Memoryful GoI with Recursion

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***Abstract*—In this preliminary report we extend our framework of *memoryful Geometry of Interaction (mGoI)* [Hoshino, Muroya & Hasuo, CSL-LICS 2014] by recursion. The mGoI framework provides a sound translation from λ-terms to *transducers*; notably it accommodates algebraic effects introduced by Plotkin and Power; and the translation, defined in terms of a coalgebraic component calculus, is extracted from categorical semantics (hence *correct-by-construction*). In our current extension, recursion is additionally accommodated by introducing a new "fixed point" operator in the component calculus.**

I. GoI Interpretation

Girard's *Geometry of Interaction (GoI)* [1] is originally introduced as semantics of linear logic proofs and, via the Curry-Howard correspondence (and the Girard transformation), it has been successfully applied to denotational semantics of higher-order functional programs. The resulting semantics give so-called "*GoI interpretation*" of programs; one of its notable features is that GoI interpretation of function application is given by interactions of a function and its arguments.

Many representations of GoI interpretation have been studied so far: the original one by elements of a C^*-algebra (or a dynamic algebra) that can be seen as "valid paths" on type derivation trees [1]; the one by token machines [2]; and the categorical one by arrows in a traced symmetric monoidal category [3]. The second one by token machines plays an important role in bridging the gap between mathematical interpretation and low-level implementation. Namely it provides techniques of compilation and high-level synthesis, such as a compilation technique [2] and a high-level synthesis technique [4] that enables hardware acceleration of programs by FPGA.

We wish to contribute to this sequence of work by enable GoI interpretation to accommodate *computational effects*.

II. Memoryful GoI

In the previous work [5] we developed the *memoryful GoI* (mGoI) framework that extends GoI interpretation of programs. Notably it accommodates *algebraic effects*—computational effects with algebraic operations as a syntactic interface, introduced by Plotkin and Power [6], [7]. Their examples include: nondeterminism, with a nondeterministic choice operation $\sqcup$ as an algebraic operation; probability, with a probabilistic choice operation $\sqcup_p$ for any $p \in [0,1]$; and global states, with operations lookup and update.

A. Component Calculus over Transducers

The mGoI interpretation of a program is given by *T-transducers*—an extension of *Mealy machines* (or *sequential machines*) by effects specified by a monad T. Here we follow [8] and model algebraic effects by a monad T on the category $\mathbf{Set}$ of sets and functions.

Definition II.1 (T-transducers [5, Definition 4.1])**.** For sets A and B, a *T-transducer* (X,c,x) *from A to B* (written as $(X,c,x)\colon A \mathrel{-\!\!\triangleright} B$) consists of a set X, a function $c\colon X \times A \to T(X \times B)$ and an element $x \in X$.

A T-transducer $(X,c,x)\colon A \mathrel{-\!\!\triangleright} B$ can be seen as an (T-effectful) transition function c with input A, output B, a set of internal states X and an initial state x. It shall be presented, in diagrams, as in Fig. 1.

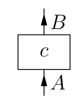
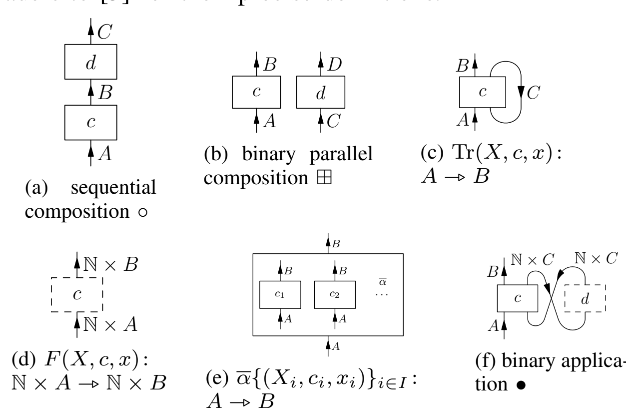


Fig. 1. a T-transducer $(X,c,x)\colon A \mathrel{-\!\!\triangleright} B$

In the mGoI framework, T-transducers are combined via a component calculus over them. It consists of primitive T-transducers (as basic building blocks) and the following operators on T-transducers: a) sequential composition $\circ$; b) binary parallel composition $\boxplus$; c) the trace operator Tr; d) the countable copy operator F; e) the operator $\overline{\alpha}$ for each algebraic operation α on T.[1] On top of these operators an auxiliary operator is defined: f) binary application $\bullet$.[2] The last is a well-known construction called *parallel composition and hiding* and is used here to translate function application. In Fig. 2 are graphical presentation of these operators; we refer readers to [5] for their precise definitions.



