PRINCIPAL Γ-CONE FOR A TREE Γ

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With an Appendix by Yoshio Sano

ABSTRACT. For a graph Γ, we introduce Γ-cones, which are polyhedral cones decomposing the real vector space of type $A_{\#\Gamma-1}$ and are subdivided into chambers. If, further, Γ is a tree, we introduce the principal Γ-cone E_Γ among all Γ-cones and characterize it by the maximality of the number of chambers contained in it. We give a formula enumerating chambers in the Γ-cone. Appendix gives the generating functions for the series of types A_l, D_l and E_l.

The principal Γ-cone, where Γ is a Coxeter diagram $\Gamma(W)$ of finite type, is introduced in [S1-2] in the study of real bifurcation set. The principal Γ-cones for any tree Γ in the present paper is its generalization. As we shall see in the present paper, the characterization of the principal Γ-cone (§3 Theorem) and the enumeration of chambers in the principal Γ-cone (§4 Theorem) can be formulated and proven for any tree independent of the study of bifurcation set. Because of their combinatorial nature, we separate the results in the present paper.

The contents of the present paper is as follows. In §1, we fix basic notation related to Γ-cones. In §2, we prepare two assertions to count chambers in a Γ-cone. The principal Γ-cone for a tree Γ is introduced in §3. The first main Theorem of the present paper in §3 states that the principal Γ-cone contains strictly maximal number of chambers among all Γ-cones. In §4, as the second main Theorem, we give the formula enumerating chambers in the principal Γ cone in terms of the tree Γ. In §5, we explain a motivation for the principal Γ-cones from a study of bifurcation set [S1-2]. We compare Γ-cones with Springer cones [Ar1]. The generating functions for the series of the numbers of chambers for the types A_l $(l \geq 1)$, D_l $(l \geq 3)$ and E_l $(l \geq 4)$ are given in Appendix.

1. The Γ-cones and their chamber decomposition

In this section, for a graph Γ, we introduce Γ-cones and their decompositions into chambers. The Γ-cones are in one to one correspondence with the orientations on the graph Γ. The chambers contained in a Γ-cone are bijective to the orderings on Γ compatible with the corresponding orientation.

Let Π be a finite set with $\#\Pi = l \in \mathbb{Z}_{\geq 1}$. Consider a vector space:

$$V_\Pi := \oplus_{\alpha\in\Pi}\mathbb{R}v_\alpha/\mathbb{R}\cdot v_\Pi \tag{1}$$

of rank $l-1$, where $\{v_\alpha\}_{\alpha\in\Pi}$ is a generator system of V_Π satisfying a single relation $v_\Pi = 0$ with $v_\Pi := \sum_{\alpha\in\Pi} v_\alpha$. The permutation group $\mathfrak{S}(\Pi)$ acts on $\{v_\alpha\}_{\alpha\in\Pi}$ fixing v_Π, and, hence, the action extends linearly on V_Π (the reflection group action of type A_{l-1}). Let $\{\lambda_\alpha\}_{\alpha\in\Pi}$ be the dual basis of $\{v_\alpha\}_{\alpha\in\Pi}$, so that the difference $\lambda_{\alpha\beta} := \lambda_\alpha - \lambda_\beta$ for $\alpha,\beta\in\Pi$ is a well defined linear form on V_Π, forming the root system of type A_{l-1}. The zero locus $H_{\alpha\beta}$ of $\lambda_{\alpha\beta}$ ($\alpha\neq\beta$) in V_Π is a reflection hyperplane of the transposition (α,β) action. The union of $H_{\alpha\beta}$ for all $\alpha,\beta\in\Pi$ with $\alpha\neq\beta$ cuts V_Π into $(\#\Pi)!$ number of connected components, called chambers of type A_{l-1}. The set of chambers is naturally bijective to the set $Ord(\Pi)$ of all linear ordering on the set Π by the correspondence: $c := \{\alpha_1 <_c \ldots <_c \alpha_l\} \in Ord(\Pi) \;\leftrightarrow\; C_c := \cap_{i=1}^{l-1}\{v\in V_\Pi \mid \lambda_{\alpha_{i+1}\alpha_i}(v) > 0\}$. Here, the order-relation with respect to c is denoted by $<_c$, and the corresponding chamber is denoted by C_c. If we denote by $-c$ the reversed ordering of c, then one has $C_{-c} = -C_c$.

A *graph* Γ on Π is a one-dimensional simplicial complex whose set of vertices is Π. An edge connecting vertices α and β (if it exists) is denoted by $\overline{\alpha\beta}$. The set of all edges of Γ is denoted by $Edge(\Gamma)$. For an abuse of notation, we sometimes denote the set of vertices by Γ, and we say "a vertex $\alpha\in\Gamma$" instead of "a vertex $\alpha\in\Pi$".

In the present paper, we mean by an *orientation* o on Γ a collection of orientations $\alpha >_o \beta$ for all edges $\overline{\alpha\beta}\in Edge(\Gamma)$ with the constraint that the oriented graph (Γ, o) does not contain an oriented cycle (in the natural sense). The set of all orientations on Γ is denoted by $Or(\Gamma)$.

Definition. Let a graph Γ on Π be given. A Γ-*cone* is a connected component of $V_\Pi \setminus \cup_{\overline{\alpha\beta}\in Edge(\Gamma)} H_{\alpha\beta}$.

The list of all Γ-cones is given by the following assertion.

Assertion 1.1. *1. For an orientation $o\in Or(\Gamma)$, define*

$$E_o := \cap_{\overline{\alpha\beta}\in Edge(\Gamma)\ \text{oriented as}\ \alpha>_o\beta}\{v\in V_\Pi \mid \lambda_{\alpha\beta}(v) > 0\}. \tag{2}$$

Then E_o is a Γ-cone. The correspondence $o\mapsto E_o$ induces a bijection

$$Or(\Gamma) \simeq \{\Gamma\text{-cones}\}. \tag{3}$$

2. A chamber C_c for $c\in Ord(\Pi)$ is contained in the Γ-cone E_o for $o\in Or(\Gamma)$ if and only if $o = c|Edge(\Gamma)$.

Proof. 1. For any $o\in Or(\Gamma)$, let us show $E_o\neq\emptyset$, that is: there exists a map $v:\Pi\to\mathbb{R}$ such that $v(\alpha) > v(\beta)$ if $\alpha >_o \beta$. This is achieved by an induction on $\#\Pi$. Since there is no oriented cycle in (Γ, o), there exists a *maximal* vertex $\alpha \in \Pi$, that is: for any edge $\overline{\alpha\beta} \in Edge(\Gamma)$, one has $\alpha >_o \beta$. Put $\Pi' := \Pi \setminus \{\alpha\}$. Then clearly $o' := o|\Pi'$ is an orientation on the graph $\Gamma' := \Gamma|\Pi'$. Therefore, by the induction hypothesis, there exists a map $v' : \Pi' \to \mathbb{R}$ preserving the sub-orientation o'. Then, v is defined by an extension of v' by choosing the value $v(\alpha)$ from the non-empty set $\mathbb{R} \setminus \cup_{\beta \in \Pi', \overline{\alpha\beta} \in Edge(\Gamma)} (-\infty, v'(\beta)]$.

