Convergence of symmetric Markov chains on $\mathbb{Z}^d$

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Abstract

For each n let $Y_t^{(n)}$ be a continuous time symmetric Markov chain with state space $n^{-1}\mathbb{Z}^d$. Conditions in terms of the conductances are given for the convergence of the $Y_t^{(n)}$ to a symmetric Markov process Y_t on $\mathbb{R}^d$. We have weak convergence of $\{Y_t^{(n)} : t \le t_0\}$ for every t_0 and every starting point. The limit process Y has a continuous part and may also have jumps.

1 Introduction

For each n, let $Y_t^{(n)}$ be a continuous time symmetric Markov chain with state space $\mathcal{S}_n = n^{-1}\mathbb{Z}^d$ and conductances $C^n(x, y)$. This means that $Y^{(n)}$ stays at a state x for an exponential length of time with parameter $\sum_{z\neq x} C^n(x, z)$ and then jumps to the next state y with probability $C^n(x, y)/\sum_{z\neq x} C^n(x, z)$. It is natural to expect that one can give conditions on the conductances such that for each starting point and each t_0, the processes $\{Y_t^{(n)}; t \le t_0\}$ converge weakly to a limiting process and that the limiting process be a symmetric Markov process. The purpose of this paper is to give such a theorem.

The earliest convergence theorem of this type is that of [DFGW] in the context of a central limit theorem for random walks in random environment. A more general result is implicit in [SZ]. In [BKu08] the first two authors of the current paper extended the theorem in [SZ] in two ways: chains with unbounded range were allowed and the rather stringent continuity conditions in [SZ] were weakened. A chain with unbounded range is one where there is no bound on the size of the jumps. In all of these papers the limit process is a symmetric diffusion on $\mathbb{R}^d$.

The paper [HK07] considered conductances that were comparable to the distribution of a stable law and the limit process is what is known as a stable-like process. Here the limit process has paths that have no continuous part. A theorem for convergence of pure jump symmetric processes on $\mathbb{R}^d$ can be found in [BKK]; as noted there the methods can be readily modified to give a result on the convergence of symmetric Markov chains whose limiting process has a more

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general jump structure than stable-like. Finally, we should mention the well-known results of [SV, Chap. 11] on non-symmetric Markov chains.

The current paper is devoted to proving a fairly general convergence theorem for symmetric Markov chains. We point out three significant differences from earlier work.

- Our Markov chains can have unbounded range and the limit process is associated with a Dirichlet form with both local and non-local components. This means the limit process has a continuous part and may also have a discontinuous part.
- We dispense with any continuity conditions on the conductances. Instead only convergence locally in L^1 is needed.
- The proofs are considerably simpler than previous work.

Let us give a heuristic description of our results, with the main theorem stated precisely in Section 5 as Theorem 5.5. First of all, the limiting symmetric Markov process is associated to the Dirichlet form

$$\mathcal{E}(f,f) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla f(x)\cdot a(x)\nabla f(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x)-f(y))^2 j(x,y)\,dx\,dy.$$

Here $a(x) = (a_{ij}(x))_{ij}$ is a symmetric uniformly positive definite and bounded matrix function. The first term on the right hand side represents the continuous part of the limit process; if the second term on the right hand side were not present, one would have a symmetric diffusion, and the Dirichlet form would be the one arising from elliptic operators on $\mathbb{R}^d$ in divergence form. The double integral on the right hand side represents the jump part, and very roughly says that the process jumps from x to y with jump intensity $j(x,y)$.

We write our conductances as $C^n = C^n_C + C^n_J$, where C^n_C and C^n_J are the local (continuous) and non-local (jump) parts, resp. Let us discuss the local part first. If one wants to understand the behavior of the limiting process at a point x, say, to look at $a(x)$, a bit of thought leads to the realization that jumps by the Markov chains that jump over but do not land on x contribute. Thus, in one dimension, one looks at a quantity $a^n(x)$ involving sums of terms involving $C^n_C(y,z)$ with $y\le x\le z$. In higher dimensions one uses a similar idea: one looks at the contribution of $C^n_C(y,z)$ where x lies on the shortest path from y to z; a path here means that at each step the path goes from a point to one of its nearest neighbors. There is no single shortest path in general, so we form $(a^n_{ij}(x))_{ij}$ in terms of an average of expressions involving $C^n_C(y,z)$, the average being over all shortest paths from y to z that pass through x. There are some very mild regularity conditions on C^n, but the main hypothesis is that the $a^n_{ij}(x)$ are uniformly bounded and converge to $a_{ij}(x)$ locally in L^1.

The conditions on the jump part are even weaker. We form a measure $j^n(x,y)\,dx\,dy$ in terms of the C^n_J. We then require that for each N, the measure $j^n(x,y)\,dx\,dy$ restricted to $B_N = (B(0,N)\times B(0,N))\setminus(B(0,N^{-1})\times B(0,N^{-1}))$ converges weakly to the measure $j(x,y)\,dx\,dy$ restricted to B_N, where $B(0,r)$ is the ball of radius r centered at 0.

Our results can be applied to the homogenization problem for random media. When the conductances $C^n(x,y)$ are random, we can obtain tightness of the laws of $\{Y^{(n)}\}$ and the convergence of the sequence $\{Y^{(n)}\}$ under mild conditions using Proposition 6.1 in this paper. We give an example of this at the end of Section 5.

The organization of this paper is as follows. After giving some definitions and setting up the framework in Section 2, we obtain upper and lower bounds and regularity results for the heat kernels for $Y^{(n)}$ in Sections 3 and 4. The formulation of the main theorem is given in Section 5 and the proof is given in Section 6.

2 Framework

For $n \in \mathbb{N}$, let $\mathcal{S}_n = n^{-1}\mathbb{Z}^d$. Let $|\cdot|$ be the Euclidean norm and $B_n(x,r) := \{y \in \mathcal{S}_n : |x-y| < r\}$.

For $n \in \mathbb{N}$, let $C^n(\cdot,\cdot)$ be a symmetric function defined on $(\mathcal{S}_n \times \mathcal{S}_n) \setminus \Delta$ into $\mathbb{R}_+$, where $\Delta = \{(x,x) : x \in \mathcal{S}_n\}$. Here symmetric means $C^n(x,y) = C^n(y,x)$ for all $x \neq y$. We call $C^n(x,y)$ the *conductance* between x and y. Throughout the paper, we assume the following;

(A1) *There exist $c_1, c_2 > 0$ independent of n such that*

$$c_1 \le \nu_x^n := \sum_{y \in \mathcal{S}_n} C^n(x,y) \le c_2 \quad \text{for all} \;\; x \in \mathcal{S}_n.$$

(A2) *There exist $M_0 \ge 1, \delta > 0$ independent of n such that the following holds: for any $x, y \in \mathcal{S}_n$ with $|x-y| = n^{-1}$, there exist $N \ge 2$ and $x_1, \cdots, x_N \in B_n(x, n^{-1}M_0)$ such that $x_1 = x$, $x_N = y$ and $C^n(x_i, x_{i+1}) \ge \delta$ for $i = 1, \cdots, N-1$.*

(A3) $M := \sup_n \sup_{x \in \mathcal{S}_n} \Big(n^2 \sum_{y \in \mathcal{S}_n} \big(1 \wedge |y|^2\big) C^n(x, x+y) \Big) < \infty.$

An example of $C^n(x,y)$ that satisfies (A1), (A2) and (A3) is the following:

$$c_1 1_{\{|x-y|=n^{-1}\}} \le C^n(x,y) \le \frac{c_2 1_{\{1 \ge |x-y| \ge n^{-1}\}}}{n^{d+2}|x-y|^{d+\beta}} + c_3 1_{\{|x-y|=n^{-1}\}} + \frac{c_4 1_{\{|x-y|>1\}}}{n^{d+2}|x-y|^{d+\alpha}},$$

where $\alpha, \beta \in (0,2)$.

Let $\mu_x^n \equiv n^{-d}$ for all $x \in \mathcal{S}_n$ and for each $A \subset \mathcal{S}_n$, define $\mu^n(A) = \sum_{y \in A} \mu_y^n$ and $\nu^n(A) = \sum_{y \in A} \nu_y^n$. Note that $L^2(\mathcal{S}_n, \mu^n) = L^2(\mathcal{S}_n, \nu^n)$ by (A1). Now, for each $f \in L^2(\mathcal{S}_n, \mu^n)$, define

$$\mathcal{E}^n(f,f) = \frac{n^{2-d}}{2} \sum_{x,y \in \mathcal{S}_n} (f(y) - f(x))^2 C^n(x,y), \tag{2.1}$$

$$\mathcal{F}^n = \{f \in L^2(\mathcal{S}_n, \mu^n) : \mathcal{E}^n(f,f) < \infty\}. \tag{2.2}$$

For $p \ge 1$, define $\|f\|_{p,n}^p = \sum_{x \in \mathcal{S}_n} \big|f(x)\big|^p \mu_x^n$. The following lemma is standard.

Lemma 2.1 *For each n, $\mathcal{F}^n = L^2(\mathcal{S}_n, \mu^n)$, and for $f \in L^2(\mathcal{S}_n, \mu^n)$, we have*

$$\mathcal{E}^n(f,f) \le 2n^2 M ||f||_{2,n}^2,$$

where M is the constant appearing in (A3)

PROOF. Let $f \in L^2(\mathcal{S}_n, \mu^n)$. Since $|x-y| \geq n^{-1}$ for any $x, y \in \mathcal{S}_n$ with $x \neq y$, we have

$$\begin{aligned}
\frac{n^{2-d}}{2} \sum_{\substack{x,y\in\mathcal{S}_n \\ x\neq y}} \big(f(x)-f(y)\big)^2 C^n(x,y) &\leq n^{2-d} \sum_{\substack{x,y\in\mathcal{S}_n \\ |x-y|\geq n^{-1}}} \big(f(x)^2+f(y)^2\big) C^n(x,y) \\
&\leq 2n^{2-d} \sum_{x\in\mathcal{S}_n} f(x)^2 \sum_{\substack{y\in\mathcal{S}_n \\ |x-y|\geq n^{-1}}} C^n(x,y) \\
&\leq 2n^{4-d} \sum_{x\in\mathcal{S}_n} f(x)^2 \sum_{\substack{y\in\mathcal{S}_n \\ |x-y|\geq n^{-1}}} \big(1\wedge|x-y|^2\big) C^n(x,y) \\
&\leq 2n^{2-d} M \sum_{x\in\mathcal{S}_n} f(x)^2 = 2n^2 M ||f||_{2,n}^2.
\end{aligned}$$

□

Using Lemma 2.1, it is easy to check that $(\mathcal{E}^n, \mathcal{F}^n)$ is a regular Dirichlet form on $L^2(\mathcal{S}_n, \mu^n)$. Further, $\mathcal{F}^n = L^2(\mathcal{S}_n, \mu^n)$ is equal to the closure of the space of compactly supported functions on $\mathcal{S}_n$ with respect to the norm $(\mathcal{E}^n(\cdot,\cdot) + ||\cdot||_{2,n}^2)^{1/2}$. Let $Y_t^{(n)}$ be the corresponding continuous time Markov chains on $\mathcal{S}_n$ and let $p^n(t,x,y)$ be the transition density for $Y_t^{(n)}$ with respect to μ^n. The infinitesimal generator of $Y_t^{(n)}$ can be written as

$$\mathcal{A}^n f(x) = \sum_{y\in\mathcal{S}_n} (f(y)-f(x)) C^n(x,y) n^2 = \sum_{y\in\mathcal{S}_n} (f(y)-f(x)) \frac{C^n(x,y) n^{2-d}}{\mu_x^n},$$

for each $f \in L^2(\mathcal{S}_n, \mu^n)$. We remark that the fact that we take the reversing measure to be constant simplifies matters considerably.

