Mean value inequalities for jump processes

Zhen-Qing Chen, Takashi Kumagai and Jian Wang

Dedicated to Professor Michael Röckner on the occasion of his 60th birthday.

Abstract Parabolic Harnack inequalities are one of the most important inequalities in analysis and PDEs, partly because they imply Hölder regularity of the solutions of heat equations. Mean value inequalities play an important role in deriving parabolic Harnack inequalities. In this paper, we first survey the recent results obtained in [14, 15] on the study of stability of heat kernel estimates and parabolic Harnack inequalities for symmetric jump processes on general metric measure spaces. We then establish the L^p-mean value inequalities for all $p \in (0,2]$ for these processes.

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1 Introduction

Consider a divergence operator $\mathscr{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ acting on functions on $\mathbb{R}^d$, where $(a_{ij}(x))_{i,j=1}^d$ is bounded, measurable, and uniform elliptic. In 1964, Moser [28] proved the parabolic Harnack inequalities (PHI(2); see Definition 6 with $\phi(r) = r^2$) for non-negative solutions to the heat equation

$$\frac{\partial u}{\partial t} = \mathscr{L}u. \tag{1}$$

In 1967, Aronson [2] obtained Gaussian type bounds (i.e. (2) with $\mu(B(x,t^{1/2})) = t^{d/2}$ and $d(\cdot,\cdot)$ being the Euclidean metric) for the fundamental solution to (1). These theorems had a profound influence on analysis and differential geometry. An impor-

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tant consequence of the results is that the non-negative solutions to (1) enjoy Hölder regularity (i.e. (16) with $\phi^{-1}(t) = t^{1/2}$). In deriving PHI(2), mean value inequalities (i.e. (18) and (19) without the tail term) play essential roles. In fact, such mean value inequalities were extended in various linear and non-linear PDEs to derive Harnack inequalities (see, for instance [7, 20, 31, 33]).

There are further significant developments later in the last century. Consider a complete Riemannian manifold M with the Riemannian metric $d(\cdot,\cdot)$ and with the Riemannian measure μ. Let $\mathscr{L}$ be the Laplace-Beltrami operator on M. In 1986, Li-Yau [26] proved the following remarkable fact – if M has non-negative Ricci curvature, then the heat kernel $p_t(x,y)$ enjoys the following estimates

$$\frac{c_1}{\mu(B(x,t^{1/2}))}\exp\Big(-\frac{c_2 d(x,y)^2}{t}\Big) \le p(t,x,y) \le \frac{c_3}{\mu(B(x,t^{1/2}))}\exp\Big(-\frac{c_4 d(x,y)^2}{t}\Big). \tag{2}$$

A few years later, Grigor'yan [21] and Saloff-Coste [30] refined the result and proved that PHI(2) is equivalent to a volume doubling condition (VD; see Definition 1 (i)) plus Poincaré inequalities (PI(2); see Definition 8 (iii) with $\phi(r) = r^2$). Later, these results were extended to the framework of strongly local Dirichlet forms on metric measure spaces by Sturm [32] and on graphs by Delmotte [17]. It was also known around 80s that (2) is equivalent to PHI(2), so the following equivalence holds:

$$(2) \Leftrightarrow \mathrm{VD} + \mathrm{PI}(2) \Leftrightarrow \mathrm{PHI}(2). \tag{3}$$