Conversely, for a given Γ-cone E, define an orientation $\alpha >_E \beta$ on an edge $\overline{\alpha\beta} \in Edge(\Gamma)$ if $\lambda_{\alpha\beta}|E > 0$. This defines an orientation o_E on Γ.

2. The inclusion $C_c \subset E_o$ is equivalent to that for any oriented edge $\overline{\alpha\beta}$ with $\alpha >_o \beta$, one has the inclusion $C_c \subset \{\lambda_{\alpha\beta} > 0\}$, which is equivalent to $\alpha >_c \beta$. □

According to the previous assertion, we put

$$\Sigma(o) := \{\ c \in Ord(\Pi)\ \mid\ o = c|Edge(\Gamma)\ \}, \tag{4}$$

and introduce a numerical invariant for the orientation $o \in Or(\Gamma)$:

$$\sigma(o) := \#\Sigma(o) = \#\{\text{chambers contained in } E_o\}. \tag{5}$$

If we denote by $-o$ the reversed orientation of o, one has $E_{-o} = -E_o$ and, therefore, $\Sigma(-o) = -\Sigma(o)$ and $\sigma(-o) = \sigma(o)$.

If $\#\Pi = 1$, then $V_\Pi = \{0\}$ has only one chamber $O := \{0\}$. There is only one graph (tree) structure on Π, denoted by $\Gamma(A_1)$, which admits only a trivial orientation denoted by o_{A_1}: $\Sigma(o_{A_1}) = \{O\}$ and $\sigma(o_{A_1}) = 1$.

Let us call $(\sigma(o))_{o \in Or(\Gamma)}$ the *decomposition vector.* One, obviously, has $\sum_{o \in Or(\Gamma)} \sigma(o) = l!$. Even though the vector is algorithmically determined from the graph Γ, it is non-trivial to calculate it in general.

Remark 1. Two chambers are said to be adjacent if they have a common $l-2$-dimensional face. The adjacency relation defines a graph structure on the set of chambers (i.e. two vertices are connected by an edge if the corresponding two chambers are adjacent). Thus, the set $\Sigma(o)$, identified with the set of chambers contained in E_o, naturally inherit the adjacency relation and carries a graph structure. The graph on $\Sigma(o)$ seems to be hard to describe. However, the graph on the set $ord(\Pi)$ of all chambers is easily described by the Caylay graph of $\mathfrak{S}(\Pi)$:

Let us fix a chamber C and let Φ be the set of transpositions in $\mathfrak{S}(\Pi)$ corresponding to the faces of C. It is well-known that the correspondence: $g \in \mathfrak{S}(\Pi) \mapsto gC \in \{chambers\}$ is a bijection. Then two chambers gC and hC are adjacent, if and only if $g^{-1}h \in \Phi$.

2. A decomposition formula

We prepare two Assertions to calculate $\sigma(o)$, which are used in the proof of Theorems in §3 and 4. The idea is to choose a base point $\alpha \in \Gamma$

and to separate (Γ, o) to the right and left sides of α. Some readers may be suggested to skip this section until it is necessary.

For any $o \in Or(\Gamma)$, $\alpha \in \Pi$ and $r \in \mathbb{Z}_{\geq 0}$, we put

$$\Sigma(o, \alpha, r) := \{c \in \Sigma(o) \mid \#\{\beta \in \Pi \mid \beta >_c \alpha\} = r\}, \tag{6}$$

$$\sigma(o, \alpha, r) := \#\Sigma(o, \alpha, r). \tag{7}$$

Obviously, one has the disjoint decomposition $\Sigma(o) = \coprod_{r=0}^{l-1} \Sigma(o, \alpha, r)$ for any $\alpha \in \Pi$ so that $\sigma(o) = \sum_{r=0}^{l-1} \sigma(o, \alpha, r)$.

1. Consider a setting that the complement $\Gamma \setminus \{\alpha\}$ of Γ at a vertex $\alpha \in \Pi$ decomposes into components. More precisely, let $\Gamma_1, \cdots, \Gamma_k$ be graphs, which contain the same named vertex α. Let us denote by

$$\Gamma_1 \coprod\nolimits_\alpha \cdots \coprod\nolimits_\alpha \Gamma_k,$$

a graph obtained by a disjoint union of the graphs Γ_i $(i=1, \cdots, k)$ up to an identification of the common vertex α.

Assertion 2.1. *Let* $\Gamma = \Gamma_1 \coprod_\alpha \cdots \coprod_\alpha \Gamma_k$ *be a decomposition as above. For an orientation* $o \in Or(\Gamma)$, *put* $o_i := o|_{\Gamma_i} \in Or(\Gamma_i)$ $(i = 1, \cdots, k)$. *Then one has formulae:*

$$\sigma(o, \alpha, r) = \sum_{\substack{r_1, \cdots, r_k \in \mathbb{Z}_{\geq 0} \\ r_1 + \cdots + r_k = r}} \sigma(o_1, \alpha, r_1) \cdots \sigma(o_k, \alpha, r_k) C_{r_1, \cdots, r_k} C_{l_1 - r_1 - 1, \cdots, l_k - r_k - 1},$$

$$\sigma(o) = \sum_{r_1, \cdots, r_k \in \mathbb{Z}_{\geq 0}} \sigma(o_1, \alpha, r_1) \cdots \sigma(o_k, \alpha, r_k) C_{r_1, \cdots, r_k} C_{l_1 - r_1 - 1, \cdots, l_k - r_k - 1}.$$

where $l_i := \#\Gamma_i$ $(i=1, \cdots, k)$ *and* $C_{r_1, \cdots, r_k} := (r_1 + \cdots + r_k)!/(r_1! \cdots r_k!)$ *is the combination number.*

Proof of Formulae. Consider the projection $\Sigma(o) \to \Sigma(o_1) \times \cdots \times \Sigma(o_k)$, $c \mapsto (c|_{\Gamma_i})_{i=1, \cdots, k}$. The projection decomposes into projections

$$\Sigma(o, \alpha, r) \to \coprod_{\substack{r_1, \cdots, r_k \in \mathbb{Z}_{\geq 0} \\ r_1 + \cdots + r_k = r}} \Sigma(o_1, \alpha, r_1) \times \cdots \times \Sigma(o_k, \alpha, r_k)$$

for $r \in \mathbb{Z}_{\geq 0}$. Let us see that the cardinality of the inverse image of a point $(c_1, \cdots, c_k) \in \Sigma(o_1, \alpha, r_1) \times \cdots \times \Sigma(o_k, \alpha, r_k)$ depends only on $(r_1, \cdots, r_k)$. An ordering $c \in \Sigma(o, \alpha, r)$ is in the inverse image, if c defines the ordering of $r = r_1 + \cdots + r_k$ elements in RHS of α and the ordering of $l - r - 1 = (l_1 - r_1 - 1) + \cdots + (l_k - r_k - 1)$ elements in LHS of α, where the sub-orderings among $r_1, \cdots, r_k$ elements in RHS are prefixed by $c_1, \cdots, c_k$ and the sub-ordering among $l_1 - r_1 - 1, \cdots l_k - r_k - 1$ elements in LHS are also prefixed by $c_1, \cdots, c_k$. The number of such possibilities of c is given by the combination $C_{r_1, \cdots, r_k} C_{l_1 - r_1 - 1, \cdots, l_k - r_k - 1}$ and is independent of $(c_1, \cdots, c_k)$. $\square$

2. A vertex $\alpha \in \Pi$ is called *maximal* (resp. *minimal*) with respect to $o \in Or(\Gamma)$, if $\alpha >_o \beta$ (resp. $\alpha <_o \beta$) for any edge $\overline{\alpha\beta} \in Edge(\Gamma)$ at α.