(a) sequential composition $\circ$

(b) binary parallel composition $\boxplus$

(c) $\mathrm{Tr}(X,c,x)\colon A \mathrel{-\!\!\triangleright} B$

(d) $F(X,c,x)\colon \mathbb{N}\times A \mathrel{-\!\!\triangleright} \mathbb{N}\times B$

(e) $\overline{\alpha}\{(X_i,c_i,x_i)\}_{i\in I}\colon A \mathrel{-\!\!\triangleright} B$

(f) binary application $\bullet$

Fig. 2. Operators on T-transducers

[1] We identify algebraic operations with their interpretations, as in [6].

[2] Binary application $\bullet$ presented here is an adaptation of that in [5].

B. Translation from Terms to Transducers

In our mGoI framework, to be precise, the provided interpretation $(\!|-|\!)$ is from a type judgment $\Gamma \vdash M : \tau$ to a T-transducer

$$(\!|\Gamma \vdash M : \tau|\!) : \coprod_{i=0}^{m} \mathbb{N} \rightharpoonup \coprod_{i=0}^{m} \mathbb{N} \ .$$

Here $\mathbb{N}$ is the set of natural numbers. The interpretation is defined inductively on the type derivations, using the component calculus introduced in the above.

In [9] we presented a prototype implementation—*TtT*, short for "Terms to Transducers"—of the translation $(\!|-|\!)$. Given a closed term M of type τ, the tool first generates a Haskell program that implements a transition function of the T-transducer $(\!|\vdash M : \tau|\!)$; and then it produces a simulation result of the execution of the transducer. We believe that the tool serves as a first step towards high-level synthesis (that translates a λ-term to hardware design like on FPGA)—much like in [4] but now with algebraic effects.

Some further comments are in order on: 1) a categorical model behind the translation $(\!|-|\!)$; and 2) prospects of accommodating recursion. In fact the translation $(\!|-|\!)$ is extracted from a categorical model $\mathbf{Per}_\Phi$—a Kleisli category of a strong monad Φ on a cartesian closed category $\mathbf{Per}$—built on T-transducers and the component calculus. It is an instance of the class of models, that is provided in [6], of the Moggi's computational λ-calculus [8] with algebraic operations and arithmetic primitives. In [6] a class of models that accommodates recursion is studied as well; the key is a fixed point operator on a categorical model. However it was not clear, at the time of writing our previous paper [5], how to obtain a fixed point operator on the categorical model $\mathbf{Per}_\Phi$ and extend the translation $(\!|-|\!)$ to recursion.

III. Translation of Recursion

Here we report our ongoing work that introduces recursion to the mGoI framework in [5].

A. Extension of Component Calculus and Translation

Our approach is to extend the component calculus shown in Fig. 2: binary parallel composition $\boxplus$ is extended to a countable one $\boxplus_{i \in I}$; and on top of the calculus, a "fixed point" operator Fix is introduced. It is presented in Fig. 3.

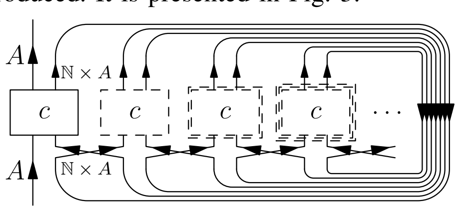


Fig. 3. $\mathrm{Fix}(X, c, x) : A \rightharpoonup A$. Here one dashed box means countable duplication of a component.

It indeed gives a fixed point with respect to binary application $\bullet$.

Lemma III.1. *Let* $(X, c, x) : A + \mathbb{N} \times A \rightharpoonup A + \mathbb{N} \times A$ *be a* T*-transducer. The* T*-transducer* $\mathrm{Fix}(X, c, x) : A \rightharpoonup A$ *satisfies the behavioral equivalence*

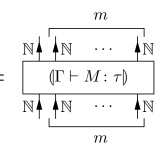
$$(X, c, x) \bullet \mathrm{Fix}(X, c, x) \ \simeq \ \mathrm{Fix}(X, c, x).$$

Here the behavioral equivalence $\simeq$ [5, Definition 5.2] is used for (equational) reasoning on T-transducers; it enables us to abstract away from internal state spaces of T-transducers.

With this extension of the component calculus the translation $(\!|-|\!)$ can be extended to recursion: the following definition is precisely what is given in [5], except recursion that is new.