One of the important consequence of the equivalence is that (2) and PHI(2) are stable under perturbations, since both VD and PI(2) are stable under the perturbations of rough isometries. Such an equivalence was generalized to the so-called sub-Gaussian heat kernel estimates for symmetric diffusions:

$$\begin{aligned}\frac{c_1}{\mu(B(x,t^{1/d_w}))}&\exp\left(-c_2\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\right)\\ &\le p(t,x,y) \le \frac{c_3}{\mu(B(x,t^{1/d_w}))}\exp\left(-c_4\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\right)\end{aligned} \tag{4}$$

for some $d_w \ge 2$. When $d_w = 2$, it is just the Aronson Gaussian estimates (2); and when $d_w > 2$, the behaviors of the corresponding diffusions are anomalous. Diffusions on fractals are typical examples that enjoy (4) for some $d_w > 2$. It turns out (see [1, 3, 4, 24]) that there is an inequality CSA(d_w), a version of the so-called cut-off Sobolev inequality, such that the following equivalence holds:

$$(4) \Leftrightarrow \mathrm{VD} + \mathrm{PI}(d_w) + \mathrm{CSA}(d_w) \Leftrightarrow \mathrm{PHI}(d_w). \tag{5}$$

See Definition 6 and Definition 8 (iii) with $\phi(r) = r^{d_w}$ for definitions of PHI(d_w) and PI(d_w), respectively. We will not give the precise definition of CSA(d_w) (see Definition 4 for the corresponding inequality for symmetric jump processes). Instead, we note that $\mathrm{CSA}(2)$ always holds (so that (5) is indeed a generalization of (3)), and that $\mathrm{CSA}(d_w)$ is stable under rough isometries (and, consequently, (4) and $\mathrm{PHI}(d_w)$ are stable under rough isometries).

For symmetric jump processes, the corresponding results have been obtained only recently. Suppose that a metric measure space (M,μ) is an Alhfors d-regular set on $\mathbb{R}^n$; namely, $\mu(B(x,r)) \asymp r^d$, and a regular Dirichlet form $(\mathscr{E},\mathscr{F})$ on $L^2(M;\mu)$ is defined by

$$\mathscr{E}(f,g) := \int_{M\times M\setminus\Delta} \frac{(f(x)-f(y)(g(x)-g(y))}{|x-y|^{d+\alpha}} c(x,y)\mu(dx)\mu(dy),$$

where $c(\cdot,\cdot)$ is a measurable symmetric function that is bounded between two strictly positive constants and $0<\alpha<2$. The Hunt process X associated with $(\mathscr{E},\mathscr{F})$ is called a symmetric α-stable-like process on M. It was proved in [12] that the corresponding heat kernel of the Dirichlet form (or equivalently, of X) enjoys the following estimates for all $t>0$ and $x,y\in M$

$$c_1\Big(t^{d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\Big) \le p(t,x,y) \le c_2\Big(t^{d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\Big).$$

In that paper, α-order parabolic Harnack inequalities ($\mathrm{PHI}(\alpha)$; see Definition 6 with $\phi(r)=r^\alpha$) were also proved. In the subsequent paper [13], the results were extended to more general time-scale functions, and in [5] some equivalence criteria were given concerning the heat kernel estimates and parabolic Harnack inequalities for symmetric α-stable-like processes with $0<\alpha<2$ on Alhfors regular graphs. In the very recent papers [14, 15], complete equivalences and stability for heat kernel estimates and parabolic Harnack inequalities have been established for symmetric jump processes of variable order on general metric measure spaces. An important ingredient in our approach in these two papers is the L^2 and L^1 mean value inequalities for subharmonic functions of symmetric finite range jump processes.

The aim of this paper is twofold. Firstly, we present the main results obtained in our recent papers [14, 15] on equivalent characterizations of heat kernel estimates and parabolic Harnack inequalities. Secondly, we show that the L^p-mean value inequalities hold not only for $p=2$ but also for all $p\in(0,2]$ for a large class of symmetric jump processes. There are done in Sections 2 and 3, respectively.

2 Stability of heat kernel estimates and parabolic Harnack inequalities for symmetric non-local Dirichlet forms

2.1 Setting

Let (M,d) be a locally compact separable metric space, and μ a positive Radon measure on M with full support. The triple (M,d,μ) is called a *metric measure* *space*. Throughout the paper, we assume for simplicity that $\mu(M)=\infty$. Note that we do not assume M to be connected nor (M,d) to be geodesic.