Assertion 2.2. *If α is maximal with respect to o, then one has*

$$\sigma(o,\alpha,0) \geq \sigma(o,\alpha,1) \geq \cdots \geq \sigma(o,\alpha,l-2) \geq \sigma(o,\alpha,l-1).$$

If α is minimal with respect to o, then one has

$$\sigma(o,\alpha,0) \leq \sigma(o,\alpha,1) \leq \cdots \leq \sigma(o,\alpha,l-2) \leq \sigma(o,\alpha,l-1).$$

If α is non-isolated in Γ, the smallest terms in the sequences are zero.

Proof. We show only the first case. The latter case is shown parallely.

It is sufficient to show that there is an injection map $\Sigma(o,\alpha,r) \to \Sigma(o,\alpha,r-1)$ for $r > 0$. In fact the map is constructed as follows: let $c = \{A <_c \alpha <_c \beta <_c B\} \in \Sigma(o,\alpha,r)$ where $\beta \in \Pi$ and A and B are linear sequence of inequalities of elements of Π such that the length of B is equal to $r-1$ (this is possible since $r > 1$). Then to c, we attach $c' := \{A <_c \beta <_c \alpha <_c B\} \in \Sigma(o,\alpha,r-1)$ where c' is well defined since α is maximal. The correspondence $c \mapsto c'$ is clearly injective.

If α is non-isolated, then the set $\Sigma(o,\alpha,l-1)$ is empty, since there exists a vertex $\beta \in \Pi$ such that $\overline{\beta\alpha} \in Edge(\Gamma)$ and $\beta <_o \alpha$ and hence for any $c \in \Sigma(o)$ one has $\beta <_c \alpha$ and $c \notin \Sigma(o,\alpha,l-1)$. □

3. Trees and the principal Γ-cones

A graph Γ is called a *tree* if it is connected and simply connected. If Γ is a tree, we introduce particular Γ-cones (unique up to sign), called the principal Γ-cones. The first main result of the present paper is to characterize the principal Γ-cones, and is formulated in Theorem.

The following is a characterization of trees in terms of Γ-cones.

Assertion 3.1. *Let Γ be a graph on Π. Then, $\{H_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ forms a system of coordinate hyperplanes of V_Π if and only if Γ is a tree.*

Proof. For each edge $\overline{\alpha\beta}$ of Γ, we choose one of $\lambda_{\alpha\beta}$ or $\lambda_{\beta\alpha}$. Then, it is immediate to show that i) $\{\lambda_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ is linearly independent if and only if Γ does not contain a cycle, and ii) $\{\lambda_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ spans the dual space of V_Π if and only if Γ is connected. □

From now on in the present paper, we assume that Γ is a tree on Π. Then the system of coordinate hyperplanes $\{H_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ cuts the vector space V_Π into $2^{\#\Pi-1}$-number of quadrants, each of which is a Γ-cone. Therefore, the Γ-cones are simplicial (i.e. are cones over simplices), and the size of a decomposition vector for a tree Γ is equal to 2^{l-1}. The other distinguished property of the decomposition vector for a tree is that it contains a unique (up to an involution, c.f. bbelow) maximal entry, which we explain now.

If Γ is a tree, which is not of type A_1, then there is a decomposition, unique up to a transposition, of the set Π of vertices in two parts:

$$\Pi = \Pi_1 \amalg \Pi_2 \tag{8}$$

such that each Π_i is totally disconnected in Γ. Using a decomposition, we define a *principal orientation* ($\in Or(\Gamma)$) on Γ:

$$o_{\Pi_1,\Pi_2} := \{\ \alpha >_{o_{\Pi_1,\Pi_2}} \beta \ \text{ for } \overline{\alpha\beta} \in Edge(\Gamma) \text{ with } \alpha \in \Pi_1, \beta \in \Pi_2\}. \tag{9}$$

Changing the role of Π_1 and Π_2, $o_{\Pi_2,\Pi_1} = -o_{\Pi_1,\Pi_2}$ is also a principal orientation. There are only two principal orientations.

Definition. A *principal* Γ-*cone* for a tree Γ is a Γ-cone attached to a principal orientation. That is: it is one of the following two cones:

$$E_{\Pi_1,\Pi_2} := E_{o_{\Pi_1,\Pi_2}} \quad \text{and} \quad E_{\Pi_2,\Pi_1} := E_{o_{\Pi_2,\Pi_1}}. \tag{10}$$

Since $o_{\Pi_2,\Pi_1} = -o_{\Pi_1,\Pi_2}$ and $E_{\Pi_2,\Pi_1} = -E_{\Pi_1,\Pi_2}$, two principal Γ-cones are isomorphic to each other as abstract cones. The isomorphisms class:

$$E_\Gamma := E_{\Pi_1,\Pi_2} \simeq E_{\Pi_2,\Pi_1} \tag{11}$$

is called the *principal* Γ-*cone*.

If $\Gamma = \Gamma(A_1)$, the trivial orientation o_{A_1} on $\Gamma(A_1)$ is called the *principal orientation on* $\Gamma(A_1)$. Thus the principal cone $E_{\Gamma(A_1)} = \{0\}$ consists of a single chamber O, i.e. $\Sigma(o_{A_1}) = \{O\}$ and $\sigma(o_{A_1}) = 1$.

We characterize the principal Γ-cone from all the other Γ-cones.

Theorem 3.2. *Let Γ be a tree on Π. The principal Γ-cone contains strictly maximal number of chambers among all Γ-cones. That is: a Γ-cone E_o is principal if and only if $\sigma(o) = \max\{\sigma(p) \mid p \in Or(\Gamma)\}$.*

Proof. After two preparations in §2, the proof is straight forward now.

Suppose $o \in Or(\Gamma)$ is not principal, that is: there exist $\alpha, \beta, \gamma \in \Pi$ with $\gamma <_o \alpha <_o \beta$. Actually, Γ decomposes as $\Gamma = \Gamma_+ \coprod_\alpha \Gamma_-$, where Γ_+ (resp. Γ_-) is a full subgraphs of Γ containing α and any connected component of $\Gamma \setminus \{\alpha\}$ which contains a vertex β s.t. $\alpha <_o \beta$ (resp. $\alpha >_o \beta$). By the assumption on o, one has $\Gamma_\pm \neq \emptyset$.

Put $o_+ := o|\Gamma_+ \in Or(\Gamma_+)$ and $o_- := o|\Gamma_- \in Or(\Gamma_-)$.