Remark 2.2 Note that under (A1), $\{Y_t^{(n)}\}$ is conservative. Indeed, define a symmetric Markov chain $\{X_m^{(n)}\}$ by

$$\mathbb{P}^x(X_1^{(n)} = y) = \frac{C^n(x,y)}{\nu_x^n} \quad \text{for all} \quad x, y \in \mathcal{S}_n.$$

Then the corresponding semigroup satisfies $P_1^{X,n} 1(x) = \sum_{y\in\mathcal{S}_n} \mathbb{P}^x(X_1^{(n)} = y) = 1$ by (A1), so inductively we have $P_m^{X,n} 1 = 1$ for all $m \in \mathbb{N}$, so that $\{X_m^{(n)}\}$ is conservative. But $\{Y_t^{(n)}\}$ is a time changed process of $\{X_m^{(n)}\}$. To see this, let $\{U_i^{x,n} : i \in \mathbb{N}, x \in \mathcal{S}_n\}$ be an independent sequence of exponential random variables, where the parameter for $U_i^{x,n}$ is ν_x^n, that is independent of $X_m^{(n)}$, and define $T_0^{(n)} = 0, T_m^{(n)} = \sum_{k=1}^m U_k^{X_{k-1}^{(n)},n}$. Set $\widetilde{Y}_t^{(n)} = X_m$ if $T_m^{(n)} \leq t < T_{m+1}^{(n)}$; then the laws of $\widetilde{Y}^{(n)}$ and $Y^{(n)}$ are the same, and hence $\widetilde{Y}^{(n)}$ is a realization of the continuous time Markov chain corresponding to (a time change of) $X_m^{(n)}$. Note that by (A1), the mean exponential holding time at each point for $\widetilde{Y}^{(n)}$ can be controlled uniformly from above and below by a positive constant, so we conclude $P_t^n 1 = 1$ for all $t > 0$, where P_t^n is the semigroup corresponding to $\{Y_t^{(n)}\}$.

3 Heat kernel estimates

In this section we derive some heat kernel and exit time probability estimates. Our methods are by now fairly standard. We derive a Nash inequality by comparison with a nearest neighbor random walk. We omit the large jumps from our Dirichlet form, and use a theorem of [CKS] to obtain heat kernel and exit time probability estimates for the process without the large jumps. Then we use a construction of Meyer to reintroduce the large jumps.

3.1 Nash inequality

For $f \in L^2(\mathcal{S}_n, \mu^n)$, let

$$\mathcal{E}^n_{NN}(f,f) = \frac{n^{2-d}}{2} \sum_{\substack{x,y\in\mathcal{S}_n \\ |x-y|=n^{-1}}} (f(x)-f(y))^2, \tag{3.1}$$

which is the Dirichlet form for the simple symmetric random walk in $\mathcal{S}_n$. We set $C^n_{NN}(x,y)$ to be 1 if $x,y \in \mathcal{S}_n$ with $|x-y| = n^{-1}$ and 0 otherwise. By [BKu08, Proposition 3.1] there exists $c_1 > 0$ independent of n such that for any $f \in L^2(\mathcal{S}_n, \mu^n)$,

$$\|f\|_{2,n}^{2(1+2/d)} \le c\mathcal{E}^n_{NN}(f,f)\|f\|_{1,n}^{4/d} \le c_1 \mathcal{E}^n(f,f)\|f\|_{1,n}^{4/d}, \tag{3.2}$$

and

$$p^n(t,x,y) \le c_1 t^{-d/2} \qquad \text{for all } \ x,y \in \mathcal{S}_n, t>0. \tag{3.3}$$

For $r \in (n^{-1}, 1]$, let $\mathcal{E}^{n,r}$ be the Dirichlet form corresponding to $\{Y^{(n),r}_t := r^{-1}Y^{(n)}_{r^2 t}, t \ge 0\}$. By simple computations, we have

$$\mathcal{E}^{n,r}(f,f) = \frac{(nr)^{2-d}}{2} \sum_{x,y\in\mathcal{S}_{nr}} (f(y)-f(x))^2 C^n(rx,ry),$$

where $\mathcal{S}_{nr} = \{x/r : x \in \mathcal{S}_n\} = (nr)^{-1}\mathbb{Z}^d$. Define

$$p^{n,r}(t,x,y) := r^d p^n(r^2 t, rx, ry). \tag{3.4}$$

Then $p^{n,r}(t,x,y)$ is the heat kernel for $\mathcal{E}^{n,r}$. By (3.3), we have

$$p^{n,r}(t,x,y) \le c_1 t^{-d/2} \qquad \text{for all } \ x,y \in \mathcal{S}_{nr}, t>0. \tag{3.5}$$

For $\lambda \ge 1$, let $Y^{(n),r,\lambda}_t$ be a process on $\mathcal{S}_{nr}$ with the large jumps of $Y^{(n)}_t$ removed. More precisely, $Y^{(n),r,\lambda}_t$ is a process whose Dirichlet form is

$$\mathcal{E}^{n,r,\lambda}(f,f) = \frac{1}{2} \sum_{\substack{x,y\in\mathcal{S}_{nr} \\ |x-y|\le\lambda}} (f(x)-f(y))^2 (nr)^{2-d} C^n(rx,ry),$$

for each $f \in L^2(\mathcal{S}_{nr}, \mu^{nr})$. We denote the heat kernel for $Y^{(n),r,\lambda}_t$ by $p^{n,r,\lambda}(t,x,y)$, $x,y \in \mathcal{S}_{nr}$.

3.2 Exit time probability estimates

In this subsection, we will obtain some exit time estimates. Note that similar estimates are obtained in [Foo, Proposition 3.7] and [CK09].

Proposition 3.1 *For $A > 0$ and $0 < B < 1$, there exists $t_0 = t_0(A, B) \in (0, 1)$ such that for every $n \in \mathbb{N}$, $r \in (0, 1]$ and $x \in \mathcal{S}_n$,*

$$\mathbb{P}^x\left(\sup_{t\le r^2 t_0} |Y_t^{(n)} - Y_0^{(n)}| > rA\right) = \mathbb{P}^x\left(\sup_{t\le t_0} |Y_t^{(n),r} - Y_0^{(n),r}| > A\right) \le B. \tag{3.6}$$

Proof. Let $\lambda > 0$. Since we have (3.5) and $p^{n,r,\lambda}(t, x, y) \le p^{n,r}(t, x, y)$, by Theorem (3.25) of [CKS], we have

$$p^{n,r,\lambda}(t, x, y) \le c_1\, t^{-\frac{d}{2}}\, \exp\left(-E(2t, x, y)\right) \tag{3.7}$$

for all $t \le 1$ and $x, y \in \mathcal{S}_{nr}$, where

$$\begin{aligned} E(t, x, y) &= \sup\{|\psi(y) - \psi(x)| - t\,\Lambda(\psi)^2 : \Lambda(\psi) < \infty\}, \\ \Lambda(\psi)^2 &= \|e^{-2\psi}\Gamma_{\lambda,r}[e^{\psi}]\|_\infty \vee \|e^{2\psi}\Gamma_{\lambda,r}[e^{-\psi}]\|_\infty, \end{aligned}$$

and $\Gamma_{\lambda,r}$ is defined by

$$\Gamma_{\lambda,r}[v](\xi) = \sum_{\substack{\eta,\xi\in\mathcal{S}_{nr}\\ |\xi-\eta|\le\lambda}} (v(\eta) - v(\xi))^2 C^n(r\eta, r\xi)(nr)^2, \qquad \xi \in \mathcal{S}_{nr}. \tag{3.8}$$

Now let $R = |x - y|$ and let $\psi(\xi) = s(|\xi - x| \wedge R)$. Then, $|\psi(\eta) - \psi(\xi)| \le s|\eta - \xi|$, so that

$$(e^{\psi(\eta)-\psi(\xi)} - 1)^2 \le |\psi(\eta) - \psi(\xi)|^2 e^{2|\psi(\eta)-\psi(\xi)|} \le cs^2|\eta - \xi|^2 e^{2|\psi(\eta)-\psi(\xi)|}$$

for $\eta, \xi \in \mathcal{S}_{nr}$ where $|\eta - \xi| \le \lambda$. Hence

$$\begin{aligned} e^{-2\psi(\xi)}\Gamma_{\lambda,r}[e^{\psi}](\xi) &= \sum_{\substack{\eta\in\mathcal{S}_{nr}\\ |\xi-\eta|\le\lambda}} (e^{\psi(\eta)-\psi(\xi)} - 1)^2 C^n(r\eta, r\xi)(nr)^2 \\ &\le c_1 s^2 e^{2s\lambda} \sum_{\substack{\eta\in\mathcal{S}_{nr}\\ |\xi-\eta|\le\lambda}} |\eta - \xi|^2 C^n(r\eta, r\xi)(nr)^2 \\ &= c_1 s^2 e^{2s\lambda} \sum_{\substack{\eta'\in\mathcal{S}_{n}\\ |\xi'-\eta'|\le\lambda r}} |\eta' - \xi'|^2 C^n(\eta', \xi')n^2 \\ &\le c_1 s^2 e^{2s\lambda}\Big(\sum_{\substack{\eta'\in\mathcal{S}_{n}\\ |\xi'-\eta'|\le 1}} |\eta' - \xi'|^2 C^n(\eta', \xi')n^2 + ((\lambda r)^2 \vee 1) \sum_{\substack{\eta'\in\mathcal{S}_{n}\\ |\xi'-\eta'|\ge 1}} C^n(\eta', \xi')n^2\Big) \\ &\le c_2(\lambda^2 \vee 1)s^2 e^{2s\lambda} \le c_3 e^{3s\lambda}(1 + 1/\lambda^2) \end{aligned}$$

for all $\xi \in \mathcal{S}_{nr}$ where (A3) and $r \le 1$ are used in the third inequality. We have the same bound when ψ is replaced by $-\psi$, so $\Lambda(\psi)^2 \le c_3 e^{3s\lambda}(1 + 1/\lambda^2)$. Now, let $\lambda = A/(6d)$, $t_0 \le 1 \wedge \lambda^4 =$

$1 \wedge (A^4/(6d)^4)$ and $s = (3\lambda)^{-1} \log(1/t^{1/2}) > 0$. Then, for each $t \le t_0$ and $R \ge A$,

$$\begin{aligned} p^{n,r,\lambda}(t,x,y) &\le c_4 t^{-\frac{d}{2}} \exp\left(-sR + c_3 t e^{3s\lambda}(1+1/\lambda^2)\right) \\ &\le c_5 \exp\left(\left(d - \frac{2Rd}{A}\right)\log\left(\frac{1}{t^{1/2}}\right)\right) \le c_5 \exp\left(-\frac{Rd}{A}\log\left(\frac{1}{t^{1/2}}\right)\right). \end{aligned} \tag{3.9}$$

Thus,

$$\begin{aligned} \sum_{B_{nr}(x,A)^c} p^{n,r,\lambda}(t,x,y)\mu_y^{nr} &\le c\int_A^\infty R^{d-1} \exp\left(-\frac{Rd}{A}\log\left(\frac{1}{t^{1/2}}\right)\right)dR \\ &= cA^d \int_1^\infty R'^{d-1} \exp\left(-R'd\log\left(\frac{1}{t^{1/2}}\right)\right)dR' < B/4 \end{aligned} \tag{3.10}$$

for all $t \le t_0$ if we choose t_0 small, depending on A and B. Thus, applying [BBCK, Lemma 3.8], we obtain

$$\mathbb{P}^x\left(\sup_{t\le t_0} |Y_t^{(n),r,\lambda} - Y_0^{(n),r,\lambda}| > A\right) \le B/2. \tag{3.11}$$

We now use a construction of Meyer to obtain the estimate for $Y^{(n),r}$. Note that for any $x \in \mathcal{S}_{nr}$,

$$\begin{aligned} \mathcal{J}(x) &:= \sum_{\substack{y\in\mathcal{S}_{nr} \\ |x-y|\ge\lambda}} C^n(rx,ry)(nr)^2 \le \sum_{\substack{y\in\mathcal{S}_{nr} \\ |x-y|\ge\lambda}} \frac{(r^2|x-y|^2)\wedge 1}{\lambda^2 r^2} C^n(rx,ry)(nr)^2 \\ &= \frac{1}{\lambda^2}\sum_{y'\in\mathcal{S}_n}\left(|x'-y'|^2\wedge 1\right)C^n(x',y')n^2 \le \frac{M}{\lambda^2} = \frac{(6d)^2 M}{A^2}, \end{aligned}$$

where (A3) is used in the last inequality. So, if we let $U_1 := \inf\{t > 0 : \int_0^t \mathcal{J}(Y_s^{(n),r})ds > S_1\}$, where S_1 is the independent exponential distribution with mean 1, we have

$$P(U_1 \le t_0) \le 1 - e^{-(6d)^2 t_0/A^2} < B/2 \tag{3.12}$$

by taking t_0 small. Using an argument that relies on a construction of Meyer (see, for example, Section 4.1 in [CK08]), we obtain

$$\begin{aligned} \mathbb{P}^x\left(\sup_{t\le t_0}|Y_t^{(n),r} - Y_0^{(n),r}| > A\right) &= \mathbb{P}^x\left(\sup_{t\le t_0}|Y_t^{(n),r} - Y_0^{(n),r}| > A, U_1 > t_0\right) \\ &\quad + \mathbb{P}^x\left(\sup_{t\le t_0}|Y_t^{(n),r} - Y_0^{(n),r}| > A, U_1 \le t_0\right) \\ &\le \mathbb{P}^x\left(\sup_{t\le t_0}|Y_t^{(n),r,\lambda} - Y_0^{(n),r,\lambda}| > A\right) + \mathbb{P}^x\left(U_1 \le t_0\right) \\ &\le B/2 + B/2 = B, \end{aligned}$$

where (3.11) and (3.12) are used in the last inequality. □

Corollary 3.2 *For $0 < A', B' < 1$, there exists $R_0 = R_0(A', B') > 0$, such that for every $n \in \mathbb{N}$, $r \in (0,1]$ and $x \in \mathcal{S}_n$,*

$$\mathbb{P}^x \left(\sup_{t \le r^2 A'} |Y_t^{(n)} - Y_0^{(n)}| > rR_0 \right) = \mathbb{P}^x \left(\sup_{t \le A'} |Y_t^{(n),r} - Y_0^{(n),r}| > R_0 \right) \le B'. \tag{3.13}$$