Let $(\mathscr{E},\mathscr{F})$ be a regular *Dirichlet form* on $L^2(M;\mu)$ of pure-jump type; namely,

$$\mathscr{E}(f,g)=\int_{M\times M\setminus\Delta}(f(x)-f(y)(g(x)-g(y))\,J(dx,dy),\quad f,g\in\mathscr{F}, \tag{6}$$

where $\Delta:=\{(x,x):x\in M\}$ and $J(\cdot,\cdot)$ is a symmetric Radon measure on $M\times M\setminus\Delta$. In the paper, we will abuse notation and always take the quasi-continuous version for an element of $\mathscr{F}$ (note that since $(\mathscr{E},\mathscr{F})$ is regular, each function in $\mathscr{F}$ admits a quasi-continuous version). Let $\mathscr{L}$ be the (negative definite) L^2-generator of $(\mathscr{E},\mathscr{F})$ and $\{P_t\}$ be the associated semigroup on $L^2(M;\mu)$. There exists an μ-symmetric *Hunt process* $X=\{X_t,t\ge 0,\mathbb{P}^x,x\in M\setminus\mathscr{N}\}$ which is associated with the regular Dirichlet form $(\mathscr{E},\mathscr{F})$ on $L^2(M;\mu)$. Here $\mathscr{N}$ is a properly exceptional set for $(\mathscr{E},\mathscr{F})$ in that $\mu(\mathscr{N})=0$ and $\mathbb{P}^x(X_t\in\mathscr{N}$ for some $t>0)=0$ for all $x\in M\setminus\mathscr{N}$. It is known that this Hunt process is uniquely determined up to a properly exceptional set (see [18, Theorem 4.2.8] or [27, Chapter IV, Theorem 6.4]). Furthermore, we can obtain a more precise version of $\{P_t\}$ with better regularity properties as follows:

$$P_tf(x)=\mathbb{E}^xf(X_t),\quad x\in M_0:=M\setminus\mathscr{N}$$

for any bounded Borel measurable function f on M.

A measurable function $p(t,x,y):(0,\infty)\times M_0\times M_0\to(0,\infty)$ is called a *heat kernel* associated with $\{P_t\}$ if the following hold:

$$\begin{aligned}
\mathbb{E}^xf(X_t)=P_tf(x)&=\int p(t,x,y)f(y)\,\mu(dy), && \forall x\in M_0,\ f\in L^\infty(M,\mu),\\
p(t,x,y)&=p(t,y,x), && \forall t>0,\ x,y\in M_0,\\
p(s+t,x,z)&=\int p(s,x,y)p(t,y,z)\,\mu(dy), && \forall s,t>0,\ x,z\in M_0.
\end{aligned}$$

We may extend $p(t,x,y)$ to all x, $y\in M$ by setting $p(t,x,y)=0$ if x or y is outside M_0.

Definition 1. Let $B(x,r)$ be the ball in (M,d) centered at x with radius r, and set

$$V(x,r)=\mu(B(x,r)).$$

(i) We say that (M,d,μ) satisfies the *volume doubling property* (VD) if there exist constants $L_\mu>1$ and $C_\mu\ge 1$ so that for all $x\in M$ and $r>0$,

$$V(x,L_\mu r)\le C_\mu V(x,r). \tag{7}$$

(ii) We say that (M,d,μ) satisfies the *reverse volume doubling property* (RVD) if there exist constants $l_\mu,c_\mu>1$ so that for all $x\in M$ and $r>0$,

$$V(x,l_\mu r)\ge c_\mu V(x,r).$$

VD condition (7) is equivalent to the following: there exist positive constants d_2 and $\widetilde{C}_\mu$ so that

$$\frac{V(x,R)}{V(x,r)} \le \widetilde{C}_\mu \left(\frac{R}{r}\right)^{d_2} \quad \text{for all } x \in M \text{ and } 0 < r \le R. \tag{8}$$

It is known that VD implies RVD if M is connected and unbounded (see, for example [22, Proposition 5.1 and Corollary 5.3]).

