [Ha1] treats arbitrary hypersurfaces (of order p).

In his paper on local uniformization, Moh investigates the possible increase of the shade of f at equiconstant points [Mo]. Let $(W', a') \to (W, a)$ be a local blowup with smooth center Z contained in the locus of order p^e of $f = x^{p^e} + y^r g(y)$ and transversal to the already existing exceptional locus $y^r = 0$. Assume that a' is an equiconstant point for f at a and let f' denote the transform of f at a'. Then Moh shows the inequality[△]

$$\mathrm{shade}_{a'} f' \leq \mathrm{shade}_a f + p^{e-1}.$$

In case $e = 1$, the inequality reads $\mathrm{shade}_{a'} f' \leq \mathrm{shade}_a f + 1$. This allows just a small increase of the shade, but is sufficient to destroy any straightforward induction. Our objective will be to understand the situations where the increase actually happens.

Kangaroo points: These are the points a' above a where $\mathrm{ord}_{a'} f' = \mathrm{ord}_a f$ and $\mathrm{shade}_{a'} f' > \mathrm{shade}_a f$. The point a prior to a kangaroo point a' is the antelope point. We shall work here only at closed points and with formal power series. Moreover, we confine to point blowups, since these entail the most delicate problems. Most of the concepts and results go through for more general situations, cf. [Ha1]. For an integral vector $r \in \mathbb{N}^m$ and a number $c \in \mathbb{N}$, let $\phi_c(r)$ denote the number of components of r which are not divisible by c,

$$\phi_c(r) = \#\{i \leq m,\, r_i \not\equiv 0 \text{ mod } c\}.$$

Define $\overline{r}^c = (\overline{r}^c_m, \dots, \overline{r}^c_1)$ as the vector of the residues $0 \leq \overline{r}^c_i < c$ of the components of r modulo c, and let $|r| = r_m + \ldots + r_1$.

The following result was proven in [Ha1] and is the one cited by Hironaka in his lectures. It characterizes completely the shape of (the tangent cone of) polynomials at an antelope point preceding a kangaroo point in a sequence of blowups. The assertions extend naturally to non purely inseparable equations of order p. For these one has to take the correct definition of coefficient ideal as above. Also, higher dimensional centers are allowed.

Theorem. *Let $(W', a') \to (W, a)$ be a local point blowup of $W = \mathbb{A}^{1+m}$ with center $Z = \{a\}$ the origin. Let be given local coordinates $(x, y_m, \dots, y_1)$ at a so that $f(x, y) = x^p + y^r \cdot g(y) \in \widehat{\mathcal{O}}_{W,a}$ has order p and $\mathrm{shade}_a f = \mathrm{ord}_a g$ at a with exceptional divisor $y^r = 0$. Let f' be the strict transform of f at a'. Then, for a' to be a kangaroo point for f, the following conditions must hold at a:*

(1) *The order $|r| + \mathrm{ord}_a g$ of $y^r g(y)$ is a multiple of p.*
(2) *The exceptional multiplicities r_i at a satisfy*

$$\overline{r}^p_m + \ldots + \overline{r}^p_1 \leq (\phi_p(r) - 1) \cdot p.$$

↺ Hironaka calls the passage to equivalence classes *cleaning*, Włodarczyk *virtual ideals*. Accordingly, the shade is called *residual order* by the first and *virtual order* by the second.

△ Abhyankar informed the author that he had been aware of the inequality.

(3) *The point a' is determined by the expansion of f at a. It lies on none of the strict transforms of the exceptional components $y_i = 0$ for which r_i is not a multiple of p.*
(4) *The tangent cone of g equals, up to linear coordinate changes and multiplication by p-th powers, a specific homogeneous polynomial, called* oblique*, which is unique for each choice of p, r and degree.*[⊕]

For the general statement of the characterization of kangaroo points and the proof of the various assertions, we refer to [Ha1, Thm. 1, sec. 5, and Thm. 2, sec. 12].