Proof. In the proof of Proposition 3.1, take $A \ge 1$, $\lambda = A^{1/2}/(6d)$ (instead of $\lambda = A/(6d)$) and $A \ge 1$. Then, since $A^{1/2} \le A \le R$, we have (3.9) by changing A to $A^{1/2}$. So as in (3.10), there exists R_0 large such that for $t \le t_0 =: A'$ and $A \ge R_0$, we have

$$\begin{aligned} \sum_{B_{nr}(x,A)^c} p^{n,r,\lambda}(t,x,y)\mu_y^{nr} &\le cA^{d/2} \int_{A^{1/2}}^{\infty} R'^{d-1} \exp\Big(- R' d \log\Big(\frac{1}{t^{1/2}}\Big)\Big) dR' \\ &\le cA^{d/2} \exp\Big(- \frac{A^{1/2}}{2} d \log\Big(\frac{1}{t^{1/2}}\Big)\Big) \int_{A^{1/2}}^{\infty} R'^{d-1} \exp\Big(- \frac{R'}{2} d \log\Big(\frac{1}{t^{1/2}}\Big)\Big) dR' \\ &< B/4. \end{aligned}$$

Also, similarly to (3.12), we have

$$P(U_1 \le t_0) \le 1 - e^{-(6d)^2 t_0/A} < B/2$$

for all $A \ge R_0$, by taking R_0 large. With these changes, we can obtain the result similarly to the proof of Proposition 3.1. □

4 Lower bounds and regularity for the heat kernel

Our methods in this section are also fairly standard. We use a weighted Poincaré inequality and a differential inequality along the lines of [SZ] and [BKu08].

We introduce the space-time process $Z_s^{(n)} := (U_s, Y_s^{(n)})$, where $U_s = U_0 + s$. The filtration generated by $Z^{(n)}$ satisfying the usual conditions will be denoted by $\{\widetilde{\mathcal{F}}_s;\ s \ge 0\}$. The law of the space-time process $s \mapsto Z_s^{(n)}$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. We say that a non-negative Borel measurable function $q(t,x)$ on $[0,\infty) \times \mathcal{S}_n$ is *parabolic* in a relatively open subset B of $[0,\infty) \times \mathcal{S}_n$ if for every relatively compact open subset B_1 of B, $q(t,x) = \mathbb{E}^{(t,x)} \left[q(Z^{(n)}_{\tau^n_{B_1}}) \right]$ for every $(t,x) \in B_1$, where $\tau^n_{B_1} = \inf\{s > 0 :\ Z_s^{(n)} \notin B_1\}$.

We denote $T_0 := t_0(1/2, 1/2) < 1$ the constant in (3.6) corresponding to $A = B = 1/2$. For $t \ge 0$ and $r > 0$, we define

$$Q^n(t,x,r) := [t, t + T_0 r^2] \times B_n(x,r),$$

where $B_n(x,r) = \{y \in \mathcal{S}_n : |x-y| < r\}$.

It is easy to see the following (see, for example, Lemma 4.5 in [CK03] for the proof).

Lemma 4.1 *For each $t_0 > 0$ and $x_0 \in \mathcal{S}_n$, $q^n(t,x) := p^n(t_0 - t, x, x_0)$ is parabolic on $[0, t_0) \times \mathcal{S}_n$.*

For $A \subset \mathcal{S}_n$ and a process Z_t on $\mathcal{S}_n$, let

$$\tau^n = \tau^n_A(Z) := \inf\{t \ge 0 : Z_t \notin A\}, \qquad T^n_A = T^n_A(Z) := \inf\{t \ge 0 : Z_t \in A\}.$$

The next proposition provides a lower bound for the heat kernel and is the key step for the proof of the Hölder continuity of $p^n(t,x,y)$.

Proposition 4.2 *There exist $c_1 > 0$ and $\theta \in (0,1)$ such that for each $n \in \mathbb{N}$, if $|x - x_0|, |y - x_0| \le t^{1/2}$, $x, y, x_0 \in \mathcal{S}_n$, $t \in (n^{-1}, 1]$ and $r \ge t^{1/2}/\theta$, then*

$$\mathbb{P}^x(Y^{(n)}_t = y, \tau^n_{B(x_0,r)} > t) \ge c_1 t^{-d/2} n^{-d}.$$

To prove this we first need some preliminary lemmas. The proof of the following weighted Poincaré inequality can be found in [SZ, Lemma 1.19] and [BKu08, Lemma 4.3].

Lemma 4.3 *Let*

$$g_n(x) = c_1 \prod_{i=1}^d e^{-|x_i|} \qquad x \in \mathcal{S}_n,$$

where c_1 is determined by the equation $\sum_{l \in \mathcal{S}_n} g_n(x)\mu^n_x = n^d$. Then there exists $c_2 > 0$ such that

$$c_2 \Big\langle (f - \langle f \rangle_{g_n})^2 \Big\rangle_{g_n} \le n^{2-d} \sum_{l \in \mathcal{S}_n} g_n(l) \sum_{i=1}^d \Big(f(l + \frac{e^i}{n}) - f(l)\Big)^2, \qquad f \in L^2(\mathcal{S}_n),$$

where

$$\langle f \rangle_{g_n} = \sum_{l \in \mathcal{S}_n} f(l) g_n(l) \mu^n_l$$

and e^i is the element of $\mathbb{Z}^d$ whose j-th component is 1 if $j = i$ and 0 otherwise.

We will need the following inequality.

$$\Big(\frac{d}{b} - \frac{c}{a}\Big)(b - a) \le -(c \wedge d)\Big(\log \frac{b}{d^{1/2}} - \log \frac{a}{c^{1/2}}\Big)^2 + (d^{1/2} - c^{1/2})^2, \qquad a, b, c, d > 0. \qquad (4.1)$$

To prove this, first note that by simultaneously interchanging a with b and c with d if necessary, we may assume $c \le d$. By homogeneity of the inequality in d/c and b/a, we may assume $a = c = 1$. Using the inequality $A + 1/A - 2 \ge (\log A)^2$ with $A = b/d^{1/2}$ and the fact that $d \ge 1$,

$$\begin{aligned}\Big(\frac{d}{b} - 1\Big)(b - 1) &= (1 - d^{1/2})^2 - d^{1/2}\Big(\frac{b}{d^{1/2}} + \frac{d^{1/2}}{b} - 2\Big) \\ &\le (d^{1/2} - 1)^2 - \Big(\log \frac{d^{1/2}}{b}\Big)^2,\end{aligned}$$

which is (4.1).

We now give a key lemma.

Lemma 4.4 *There is an $\varepsilon > 0$ such that*

$$p^n(t,x,y) \geq \varepsilon t^{-d/2}, \tag{4.2}$$

for all $n \in \mathbb{N}$, $(t,x,y) \in (n^{-1},1] \times \mathcal{S}_n \times \mathcal{S}_n$ with $|x-y| \leq 2t^{1/2}$.

Proof. It is enough to prove the following: there is an $\varepsilon > 0$ such that

$$(nr)^{-d} \sum_{l \in \mathcal{S}_{nr}} \log\Big(p^{n,r}(\tfrac{1}{2},k,l+m)\Big) g_{nr}(l) \geq \tfrac{1}{2}\log\varepsilon, \tag{4.3}$$

for any $n \in \mathbb{N}$, $r \in (n^{-1},1]$ and $k,m \in \mathcal{S}_n$ with $|k-m| \leq 2$. Indeed, by the Chapman-Kolmogorov equation, symmetry, and the fact that $g_{nr}(j) \leq 1$ for all $k,m \in \mathcal{S}_{nr}$,

$$p^{n,r}(1,k,m) \geq (nr)^{-d} \sum_{j \in \mathcal{S}_{nr}} p^{n,r}(\tfrac{1}{2},k,j+k)p^{n,r}(\tfrac{1}{2},m,j+k)g_{nr}(j).$$

Thus, by Jensen's inequality, (4.3) yields

$$r^d p^n(r^2,rk,rl) = p^{nr}(1,k,l) \geq \varepsilon \qquad D \geq 1, |k-l| \leq 2.$$

Taking $t = r^2$, this gives (4.2).

So we will prove (4.3). Let $k,m \in \mathcal{S}_n$ satisfy $|k-m| \leq 2$ and set $u_t(l) = p^{n,r}(t,k,l+m)$. Define

$$G(t) = (nr)^{-d} \sum_{l \in \mathcal{S}_{nr}} \log(u_t(l)) g_{nr}(l).$$

By Jensen's inequality, we see that $G(t) \leq 0$. Further,

$$G'(t) = (nr)^{-d} \sum_{l \in \mathcal{S}_{nr}} \frac{\partial u}{\partial t}(l) \frac{g_{nr}(l)}{u_t(l)} = -\mathcal{E}^{(n),r}(u_t, \frac{g_{nr}}{u_t}).$$

Applying (4.1) with $a = u_t(l)$, $b = u_t(l+m)$, $c = g_{nr}(l)$, $d = g_{nr}(l+m)$, we have

$$\begin{aligned}
&G'(t)\\
&= -(nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} \Big(\frac{g_{nr}(l+m)}{u_t(l+m)} - \frac{g_{nr}(l)}{u_t(l)}\Big)\Big(u_t(l+m) - u_t(l)\Big) C^n(rl, r(l+m))\\
&\geq (nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} (g_{nr}(l+m) \wedge g_{nr}(l)) \Big(\log\frac{u_t(l+m)}{g_{nr}(l+m)^{1/2}} - \log\frac{u_t(l)}{g_{nr}(l)^{1/2}}\Big)^2 C^n(rl, r(l+m))\\
&\quad - (nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} (g_{nr}(l+m)^{1/2} - g_{nr}(l)^{1/2})^2 C^n(rl, r(l+m))\\
&\geq c(nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{j=1}^{d} g_{nr}(l) \Big(\log u_t\Big(l + \frac{e^j}{nr}\Big) - \log u_t(l) + \frac{1}{2}\Big(\Big|l_j + \frac{1}{nr}\Big| - |l_j|\Big)\Big)^2\\
&\quad - (nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} (g_{nr}(l+m)^{1/2} - g_{nr}(l)^{1/2})^2 C^n(rl, r(l+m)) =: I - II,
\end{aligned}$$

where the last inequality is due to (A2) and the definition of g_{nr} (here e^j is the element of $\mathbb{Z}^d$ whose k-th component is 1 if $k = j$ and 0 otherwise). Note that

$$(g_{nr}(l+m)^{1/2} - g_{nr}(l)^{1/2})^2 \le c_1(|m|^2 \wedge 1)(g_{nr}(l+m) + g_{nr}(l)).$$