Definition III.2 (translation $(\!|-|\!)$)**.** For each type judgment $\Gamma \vdash M : \tau$ where $\Gamma = x_1 : \tau_1, \ldots x_m : \tau_m$, we inductively define a T-transducer

$$(\!|\Gamma \vdash M : \tau|\!) =$$



$$: \coprod_{i=0}^{m} \mathbb{N} \rightharpoonup \coprod_{i=0}^{m} \mathbb{N}$$

as in Fig. 4. In Fig. 4, α is an n-ary algebraic operation on T that is the interpretation of op; and all the T-transducers other than those in the form $(\!|\Gamma \vdash M : \tau|\!)$ are primitives (see [5] for their definitions).

The translation $(\!|-|\!)$ is sound with respect to the equational theory given in [6]. The latter is (an almost full fragment of) the Moggi's equational theory of computational λ-calculus, extended by algebraic operations, arithmetic primitives and recursion.

Theorem III.3 (soundness of $(\!|-|\!)$)**.** *For closed terms* M *and* N *of the base type* `nat`, $\vdash M = N : \texttt{nat}$ *implies* $(\!|\vdash M : \texttt{nat}|\!) \simeq (\!|\vdash N : \texttt{nat}|\!)$.

For simplicity we have restricted to algebraic operations with finite arities; accommodating countable arities is straightforward (much like in [5], [10]). On top of soundness, we expect adequacy to hold too, against the operational semantics in [6]. Extension of our implementation tool TtT with recursion is future work, too.

B. The Categorical Model

The translation $(\!|-|\!)$ extended with recursion (Def. III.2) is backed up by a categorical model, too—this fact underlies Thm. III.3. Starting from the model $\mathbf{Per}_\Phi$ used in [5], we use its modification $\mathbf{Per}_{\Phi'}$ (whose details we do not describe here); then we can show that the construction Fix in Lem. III.1 indeed yields a (categorical) fixed point operator in $\mathbf{Per}_{\Phi'}$. In showing the latter, the following is a key technical lemma.

Lemma III.4. *Let* $\mathbf{Cppo}$ *be the category of pointed* ω*-cpo's (i.e. with the least element* $\bot$*) and continuous maps. Assume that the Kleisli category* $\mathbf{Set}_T$ *satisfies the following:*

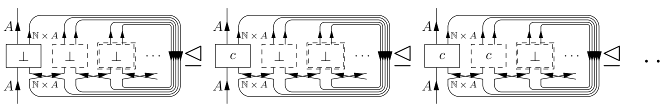
- *it is* $\mathbf{Cppo}$*-enriched (with a partial order* $\sqsubseteq$*) and has* $\mathbf{Cppo}$*-enriched (countable) cotupling;*
- *its compositions* $\circ_T$ *is strict, in the restricted sense as in [5, Lem. 4.3];*

- *its premonoidal structures $X \otimes -, - \otimes X$ are locally continuous and strict, for any $X \in \mathbf{Set}$.*

The $\mathbf{Cppo}$-enrichment of $\mathbf{Set}_T$ induces the following ω-cpo structure on T-transducers. A partial order $\trianglelefteq$ on T-transducers $(X, c, x), (Y, d, y) \colon A \multimap B$ is defined by

$$(X, c, x) \trianglelefteq (Y, d, y) \overset{\text{def.}}{\iff} X = Y \wedge x = y \wedge c \sqsubseteq d \ .$$

Minimal T-transducers with respect to $\trianglelefteq$ are given by $(Z, \bot, z)$ for any set Z. Now for a T-transducer $(X, c, x) \colon A + \mathbb{N} \times A \multimap A + \mathbb{N} \times A$, the T-transducer $\mathrm{Fix}(X, c, x) \colon A \multimap A$ is a supremum of the following ω-chain.



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REFERENCES

[1] J.-Y. Girard, "Geometry of Interaction I: interpretation of system F," in *Logic Colloquium 1988*, ser. Studies in Logic & Found. Math., vol. 127. Elsevier, 1989, pp. 221–260.

[2] I. Mackie, "The Geometry of Interaction machine," in *POPL 1995*. ACM, 1995, pp. 198–208.

[3] S. Abramsky, E. Haghverdi, and P. J. Scott, "Geometry of Interaction and linear combinatory algebras," *Math. Struct. in Comp. Sci.*, vol. 12, no. 5, pp. 625–665, 2002.

[4] D. Ghica, "Geometry of Synthesis: a structured approach to VLSI design," in *POPL 2007*. ACM, 2007, pp. 363–375.