The necessity of condition (1) is easy to see and already appears in [Mo]. The arithmetic inequality in condition (2) is related to counting the number of p-multiples in convex polytopes and their r-translates in $\mathbb{R}^m$. It implies that *at least two* exponents r_i must be prime to p. For surfaces ($m = 2$), condition (2) reads $r_2, r_1 \not\equiv 0$ mod p and $\overline{r}_2 + \overline{r}_1 \leq p$. Condition (3) implies that the reference point a has to jump off all exceptional components with $r_i \not\equiv 0$ mod p in order to arrive at a kangaroo point. So it has to leave at least two exceptional components.[≠]

The uniqueness assertion in condition (4) will be explained for the purely separable case in the section on oblique polynomials below. If f is not purely inseparable, there is an analogous description as in (4) characterizing completely the *weighted tangent cone* of f.[±] The uniqueness proof becomes much more involved, cf. [Ha1, Ha2]. The assertion of (4) can be interpreted as a "modulo p-th power version" of the Bernstein-Koushnirenko Theorem on the number of solutions of systems of polynomial equations (which can be computed as the mixed volume of the associated polytopes).

If g is homogeneous (or if f is weighted homogeneous), the increase of the shade is not a serious obstacle since the coefficient ideal of f at the kangaroo point has become a monomial ideal. The intricacy of the resolution in positive characteristic occurs when g is *not homogeneous*. It is then necessary to control the higher order terms of g, and this seems to be delicate. Aside the surface case, it is not clear how to define local invariants for f which do not increase under blowup and thus allow an induction argument. For surfaces, various invariants built from modifications of the pair $(\mathrm{ord}_a f, \mathrm{shade}_a f)$ are possible. They are described in [HWZ].

From a more distanced perspective, the increase of the shade under certain blowups suggests to change radically our approach to resolution. Orders of ideals and especially the concept of shade as the order of a coefficient ideal just seem to be too simple-minded to catch accurately the complexity of singularities in positive charateristic (though they might work after all by applying suitable extra-arguments). One possibility consists in replacing blowups by more sophisticated modifications (e.g., weighted blowups, higher Nash-modifications or a generalization of normalization) or to look out for substantially new invariants. Attempts in this latter direction have been made by Giraud (the order ν of the jacobian ideal of a polynomial f, cf. Cossart's lecture), Youssin, or Hauser. Valuable proposals which really work are still to await.

⊕ The possibility of multiplication with p-th powers was not properly indicated in the original version of [Ha1] (though it was proven there).

≠ This fact seems to be kind of folklore in the field. It was apparently observed by several people, among them Cossart, Spivakovsky and F. Cano.

± If f is of order c at 0, write it in Weierstrass form $f = x^c + \sum_{i=0}^{c-1} a_i(y) x^i$, set $e = \min_i \frac{c}{c-i} \cdot \mathrm{ord}\, a_i$ with $e \geq c$ and then take the tangent cone of f with respect to the weight vector $(e/c, 1, \ldots, 1)$, see [Ha1].

Oblique polynomials: We now describe the tangent cone of the polynomials g appearing in $f = x^p + y^r g(y)$ at antelope points preceding a kangaroo point. In [Ha1], the uniqueness assertion (4) in the theorem above was established for the tangent cone of arbitrary hypersurfaces of order p, and oblique polynomials were characterized in various specific situations. In [Hi1], a general description of oblique polynomials is given, and Schicho found independently a similar formula. Below we combine all viewpoints to a conjoint presentation.

Fix variables $y = (y_m, \ldots, y_1)$. Set $\ell = m - 1$, and let p be the characteristic of the ground field K. A non-zero polynomial $P = y^r g(y)$ with $r \in \mathbb{N}^m$ and g homogeneous of degree k is called *oblique with parameters* p, r and k if P has no non-trivial p-th power polynomial factor and if there is a vector $t = (0, t_\ell, \ldots, t_1) \in (K^*)^m$ so that the polynomial $P^+(y) = (y + ty_m)^r g(y + ty_m)$ has, after deleting all p-th power monomials from it, order $k + 1$ with respect to the variables $y_\ell, \ldots, y_1$. Without loss of generality, the vector t can and will be taken equal to $(0, 1, \ldots, 1)$. We shall write $\mathrm{ord}_z^p P^+$ to denote the order of P^+ with respect to $z = (y_\ell, \ldots, y_1)$ modulo p-th powers.