Assertion 3.3. *Define a new orientation $\tilde{o} \in Or(\Gamma)$ by the rule: $\tilde{o}$ agrees with o_+ on Γ_+ and with $-o_-$ on Γ_-. Then $\sigma(\tilde{o}) > \sigma(o)$.*

Proof. For a proof of Assertion, we apply the second formula in §2 Assertion 2.1 to the decomposition $\Gamma = \Gamma_+ \coprod_\alpha \Gamma_-$and to $o, \tilde{o} \in Or(\Gamma)$:

$$\sigma(o) = \sum_{r_+=0}^{l_+} \sum_{r_-=0}^{l_-} \sigma(o_+, \alpha, r_+)\sigma(o_-, \alpha, r_-) C_{r_+,r_-} C_{l_+-r_+,l_--r_-},$$

$$\sigma(\tilde{o}) = \sum_{r_+=0}^{l_+} \sum_{r_-=0}^{l_-} \sigma(o_+, \alpha, r_+)\sigma(o_-, \alpha, r_-) C_{r_+,l_--r_-} C_{l_+-r_+,r_-},$$

where $l_+ := \#\Gamma_+ - 1 > 0$ and $l_- := \#\Gamma_- - 1 > 0$.

We want to calculate the difference $\sigma(\tilde{o}) - \sigma(o)$ term-wisely. Observe that the terms for $r_+ = l_+/2$ (if l_+ is even) and the terms for $r_- = l_-/2$ (if l_- is even) in the two formulae give the same value and so that their difference cancel to each other. Therefore, we decompose the region $[0, l_+] \times [0, l_-]$ of the summation index (r_+, r_-) into 4 regions according as r_+ is larger or less than $l_+/2$ and r_- is larger or less than $l_-/2$.

For an index (r_+, r_-) in the region $[0, l_+/2) \times [0, l_-/2)$, we consider 4 indices (r_+, r_-), (r_+, r_-^*), (r_+^*, r_-) and (r_+^*, r_-^*) in the 4 regions simultaneously, where $r_+^* := l_+ - r_+$ and $r_-^* := l_- - r_-$. Let us explicitly write down the difference of the 4 terms in $\sigma(\tilde{o})$ and that in $\sigma(o)$:

$$\begin{aligned} &\sigma(r_+)\sigma(r_-)C_{r_+,r_-^*}C_{r_+^*,r_-} + \sigma(r_+)\sigma(r_-^*)C_{r_+,r_-}C_{r_+^*,r_-^*} \\ &+ \sigma(r_+^*)\sigma(r_-)C_{r_+^*,r_-^*}C_{r_+,r_-} + \sigma(r_+^*)\sigma(r_-^*)C_{r_+^*,r_-}C_{r_+,r_-^*} \\ &- \sigma(r_+)\sigma(r_-)C_{r_+,r_-}C_{r_+^*,r_-^*} - \sigma(r_+)\sigma(r_-^*)C_{r_+,r_-^*}C_{r_+^*,r_-} \\ &- \sigma(r_+^*)\sigma(r_-)C_{r_+^*,r_-}C_{r_+,r_-^*} - \sigma(r_+^*)\sigma(r_-^*)C_{r_+^*,r_-^*}C_{r_+,r_-}, \end{aligned}$$

where we used the simplified notation $\sigma(r_+) := \sigma(o_+, \alpha, r_+)$, $\sigma(r_+^*) := \sigma(o_+, \alpha, r_+^*)$, $\sigma(r_-) := \sigma(o_-, \alpha, r_-)$ and $\sigma(r_-^*) := \sigma(o_-, \alpha, r_-^*)$.

Marvelously, one can factorize this difference as follows:

$$(\sigma(r_+) - \sigma(r_+*))(\sigma(r_-^*) - \sigma(r_-))(C_{r_+,r_-}C_{r_+^*,r_-^*} - C_{r_+,r_-^*}C_{r_+^*,r_-}).$$

Let us examine the sign of the factors so that the product turns out to be non-negative. First, recall that the vertex α is minimal in Γ_+ and maximal in Γ_- by definition. Note also $r_+ < l_+/2 < r_+^*$ and $r_- < l_-/2 < r_-^*$. Therefore, applying §2 Assertion 2.2, we observes that $(\sigma(r_+) - \sigma(r_+*))(\sigma(r_-^*) - \sigma(r_-)) \geq 0$. Next, let us examine the last factor. For the purpose, we use the proportion of the two terms:

$$\frac{C_{r_+,r_-}C_{r_+^*,r_-^*}}{C_{r_+,r_-^*}C_{r_+^*,r_-}} = \frac{(r_+^* + r_-^*)!}{(r_+ + r_-^*)!} \cdot \frac{(r_+ + r_-)!}{(r_+^* + r_-)!}.$$

Using the fact $r_+ < r_+^*$, one has $r_+^* + r_-^* > r_+ + r_-^*$ and $r_+ + r_- < r_+^* + r_-$. Hence, the expression can be reduced to $\prod_{k=r_++1}^{r_+^*} \frac{r_-^*+k}{r_-+k}$, where each factor is larger than 1 for $r_- + k < r_-^* + k$ and the number of the factors is $r_+^* - r_+ > 0$ so that the result is always larger than 1. These together imply that the difference of the 4 terms is non-negative.

By summing up terms for all indices (r_+, r_-) in the region $[0, l_+/2) \times [0, l_-/2)$, we see that the difference $\sigma(\tilde{o}) - \sigma(o)$ is non-negative. To show that it is strictly positive, let us calculate the term for $(r_+, r_-) = (0, 0)$. Then, again by §2, Assertion 2.2, one has $\sigma(r_+) = \sigma(r_-^*) = 0$. Since $l_+, l_- > 0$ (non-principality of σ), one obtains a rather big number:

$$\sigma(o_+, \alpha, l_+)\sigma(o_-, \alpha, 0)(C_{l_+,l_-} - 1) \neq 0.$$

This completes a proof of Assertion. $\square$

Assertion says that if an orientation o on Γ is not principal, it can not attain the maximal value of σ. In fact, starting from any orientation $o \in Or(\Gamma)$, and by a successive application of the construction in Assertion, one arrive at one of the principal orientations. This implies that any non-principal Γ-cone attains strictly smaller values of σ than a principal Γ-cone. These completes a proof of Theorem. $\square$

We are interested in the chamber decomposition $\Sigma(o_{\Pi_1,\Pi_2})$ of a principal Γ-cone E_{Π_1,Π_2} and its adjacency relations..

Problem 1. 1. Give an explicit formula of the number $\sigma(\Gamma) := \sigma(o_\Gamma)$ of chambers in the principal Γ-cone E_Γ. Describe the graph of adjacency relation among the chambers in $\Sigma(o_\Gamma)$ from the tree Γ.

2. Give an explicit formula of the decomposition vector $(\sigma(o))_{o \in Or(\Gamma)}$ from the tree Γ. Characterize the decomposition vector $(\sigma(o))_{o \in Or(\Gamma)}$ coming from a tree among all numerical vectors.