Let $\mathbb{R}_+ := [0,\infty)$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing continuous function with $\phi(0) = 0$, $\phi(1) = 1$ that satisfies the following: there exist constants $c_1, c_2 > 0$ and $\beta_2 \ge \beta_1 > 0$ such that

$$c_1 \left(\frac{R}{r}\right)^{\beta_1} \le \frac{\phi(R)}{\phi(r)} \le c_2 \left(\frac{R}{r}\right)^{\beta_2} \quad \text{for all } 0 < r \le R. \tag{9}$$

Definition 2. We say J_ϕ holds if there exists a non-negative symmetric function $J(\cdot,\cdot)$ so that for $\mu \times \mu$-almost all $x, y \in M$,

$$J(dx,dy) = J(x,y)\,\mu(dx)\,\mu(dy), \tag{10}$$

and

$$\frac{c_1}{V(x,d(x,y))\phi(d(x,y))} \le J(x,y) \le \frac{c_2}{V(x,d(x,y))\phi(d(x,y))} \tag{11}$$

for some constants $c_2 \ge c_1 > 0$. We say that $\mathrm{J}_{\phi,\le}$ (resp. $\mathrm{J}_{\phi,\ge}$) if (10) holds and the upper bound (resp. lower bound) in (11) holds.

For a non-local Dirichlet form $(\mathscr{E},\mathscr{F})$, we define the carré du-Champ operator $\Gamma(f,g)$ for $f,g \in \mathscr{F}$ by

$$\Gamma(f,g)(dx) = \int_{y\in M} (f(x)-f(y))(g(x)-g(y))\,J(dx,dy).$$

Clearly $\mathscr{E}(f,g) = \Gamma(f,g)(M)$. Note that for any $f \in \mathscr{F}_b := \mathscr{F} \cap L^\infty(M,\mu)$, $\Gamma(f,f)$ is the unique Borel measure (called the *energy measure*) on M satisfying

$$\int_M g\,d\Gamma(f,f) = \mathscr{E}(f,fg) - \frac{1}{2}\mathscr{E}(f^2,g), \quad f,g \in \mathscr{F}_b.$$

2.2 Heat kernel estimates

Definition 3. We say that $\mathrm{HK}(\phi)$ holds if there exists a kernel $p(t,x,y)$ with respect to the measure μ of the semigroup $\{P_t\}$ for $(\mathscr{E},\mathscr{F})$ so that the following estimates hold for all $t > 0$ and all $x, y \in M_0$,

$$
\begin{aligned}
c_1\Big(\frac{1}{V(x,\phi^{-1}(t))}\wedge\frac{t}{V(x,d(x,y))\phi(d(x,y))}\Big)&\le p(t,x,y)\\
&\le c_2\Big(\frac{1}{V(x,\phi^{-1}(t))}\wedge\frac{t}{V(x,d(x,y))\phi(d(x,y))}\Big),
\end{aligned}\tag{12}
$$

where $c_1,c_2>0$ are constants independent of $x,y\in M_0$ and $t>0$. Here $\phi^{-1}(t)$ is the inverse function of $t\mapsto\phi(t)$. We say UHK(ϕ) (resp. LHK(ϕ)) holds if the upper bound (resp. the lower bound) in (12) holds for $p(t,x,y)$.

Remark 1. (i) We can replace $V(x,d(x,y))$ by $V(y,d(x,y))$ in (12) by modifying the values of c_1 and c_2. Indeed, the following holds (see [14, Remark 1.12]):

$$\frac{1}{V(y,\phi^{-1}(t))}\wedge\frac{t}{V(y,d(x,y))\phi(d(x,y))}\asymp\frac{1}{V(x,\phi^{-1}(t))}\wedge\frac{t}{V(x,d(x,y))\phi(d(x,y))}.$$

Here for two functions f and g, notation $f\asymp g$ means f/g is bounded between two positive constants.

