On Abel's Theorem

(Solution of Exercise A)

Integral 1. Division with remainder gives

$$\frac{x^3}{x^2-3x+2} \;=\; x+3+\frac{7x-6}{x^2-3x+2}.$$

Factorizing the denominator is easy

$$x^2-3x+2 \;=\; (x-1)(x-2).$$

So, by the partial fraction method, there are numbers a and b with

$$\frac{7x-6}{(x-1)(x-2)} \;=\; \frac{a}{x-1}+\frac{b}{x-2}.$$

This leads to two linear equations for a and b with the solution $a=-1$ and $b=8$. Therefore, we get from our previous computation

$$\frac{x^3}{x^2-3x+2} \;=\; x+3-\frac{1}{x-1}+\frac{8}{x-2}$$

and the integral is

$$\int\frac{x^3\,dx}{x^2-3x+2} \;=\; \frac{1}{2}x^2+3x-\log(x-1)+8\log(x-2)+C$$

Integral 2. Put $y=\sqrt{x^2-3x+4}$ and define $t=(y-2)/x$. Then, solving for x and y one finds

$$x \;=\; \frac{3+4t}{1-t^2} \quad\text{and}\quad y \;=\; \frac{2+3t+2t^2}{1-t^2}.$$

Substituting this into the integral gives

$$\int\frac{dx}{y} \;=\; 2\int\frac{dt}{1-t^2} \;=\; \log(1+t)-\log(1-t)+C.$$

The final result is now obtained by substituting the definition of t and simplifying

$$\int\frac{dx}{\sqrt{x^2-3x+4}} \;=\; \log\left(2x-3+2\sqrt{x^2-3x+4}\right)+C.$$