On the regularity problem for the $3D$ incompressible Navier-Stokes equations

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The $3D$ incompressible Navier-Stokes equations on the periodic domain $\Omega = [0,\, L]^3_{per}$ can be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f}(\boldsymbol{x})$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ is a divergence-free, mean zero velocity field. The forcing function $\boldsymbol{f}(\boldsymbol{x})$ is divergence-free and time-independent.

The issue of whether solutions of these equations remain regular for arbitrarily long times has been an open problem in applied mathematics since Leray's first investigations into weak solutions in the 1930's. This seminar will initially review where some of the problems lie in bounding the global enstrophy ($\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$)

$$H_1 = \int_\Omega |\boldsymbol{\omega}|^2\, dV.$$

Failure to control this quantity denies us the chance to prove the existence of a global attractor (this does exist in the $2D$ case). Ladyzhenskaya's small initial data result (1960), among others, is normally the best that can be extracted from this approach.

However, in 1982, Foias, Guillopé & Temam found an excellent weak solution result. They showed that an infinite set of time-averaged quantities are bounded a priori, namely for $n \geq 1$

$$\left\langle H_n^{\frac{1}{2n-1}} \right\rangle < \infty,$$

where

$$H_n = \int_\Omega |\nabla^n \boldsymbol{u}|^2\, dV$$

and $< \cdot >$ stands for time average. This seminar will show how these ideas can be extended such that an infinite set of time-dependent ratios of norms

$$\kappa_{n,r}(t) = \left(\frac{H_n}{H_r}\right)^{\frac{1}{2(n-r)}}$$

can be proven to have bounded time averages

$$\langle \kappa_{n,r} \rangle \leq \left(\frac{L}{\lambda_k} \right)^4$$

for $0 \leq r < n$. λ_k is the Kolmogorov length which is bounded below, a priori. The quantities $\kappa_{n,r}$ that can be thought of as dynamical wavenumbers of the system; the case of $\kappa_{n,0}$ is special and a better upper bound can be found for this case. In the latter case, a tentative connection is made with the nature of the energy spectrum for weak solutions.