Example. The principal cone $E_{\Gamma(D_4)}$ consists of 6 chambers forming a hexagon. The decomposition vector is $(\sigma(o))_{o \in Or(\Gamma(D_4))} = 2(6, 2, 2, 2)$.

The principal cone $E_{\Gamma(A_4)}$ consists of 5 chambers forming a spoon graph. The decomposition vector is $(\sigma(o))_{o \in Or(\Gamma(A_4))} = 2(5, 3, 3, 1)$.

Let Γ be a 4-cycle. Even though Γ is not a tree, the decomposition vector contains the maximal entry: $(\sigma(o))_{o \in Or(\Gamma)} = 2(4, 2, 2, 1, 1, 1, 1)$.

We answer to the first part of Problem 1. in the next section.

4. Enumeration of chambers in the principal Γ cone

As the second main result of the present paper, we give a formula of $\sigma(\Gamma)$ in terms of the tree Γ. It is a sum whose summation index runs in a suitable equivalence classes $Ord(\Pi_1)/\!\sim$ of all linear orderings on Π_1, where we recall that Π_1 is a component of the decomposition (8). Therefore, we, first, define the equivalence $\sim$ on the set $Ord(\Pi_1)$.

Let $c \in Ord(\Pi_1)$ be an ordering on Π_1. For $v \in \Pi_1$, put

$$\Gamma_{c,v} := \text{the connected component of } \Gamma \setminus \{w \in \Pi_1 \mid w <_c v\} \text{ containing } v.$$

In particular, one has $\Gamma_{c,v} = \Gamma$ for the smallest element v of Π_1.

Definition. Two orderings $c, c' \in Ord(\Pi_1)$ are called *equivalent* if $\Gamma_{c,v} = \Gamma_{c',v}$ for all $v \in \Pi_1$. The equivalence class of c is denoted by $\tilde{c}$ and the set of equivalence classes is denoted by $Ord(\Pi_1)/\sim$.

We state the second main result of the present paper.

Theorem 4.1. *Let* Γ *be a tree. The number of chambers in the principal* Γ*-cone is given by the formula:*

$$\sigma(\Gamma) = (\#\Gamma)! \sum_{\tilde{c} \in Ord(\Pi_1)/\sim} \frac{1}{\prod_{v\in\Pi_1} \#\Gamma_{c,v}}, \tag{12}$$

where RHS is well-defined since $\Gamma_{c,v}$ *depends only on the equivalence class* $\tilde{c}$ *of* $c \in Ord(\Pi_1)$ *and on* $v \in \Pi_1$.

Proof. Before we start with the proof of the formula, we reformulate the equivalence $\sim$ in terms of partial orderings on the set Π_1.

Fact i) *For any two indices* $v, v' \in \Pi_1$*, one has three cases:*

$$\Gamma_{c,v} \cap \Gamma_{c,v'} = \begin{cases} \emptyset \\ \Gamma_{c,v} \\ \Gamma_{c,v'}. \end{cases}$$

Fact ii) *If* $\Gamma_{c,v} \cap \Gamma_{c,v'} = \Gamma_{c,v}$ *then* $v >_c v'$.

Proof. Since c is a linear ordering, we may assume $v >_c v'$. The fact $\Gamma \setminus \{w \in \Pi_1 \mid w <_c v\} \subset \Gamma \setminus \{w \in \Pi_1 \mid w <_c v'\}$ implies that the component $\Gamma_{c,v}$ is either contained in the component $\Gamma_{c,v'}$ or they are disjoint. Accordingly, the intersection is either $\Gamma_{c,v}$ or an empty set. □

To the equivalence class $\tilde{c}$ of c, we attach a partial ordering on Π_1:

$$v \geq_{\tilde{c}} v' \text{ for } v, v' \in \Pi_1 \quad \overset{\text{def}}{\Leftrightarrow} \quad \Gamma_{c,v} \cap \Gamma_{c,v'} = \Gamma_{c,v}, \tag{13}$$

i.e. there is no relation for v, v' if $\Gamma_{c,v} \cap \Gamma_{c,v'} = \emptyset$, else $\tilde{c}$ agrees with c.

Fact iii) *For any* $v \in \Pi_1$*, there is a one to one correspondence:*

$$\begin{array}{rl} v' & \in \{v' \in \Pi_1 \mid v <_{\tilde{c}} v', v' \text{ is an immediate successor of } v\} \\ \leftrightarrow \ \Gamma_{c,v'} & \in \{\text{connected components of } \Gamma_{c,v} \setminus \{v\} \text{ containing a point } \Pi_1\}. \end{array}$$

Proof of iii). By the definition of $\Gamma_{c,v}$, any point $w \in \Gamma_{c,v} \cap \Pi_1$ satisfies $v <_c w$. Because of the equivalence: $v \leq_{\tilde{c}} w \ \Leftrightarrow \Gamma_{c,v} \supset \Gamma_{c,w}$, w is an immediate successor of v if and only if $\Gamma_{c,w}$ is a connected component of $\Gamma_{c,v} \setminus \{v\}$. Contrary, if a connected component Γ' of $\Gamma_{c,v} \setminus \{v\}$ contains a point Π_1, then the smallest (with respect to the ordering c) element, say w, of $\Gamma' \cap \Pi_1$ is, clearly, an immediate successor of v such that $\Gamma' = \Gamma_{c,w}$. This completes a proof of Fact iv). $\square$

Finally we show the following equivalence.

Assertion 4.2. *For two orderings $c, c' \in Ord(\Pi_1)$, the following two conditions are equivalent.*

a) One has the equality $\Gamma_{c,v} = \Gamma_{c',v}$ for all $v \in \Pi_1$, i.e. $c \sim c'$.

b) The partial orderings $\tilde{c}$ and $\tilde{c}'$ on the set Π_1 coincides.

Proof. It is sufficient to show $\Gamma_{c,v} = \Gamma_{\tilde{c},v}$ for all $v \in \Pi_1$, where

$\Gamma_{\tilde{c},v}$:=the connected component of $\Gamma \setminus \{w \in \Pi_1 \mid w <_{\tilde{c}} v\}$ containing v.

This is shown by induction on the number $\#\{w \in \Pi_1 \mid w <_{\tilde{c}} v\}$ of predecessors of v. The smallest element v with respect to c is also the smallest with respect to $\tilde{c}$ and $\Gamma_{\tilde{c},v} = \Gamma_{c,v} = \Gamma$. For a non smallest element $v \in \Pi_1$, let $w \in \Pi_1$ be an immediate predecessor of v, i.e. v is an immediate successor of w. By induction hypothesis, $\Gamma_{\tilde{c},w} = \Gamma_{c,w}$. Put $\{v_1, \cdots, v_k\} := \{v' \in \Pi_1 \mid w <_c v' <_c v\}$. By definition, $\Gamma_{\tilde{c},v}$ is the connected component of $\Gamma_{c,w} \setminus \{w\}$ containing v and $\Gamma_{c,v}$ is the connected component of $\Gamma_{c,w} \setminus \{w, v_1, \cdots, v_k\}$ containing v. However, due to the description in Fact iii), $v_1, \cdots, v_k$ can not belong to $\Gamma_{\tilde{c},v}$. This implies the equality $\Gamma_{\tilde{c},v} = \Gamma_{c,v}$. $\square$

We return to a proof of Theorem. The formula (12) is shown by an induction on $\#\Gamma$. We first prepare an induction formula.

