# The Basel Problem and the Riemann Hypothesis

**Exercise 1** (easy). In his paper [E020], Euler proved the remarkable identitiy

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \;=\; (\ln 2)^2 \;+\; \sum_{k=1}^{\infty} \frac{1}{2^{k-1} \cdot k^2}.$$

He knew that the square of the natural logarithm of 2 is approximately 0.480453. The sum on the right hand side of his identity converges much faster than the left had side and he tells us that this sum is approximately 1.164481, so that the left hand side is approximately 1.644934. But he doesn't tell us how exactly he computed this sum. Calculate the first few terms of this sum,

$$1, \quad 1+\frac{1}{8}, \quad 1+\frac{1}{8}+\frac{1}{36}, \quad 1+\frac{1}{8}+\frac{1}{36}+\frac{1}{128}, \quad 1+\frac{1}{8}+\frac{1}{36}+\frac{1}{128}+\frac{1}{400}, \ldots$$

until you reach to the approximate value 1.164481 (you may use a calculator or a computer for this). How many terms Euler computed at least (by hand!)?

**Exercise 2** (not so easy). When Euler wrote his paper [E041], the Taylor series of the sine was already well-known,

$$\sin x \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1}.$$

He somehow derived in a mysteries way another formula for the sine, its product expansion

$$\frac{\sin x}{x} \;=\; \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right),$$

which turns out to be correct. Now, by comparing the coefficients of the term $x^3$ in those two expansions, Euler obtained the solution to the Basel Problem as

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \;=\; \frac{\pi^2}{6}.$$

In fact, he derived many more interesting formulas in this paper. One of them is obtained by comparing the coefficients of the term $x^5$ in the two expansions of the sine, which gives the precise value of the sum

$$\sum_{k=1}^{\infty} \frac{1}{k^4}.$$

Following Euler's path, can you compute this sum too?

## References

1. Leonhard Euler (1707–1783). Scans of Euler's original papers, transcripts and translations can be obtained online for example at `http://eulerarchive.maa.org`. The following is a list of publications related to the Basel Problem. In brackets is the Eneström Index of the paper. The reference includes both, the original journal as well as its place in Euler's collected works, the Opera Omnia, where all of the references except the last are contained in the same volume.

[E019] *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt*, Commentarii academiae scientiarum Petropolitanae **5** (1738), 36–57. Opera Omnia: Series 1, Volume **14**, 1–24.

[E020] *De summatione innumerabilium progressionum*, Commentarii academiae scientiarum Petropolitanae **5** (1738), 91–105. Opera Omnia: Series 1, Volume **14**, 25–41.

[E025] *Methodus generalis summandi progressiones*, Commentarii academiae scientiarum Petropolitanae **6** (1738), 68–97. Opera Omnia: Series 1, Volume **14**, 42–72.

[E041] *De summis serierum reciprocarum written*, Commentarii academiae scientiarum Petropolitanae **7** (1740), 123–134. Opera Omnia: Series 1, Volume **14**, 73–86.

[E063] *Demonstration de la somme de cette suite* $1 + 1/4 + 1/9 + 1/16 \ldots$, Journ. lit. d'Allemange, de Suisse et du Nord, **2:1** (1743), 115–127. Opera Omnia: Series 1, Volume **14**, 177–186.

[E072] *Variae observationes circa series infinitas*, Commentarii academiae scientiarum Petropolitanae **9** (1744), 160–188. Opera Omnia: Series 1, Volume **14**, 217–244.

[E352] *Remarques sur un beau rapport entre les series des puissances tant directes que reciproques*, Memoires de l'academie des sciences de Berlin **17** (1768), 83–106. Opera Omnia: Series 1, Volume **15**, 70–90.

2. Karl Weierstrass (1815–1897).

[W1] *Über die Theorie der analytischen Facultäten*, Crelle's Journal Bd. **51** (1856).

[W2] *Zur Theorie der eindeutigen analytischen Functionen*, Abh. der Königlichen Akademie der Wissenschaften (1876).

3. Bernhard Riemann (1826–1866). The original manuscript, a transcription and an English translation of the following famous paper by Riemann can be found online at `http://www.claymath.org/publications/riemanns-1859-manuscript`.

[R] *Über die Anzahl der Primzahlen unter einer gegebenen Größe*, (19. Oktober 1859). In: Monatsberichte der Königlichen Preußischen Akademie der Wissenschaften zu Berlin, 1860, 671–680.