Thus

$$\begin{aligned}
II \le & c_2(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{m\in\mathcal{S}_{nr}} (g_{nr}(l+m) + g_{nr}(l))(|m|^2 \wedge 1)C^n(rl, r(l+m)) \\
= & 2c_2(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{m\in\mathcal{S}_{nr}} g_{nr}(l)(|m|^2 \wedge 1)C^n(rl, r(l+m)) \\
\le & c_3\Big(\sup_{l\in\mathcal{S}_{nr}} n^2 \sum_{m\in\mathcal{S}_{nr}} (r^2|m|^2 \wedge r^2)C^n(rl, r(l+m))\Big) \cdot (nr)^{-d} \sum_{l\in\mathcal{S}_{nr}} g_{nr}(l) \\
\le & c_4\Big(\sup_{l'\in\mathcal{S}_{n}} n^2 \sum_{m'\in\mathcal{S}_{n}} (|m'|^2 \wedge 1)C^n(l', l'+m')\Big) \le c_5,
\end{aligned}$$

where we used $r \le 1$ in the third inequality and (A3) in the last inequality. Further, since $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$,

$$\begin{aligned}
I \ge & c(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{j=1}^{d} g_{nr}(l)\Big\{\frac{1}{2}\Big(\log u_t\Big(l + \frac{e^j}{nr}\Big) - \log u_t(l)\Big)^2 - \Big(\frac{1}{2}\Big(\Big|l_j + \frac{1}{nr}\Big| - |l_j|\Big)\Big)^2\Big\} \\
\ge & \frac{c}{2}(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{j=1}^{d} g_{nr}(l)\Big(\log u_t\Big(l + \frac{e^j}{nr}\Big) - \log u_t(l)\Big)^2 - \frac{cd}{4}(nr)^{-d} \sum_{l\in\mathcal{S}_{nr}} g_{nr}(l) \\
\ge & \frac{c}{2}(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{j=1}^{d} g_{nr}(l)\Big(\log u_t\Big(l + \frac{e^j}{nr}\Big) - \log u_t(l)\Big)^2 - c'
\end{aligned}$$

Combining these, we have

$$\begin{aligned}
G'(t) \ \ge \ & c_6(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{j=1}^{d} \Big(\log u_t\Big(l + \frac{e^j}{nr}\Big) - \log u_t(l)\Big)^2 g_{nr}(l) - c_5 \\
\ge \ & c_7(nr)^{-d} \sum_{l\in\mathcal{S}_{nr}} (\log u_t(l) - G(t))^2 g_{nr}(l) - c_5,
\end{aligned}$$

where we used Lemma 4.3 in the last inequality. Given these estimates, the rest of the proof is very similar to that of [BKu08, Lemma 4.4]. □

Remark 4.5 There is an error in the proof of [BKu08, Lemma 4.4]. The estimate $|g_D(l+e) - g_D(l)| \le c_1 D^{-1}|e|(g_D(l+e) \wedge g_D(l))$ in page 2051, line 23, is not true when $D \ll |e|$. However, one can easily fix the proof by arguing as in the proof here.

The next lemma can be proved exactly in the same way as [BKu08, Lemma 4.5].

Lemma 4.6 *Given $\delta > 0$ there exists κ such that for each $n \in \mathbb{N}$, if $x, y \in \mathcal{S}_n$ and $C \subset \mathcal{S}_n$ with* $\operatorname{dist}(x, C)$ *and* $\operatorname{dist}(y, C)$ *both larger than $\kappa t^{1/2}$ where $t \in (n^{-1}, 1]$, then*

$$\mathbb{P}^x(Y_t^{(n)} = y, T_C^n \le t) \le \delta t^{-d/2} n^{-d}.$$

Proof of Proposition 4.2. We have from Lemma 4.4 that there exists ε such that

$$\mathbb{P}^x(Y_t^{(n)} = y) = p^n(t, x, y)\mu_y^n \ge \varepsilon t^{-d/2} n^{-d}$$

if $|x - y| \le 2t^{1/2}$. If we take $\delta = \varepsilon/2$ in Lemma 4.6, then provided $r > (\kappa + 1)t^{1/2}$, we have

$$\mathbb{P}^x(Y_t^{(n)} = y, \tau^n_{B_n(x_0,r)} \le t) \le \frac{\varepsilon}{2} t^{-d/2} n^{-d}.$$

Subtracting,

$$\mathbb{P}^x(Y_t^{(n)} = y, \tau^n_{B_n(x_0,r)} > t) \ge \frac{\varepsilon}{2} t^{-d/2} n^{-d}$$

if $|x - y| \le t^{1/2}$, which is equivalent to what we want. □

For $(t, x) \in [0, 1] \times \mathcal{S}_n$ and $r > 0$ let $Q^n(t, x, r) := [t,\, t + \gamma r^2] \times B_n(x, r)$, where $\gamma := \gamma(1/2, 1/2) < 1$. Here $\gamma(1/2, 1/2)$ is the constant in (3.6) corresponding to $A = B = 1/2$.

Given the above estimates, we can prove the uniform Hölder continuity of the heat kernel $p^n(t, x, y)$ similarly to [BKu08, Theorem 4.9].

Theorem 4.7 *There are constants $c > 0$ and $\beta > 0$ (independent of R, n) such that for every $0 < R \le 1$, every $n \ge 1$, and every bounded parabolic function q in $Q^n(0, x_0, 4R)$,*

$$|q(s, x) - q(t, y)| \le c\, \|q\|_{\infty,R}\, R^{-\beta} \left(|t - s|^{1/2} + |x - y|\right)^\beta \tag{4.4}$$

holds for (s, x), $(t, y) \in Q^n(0, x_0, R)$, where $\|q\|_{\infty,R} := \sup_{(t,y)\in[0,\,\gamma(4R)^2]\times\mathcal{S}_n} |q(t, y)|$. In particular, for the transition density function $p^n(t, x, y)$ of $Y^{(n)}$,

$$|p^n(s, x_1, y_1) - p^n(t, x_2, y_2)| \le c\, t_0^{-(d+\beta)/2} \left(|t - s|^{1/2} + |x_1 - x_2| + |y_1 - y_2|\right)^\beta, \tag{4.5}$$

for any $n^{-1} < t_0 < 1$, $t,\ s \in [t_0,\ 1]$ and $(x_i, y_i) \in \mathcal{S}_n \times \mathcal{S}_n$ with $i = 1, 2$.

Proof. Given the above estimates, we can prove the analogues of Corollary 4.6 and Lemma 4.7 in [BKu08] exactly in the same way as is done there. Thus the proof of Theorem 4.7 is almost the same as that of [BKu08, Theorem 4.9] except for the following small change.

The following computation is needed to obtain the first inequality of (4.13) in [BKu08]:

$$\sup_{z\in B_n(x,r)} n^2 \sum_{y\in\mathcal{S}_n\setminus\overline{B_n(x,s)}} C^n(z, y) \le \left(\frac{s}{2}\right)^{-2} \sup_{z\in B_n(x,r)} \sum_{y\in\mathcal{S}_n} \left(|z - y|^2 \wedge 1\right) C^n(z, y) n^2 \le \frac{c_2}{s^2}$$

where (A3) is used in the last inequality (note that $2r \le s \le 1$). □

5 Weak convergence of the process

Recall that $Y_t^{(n)}$ are the continuous time Markov chains on $\mathcal{S}_n$ corresponding to $(\mathcal{E}^n, \mathcal{F}^n)$ in (2.1) and (2.2). Since the state space of $Y^{(n)}$ is $\mathcal{S}_n$ while the limit process will have $\mathbb{R}^d$ as its state space, we need to exercise some care with the domains of the functions we deal with. First, if g is defined on $\mathbb{R}^d$, we define $R_n(g)$ to be the restriction of g to $\mathcal{S}_n$:

$$R_n(g)(x) = g(x), \qquad x \in \mathcal{S}_n.$$

If g is defined on $\mathcal{S}_n$, we define $E_n g$ to be the extension of g to $\mathbb{R}^d$ defined by

$$E_n g(x) = g([x]_n),$$

where $[x]_n = ([nx_1]/n, [nx_2]/n, \ldots, [nx_d]/n)$ for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

In order to consider the convergence of the processes and to identify the limit process, we need to show the convergence of the semigroups of the Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ in an appropriate sense.

Let us first give a rough idea of the conditions we impose, and after that we will give a precise statement. We take a sequence ε_n decreasing to 0, and arrange for $C^n_C(x,y)$, the part of the Dirichlet form determined by $C^n(x,y)$ with $|x-y| \le \varepsilon_n$, to converge to the local part of the limiting Dirichlet form, and for $C^n_J(x,y)$, the part of the Dirichlet form determined by $C^n(x,y)$ with $|x-y| > \varepsilon_n$, to converge to the non-local part of the limiting Dirichlet form. The conditions on C^n_J are reasonably intuitive, namely, that a measure derived from the C^n_J converge weakly in the appropriate sense.

The conditions on C^n_C are much more complicated. Let us first consider the case $d = 1$. Suppose f, g are C_c^∞, the smooth functions with compact support, let $f_n(x) = f([x]_n)$, $g_n(x) = g([x]_n)$, and $u_n(x) = U_n^\lambda f_n$, where U_n^λ is the resolvent corresponding to $\mathcal{E}^n$. We write

$$\begin{aligned}\mathcal{E}^n_C(u_n, g_n) &= \frac{n}{2} \sum_{x\in\mathcal{S}_n} \sum_{y\in\mathcal{S}_n} [u_n(x) - u_n(y)] C^n_C(x,y) [g_n(y) - g_n(x)] \\ &= \frac{n}{2} \sum_{x,y} \sum_{s,t} P^{(x,y)}(s,t)[u_n(t) - u_n(s)] C^n_C(x,y) \sum_{w,z} P^{(x,y)}(w,z)[g(z) - g(w)],\end{aligned}$$

where $P^{(x,y)}(w,z)$ is 1 if w and z are nearest neighbors in $\mathcal{S}_n$ and the line segment connecting w and z is contained in the line segment connecting x and y, and 0 otherwise. Writing

$$\Delta_{1/n} f(x) = n(f(x+1/n) - f(x)), \qquad \Delta_{-1/n} f(x) = n(f(x) - f(x-1/n)),$$

$$\begin{aligned}\mathcal{E}^n_C(u_n, g_n) &= \frac{n^{-1}}{2} \sum_w \sum_z \Big\{ \sum_{x,y} P^{(x,y)}(w, w\pm 1/n) P^{(x,y)}(z, z \pm 1/n) C^n_C(x,y) \Big\} \\ &\qquad \times \Delta_{\pm 1/n} u_n(z) \Delta_{\pm 1/n} g_n(w) \\ &= \frac{n^{-1}}{2} \sum_w \sum_z G^n_{11}(w,z) \Delta_{\pm 1/n} u_n(z) \Delta_{\pm 1/n} g_n(w),\end{aligned}$$

where G^n_{11} is defined by the second equality. If we let $F^n_{11}(z) = \sum_w G^n_{11}(w, z)$ and realize that

$$\Delta_{\pm 1/n} g_n(w) \approx g'(w) \approx g'(z)$$

(since $|w - z| \le \varepsilon_n \to 0$), then we have that $\mathcal{E}^n_C(u_n, g_n)$ is approximately equal to

$$\frac{1}{2} \sum_z n^{-1} F^n_{11}(z) \Delta_{\pm 1/n} u_n(z) g'(z).$$

We show that $\Delta_{\pm 1/n} u_n(z)$ converges weakly in L^2, and if F^n_{11} converges almost everywhere, then $F^n_{11}(x)\Delta_{\pm 1/n} u_n(z)$ converges weakly in L^2, and hence the sum converges. Therefore the natural hypothesis for our theorem is that F^n_{11} converges a.e. (In fact we can relax this condition a bit; see (A4) below.)

When $d > 1$, there is no canonical path connecting two points x and y. We form G^n_{ij} similarly to the above, but we take the collection of shortest rectilinear paths connecting x and y, and take an average over all such paths.

We now specify some notation in order to make a precise statement of our convergence theorem. For $n \in \mathbb{N}$, set

$$|x - y|_n := n|x_1 - y_1| + n|x_2 - y_2| + \cdots + n|x_d - y_d| \quad \text{for } x, y \in \mathcal{S}_n.$$

Note that $1 \le |x - y|_n \le dn|x - y|$ holds for any $x, y \in \mathcal{S}_n$ with $x \ne y$, where $|x - y|$ is the Euclidean distance between x and y. Clearly $|x - y|_n$ is always a non-negative integer.