Example. Take $m = 2$, $p = 2$ and $P(y) = y_2 y_1 (y_2^2 + y_1^2)$ with $k = 2$. Then $P^+(y) = P(y_2, y_1 + y_2) = y_2 y_1^2 (y_1 + y_2)$ has modulo squares order 3 with respect to y_1.

It is checked by computation that the condition $\mathrm{ord}_z^p P^+ \geq k+1$ on P^+ is a prerequisite for the occurence of a kangaroo point as in the theorem. The result of Moh implies $\mathrm{ord}_z^p P^+ \leq k+1$, so that equality must hold. Condition (4) of the theorem tells us that there is, up to addition of p-th powers, *at most one* oblique polynomial for each choice of the parameters p, r and k. In order that P is indeed oblique it is then also necessary that the degree of P is a multiple of p and that r satisfies $\overline{r}_m^p + \ldots + \overline{r}_1^p \leq (\phi_p(r) - 1) \cdot p$ (again by the theorem).

The following trick for characterizing oblique polynomials appears in [Ha1] for surfaces and is extended in [Hi1] to arbitrary dimension.[◇] We dehomogenize P with respect to y_m. This clearly preserves p-th powers. Moreover, when applied to monomials of total degree divisible by p (as is the case for the monomials of the expansion of P), the dehomogenization creates no new p-th powers. It is thus an "authentic" transformation in our context, i.e., the characterization of oblique polynomials can be transcribed entirely to the dehomogenized situation. Setting $y_m = 1$ and $z = (y_\ell, \ldots, y_1)$ we get $Q(z) = P(1, z) = z^s \cdot h(z)$ with $s = (r_\ell, \ldots, r_1) \in \mathbb{N}^\ell$ and $h(z) = g(1, z)$ a polynomial of degree $\leq k$. The translated polynomial is $Q^+(z) = Q(z + \mathbb{I}) = (z + \mathbb{I})^s \cdot h(z + \mathbb{I})$, where $\mathbb{I} = (1, \ldots, 1) \in \mathbb{N}^\ell$. The condition $\mathrm{ord}_z^p P^+ \geq k+1$ now reads $\mathrm{ord}_z^p Q^+ \geq k+1$ or, equivalently, $Q^+ \in \langle z_\ell, \ldots, z_1 \rangle^{k+1} + K[z^p]$. Let us write this as

$$(z + \mathbb{I})^s \cdot h(z + \mathbb{I}) - v(z)^p \in \langle z_\ell, \ldots, z_1 \rangle^{k+1}$$

for some polynomial $v \in K[z]$. As h has degree $\leq k$, the polynomial v cannot be zero. In addition, we see that the condition $\mathrm{ord}_z^p Q^+ \geq k + 1$ is stable under multiplication with homogeneous p-th power polynomials $w(z)$, in the sense that $\mathrm{ord}_z^p \, (w^p \cdot Q^+) \geq k + 1 + p \cdot \deg w$. Using that $(z + \mathbb{I})^s$ is invertible in the completion $K[[z]]$ we get

$$h(z + \mathbb{I}) = \lfloor (z + \mathbb{I})^{-s} \cdot v(z)^p \rfloor_k,$$

where $\lfloor u(z) \rfloor_k$ denotes the k-jet (= expansion up to degree k) of a formal power series $u(z)$. From Moh's inequality we know that $(z + \mathbb{I})^s \cdot h(z + \mathbb{I}) - v(z)^p$ cannot belong to $\langle z_\ell, \ldots, z_1 \rangle^{k+2}$. Therefore, in case that $v(z)$ is a constant, the homogeneous form of degree

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[◇] We are indebted to R. Blanco, D. Wagner and E. Faber for computing several significative examples.

$k+1$ in $(z+\mathbb{I})^{-s}$ must be non-zero. This form equals $\sum_{\alpha\in\mathbb{N}^\ell,|\alpha|=k+1}\binom{-s}{\alpha}z^\alpha$. We conclude that if all $\binom{-s}{\alpha}$ with $|\alpha| = k+1$ are zero in K, then v was not a constant.[∂] Inverting the translation $\tau(z) = z+\mathbb{I}$ we get the following formula for the dehomogenized tangent cone at antelope points preceding kangaroo points,

$$z^s \cdot h(z) = z^s \cdot \tau^{-1}\{\lfloor (z+\mathbb{I})^{-s} \cdot v(z)^p \rfloor_k\}.$$

The homogenization of this polynomial with respect to y_m followed by the multiplication with $y_m^{r_m}$ then yields the actual oblique polynomial $P(y) = y^r g(y)$.