[5] N. Hoshino, K. Muroya, and I. Hasuo, "Memoryful Geometry of Interaction: from coalgebraic components to algebraic effects," in *CSL-LICS 2014*. ACM, 2014, p. 52.

[6] G. Plotkin and J. Power, "Adequacy for algebraic effects," in *FoSSaCS 2001*, ser. Lect. Notes Comp. Sci., vol. 2030. Springer, 2001, pp. 1–24.

[7] ——, "Semantics for algebraic operations," *Elect. Notes in Theor. Comp. Sci.*, vol. 45, pp. 332–345, 2001.

[8] E. Moggi, "Computational lambda-calculus and monads," *Tech. Report*, pp. 1–23, 1988.

[9] K. Muroya, T. Kataoka, I. Hasuo, and N. Hoshino, "Compiling effectful terms to transducers: prototype implementation of memoryful Geometry of Interaction," in *LOLA 2014*, 2014.

[10] G. Plotkin and J. Power, "Algebraic operations and generic effects," *Appl. Categorical Struct.*, vol. 11, no. 1, pp. 69–94, 2003.

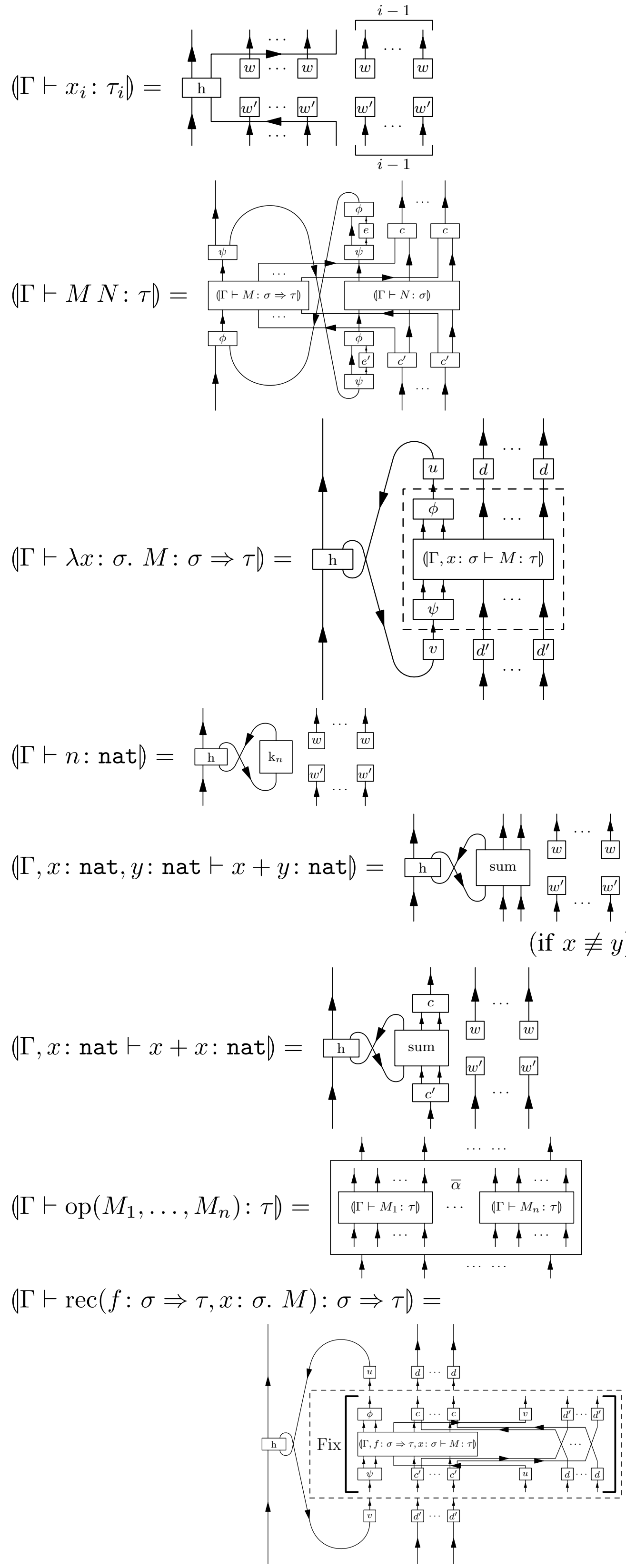

Fig. 4. inductive definition of the translation $(\!|-|\!)$