(ii) It follows from [14, Theorem 1.13 and Lemma 5.6] that if HK(ϕ) holds, then the heat kernel $p(t,x,y)$ is Hölder continuous on (x,y) for every $t>0$, so (12) holds for all $x,y\in M$ and $t>0$.

In [14], stability of heat kernel estimates has been established for symmetric pure-jump processes on a general metric measure space. Below is the precise statement.

Theorem 1. *Assume that the metric measure space (M,d,μ) satisfies* VD *and* RVD*, and ϕ satisfies* (9)*. Let $(\mathscr{E},\mathscr{F})$ be a regular (resp. regular and conservative) symmetric Dirichlet form on $L^2(M;\mu)$ of pure-jump type* (6)*. $(\widetilde{\mathscr{E}},\mathscr{F})$ Let $(\widetilde{\mathscr{E}},\mathscr{F})$ be another regular (resp. regular and conservative) symmetric Dirichlet form on $L^2(M;\widetilde{\mu})$ of pure-jump type* (6) *with jumping measure $\widetilde{J}(dx,dy)$, and there exists a constant $1\le c<\infty$ such that for all measurable sets A and B,*

$$c^{-1}\mu(A)\le\widetilde{\mu}(A)\le c\mu(A),\tag{13}$$

$$c^{-1}J(A,B)\le\widetilde{J}(A,B)\le cJ(A,B)\quad \textit{when } d(A,B)>0.\tag{14}$$

Then $(\mathscr{E},\mathscr{F})$ satisfies HK(ϕ) *(resp.* UHK(ϕ)*) if and only if so does $(\widetilde{\mathscr{E}},\mathscr{F})$.*

In [14], this theorem is a direct consequence of the stable characterization of HK(ϕ) and UHK(ϕ), which is stable under perturbations (13) and (14). Precise statements will be given in Theorems 2 and 3 below. First we need some definitions.

The following inequality CSJ(ϕ) that controls the energy of cutoff functions, introduced in [14], is a modification of CSA(ϕ) in [1] for strongly local Dirichlet forms as a weaker version of the cut-off Sobolev inequality CS(ϕ) in [3, 4]. In [24], the inequality corresponding to CSJ(ϕ) for strongly local Dirichlet forms is called a generalized capacity inequality.

Definition 4. (i) Let $U \subset V$ be open sets in M with $U \subset \overline{U} \subset V$. A non-negative bounded measurable function φ is said to be a *cutoff function for* $U \subset V$ if $\varphi = 1$ on U, $\varphi = 0$ on V^c and $0 \le \varphi \le 1$ on M.
(ii)We say that CSJ(ϕ) holds if there exist constants $c_0 \in (0,1]$ and $c_1, c_2 > 0$ such that for every $0 < r \le R$, almost all $x \in M$ and any $f \in \mathscr{F}$, there exists a cutoff function $\varphi \in \mathscr{F}_b$ for $B(x,R) \subset B(x,R+r)$ so that the following holds:

$$\begin{aligned}\int_{B(x,R+(1+c_0)r)} f^2\, d\Gamma(\varphi,\varphi) \le & c_1 \int_{U\times U^*} (f(x)-f(y))^2 J(dx,dy) \\ & + \frac{c_2}{\phi(r)} \int_{B(x,R+(1+c_0)r)} f^2\, d\mu,\end{aligned}$$

where $U = B(x,R+r) \setminus B(x,R)$ and $U^* = B(x,R+(1+c_0)r) \setminus B(x,R-c_0 r)$.

Remark 2. As is pointed out in [14, Remark 1.7], under VD, (9) and $\mathrm{J}_{\phi,\le}$, CSJ(ϕ) always holds if $\beta_2 < 2$, where β_2 is the exponent in (9). In particular, CSJ(ϕ) always holds for $\phi(r) = r^\alpha$ with $0 < \alpha < 2$.