Let Γ be a tree. For a given decomposition $\{\Pi_1, \Pi_2\}$ and attached principal orientation o_{Π_1,Π_2}, we want to enumerate the set $\Sigma(o_{\Pi_1,\Pi_2})$.

By definition, for any total ordering $c \in \Sigma(o_{\Pi_1,\Pi_2})$, the largest element belongs to Π_1. Therefore, we have a decomposition:

$$\Sigma(o_{\Pi_1,\Pi_2}) = \coprod_{v \in \Pi_1} \Sigma(o_{\Pi_1,\Pi_2}, v)$$

where $\Sigma(o_{\Pi_1,\Pi_2}, v) := \{c \in \Sigma(o_{\Pi_1,\Pi_2}) \mid v \text{ is the largest element in } c.\}$. Put $\sigma(\Gamma, v) := \sigma(o_{\Pi_1,\Pi_2}, v) := \#\Sigma(o_{\Pi_1,\Pi_2}, v)$ so that one has $\sigma(\Gamma) := \sigma(o_{\Pi_1,\Pi_2}) = \sum_{v \in \Pi_1} \sigma(o_{\Pi_1,\Pi_2}, v) = \sum_{v \in \Pi_1} \sigma(\Gamma, v)$.

For $w \in Nbd(v) := \{w \in \Pi \mid \exists \overline{vw} \in Edge(\Gamma)\} \subset \Pi_2$, let us denote by Γ_{vw} the connected component of $\Gamma \setminus \{v\}$ containing w so that one has the decomposition $\Gamma \setminus \{v\} = \coprod_{w \in Nbd(v)} \Gamma_{vw}$.

Applying the first formula in §2 Assertion 2.1 for $\alpha = v$ and $r = r_1 = \cdots = r_k = 0$, we obtain a formula:

$$\sigma(\Gamma, v) = (\#\Gamma - 1)! \prod_{w \in Nbd(v)} \frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{vw})!}. \tag{14}$$

Summing this over all vertices $v\in\Pi_1$, we obtain an induction formula

$$\frac{\sigma(\Gamma)}{(\#\Gamma)!} = \frac{1}{\#\Gamma}\sum_{v\in\Pi_1}\prod_{w\in Nbd(v)}\frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{vw})!}. \tag{15}$$

By induction hypothesis, for any $v \in \Pi_1$ and $w \in Nbd(v)$, we have already the formula

$$*)\qquad \frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{\overline{vw}})!} = \sum_{c_w\in Ord(\Gamma_{vw}\cap\Pi_1)/\sim}\frac{1}{\prod_{v_w\in\Gamma_{vw}\cap\Pi_1}\#(\Gamma_{vw})_{\tilde{c}_w,v_w}}.$$

For a collection $\tilde{c}_w \in Ord(\Gamma_{vw}\cap\Pi_1)/\sim$ for $w \in Nbd(v)$, we denote by $v \times \{\tilde{c}_w\}_{w\in Nbd(v)}$ the partial ordering of the set Π_1 defined by the rule a) v is the smallest element, b) on the set $\Gamma_{vw}\cap\Pi_1$, the partial ordering agrees with $\tilde{c}_w$ for $w \in Nbd(v)$, and c) there is no order relation between $\Gamma_{vw}\cap\Pi_1$ and $\Gamma_{vw'}\cap\Pi_1$ for different $w, w'\in Nbd(v)$. In fact this partial ordering corresponds to the system $\{\Gamma\}\coprod_{w\in Nbd(v)}\{(\Gamma_{vw})_{\tilde{c}_w,v_w}\}_{v_w\in\Gamma_{vw}\cap\Pi_1}$ of subgraphs index by the set $\{v\}\coprod_{w\in Nbd(v)}\{\Gamma_{vw}\cap\Pi_1\} = \Pi_1$. Since $v \times \{\tilde{c}_w\}_{w\in Nbd(v)}$ is the equivalent class of any of its linear extension to Π_1, and any element of $Ord(\Gamma_{vw}\cap\Pi_1)/\sim$ having v as the smallest element has such expression, taking the union for the index $v \in \Pi_1$, one obtains the bijection

$$\bigcup_{v\in\Pi_1}\big(v\times\prod_{w\in Nbd(v)}(Ord(\Gamma_{vw}\cap\Pi_1)/\sim)\big) \simeq Ord(\Pi_1)/\sim .$$

This means that the substitution of the formulae $*$) in RHS of (15) gives RHS of the formula (12). This completes a proof of Theorem. □

Remark 2. The term $\frac{(\#\Gamma)!}{\prod_{v\in\Pi_1}\#\Gamma_{\tilde{c},v}}$ of the formula (12) gives the number of chambers in the principal Γ-cone corresponding to the orderings $c \in Ord(\Pi)$ whose restriction $c|_{\Pi_1}$ to Π_1 agrees with $\tilde{c}$.

Remark 3. By changing the role of Π_1 and Π_2, we get the equality

$$\sum_{\tilde{c}\in Ord(\Pi_1)/\sim}\frac{(\#\Gamma)!}{\prod_{v\in\Pi_1}\#\Gamma_{c,v}} = \sum_{\tilde{c}\in Ord(\Pi_2)/\sim}\frac{(\#\Gamma)!}{\prod_{v\in\Pi_2}\#\Gamma_{c,v}}. \tag{16}$$

For some choices of the tree Γ, the equality seems to give some combinatorial identities.

5. Geometric backgrounds.

We recall briefly a theorem [S1§3, S2§12], which combines the principal Γ-cones with some geometry of real bifurcation set in case Γ is a Coxeter graph of finite type. For details, one is refereed to [ibid].

Let W be a finite reflection group acting irreducibly on an $\mathbb{R}$-vector space V of rank l. Due to Theorem of Chevalley, the quotient variety $S_W := V /\!\!/ W$, as a scheme, is a smooth affine variety,which contains the discriminant divisor D_W consisting of irregular orbits. The integration $\tau = \exp(D)$ of the lowest degree vector field D on S_W, which is unique up to a constant factor and is called the *primitive vector field*, defines a $\mathbb{G}_a$-action on S_W. The quotient $T_W := S_W /\!\!/ \tau(\mathbb{G}_a)$ is an $l-1$-dimensional affine variety. The restriction to D_W of the projection map $S_W \to T_W$ is a l-fold flat covering, whose ramification divisor in T_W is denoted by B_W and called the *bifurcation divisor*. The B_W decomposes into the ordinary part $B_{W,2}$ and the higher part $B_{W,\geq 3}$ according to the ramification index.