Let $\boldsymbol{\alpha}_i = \boldsymbol{e}_i$ if $i = 1, 2, \ldots, d$ and $\boldsymbol{\alpha}_i = -\boldsymbol{e}_{i-d}$ if $i = d + 1, \ldots, 2d$. *A shortest path σ from x to y* is a sequence of points $p_i \in \mathcal{S}_n$ for $i = 0, 1, 2, \ldots, k = |x - y|_n$, which we denote by $\sigma = \sigma(p_0, \ldots, p_k)$, so that $p_0 = x, p_k = y$ and for any $\ell = 0, 1, \ldots, k - 1$, there exists $j \in \{1, 2, \ldots, 2d\}$ such that

$$p_\ell = p_{\ell+1} + \frac{1}{n}\boldsymbol{\alpha}_j.$$

Let $\mathcal{P}(x, y)$ be the set of all shortest paths σ from x to y. The number of all such shortest paths σ is

$$\Pi(x, y) := \frac{(|x - y|_n)!}{\big(n|x_1 - y_1|\big)!\big(n|x_2 - y_2|\big)! \cdots \big(n|x_d - y_d|\big)!}.$$

For $\sigma \in \mathcal{P}(x, y)$, define a function D_σ defined on $\mathcal{S}_n \times \mathcal{S}_n$ as follows:

$$D_\sigma(w, z) := \begin{cases} 1, & \text{if there exists } \ell \text{ such that } w = p_\ell \text{ and } z = p_{\ell+1}, \\ 0, & \text{otherwise.} \end{cases}$$

For any function u defined on $\mathcal{S}_n$ and for any $x, y \in \mathcal{S}_n$, we easily see that

$$u(x) - u(y) = \frac{1}{\Pi(x, y)} \sum_{\sigma \in \mathcal{P}(x,y)} \sum_{z, w \in \mathcal{S}_n} D_\sigma(w, z)\big(u(w) - u(z)\big).$$

Now let

$$P^{x,y}(w, z) = \frac{1}{\Pi(x, y)} \sum_{\sigma \in \mathcal{P}(x,y)} D_\sigma(w, z).$$

For $h \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $i = 1, 2, \ldots, d$, let

$$\nabla_h^i u(x) = \frac{u(x + h\boldsymbol{e}_i) - u(x)}{h}.$$

We then have the following.

Lemma 5.1

$$u(x) - u(y) = \frac{1}{n} \sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) - P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) \nabla_{1/n}^i u(z).$$

Proof. We have

$$\begin{aligned}
&\sum_{w \in \mathcal{S}_n} D_\sigma(w, z)\big(u(w) - u(z)\big) \\
&\qquad = \sum_{i=1}^{2d} D_\sigma(z + \boldsymbol{\alpha}_i/n, z)\big(u(z + \boldsymbol{\alpha}_i/n) - u(z)\big) \\
&\qquad = \sum_{i=1}^{d} \Big\{ D_\sigma(z + \boldsymbol{e}_i/n, z)\big(u(z + \boldsymbol{\alpha}_i/n) - u(z)\big) \\
&\qquad\qquad + D_\sigma(z - \boldsymbol{e}_i/n, z)\big(u(z - \boldsymbol{e}_i/n) - u(z)\big) \Big\} \\
&\qquad = \frac{1}{n} \sum_{i=1}^{d} \Big\{ D_\sigma(z + \boldsymbol{e}_i/n, z) \nabla_{1/n}^i u(z) - D_\sigma(z - \boldsymbol{e}_i/n, z) \nabla_{-1/n}^i u(z) \Big\}.
\end{aligned}$$

So

$$\begin{aligned}
&u(x) - u(y) \\
&\quad = \sum_{z \in \mathcal{S}_n} \frac{1}{\Pi(x, y)} \sum_{\sigma \in \mathcal{P}(x,y)} \sum_{w \in \mathcal{S}_n} D_\sigma(w, z)\big(u(w) - u(z)\big) \\
&\quad = \frac{1}{n} \sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \frac{1}{\Pi(x, y)} \sum_{\sigma \in \mathcal{P}(x,y)} \Big(D_\sigma(z + \boldsymbol{e}_i/n, z) \nabla_{1/n}^i u(z) - D_\sigma(z - \boldsymbol{e}_i/n, z) \nabla_{-1/n}^i u(z) \Big) \\
&\quad = \frac{1}{n} \sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) \nabla_{1/n}^i u(z) - P^{x,y}(z - \boldsymbol{e}_i/n, z) \nabla_{-1/n}^i u(z) \Big).
\end{aligned}$$

Moreover, for each $i = 1, 2, \ldots, d$, and $x, y \in \mathcal{S}_n$,

$$\begin{aligned}\sum_{z\in\mathcal{S}_n} P^{x,y}(z-\boldsymbol{e}_i/n, z)\nabla^i_{-1/n}u(z) &= \sum_{z\in\mathcal{S}_n} P^{x,y}(z, z+\boldsymbol{e}_i/n)\nabla^i_{-1/n}u(z+\boldsymbol{e}_i/n)\\ &= -n\sum_{z\in\mathcal{S}_n} P^{x,y}(z, z+\boldsymbol{e}_i/n)\big(u(z)-u(z+\boldsymbol{e}_i/n)\big)\\ &= \sum_{z\in\mathcal{S}_n} P^{x,y}(z, z+\boldsymbol{e}_i/n)\nabla^i_{1/n}u(z).\end{aligned}$$

We thus obtain the desired equality. □

Remark 5.2 Here $P^{x,y}(\cdot,\cdot)$ is defined by averaging over the set of all shortest paths between x and y. However, we could take an average over other collections of paths. Let $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d)$. Other possible collections of paths are the following:

(i) Let $H(x, y)$ be the d-dimensional cube whose vertices consist of $\{(z_1, \cdots, z_d) : z_i$ is either x_i or y_i for $i = 1, \cdots, d\}$. Let $\mathcal{P}(x, y)$ be the set of shortest paths between x and y that consist of a union of the edges of $H(x, y)$, and take the average over $\mathcal{P}(x, y)$. In this case $\Pi(x, y)$ in the definition of $P^{x,y}(\cdot,\cdot)$ is $d!$.

(ii) Let $L_{x,y}$ be the union of the line segment from x to $(y_1, x_2, \cdots, x_d)$, the line segment from $(y_1, x_2, \cdots, x_d)$ to $(y_1, y_2, x_3 \cdots, x_d)$, $\cdots$, and the line segment from $(y_1, \cdots, y_{d-1}, x_d)$ to y. Set $\mathcal{P}(x, y) = \{L_{xy}\}$ and $\Pi(x, y) = 1$. This was used in [BKu08].

Next, let us fix a decreasing sequence $\{\varepsilon_n\}$ such that $1 \geq \varepsilon_n \searrow 0$, and define functions $C^n_C(x, y)$, $C^n_J(x, y)$ on $\mathcal{S}_n \times \mathcal{S}_n$ as follows:

$$C^n_C(x,y) := \begin{cases} C^n(x,y), & \text{if} \quad |x-y| \leq \varepsilon_n, \\ 0, & \text{otherwise,}\end{cases}$$

and $C^n_J(x, y) := C^n(x, y) - C^n_C(x, y), \quad x, y \in \mathcal{S}_n$.

Now define the following Dirichlet forms corresponding to the conductances $C^n_C(x, y)$ and $C^n_J(x, y)$, which we consider as the 'continuous part' and the 'jump part' of the Dirichlet form $(\mathcal{E}^n, \mathcal{F}^n)$; for $f \in L^2(\mathcal{S}_n, \mu^n)$,

$$\begin{cases} \mathcal{E}^n_C(f,g) &:= \dfrac{n^{2-d}}{2}\displaystyle\sum_{x,y\in\mathcal{S}_n}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)C^n_C(x,y), \\ \mathcal{E}^n_J(f,g) &:= \dfrac{n^{2-d}}{2}\displaystyle\sum_{x,y\in\mathcal{S}_n}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)C^n_J(x,y).\end{cases}$$

Then clearly $\mathcal{E}^n(f, g) = \mathcal{E}^n_C(f, g) + \mathcal{E}^n_J(f, g)$.

Using Lemma 5.1, we can write $\mathcal{E}_C^n(u,v)$ as follows:

$$\begin{aligned}\mathcal{E}_C^n(u,v) &= \frac{n^{2-d}}{2}\sum_{x,y\in\mathcal{S}_n}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)C_C^n(x,y)\\ &= \frac{1}{2n^d}\sum_{x,y\in\mathcal{S}_n}\sum_{i,j=1}^{d}\sum_{z,w\in\mathcal{S}_n}\Big(P^{x,y}(z+\boldsymbol{e}_i/n,z)-P^{x,y}(z,z+\boldsymbol{e}_i/n)\Big)\\ &\quad\times\Big(P^{x,y}(w+\boldsymbol{e}_j/n,w)-P^{x,y}(w,w+\boldsymbol{e}_j/n)\Big)\nabla^i_{1/n}u(z)\nabla^j_{1/n}v(w)C_C^n(x,y).\end{aligned} \tag{5.1}$$

For $i,j=1,2,\dots,d$ and $w,z\in\mathcal{S}_n$, set

$$\begin{aligned}G_{ij}^n(w,z) &:= \sum_{x,y\in\mathcal{S}_n}\Big(P^{x,y}(z+\boldsymbol{e}_i/n,z)-P^{x,y}(z,z+\boldsymbol{e}_i/n)\Big)\\ &\qquad\times\Big(P^{x,y}(w+\boldsymbol{e}_j/n,w)-P^{x,y}(w,w+\boldsymbol{e}_j/n)\Big)C_C^n(x,y);\end{aligned}$$

then we see that

$$\mathcal{E}_C^n(u,v)=\frac{1}{2n^d}\sum_{i,j=1}^{d}\sum_{w,z\in\mathcal{S}_n}\nabla^i_{1/n}u(z)\nabla^j_{1/n}v(w)G_{ij}^n(w,z). \tag{5.2}$$

Let

$$F_{ij}^n(z)=\sum_{w\in\mathcal{S}_n}G_{ij}^n(w,z),\quad z\in\mathcal{S}_n,\ i,j=1,2,\dots,d. \tag{5.3}$$

Note that if (A4) below holds, then by the fact that $C_C^n(x,y)=0$ for $|x-y|>\varepsilon_n$, we have $F_{ij}^n\in L^1(\mathcal{S}_n,\mu^n)$.

From now on, we extend the conductances $C^n(x,y)$ to $\mathbb{R}^d\times\mathbb{R}^d$ as follows:

$$C^n(x,y)=C^n([x]_n,[y]_n)\quad\text{for } x,y\in\mathbb{R}^d.$$

We extend $C_C^n(\cdot,\cdot), C_J^n(\cdot,\cdot)$ to $\mathbb{R}^d\times\mathbb{R}^d$ and extend $F_{ij}^n(\cdot)$ to $\mathbb{R}^d$ similarly.

We now give an assumption needed to obtain weak convergence of the processes.

(A4) *There exist a decreasing sequence $\{\varepsilon_n\}$ satisfying $1/n\le\varepsilon_n\le 1$ and $\varepsilon_n\searrow 0$, symmetric matrix-valued functions $a(x)=(a_{ij}(x))$ on $\mathbb{R}^d$, and symmetric functions $j(x,y)$ on $\mathbb{R}^d\times\mathbb{R}^d\setminus D$ so that for any $i,j=1,2,\dots,d$, the functions $F_{ij}^n(x)$ are uniformly bounded and converge to $a_{ij}(x)$ locally in $L^1(\mathbb{R}^d)$, and*

$$\lambda^{-1}|\xi|^2\le\sum_{i,j=1}^{d}\xi_i\xi_j a_{ij}(x)\le\lambda|\xi|^2,\quad x,\xi\in\mathbb{R}^d,$$

for some $\lambda > 0$. Further, there exists a function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ so that for any $n \in \mathbb{N}$,

$$C_J^n(x,y) \le n^{-(d+2)}\varphi\big(|x-y|\big), \quad x,y \in \mathcal{S}_n \quad \text{and} \quad \int_0^\infty \big(1 \wedge t^2\big)\, t^{d-1}\varphi(t)dt < \infty, \tag{5.4}$$

and for each $N > 1$, the measures

$$n^{d+2}C_J^n(x,y)\mathbf{1}_{[N^{-1},N]}(|x-y|)dx\,dy \longrightarrow j(x,y)\mathbf{1}_{[N^{-1},N]}(|x-y|)dx\,dy \tag{5.5}$$

weakly as $n \to \infty$.

Remark 5.3 1) Here (5.5) refers to the weak convergence of the measures on the left to the measures on the right. Saying that the F_{ij}^n are uniformly bounded and converge locally in L^1 means that $\sup_{i,j,n} ||F_{ij}^n||_\infty < \infty$ and for every compact set B,

$$\int_B |F_{ij}^n(x) - a_{ij}(x)|\, dx \to 0.$$

Since the F_{ij}^n are uniformly bounded, the convergence locally in L^1 is equivalent to the convergence in measure on each compact set. In particular, a subsequence will converge almost everywhere.

2) One may consider the weaker condition that the F_{ij}^n are uniformly bounded and converge to a_{ij} weakly. However, this condition is not sufficient for Theorem 5.5 to hold (see the example at the end of the introduction in [SZ]).