For any open set $D \subset M$, $\mathscr{F}_D$ is defined to be the $\mathscr{E}_1$-closure in $\mathscr{F}$ of $\mathscr{F} \cap C_c(D)$, where $\|\cdot\|^2_{\mathscr{E}_1} = \|\cdot\|^2_{\mathscr{E}} + \|\cdot\|_2^2$, and $C_c(D)$ is the space of continuous functions on M with compact support in D. Define

$$\lambda_1(D) = \inf\left\{\mathscr{E}(f,f) : f \in \mathscr{F}_D \text{ with } \|f\|_2 = 1\right\},$$

the bottom of the Dirichlet spectrum of $-\mathscr{L}$ on D. For a set $A \subset M$, define its exit time $\tau_A = \inf\{t > 0 : X_t \in A^c\}$.

Definition 5. (i) We say that the *Faber-Krahn inequality* FK(ϕ) holds if there exist constants $c, \nu > 0$ such that for any ball $B(x,r)$ and any open set $D \subset B(x,r)$,

$$\lambda_1(D) \ge \frac{c}{\phi(r)} (V(x,r)/\mu(D))^\nu.$$

(ii) We say that E_ϕ holds if there is a constant $c_1 > 1$ such that for all $r > 0$ and all $x \in M_0$,

$$c_1^{-1}\phi(r) \le \mathbb{E}^x[\tau_{B(x,r)}] \le c_1 \phi(r).$$

We say that $\mathrm{E}_{\phi,\le}$ (resp. $\mathrm{E}_{\phi,\ge}$) holds if the upper bound (resp. lower bound) in the above display holds for $\mathbb{E}^x[\tau_{B(x,r)}]$.
(iii) We say UHKD(ϕ) holds if there is a constant $c > 0$ such that

$$p(t,x,x) \le \frac{c}{V(x,\phi^{-1}(t))} \quad \text{for all } t > 0 \text{ and } x \in M_0.$$

(iv) We say $(\mathscr{E},\mathscr{F})$ *is conservative* if its associated Hunt process X has infinite lifetime. This is equivalent to $P_t 1 = 1$ a.e. on M_0 for every $t > 0$.

The following are the main results of [14].

Theorem 2. ([14, Theorem 1.13]) *Assume that the metric measure space* (M,d,μ) *satisfies* VD *and* RVD*, and* ϕ *satisfies* (9). *Then the following are equivalent:*
(1) $\mathrm{HK}(\phi)$.
(2) J_ϕ *and* E_ϕ.
(3) J_ϕ *and* $\mathrm{CSJ}(\phi)$.

Theorem 3. ([14, Theorem 1.15]) *Assume that the metric measure space* (M,d,μ) *satisfies* VD *and* RVD*, and* ϕ *satisfies* (9). *Then the following are equivalent:*
(1) $\mathrm{UHK}(\phi)$ *and* $(\mathscr{E},\mathscr{F})$ *is conservative.*
(2) $\mathrm{UHKD}(\phi)$*,* $\mathrm{J}_{\phi,\le}$ *and* E_ϕ.
(3) $\mathrm{FK}(\phi)$*,* $\mathrm{J}_{\phi,\le}$ *and* $\mathrm{CSJ}(\phi)$.

As is remarked in [14], $\mathrm{UHK}(\phi)$ alone does not imply the conservativeness of the associated Dirichlet form $(\mathscr{E},\mathscr{F})$.

We note that there are two other independent related work around the same time. In [29], stability of discrete-time long range random walks of stable-like jumps is studied on infinite connected locally finite graphs. In [23], stability of stable-like pure-jump processes is studied on metric measure spaces. In both papers, they obtain the stability results under the condition that $\phi(r)=r^\alpha$ and that (M,d,μ) is an Alhfors d-regular set.

