On Cramer's Paradox

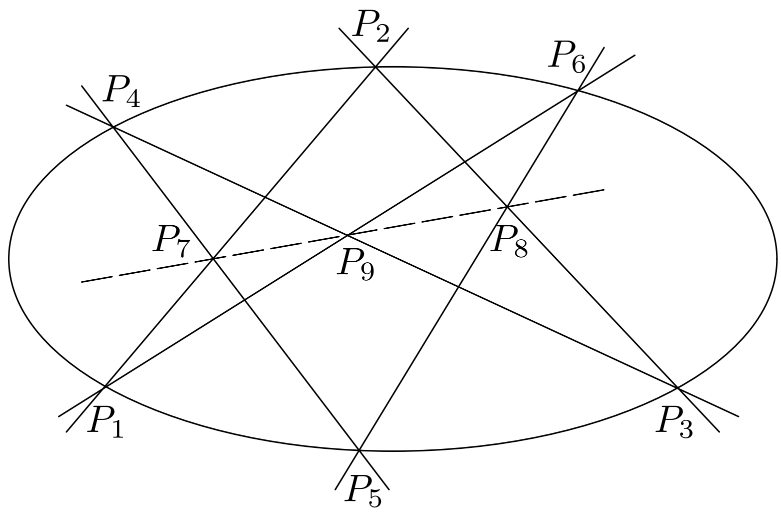
Exercise 1.
Show that $r \leq 8$ points in the plane fail to implose independent conditions on cubics, if and only if one of the following conditions is fulfilled

(i) $r \geq 5$ and 5 of the r points lie on a line.
(ii) $r = 8$ and all of the 8 points lie on a conic.

Hint. First show directly that $r \leq 4$ points always impose independent conditions on cubics. Then consider the cases $r = 5, 6, 7, 8$ one after another. At each step, one has to construct a cubic passing through exactly $r - 1$ of the r points. As long as $r \leq 7$ only unions of lines have to be considered, but for $r = 8$ one has to use the fact that for any 5 points one can find a conic passing through them.

Exercise 2.
Prove Pascal's Theorem (see the picture below): Let $P_1, \ldots, P_6$ be six points on a conic. Then, the three intersection points $P_7 = \overline{P_1P_2} \cap \overline{P_4P_5}$, $P_8 = \overline{P_2P_3} \cap \overline{P_5P_6}$ and $P_9 = \overline{P_3P_4} \cap \overline{P_6P_1}$ lie on a line.

Hint. Let $C_1 = \overline{P_1P_2} \cup \overline{P_5P_6} \cup \overline{P_3P_4}$ and $C_2 = \overline{P_4P_5} \cup \overline{P_2P_3} \cup \overline{P_6P_1}$. Note that $C_1 \cap C_2 = \{P_1, \ldots, P_9\}$ and use Exercise 1 to show that the 8 points $P_1, \ldots, P_8$ impose independent conditions on cubics. Now, the union of the conic with the line $\overline{P_7P_8}$ is another cubic passing through those 8 points $P_1, \ldots, P_8$. Finally conclude that P_9 must lie on $\overline{P_7P_8}$.



Remark. Pascal's Theorem was discovered by Blaise Pascal (1623–1662) at the age of 16. His proof is unfortunately unknown. The proof from Exercise 2 is due to Michel Chasles (1793–1880) published in his book in 1837.