3) If $C_J^n(\cdot,\cdot)$ satisfies (5.4), then we see for any $x \in \mathcal{S}_n$,

$$\begin{aligned} n^2 \sum_{y\in\mathcal{S}_n} \big(1 \wedge |y|^2\big)C_J^n(x,x+y) &\le n^2 \sum_{y\in\mathcal{S}_n} \big(1 \wedge |y|^2\big)n^{-(d+2)}\varphi(|y|) = \sum_{y\in\mathcal{S}_n} \big(1 \wedge |y|^2\big)\varphi(|y|)n^{-d} \\ &\le c_d \int_{\mathbb{R}^d} \big(1 \wedge |y|^2\big)\varphi(|y|)dy = c_d' \int_0^\infty \big(1 \wedge t^2\big)\, t^{d-1}\varphi(t)dt < \infty. \end{aligned}$$

So the condition (A3) with $C^n(\cdot,\cdot)$ replaced by $C_J^n(\cdot,\cdot)$ holds.

From (A4), we have

$$\sup_x \int_{y\neq x} \big(1 \wedge |x-y|^2\big)j(x,y)dy \le \int_{y\neq x} \big(1 \wedge |x-y|^2\big)\varphi(|x-y|)dy = \int_{h\neq 0} \big(1 \wedge |h|^2\big)\varphi(|h|)dh < \infty.$$

Since a is uniformly elliptic, if we define

$$\begin{aligned} \mathcal{E}(f,g) &:= \mathcal{E}_C(f,g) + \mathcal{E}_J(f,g) \\ &:= \frac{1}{2}\int_{\mathbb{R}^d} \nabla f(x)\cdot a(x)\nabla g(x)dx + \frac{1}{2}\iint_{x\neq y} \big(f(x)-f(y)\big)\big(g(x)-g(y)\big)j(x,y)dx\,dy, \end{aligned}$$

then $(\mathcal{E}, C_c^1(\mathbb{R}^d))$ is a closable Markovian form on $L^2(\mathbb{R}^d, dx)$. Denote the closure by $(\mathcal{E},\mathcal{F})$.

Lemma 5.4 *Let $W^{1,2}(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d, dx) : \nabla f \in L^2(\mathbb{R}^d, dx)\}$. Then,*

$$\{f \in L^2(\mathbb{R}^d, dx) : \mathcal{E}(f,f) < \infty\} = W^{1,2}(\mathbb{R}^d) = \mathcal{F}. \tag{5.6}$$

Moreover, if $\mathcal{F}'$ is a subset of $L^2(\mathbb{R}^d, dx)$ such that $(\mathcal{E}, \mathcal{F}')$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$, then $\mathcal{F}' = W^{1,2}(\mathbb{R}^d)$.

Proof. Let $f \in L^2$ be such that $\mathcal{E}(f,f) < \infty$. Then, $\mathcal{E}_C(f,f) < \infty$ and $\mathcal{E}_C(f,f)$ is comparable to $\|\nabla f\|_2^2$, so $f \in W^{1,2}(\mathbb{R}^d)$. On the other hand, suppose $f \in W^{1,2}(\mathbb{R}^d)$. Then $\mathcal{E}_J(f,f) \leq \mathcal{E}_\varphi(f,f)$, where φ is given in (A4) and $\mathcal{E}_\varphi$ is the Dirichlet form for the symmetric Lévy process with Lévy measure $\varphi(|h|)dh$. By the Lévy-Khintchine formula (see, e.g., (1.4.21) in [FOT]), the characteristic function of the process at time t is given by $e^{t\psi(u)}$, where

$$\psi(u) = \int_{\mathbb{R}^d} \Big(1 - \cos\big(u \cdot h\big)\Big)\varphi(|h|)dh, \quad u \in \mathbb{R}^d.$$

According to (A4), we have,

$$\begin{aligned}\psi(u) &= \int [1 - \cos(u \cdot h)]\varphi(|h|)\, dh \\ &\leq c_1 \int [|u|^2|h|^2 \wedge 1]\varphi(|h|)\, dh \\ &\leq c_2(|u|^2 + 1).\end{aligned}$$

Using Plancherel's theorem, for $f \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{E}_\varphi(f,f) &= \frac{1}{2}\iint_{y \neq x} \big(f(x+h) - f(x)\big)^2 \varphi(|h|)dhdx \\ &= \int \big|\widehat{f}(u)\big|^2 \psi(u)du \\ &\leq c_2 \int (1 + |u|^2)|\widehat{f}(u)|^2\, du = c_3(\|f\|_2^2 + \|\nabla f\|_2^2).\end{aligned}$$

Here $\hat{f}$ is the Fourier transform of f. A limit argument shows that

$$\mathcal{E}_\varphi(f,f) \leq c_4(\|f\|_2^2 + \|\nabla f\|_2^2) \tag{5.7}$$

for $f \in W^{1,2}(\mathbb{R}^d)$. Since $\mathcal{E}_C(f,f)$ is comparable to $\|\nabla f\|_2^2$, adding shows that $\mathcal{E}(f,f) < \infty$, and the first equality in (5.6) is proved.

Now suppose $(\mathcal{E}, \mathcal{F}')$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$; then since $W^{1,2}(\mathbb{R}^d)$ is the maximal domain (due to the first equality in (5.6)), we have $\mathcal{F}' \subset W^{1,2}(\mathbb{R}^d)$. From the above results, we know that the $(\mathcal{E}(\cdot,\cdot) + \|\cdot\|_2^2)^{1/2}$-norm is comparable to the $W^{1,2}$-norm on $W^{1,2}(\mathbb{R}^d)$. Using this, we see that $(\|\nabla \cdot\|_2^2, \mathcal{F}')$ is a regular Dirichlet form. This implies $\mathcal{F}' = W^{1,2}(\mathbb{R}^d)$ (so $W^{1,2}(\mathbb{R}^d) = \mathcal{F}$ as well) and the proof is complete. □

Under the above set-up we have the following, which is the main theorem of this paper.

Theorem 5.5 *Suppose (A1)-(A4) hold. Then for each x and each t_0 the $\mathbb{P}^{[x]_n}$-laws of $\{Y_t^{(n)}; 0 \le t \le t_0\}$ converge weakly with respect to the topology of the space $D([0,t_0],\mathbb{R}^d)$. If Z_t is the canonical process on $D([0,t_0),\mathbb{R}^d)$ and $\mathbb{P}^x$ is the weak limit of the $\mathbb{P}^{[x]_n}$-laws of $Y^{(n)}$, then the process $\{Z_t, \mathbb{P}^x\}$ is the symmetric Markov process corresponding to the Dirichlet form $\mathcal{E}$ with domain $W^{1,2}(\mathbb{R}^d)$.*

The proof of this theorem is given in the next section.

In the remainder of this section we give an example of how our results can be applied to the homogenization problem for random media. Let $\hat{\mathcal{S}}_n = 2^{-n}\mathbb{Z}^d$ and for simplicity consider the case $d = 1$. (In this example, the state space is $\hat{\mathcal{S}}_n$ instead of $\mathcal{S}_n$.) Let $\{\xi_1^n(x,y), \xi_2^n(x,y)\}_{x,y\in\hat{\mathcal{S}}_n, n\in\mathbb{N}}$ be i.i.d. with values in $[0,M]$ a.e. for some non-random $M < \infty$, and let $m = E[\xi_1^n(x,y)] = E[\xi_2^n(x,y)]$. Define

$$\hat{C}^n(x,y) = \frac{\xi_1^n(x,y)}{2^{3n/2}} 1_{\{|x-y|\le 2^{-n/2}\}} + 1_{\{|x-y|=2^{-n}\}} + \frac{\xi_2^n(x,y)}{2^{3n}|x-y|^{1+\alpha}} \quad \text{for } x,y\in\hat{\mathcal{S}}_n, \tag{5.8}$$

where $0<\alpha<2$. Then the corresponding processes $\{\hat{Y}_t^{(n)}; t\le t_0\}$ on $\hat{\mathcal{S}}_n$ converge weakly to a symmetric Markov process whose Dirichlet form is expressed by

$$(\frac{m}{3}+1)\int_{\mathbb{R}} \nabla f(x)\nabla g(x)dx + \frac{m}{2}\int\int_{x\ne y} \frac{\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)}{|x-y|^{1+\alpha}}dx\,dy. \tag{5.9}$$

Conditions (A1)-(A2) can be easily verified. Since

$$\begin{aligned}
&\sup_{x\in\hat{\mathcal{S}}_n}\Big(2^{2n}\sum_{y\in\hat{\mathcal{S}}_n}(1\wedge|y|^2)\hat{C}^n(x,x+y)\Big)\\
&\qquad\le c_1 2^{2n}\Big(\sum_{i=1}^{[2^{n/2}]}(1\wedge\frac{i^2}{2^{2n}})\frac{M}{2^{3n/2}} + 2\cdot\frac{1}{2^{2n}} + \sum_{i=1}^{\infty}(1\wedge\frac{i^2}{2^{2n}})\frac{M}{2^{(2-\alpha)n}\, i^{1+\alpha}}\Big)\\
&\qquad\le c_2\Big(2^{-3n/2}\sum_{i=1}^{[2^{n/2}]} i^2 + 2 + 2^{-(2-\alpha)n}\sum_{i=1}^{2^n} i^{1-\alpha} + 2^{n\alpha}\sum_{i=2^n+1}^{\infty} i^{-(1+\alpha)}\Big) \le c_3,
\end{aligned}$$

(A3) holds. We now prove that (A4) holds. We take $\varepsilon_n = 2^{-n/2}$ in this case. First, let $w,z\in\hat{\mathcal{S}}_n$ with $w\le z$ and $|w-z| = k2^{-n}$, $k=0,1,\cdots,[2^{n/2}]-1$. Then by a simple computation, we have, letting $\hat{G}_{11}^n(w,z) = G_{11}^{2^n}(w,z)$ for $w,z\in\hat{\mathcal{S}}_n$,

$$\begin{aligned}
&\hat{G}_{11}^n(w,z)\\
&\quad = 2\sum_{l=1}^{[2^{n/2}]-k}\sum_{i=0}^{l-1}\Big(\frac{\xi_1^n(w-i2^{-n}, z+(l-i)2^{-n})}{2^{3n/2}} + \frac{\xi_2^n(w-i2^{-n}, z+(l-i)2^{-n})}{2^{3n}(k+l)^{1+\alpha}2^{-(1+\alpha)n}}\Big) + 2\cdot 1_{\{k=0\}},
\end{aligned}$$

where the last term comes from $1_{\{|x-y|=2^{-n}\}}$ in (5.8). Since $\hat{F}_{11}^n(z) = \sum_w \hat{G}_{11}^n(w,z)$ (note that we include the case $w>z$ here), the total number of the $\xi_1^n(\cdot,\cdot)$ that are summed up (counting multiplicity) is

$$2\cdot 2\sum_{k=0}^{[2^{n/2}]-1}\sum_{l=1}^{[2^{n/2}]-k} l \sim 4\cdot 2^{3n/2}/(3!) = 2\cdot 2^{3n/2}/3,$$

while the sum of the $\xi_2^n(\cdot,\cdot)$ terms is bounded above by

$$2\cdot 2\sum_{k=0}^{[2^{n/2}]-1}\sum_{l=1}^{[2^{n/2}]-k}\frac{lM}{2^{3n}(k+l)^{1+\alpha}2^{-(1+\alpha)n}}\le\frac{c_4}{2^{(2-\alpha)n}}\sum_{l=1}^{[2^{n/2}]}\sum_{k=0}^{l}\frac{l}{(k+l)^{1+\alpha}}\le c_5 2^{-(2-\alpha)n/2},$$

which goes to 0 a.s. as $n\to\infty$. Therefore the limit of $\hat F_{11}^n(z)$ as $n\to\infty$ is almost surely the same as the limit of

$$\hat H_n=\frac{1}{2^{3n/2}}\sum_{k=-N}^{N}\sum_{l=1}^{N+1-k}\sum_{i=0}^{l-1}\xi_1^n(z+(k-i)2^{-n},z+(l-i)2^{-n})+2, \tag{5.10}$$

where we write $N=[2^{n/2}]-1$ and recall that $|w-z|=k2^{-n}$ for $|k|\le N$. Rearranging, this is equal to

$$\frac{1}{2^{3n/2}}\sum_{k=-N}^{N}\sum_{r=2k-N}^{k}\sum_{s=1}^{N+1+r-2k}\xi_1^n(z+r2^{-n},z+s2^{-n})+2. \tag{5.11}$$

We rewrite this as

$$\hat H_n=2+\frac{1}{2^{3n/2}}\sum_{r,s}b_{rs}V_{rs}, \tag{5.12}$$

where $V_{rs}=\xi_1^n(z+r2^{-n},z+s2^{-n})$, b_{rs} is the number of times V_{rs} appears in the sum (5.11), each b_{rs} is bounded by c_6N, and the sum is over a collection of pairs (r,s) of cardinality less than c_7N^2. Since

$$\text{Var }\hat H_n=\frac{1}{2^{3n}}\sum_{r,s}b_{rs}^2\text{ Var }V_{rs}\le\frac{c_8}{N^6}\sum_{r,s}(c_6N)^2\le c_9/N^2$$

and $N\sim 2^{n/2}$, by Chebyshev's inequality and the Borel-Cantelli lemma $\hat H_n$ converges a.s. to its mean. If m is the mean of V_{rs} and we let $a_{11}(z)=2m/3+2$, then we conclude $\hat F_{11}^n(z)\to a_{11}(z)$ a.s. for each z. By a Fubini argument, except for an event of probability zero, $\hat F_{11}^n(z)\to a_{11}(z)$ for almost every z. Since $\hat F_{11}^n$ is uniformly bounded, we see by the dominated convergence theorem that $\hat F_{11}^n(x)$ converges to $a_{11}(x)$ locally in L^1. (5.4) is easy to verify. Using the law of large numbers again, we can check (5.5) where $j(x,y)=m|x-y|^{-1-\alpha}$. Thus (A4) is verified and the limiting Dirichlet form is given as in (5.9). Note that one can construct similar examples for $d\ge 2$ by choosing $\mathcal{P}(x,y)$ as in Remark 5.2 (ii). One can also easily modify the above example so that the limiting process is not spatially homogeneous.