2.3 Parabolic Harnack inequalities

In this subsection, we assume that for each $x\in M$, there is a kernel $J(x,dy)$ so that

$$J(dx,dy)=J(x,dy)\,\mu(dx).$$

Let $Z:=\{V_s,X_s\}_{s\ge0}$ be the space-time process corresponding to X, where $V_s=V_0-s$. We denote by $\{\widetilde{\mathscr{F}_s};s\ge0\}$ the filtration generated by Z satisfying the usual conditions. The law of the space-time process $s\mapsto Z_s$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. Define $\tau_D=\inf\{s>0:Z_s\notin D\}$ for every open subset D of $[0,\infty)\times M$. A set $A\subset[0,\infty)\times M$ is said to be nearly Borel measurable if for any probability measure μ on $[0,\infty)\times M$, there are Borel measurable subsets A_1, A_2 of $[0,\infty)\times M$ so that $A_1\subset A\subset A_2$ and that $\mathbb{P}^\mu(Z_t\in A_2\setminus A_1$ for some $t\ge0)=0$. Nearly Borel measurable σ-field is the collection of all nearly Borel measurable subsets of $[0,\infty)\times M$.

Definition 6. (i) We say that a nearly Borel measurable function $u(t,x)$ on $[0,\infty)\times M$ is *parabolic* (or *caloric*) on $D=(a,b)\times B(x_0,r)$ for the process X if there is a properly exceptional set $\mathscr{N}_u$ of the process X so that for every relatively compact open subset U of D, $u(t,x)=\mathbb{E}^{(t,x)}u(Z_{\tau_U})$ for every $(t,x)\in U\cap([0,\infty)\times(M\setminus\mathscr{N}_u))$.
(ii) A nearly Borel measurable function u on M is said to be *subharmonic* (resp. *harmonic, superharmonic*) in D (with respect to the process X) if for any relative-

ly compact subset $U \subset D$, $t \mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable submartingale (resp. martingale, supermartingale) under $\mathbb{P}^x$ for q.e. $x \in U$.
(iii) We say that the *parabolic Harnack inequality* $\mathrm{PHI}(\phi)$ holds for the process X, if there exist constants $0 < c_1 < c_2 < c_3 < c_4$, $0 < c_5 < 1$ and $c_6 > 0$ such that for every $x_0 \in M$, $t_0 \ge 0$, $R > 0$ and for every non-negative function $u = u(t,x)$ on $[0,\infty)\times M$ that is parabolic on cylinder $Q(t_0,x_0,c_4\phi(R),R) := (t_0,t_0+c_4\phi(R))\times B(x_0,R)$,

$$\text{ess sup}_{Q_-} u \le c_6 \,\text{ess inf}_{Q_+} u, \tag{15}$$

where $Q_- := (t_0+c_1\phi(R), t_0+c_2\phi(R))\times B(x_0,c_5R)$ and $Q_+ := (t_0+c_3\phi(R), t_0+c_4\phi(R))\times B(x_0,c_5R)$.

Note that the above definition of $\mathrm{PHI}(\phi)$ is called a weak parabolic Harnack inequality in [6], in the sense that (15) holds for some $c_1,\cdots,c_5$. The definition of a parabolic Harnack inequality in [6] is (15) valid for any choice of positive constants $c_4 > c_3 > c_2 > c_1 > 0$, $0 < c_5 < 1$ with $c_6 = c_6(c_1,\dots,c_5) < \infty$. Since our underlying metric measure space may not be geodesic, we cannot deduce parabolic Harnack inequality from weak parabolic Harnack inequality.

The following stability result for parabolic Harnack inequalities for symmetric pure-jump processes has been obtained in [15].