Depending on $\varepsilon \in \{\pm 1\}$, there are real forms $T^{\varepsilon}_{W,\mathbb{R}}, B^{\varepsilon}_{W,2,\mathbb{R}}$ and $B^{\varepsilon}_{W,\geq 3,\mathbb{R}}$ of these schemes. There is a distinguished real half axis $AO^{\varepsilon} \simeq \mathbb{R}_{>0}$ (arising from eigenspaces of Coxeter elements, see [S1-2] for details) embedded in $T^{\varepsilon}_{W,\mathbb{R}} \setminus B^{\varepsilon}_{W,\geq 3,\mathbb{R}}$. The connected component of $T^{\varepsilon}_{W,\mathbb{R}} \setminus B^{\varepsilon}_{W,\geq 3,\mathbb{R}}$ containing AO^{ε} is denoted by E^{ε}_W and is called the *central region*.

Let P_l be a largest degree coordinate of S_W. Consider the l-valued algebraic correspondence $T_W \to D_W \overset{P_l|D_W}{\to} \mathbb{A}$. Its l branches at the base point AO^{ε} can be indexified by the set of a simple generator system Π of W. Let us denote them by $\{\varphi_\alpha\}_{\alpha\in\Pi}$ as a system of algebroid functions on T_W (which are branching along $B_{W,\geq 3}$). Then, one has:

Theorem 5.1. *The correspondence* $b_W := \sum_{\alpha\in\Pi} \varphi_\alpha \cdot v_\alpha$ *induces a semi-algebraic homeomorphism:*

$$b_W : \overline{E}^{\varepsilon}_W \simeq \overline{E}_{\Gamma(W)} \tag{17}$$

from the closure of the central region of W *to the closure of the principal cone for the Coxeter graph* $\Gamma(W)$ *of* W *on* Π*, and a homeomorphism:*

$$b_W \ : \ \overline{E}^{\varepsilon}_W \cap B_{W,2,\mathbb{R}} \ \simeq \ \overline{E}_{\Gamma(W)} \cap \big(\cup_{\alpha\beta\in\Pi} H_{\alpha\beta} \big). \tag{18}$$

That is: the central region E^{ε}_W *is a simplicial cone and connected components of* $E^{\varepsilon}_W \setminus B_{W,2,\mathbb{R}}$ *is in one to one correspondence with the set* $\Sigma(\Gamma(W))$ *of chambers contained in the principal* $\Gamma(W)$*-cone* $E_{\Gamma(W)}$*.*

The theorem (in some more precise form) has several important implications in the study of the topology of the configuration space S_W.

Note. 1. *The correspondence* b_W *is, up to a scaling factor, unique and does not depend on a choice of* P_l (a largest degree coordinate of S_W).
Proof. Since the largest exponent of W is unique, any other largest degree coordinate $\tilde{P}_l$ of S_W is of the form $a \cdot P_l + Q$ for a scaling constant a and a polynomial Q of lower degree coordinates. Then, $\tilde{\varphi}_\alpha = a \cdot \varphi_\alpha + Q$ $(\alpha \in \Pi)$, whose second term is independent of α, and, so, $\tilde{b}_W = a \cdot b_W$. $\square$

2. The principal cone in RHS of (12) depends only on the graph structure of the diagram $\Gamma(W)$ but not on the labels on the edges. The graphs $\Gamma(W)$ (forgetting about the labels) of types A_l, B_l, C_l, F_4, G_2, H_3, H_4 and $I_2(p)$ are linear. Hence, the central regions E_W for them are homeomorphic to the principal cones of type A.

Finally, in the present paper, we compare the concept of Γ-cones with somewhat similar concept, the *Springer cones*, explained below.
Definition ([Ar1]). Let V_W be a real vector space with an irreducible action of a finite reflection group W. Let $\{H_\alpha\}_{\alpha\in\Pi}$ be the system of walls of a chamber. A connected component of $V_W \setminus \cup_{\alpha\in\Pi} H_\alpha$ is called a *Springer cone.* A Springer cone containing the maximal number of chambers (unique up to sign [Sp1]) is called a *principal Springer cone.* The maximal number is called the *Springer number.* The Springer number has been calculated by the authors ([So],[Sp1], [Ar1]).

There are some formal similarities between the (principal) Springer cones in V_W and the (principal) Γ-cones in V_Π (see Table below). A result similar to **3.** Theorem is proven for Springer cones [Sp1, Prop.3].

	Springer cone	Γ-cone
The ambient vector space	V_W with W-chambers (depending on the group W)	V_Π with $A_{\#\Pi-1}$-chambers (depending on the set Π)
The cutting hyperplanes	$\{H_\alpha\}_{\alpha\in\Pi}$ (indexed by the vertices of $\Gamma(W)$)	$\{H_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ (indexed by the edges of the tree Γ)

However, the only case when a Γ-cone decomposition is simultaneously a Springer cone decomposition is given by the following.

Assertion 5.2. *For a tree* Γ*, the following i)–iii) are equivalent.*

i) The Γ*-cone decomposition of* V_Π *is isomorphic to the Springer cone decomposition of* V_W *for some finite Coxeter group* W*.*

ii) The smallest number of chambers contained in a Γ*-cone is equal to 1, i.e.* $\inf\{\sigma(o) \mid o \in Or(\Gamma)\} = 1$.

iii) The Γ *is a linear graph of type* A_l*, and* $W = W(A_{l-1})$ *for* $l > 1$.

Proof. i) ⇒ ii): This follows from the definition of the Springer cone.

ii) ⇒ iii): if a chamber $\overline{C} := \{\lambda_{\alpha_1} \le \cdots \le \lambda_{\alpha_l}\}$ alone consists a Γ-cone, then Γ is a linear graph α_1-α_2-$\cdots$-α_l (of type A_l) on Π.

iii) ⇒ i): If Γ is a linear graph α_1-α_2-$\cdots$-α_l, then the orientation $\alpha_1 < \alpha_2 < \cdots < \alpha_l$ on Γ corresponds to the Γ-cone consisting only of a single chamber $\overline{C} := \{\lambda_{\alpha_1} \le \cdots \le \lambda_{\alpha_l}\}$ of type A_{l-1} in $V_\Pi = V_{A_{l-1}}$. □

Due to above Fact, $\sigma(A_l) := \sigma(\Gamma(A_l))$ is equal to the Springer number a_{l-1} of type A_{l-1}. Since the Springer number a_n of type A_n is known to be given by the generating function: $1+\sum_{n=1}^{\infty} \frac{a_n}{n!} x^n = \frac{1}{1-\sin(x)}$

([Sp1,3.]), one has

$$1+\sum_{n=1}^{\infty}\frac{\sigma(A_n)}{n!}x^n = 1+\int_0^x \frac{1}{1-\sin(x)}dx = \tan(\frac{x}{2}+\frac{\pi}{4}). \tag{19}$$

Actually, this formula is directly proven in the Appendix.

Question. By an analogy to §3 Theorem, consider any system of l-reflection hyperplanes in V_W forming coordinate hyperplanes and ask a question: whether there is a unique (up to sign) quadrangle of V_W cut by the hyperplanes which contains the maximal number of chambers. The answer is apparently positive for the type A_l and $I_2(p)$ for odd $p \in 2\mathbb{Z}_{>0}$, and negative for the types B_l, C_l and $I_2(p)$ for even $p \in 2\mathbb{Z}_{>0}$.