6 Proof of Theorem 5.5

In this section, we will prove Theorem 5.5. The proof of the convergence of the jump part of the Dirichlet form is along the lines of [BKK], while the convergence of the continuous part of the Dirichlet form was briefly described in the previous section.

We first extend $\mathcal{E}^n$ and define a quadratic form on $L^2(\mathbb{R}^d,dx)$. Define

$$\mathcal{H}_n:=\Big\{E_nu:\ u\text{ is a function on }\mathcal{S}_n\Big\}\cap L^2(\mathbb{R}^d,dx).$$

For $f = E_n u \in \mathcal{H}_n$, define

$$\tilde{\mathcal{E}}^n(f,f) = \frac{n^{2+d}}{2} \iint_{x \neq y} \big(f(x) - f(y)\big)^2 C^n(x,y) dx\, dy.$$

Then we see

$$\begin{aligned} \tilde{\mathcal{E}}^n(f,f) &= \frac{n^{2+d}}{2} \sum_{w_1, w_2 \in \mathcal{S}_n} \big(u(w_1) - u(w_2)\big)^2 C^n(w_1,w_2) \big(n^{-d}\big)^2 \\ &= \frac{n^{2-d}}{2} \sum_{w_1, w_2 \in \mathcal{S}_n} \big(u(w_1) - u(w_2)\big)^2 C^n(w_1,w_2) = \mathcal{E}^n(u,u). \end{aligned} \tag{6.1}$$

Before proving Theorem 5.5, we state a proposition showing tightness of the laws of $Y^{(n)}$.

Proposition 6.1 *Suppose (A1)-(A3) hold and let $\{n_j\}$ be a subsequence. Then there exists a further subsequence $\{n_{j_k}\}$ such that*

(a) For each continuous function f on $\mathbb{R}^d$ with compact support, $E_{n_{j_k}}(P_t^{n_{j_k}} R_{n_{j_k}}(f))$ converges uniformly on compact subsets; if we denote the limit by $P_t f$, then the operator P_t is linear and extends to all continuous functions on $\mathbb{R}^d$ with compact support and is the semigroup of a symmetric strong Markov process on $\mathbb{R}^d$.

(b) For each x and each t_0 the $\mathbb{P}^{[x]_{n_{j_k}}}$ law of $\{Y_t^{(n_{j_k})}; 0 \le t \le t_0\}$ converges weakly to a probability $\mathbb{P}^x$.

Given Proposition 3.1 and Theorem 4.7, the proof of this proposition is very similar to that of [BKu08, Proposition 6.2], so we omit it.

Proof of Theorem 5.5. Let U_n^λ be the λ-resolvent for $Y^{(n)}$; this means that

$$U_n^\lambda h(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} h(Y_t^{(n)})\, dt$$

for $x \in \mathcal{S}_n$ and $h : \mathcal{S}_n \to \mathbb{R}$. First, note that any subsequence $\{n_j\}$ has a further subsequence $\{n_{j_k}\}$ such that $U_{n_{j_k}}^\lambda(R_{n_{j_k}} f)$ converges uniformly on compacts whenever $f \in C_c(\mathbb{R}^d)$, that is, when f is continuous with compact support. This can be proved similarly to Proposition 6.1, so we refer the reader to [BKu08].

Now suppose we have a subsequence $\{n'\}$ such that the $U_{n'}^\lambda(R_{n'} f)$ are equicontinuous and converge uniformly on compacts whenever $f \in C_c(\mathbb{R}^d)$. Fix such an f and let H be the limit of $U_{n'}^\lambda(R_{n'} f)$. Let $g \in C_c^2(\mathbb{R}^d)$ and write $\langle f, g \rangle := \int_{\mathbb{R}^d} f(x) g(x) dx$.

In the following, we drop the primes for legibility. Set $u_n = U_n^\lambda(R_n f)$ for $\lambda > 0$. We will prove that

$$H \in W^{1,2}(\mathbb{R}^d) \qquad \text{and} \qquad \mathcal{E}^n(u_n, g) \to \mathcal{E}(H, g) \tag{6.2}$$

along some subsequence. Once we have (6.2), then

$$\begin{aligned} \mathcal{E}(H,g) &= \lim \mathcal{E}^n(u_n, g) = \lim(\langle f, g\rangle_n - \lambda \langle u_n, g \rangle_n) \\ &= \langle f, g \rangle - \lambda \langle H, g \rangle, \end{aligned}$$

the limit being taken along the subsequence and where $\langle h_1, h_2\rangle_n = n^{-d}\sum_{x\in\mathcal{S}_n} h_1(x)h_2(x)$ for $h_1, h_2 : \mathcal{S}_n \to \mathbb{R}$. By (6.2), $H \in W^{1,2}(\mathbb{R}^d)$, and the equality

$$\mathcal{E}(H, g) = \langle f, g\rangle - \lambda\langle H, g\rangle \tag{6.3}$$

holds for all $g \in C_c^2(\mathbb{R}^d)$. By Lemma 5.4, $C_c^2(\mathbb{R}^d)$ is dense in $W^{1,2}(\mathbb{R}^d)$ with respect to the norm $(\mathcal{E}(\cdot,\cdot) + \|\cdot\|_2^2)^{1/2}$, and so (6.3) holds for all $g \in W^{1,2}(\mathbb{R}^d)$. Since $W^{1,2}(\mathbb{R}^d)$ is the maximal domain due to (5.6), this implies that H is the λ-resolvent of f for the process corresponding to $(\mathcal{E}, W^{1,2}(\mathbb{R}^d))$, that is, $H = U^\lambda f$. We can then conclude that the full sequence $U_n^\lambda(R_n f)$ (without the primes) converges to $U^\lambda f$ whenever $f \in C_c(\mathbb{R}^d)$. The assertions about the convergence of $\mathbb{P}^{[x]_n}$ then follow as in the proof of [BKu08, Proposition 6.2]. The rest of the proof will be devoted to proving (6.2).

The jump part.

This part of the proof is similar to that of [BKK, Theorem 4.1]. We know

$$\mathcal{E}^n(u_n, u_n) = \langle R_n f, u_n\rangle_n - \lambda\|u_n\|_{2,n}^2. \tag{6.4}$$

Since $||\lambda u_n||_{2,n}^2 = ||\lambda U_n^\lambda R_n f||_{2,n} \le ||R_n f||_{2,n} \le \sup_n ||R_n f||_{2,n}$ (note that $\sup_n ||R_n f||_{2,n} < \infty$ because $\lim_{n\to\infty} ||R_n f||_{2,n} = \|f\|_2$ for $f \in C_c(\mathbb{R}^d)$), the right hand side of (6.4) is bounded by

$$\left|\langle R_n f, u_n\rangle_n\right| + \lambda||u_n||_{2,n}^2 \le \frac{1}{\lambda}||R_n f||_{2,n}||\lambda u_n||_{2,n} + \frac{1}{\lambda}||\lambda u_n||_{2,n}^2 \le \frac{2}{\lambda}\sup_n ||R_n f||_{2,n}^2.$$

This tells us that $\{\mathcal{E}^n(u_n, u_n)\}_n$ is uniformly bounded.

Since the u_n are equicontinuous and converge uniformly to H on $\overline{B(0,N)}$ for $N > 0$, using (5.5), we have

$$\begin{aligned}
\int\int_{N^{-1}\le|y-x|\le N} &(H(y) - H(x))^2 j(x, y)\, dy\, dx \\
&\le \limsup_{n\to\infty} n^{2-d} \sum_{\substack{x,y\in\mathcal{S}_n \\ N^{-1}<|y-x|\le N}} (u_n(y) - u_n(x))^2 C_J^n(x, y) \\
&\le \limsup_n \mathcal{E}^n(u_n, u_n) \le c < \infty.
\end{aligned}$$

Letting $N \to \infty$, we have

$$\mathcal{E}_J(H, H) < \infty. \tag{6.5}$$

Fix a function g on $\mathcal{S}_n$ with compact support and choose M large enough so that the support of g is contained in $B(0, M)$. Then

$$\begin{aligned}
&\Big| n^{2-d} \sum_{\substack{x,y\in\mathcal{S}_n \\ |y-x|>N}} (u_n(y) - u_n(x))(g(y) - g(x))C_J^n(x, y)\Big| \\
&\le \Big(n^{2-d} \sum_{x,y\in\mathcal{S}_n} (u_n(y) - u_n(x))^2 C_J^n(x, y)\Big)^{1/2} \Big(n^{2-d} \sum_{\substack{x,y\in\mathcal{S}_n \\ |y-x|>N}} (g(y) - g(x))^2 C_J^n(x, y)\Big)^{1/2}.
\end{aligned}$$

The first factor is $(\mathcal{E}^n(u_n,u_n))^{1/2}$, while the second factor is bounded by

$$2\|g\|_\infty\Big(n^{2-d}\sum_{x\in B(0,M)\cap\mathcal{S}_n}\sum_{|y-x|>N}C_J^n(x,y)\Big)^{1/2},$$

which, in view of (5.4), will be small if N is large. Similarly,

$$\Big|n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\|y-x|<N^{-1}}}(u_n(y)-u_n(x))(g(y)-g(x))C_J^n(x,y)\Big|$$
$$\leq\Big(n^{2-d}\sum_{x,y\in\mathcal{S}_n}(u_n(y)-u_n(x))^2C_J^n(x,y)\Big)^{1/2}\cdot\Big(n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\|y-x|<N^{-1}}}(g(y)-g(x))^2C_J^n(x,y)\Big)^{1/2}.$$

The first factor is as before, while the second is bounded by

$$\|\nabla g\|_\infty\Big(n^{2-d}\sum_{x\in B(0,M)\cap\mathcal{S}_n}\sum_{|y-x|<N^{-1}}|y-x|^2C_J^n(x,y)\Big)^{1/2}.$$

In view of (5.4), the second factor will be small if N is large.

Using (6.5), we have that

$$\Big|\int\int_{|y-x|\notin[N^{-1},N]}(H(y)-H(x))(g(y)-g(x))j(x,y)\,dy\,dx\Big|$$

will be small if N is taken large enough.

By (5.5) and the fact that the $U_n^\lambda f$ are equicontinuous and converge to H uniformly on compacts, we have

$$n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\N^{-1}\leq|y-x|\leq N}}(u_n(y)-u_n(x))(g(y)-g(x))C_J^n(x,y)$$
$$\to\int\int_{N^{-1}\leq|y-x|\leq N}(H(y)-H(x))(g(y)-g(x))j(x,y)\,dy\,dx.$$

It follows that

$$\mathcal{E}_J^n(u_n,g)\to\mathcal{E}_J(H,g),\tag{6.6}$$

which takes care of the jump part of (6.2).

The continuous part.

Step 1. First we show that $H\in W^{1,2}(\mathbb{R}^d)$.