Example. In the example $P(y) = y_2^3 y_1^3 (y_2^2 + y_1^2)$ from the beginning we have characteristic $p = 2$, exponents $r_2 = r_1 = 3$ and degree $k = 2$. Therefore $\ell = 1$ and $s = 3$, which yields a binomial coefficient $\binom{-3}{\alpha} = \binom{-3}{3} = -10$ equal to 0 in K. Indeed, P has as non-monomial factor $g(y)$ the square $(y_2 + y_1)^2$. In the example $P(y) = y_2 y_1 (y_2^2 + y_1^2)$ from above with $r_2 = r_1 = s = 1$, the polynomial g is again a square, even though $\binom{-s}{\alpha} = \binom{-1}{3} = -1$ is non-zero in K.

Surfaces: In the surface case, there are several ways to overcome (or avoid) the obstruction produced by the appearance of kangaroo points. The first proof of surface resolution in positive characteristic is due to Abhyankar, using commutative algebra and field theory [Ab]. Resolution invariants for surfaces then appear, at least implicitly, in his later work on resolution of three-folds. In [Hi2], Hironaka proposes an explicit invariant for the embedded resolution of surfaces in three-space (see [Ha3] for its concise definition). It is not clear how to extend this invariant to higher dimensions.

In [Ha1], it is shown for surfaces that during the blowups prior to the jump at a kangaroo point the shade must have decreased at least by 2 (with one minor exception) and thus makes up for the later increase at the kangaroo point. To be more precise, given a sequence of point blowups in a three dimensional ambient space for which the subsequent centers are equiconstant points for some f, call *oasis point* the last point a° below the antelope point a where none of the exceptional components through a has appeared yet. The following is then a nice exercise:

The shade of f drops between the oasis point a° and the antelope point a of a kangaroo point a' at least to the integer part of its half,

$$\mathrm{shade}_a f \leq \lfloor \tfrac{1}{2} \cdot \mathrm{shade}_{a^\circ} f^\circ \rfloor.$$

In the purely inseparable case of an equation of order equal to the characteristic, this decrease thus dominates the later increase of the shade by 1 except for the case $\mathrm{shade}_{a^\circ} f^\circ = 2$ which is easy to handle separately and will be left to the reader. It seems challenging to establish a similar statement for singular three-folds in four-space.

In [HWZ], we proceed somewhat differently by considering also blowups after the occurence of a kangaroo point. A detailed analysis shows that when taking three blowups together (the one between the antelope and the kangaroo point, and two more afterwards), the shade always either decreases in total, or, if it remains constant, an auxiliary secondary shade drops. This shade can again be interpreted as the order of a suitable coefficient ideal (now in just one variable), made coordinate independent by maximizing it over all choices of hypersurfaces inside the chosen hypersurface.

[∂] The converse need not hold, see the example.

The cute thing is that one can subtract, following an idea of Dominik Zeillinger [Ze] which was made precise and worked out by Dominique Wagner, a correction term from the shade which eliminates the increases without creating new increases at other blowups. This correction term, called the *bonus*, is defined in a subtle way according to the internal structure of the defining equation. It is mostly zero, takes at kangaroo points a value between 1 and 2, and in certain well defined situations a value between $1/2$ and 1.

This bonus allows to define an invariant – a triple consisting of the order, the modified shade and the secondary shade – which now drops lexicographically after *each* blowup. The bonus is defined with respect to a *local flag*. Flags break symmetries and are stable under blowup (in a precise sense) and thus allow to define the bonus at any stage of the resolution process. We refer to [HWZ] for the details, as well as for the definition of an alternative invariant, the *height*, which profits much more from the flag than the shade and allows a simpler definition of the bonus. The invariant built from the height yields a quite systematic induction argument which may serve as a testing ground for the embedded resolution of singular three-folds.

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Fakultät für Mathematik
Universität Wien, Austria
herwig.hauser@univie.ac.at