6. Appendix: Generating functions for types A_l, D_l and E_l

By Yoshio Sano

For the three series A_l, D_l and E_l, we give the generating functions

1. A_l-type.

Let $\Gamma(A_l)$ be the tree of type A_l ($l \geq 1$) given as follows:

$$\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\cdots-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ$$

where l is the number of vertices of the graph. Put $\sigma(A_n) := \sigma(\Gamma(A_n))$.

Formula.

$$1+\sum_{n=1}^{\infty}\frac{\sigma(A_n)}{n!}x^n = \tan(\frac{x}{2}+\frac{\pi}{4}). \tag{20}$$

Proof. Put $\sigma(A_0) := 1$, and we show a formula:

$$\sigma(A_{n+1}) = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\sigma(A_i)\cdot\sigma(A_{n-i}) \qquad (n \geq 1) \tag{21}$$

Proof of (21). According to l is even or odd, apply the formula (14) in §3 and one obtains:

$$\sigma(A_{2k}) = \textstyle\sum_{i=1}^{k}\binom{2k-1}{2i-1}\sigma(A_{2i-1})\cdot\sigma(A_{2k-2i}),$$

$$\sigma(A_{2k+1}) = \textstyle\sum_{i=1}^{k}\binom{2k}{2i-1}\sigma(A_{2i-1})\cdot\sigma(A_{2k-2i+1}) \quad \square.$$

Put $f_A(x) := \sum_{n=0}^{\infty}\frac{\sigma(A_n)}{n!}x^n$. Then, (21) implies the differential equation

$$f_A' = \frac{1}{2}(f_A^2+1).$$

Together with the initial condition: $f_A(0) = 1$, one has the solution

$$f_A(x) = \tan(\frac{x}{2} + \frac{\pi}{4}). \qquad \square$$

Remark 4. The formula (20) agrees with the formula (19) in §4.

2. D_l-type.

Let $\Gamma(D_l)$ be the tree of type D_l $(l \geq 3)$ given as follows:

$$\begin{array}{l} \circ - \circ - \circ - \cdots - \circ - \circ - \circ \\ \quad\ \ | \\ \quad\ \ \circ \end{array}$$

where l is the number of vertices of the graph. Put $\sigma(D_n) := \sigma(\Gamma(D_n))$.

Formula.

$$\sum_{n=3}^{\infty} \frac{\sigma(D_n)}{n!} x^n = 2(x-1)\tan(\frac{x}{2} + \frac{\pi}{4}) + 2 - 2x^2. \tag{22}$$

Proof. Put $\sigma(D_2) := 2$, $\sigma(D_1) := 0$ and $\sigma(D_0) := 2$, and we show a formula:

$$\sigma(D_{n+1}) = \frac{1}{2} \sum_{i=0}^{n} \binom{n}{i} \sigma(A_i) \cdot \sigma(D_{n-i}) \qquad (n \geq 2) \tag{23}$$

Proof of (23). According to l is even or odd, apply the formula (14) in §3 and one obtains:

$$\sigma(D_{2k}) = \textstyle\sum_{i=1}^{k-1} \binom{2k-1}{2k-2i-1} \sigma(D_{2i}) \cdot \sigma(A_{2k-2i-1}),$$

$$\sigma(D_{2k+1}) = \textstyle\sum_{i=1}^{k} \binom{2k}{2k-2i} \sigma(D_{2i}) \cdot \sigma(A_{2k-2i}) \qquad \square.$$

Put $f_D(x) := \sum_{n=0}^{\infty} \frac{\sigma(D_n)}{n!} x^n$. Then, (23) implies the differential equation

$$f'_D = \frac{1}{2} f_D f_A + x - 1.$$

Together with the initial condition: $f_D(0) = 2$, one has the solution

$$f_D(x) = 2(x-1)\tan(\frac{x}{2} + \frac{\pi}{4}) + 4. \qquad \square$$

Remark 5. Using the relation: $f_D(x) = 2(x-1)f_A(x) + 4$, one obtains

$$\sigma(D_n) = 2(n\sigma(A_{n-1}) - \sigma(A_n)) \qquad (n \geq 1). \tag{24}$$

3. E_l-type.

Let $\Gamma(E_l)$ be the tree of type E_l $(l \geq 4)$ given as follows:

$$\begin{array}{ccccccccccccc} \circ & - & \circ & - & \circ & - & \cdots & - & \circ & - & \circ & - & \circ \\ & & & & | & & & & & & & & \\ & & & & \circ & & & & & & & & \end{array}$$

where l is the number of vertices of the graph. Put $\sigma(E_n) := \sigma(\Gamma(E_n))$.

Formula.

$$\sum_{n=4}^{\infty} \frac{\sigma(E_n)}{n!} x^n = (\frac{1}{2}x^2 - 2x + 3)\tan(\frac{x}{2} + \frac{\pi}{4}) - 3x^3 - x - 3. \tag{25}$$

Proof. Put $\sigma(E_3) := 3$, $\sigma(E_2) := 0$, $\sigma(E_1) := 3$ and $\sigma(E_0) := -1$, and we show a formula:

$$\sigma(E_{n+1}) = \frac{1}{2} \sum_{i=0}^{n} \binom{n}{i} \sigma(E_i) \cdot \sigma(A_{n-i}) \qquad (n \geq 5). \tag{26}$$

Proof of (26). According to l is even or odd, apply the formula (14) in §3 and one obtains:

$$\begin{aligned} 2\sigma(E_{2k}) &= \sigma(A_{2k-1}) + \sigma(D_{2k-1}) + \binom{2k-1}{1} \sigma(A_{2k-2}) \\ &\quad + \textstyle\sum_{i=3}^{2k-1} \binom{2k-1}{i} \sigma(E_i) \cdot \sigma(A_{2k-i-1}), \\ 2\sigma(E_{2k+1}) &= \sigma(A_{2k}) + \sigma(D_{2k}) + \binom{2k}{1} \sigma(A_{2k-1}) \\ &\quad + \textstyle\sum_{i=3}^{2k} \binom{2k}{i} \sigma(E_i) \cdot \sigma(A_{2k-i}) \end{aligned}$$

Eliminate the $\sigma(D_n)$ term by the use of , we obtain (26). $\square$.

Put $f_E(x) := \sum_{n=0}^{\infty} \frac{\sigma(E_n)}{n!} x^n$. Then, (26) implies the differential equation

$$f_E' = \frac{1}{2} f_E f_A + \frac{1}{4}x^2 - x + \frac{7}{2}.$$

Together with the initial condition: $f_E(0) = -1$, one has the solution

$$f_E(x) = (\frac{1}{2}x^2 - 2x + 3)\tan(\frac{x}{2} + \frac{\pi}{4}) + 2x - 4. \qquad \square$$

Remark 6. Using the relation: $f_E(x) = (\frac{1}{2}x^2 - 2x + 3) f_A(x) + 2x - 4$, one obtains

$$\sigma(E_n) = \frac{n(n-1)}{2} \sigma(A_{n-2}) - 2n\sigma(A_{n-1}) + 3\sigma(A_n) \qquad (n \geq 2). \tag{27}$$

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