As in the discussion of the jump part, we know $\{\mathcal{E}^n(u_n,u_n)\}_n$ is uniformly bounded. On the other hand, making use of the assumption (A2), we see

$$\tilde{\mathcal{E}}^n(E_nu_n,E_nu_n)=\mathcal{E}^n(u_n,u_n)\geq c\mathcal{E}_{NN}^n(u_n,u_n)=c\tilde{\mathcal{E}}_{NN}^n(E_nu_n,E_nu_n).$$

(Recall $\mathcal{E}^n_{NN}$ is defined in (3.1) and C^n_{NN} is defined immediately thereafter.) Therefore, for $f \in C^1_c(\mathbb{R}^d)$, the sequence $\{\tilde{\mathcal{E}}^n_{NN}(E_n u_n, E_n u_n)\}_n$ is uniformly bounded with respect to n. Letting $Q_n(w) = \prod_{i=1}^d [w_i, w_i + 1/n)$, we see that for any $i = 1, 2, \ldots, d$,

$$
\begin{aligned}
\tilde{\mathcal{E}}^n_{NN}(E_n u_n, E_n u_n) &= \frac{n^{2+d}}{2} \iint_{x \neq y} \big(E_n u_n(x) - E_n u_n(y)\big)^2 C^n_{NN}(x, y) dx dy \\
&= \frac{n^{2+d}}{2} \sum_{w \in \mathcal{S}_n} \int_{Q_n(w)} \Big(\int_{y \neq x} \big(E_n u_n(x) - E_n u_n(y)\big)^2 C^n_{NN}(w, [y]_n) dy \Big) dx \\
&\geq \frac{n^{2+d}}{2} \sum_{w \in \mathcal{S}_n} \int_{Q_n(w)} \big(E_n u_n(x) - E_n u_n(x + \boldsymbol{e}_i/n)\big)^2 \Big(\int_{Q_n(w + \boldsymbol{e}_i/n)} dy \Big) dx \\
&= \frac{n^2}{2} \sum_{w \in \mathcal{S}_n} \int_{Q_n(w)} \big(E_n u_n(x) - E_n u_n(x + \boldsymbol{e}_i/n)\big)^2 dx \\
&= \frac{n^2}{2} \int_{\mathbb{R}^d} \big(E_n u_n(x) - E_n u_n(x + \boldsymbol{e}_i/n)\big)^2 dx.
\end{aligned}
$$

In other words, $\{n\big(E_n u_n(\cdot) - E_n u_n(\cdot + \boldsymbol{e}_i/n)\big)\}_n$ is a bounded sequence in $L^2(\mathbb{R}^d, dx)$. So there exists a subsequence $\{n'\}$ and a unique $v_i \in L^2(\mathbb{R}^d, dx)$ so that $n'\big(E_{n'} u_{n'}(\cdot) - E_{n'} u_{n'}(\cdot + \boldsymbol{e}_i/n')\big)$ converges to v_i weakly in $L^2(\mathbb{R}^d, dx)$. On the other hand, if $\varphi \in C^2_c(\mathbb{R}^d)$, it follows that

$$\langle E_{n'} u_{n'}(\cdot + \boldsymbol{e}_i/n'), \varphi \rangle = \langle E_{n'} u_{n'}, \varphi(\cdot - \boldsymbol{e}_i/n') \rangle$$

by a change of variables, and then

$$n' \langle E_{n'} u_{n'}(\cdot + \boldsymbol{e}_i/n'), \varphi \rangle - n' \langle E_{n'} u_{n'}, \varphi \rangle = n' \langle E_{n'} u_{n'}, \varphi(\cdot - \boldsymbol{e}_i/n') - \varphi \rangle.$$

Since $\varphi \in C^2_c(\mathbb{R}^d)$, we see that $n'\big(\varphi(\cdot - \boldsymbol{e}_i/n') - \varphi\big)$ converges to $-\partial\varphi/\partial x_i$ uniformly and in $L^2(\mathbb{R}^d, dx)$. So we have, letting $n' \to \infty$,

$$\langle v_i, \varphi \rangle = -\langle H, \partial\varphi/\partial x_i \rangle,$$

since u_n converges to H uniformly on compact sets. This shows that $v_i = \partial H/\partial x_i$ and so $H \in W^{1,2}(\mathbb{R}^d)$.

Step 2. We next show that for some subsequence $\{n'\}$,

$$\mathcal{E}^{n'}_C(u_{n'}, g) \longrightarrow \frac{1}{2} \int_{\mathbb{R}^d} \nabla H(x) \cdot a(x) \nabla g(x) dx = \mathcal{E}_C(H, g)$$

for any $g \in C^2_c(\mathbb{R}^d)$. Recall (5.2); since $C^n_C(x, y) = 0$ if $|x - y| > \varepsilon_n$ and the w, z are on the shortest paths from x and y, it is enough to consider w's only for $|w - z| \leq \varepsilon_n$ in the sum of the right hand side of (5.2). So

$$\begin{aligned}
\mathcal{E}_C^n(u_n,g) &= \frac{1}{2n^d}\sum_{i,j=1}^d \sum_{z\in\mathcal{S}_n} \nabla^i_{1/n}u_n(z) \sum_{\substack{w\in\mathcal{S}_n\\ |w-z|\le\varepsilon_n}} \nabla^j_{1/n}g(w)G^n_{ij}(w,z)\\
&= \frac{1}{2n^d}\sum_{i,j=1}^d \sum_{z\in\mathcal{S}_n} \nabla^i_{1/n}u_n(z)\nabla^j_{1/n}g(z) \sum_{\substack{w\in\mathcal{S}_n\\ |w-z|\le\varepsilon_n}} G^n_{ij}(w,z)\\
&\qquad + \frac{1}{2n^d}\sum_{i,j=1}^d \sum_{z\in\mathcal{S}_n} \nabla^i_{1/n}u_n(z) \sum_{\substack{w\in\mathcal{S}_n\\ |w-z|\le\varepsilon_n}} \Big(\nabla^j_{1/n}g(w)-\nabla^j_{1/n}g(z)\Big)G^n_{ij}(w,z)\\
&=: I_1^n + I_2^n.
\end{aligned}$$

Let K be the support of $g\in C_c^2(\mathbb{R}^d)$. Since $1/n\le\varepsilon_n\le 1$ and $|w-z|\le\varepsilon_n$ in the summation defining I_2^n, the z's must lie in the set $K_1\cap\mathcal{S}_n$, where $K_1=\{x\in\mathbb{R}^d:\ d(K,x)\le 1\}$. By using the mean value theorem for g and the definition of $\nabla^i_{1/n}u_n$, we see that for some $0<\theta,\tilde\theta<1$ depending on z and w,

$$\begin{aligned}
2|I_2^n| &= \Big| n^{-d}\sum_{i,j=1}^d \sum_{z\in\mathcal{S}_n} \nabla^i_{1/n}u_n(z) \sum_{\substack{w\in K_1\cap\mathcal{S}_n\\ |w-z|\le\varepsilon_n}} \Big(\nabla^j_{1/n}g(w)-\nabla^j_{1/n}g(z)\Big)G^n_{ij}(w,z)\Big|\\
&= \Big| n^{1-d}\sum_{i,j=1}^d \sum_{z\in\mathcal{S}_n} \Big(u_n(z+\boldsymbol{e}_i/n)-u_n(z)\Big)\\
&\qquad \times \sum_{\substack{w\in K_1\cap\mathcal{S}_n\\ |w-z|\le\varepsilon_n}} \Big(\partial_j g(w+\theta\boldsymbol{e}_j/n)-\partial_j g(w+\tilde\theta\boldsymbol{e}_j/n)\Big)G^n_{ij}(w,z)\Big|\\
&\le \Big(\sup_{|z-z'|\le 1/n}\big|u_n(z)-u_n(z')\big|\Big)\cdot\sup_j ||\partial_{jj}g||_\infty \times \Big(n^{-d}\sum_{i,j=1}^d\sum_{z\in\mathcal{S}_n}\sum_{\substack{w\in K_1\cap\mathcal{S}_n\\ |w-z|\le\varepsilon_n}}\big|G^n_{ij}(w,z)\big|\Big)\\
&=: \Big(\sup_{|z-z'|\le 1/n}\big|u_n(z)-u_n(z')\big|\Big)\cdot\sup_j ||\partial_{jj}g||_\infty \times I_3^n.
\end{aligned}$$

We now estimate I_3^n. Let $K_2 = \{x \in \mathbb{R}^d : \ d(K_1, x) \le 1\}$. Then,

$$\begin{aligned} I_3^n &= n^{-d} \sum_{i,j=1}^{d} \sum_{z \in \mathcal{S}_n} \sum_{\substack{w \in K_1 \cap \mathcal{S}_n \\ |w-z| \le \varepsilon_n}} \Big| \sum_{\substack{x,y \in \mathcal{S}_n \\ |x-y| \le \varepsilon_n}} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) - P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) \\ &\qquad \times \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) - P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big) C_C^n(x,y) \Big| \\ &\le n^{-d} \sum_{\substack{x,y \in \mathcal{S}_n \\ |x-y| \le \varepsilon_n}} C_C^n(x,y) \sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) + P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) \\ &\qquad \times \sum_{j=1}^{d} \sum_{\substack{w \in K_1 \cap \mathcal{S}_n \\ |w-z| \le \varepsilon_n}} \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) + P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big). \\ &= n^{-d} \sum_{\substack{x \in K_2 \cap \mathcal{S}_n, y \in \mathcal{S}_n \\ |x-y| \le \varepsilon_n}} C_C^n(x,y) \sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) + P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) \\ &\qquad \times \sum_{j=1}^{d} \sum_{\substack{w \in K_1 \cap \mathcal{S}_n \\ |w-z| \le \varepsilon_n}} \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) + P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big). \end{aligned}$$

The last equality holds since the w's (belonging to K_1) lie on some shortest path between x and y in the summations for some $x, y \in \mathcal{S}_n$ with $|x - y| \le \varepsilon_n$. Noting now that

$$\begin{aligned} &\sum_{j=1}^{d} \sum_{\substack{w \in K_1 \cap \mathcal{S}_n \\ |w-z| \le \varepsilon_n}} \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) + P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big) \\ &\qquad \le \sum_{j=1}^{d} \sum_{w \in \mathcal{S}_n} \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) + P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big) = n|x-y| \end{aligned}$$

and similarly

$$\sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) + P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) = n|x-y|,$$

we see that, using (A3),

$$I_3^n \le n^{-d} \sum_{x \in K_2 \cap \mathcal{S}_n} \sum_{\substack{y \in \mathcal{S}_n \\ |x-y| \le \varepsilon_n}} n^2 |x-y|^2 C_C^n(x,y) \le M \mu^n(K_2),$$

where M is the constant in the assumption (A3). So, I_3^n is uniformly bounded in n and hence I_2^n converges to 0 as n tends to ∞ since the $\{u_n\}$ are equicontinuous.

Finally we consider the term I_1^n:

$$\begin{aligned} I_1^n &= \frac{1}{2n^d}\sum_{i,j=1}^d \sum_{z\in\mathcal{S}_n} \nabla^i_{1/n} u_n(z) \nabla^j_{1/n} g(z) F^n_{ij}(z) \\ &= \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} \nabla^i_{1/n} E_n u_n(x) \nabla^j_{1/n} E_n g(x) F^n_{ij}(x)dx. \end{aligned}$$

Observe that if f_n converges to f weakly in L^2 and g_n converges to g boundedly and almost everywhere, then f_ng_n converges to fg weakly. To see this, if $h\in L^2$,

$$\int (f_ng_n)h - \int (fg)h = \int f_n(g_n-g)h + \Big[\int f_ngh - \int fgh\Big].$$

The term inside the brackets on the right hand side goes to 0 since f_n converges to f weakly and the boundedness of g implies that gh is in L^2. The first term on the right hand side is bounded, using Cauchy-Schwarz, by $\|f_n\|_2\,\|(g_n-g)h\|_2$. The factor $\|f_n\|_2$ is uniformly bounded since f_n converges weakly in L^2, while $\|(g_n-g)h\|_2$ converges to 0 by dominated convergence.

Since some subsequence of $\nabla^i_{1/n}E_nu_n$ converges to $v_i=\partial_i H$ weakly in L^2 (as proved in Step 1), and for some further subsequence F^n_{ij} converges to a_{ij} boundedly and almost everywhere (by (A4) and Remark 5.3) and $\nabla^j_{1/n}E_ng$ converges to $\partial_j g$ uniformly on compact sets (because $g\in C^2_c(\mathbb{R}^d)$), we see that, along this further subsequence, the right hand side goes to

$$\frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_j H\, \partial_j g\, a_{ij}\, dx = \frac{1}{2}\int_{\mathbb{R}^d} \nabla H(x)\cdot a(x)\nabla g(x) dx.$$

Hence

$$\mathcal{E}_C^{n'}(u_{n'},g)\to \mathcal{E}_C(H,g).$$

This completes the proof of (6.2) and hence the theorem. □

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