Theorem 4. *Assume that the metric measure space* (M,d,μ) *satisfies* VD *and* RVD*, and* ϕ *satisfies* (9)*. Let* $(\mathscr{E},\mathscr{F})$ *be a regular Dirichlet form on* $L^2(M;\mu)$ *of pure-jump type* (6)*. Let* $(\widetilde{\mathscr{E}},\mathscr{F})$ *be another regular Dirichlet form on* $L^2(M;\widetilde{\mu})$ *of pure-jump type* (6) *with jumping measure* $\widetilde{J}(dx,dy)$ *that satisfies* (13) *and* (14)*. Then* $\mathrm{PHI}(\phi)$ *holds for* $(\mathscr{E},\mathscr{F})$ *if and only if it holds for* $(\widetilde{\mathscr{E}},\mathscr{F})$*.*

In fact the above theorem is a direct consequence of the stable characterization of $\mathrm{PHI}(\phi)$ obtained in [15], which is stable under perturbations (13) and (14). A precise statement of the latter will be given below in Theorem 5(7).

Definition 7. (i) We say that the *parabolic Harnack inequality* $\mathrm{PHI}^+(\phi)$ holds for the process X, if Definition 6 (iii) holds for some constants $c_1 > 0$, $c_k = kc_1$ for $k = 2,3,4$, $0 < c_5 < 1$ and $c_6 > 0$.
(ii) We say that the *elliptic Harnack inequality* (EHI) holds for the process X, if there exist constants $c > 0$ and $\delta \in (0,1)$ such that for every $x_0 \in M$, $r > 0$ and for every non-negative function u on M that is harmonic in $B(x_0,r)$,

$$\text{ess sup}_{B(x_0,\delta r)} h \le c\,\text{ess inf}_{B(x_0,\delta r)} h.$$

(iii) We say that the *parabolic Hölder regularity* $\mathrm{PHR}(\phi)$ holds for the process X, if there exist constants $c > 0$, $\theta \in (0,1]$ and $\varepsilon \in (0,1)$ such that for every $x_0 \in M$, $t_0 \ge 0$, $r > 0$ and for every bounded measurable function $u = u(t,x)$ that is caloric in $Q(t_0,x_0,\phi(r),r)$, there is a properly exceptional set $\mathscr{N}_u \supset \mathscr{N}$ so that

$$|u(s,x)-u(t,y)| \le c\left(\frac{\phi^{-1}(|s-t|)+d(x,y)}{r}\right)^{\theta} \text{ess sup}_{[t_0,t_0+\phi(r)]\times M}|u| \tag{16}$$

for every $s,t \in (t_0, t_0+\phi(\varepsilon r))$ and $x,y \in B(x_0,\varepsilon r)\setminus \mathscr{N}_u$.
(iv) We say that the *elliptic Hölder regularity* (EHR) holds for the process X, if there exist constants $c>0$, $\theta\in(0,1]$ and $\varepsilon\in(0,1)$ such that for every $x_0\in M$, $r>0$ and for every bounded measurable function u on M that is harmonic in $B(x_0,r)$, there is a properly exceptional set $\mathscr{N}_u \supset \mathscr{N}$ so that

$$|u(x)-u(y)| \le c\left(\frac{d(x,y)}{r}\right)^{\theta} \operatorname{ess\,sup}_M |u| \tag{17}$$

for any x, $y\in B(x_0,\varepsilon r)\setminus \mathscr{N}_u$.

Note that in the definition of PHR(ϕ) (resp. EHR) if the inequality (16) (resp. (17)) holds for some $\varepsilon\in(0,1)$, then it holds for all $\varepsilon\in(0,1)$ (with possibly different constant c). See [15, Remark 1.13 (iv)].

Clearly $\mathrm{PHI}^+(\phi) \Longrightarrow \mathrm{PHI}(\phi) \Longrightarrow$ EHI and $\mathrm{PHR}(\phi) \Longrightarrow$ EHR.

In order to discuss stability of parabolic Harnack inequalities, we need some more definitions.

Definition 8. (i) We say that *lower bound near diagonal estimates for Dirichlet heat kernel* (NDL(ϕ)) hold, i.e. there exist $\varepsilon\in(0,1)$ and $c_1>0$ such that for any $x_0\in M$, $r>0$, $0<t\le\phi(\varepsilon r)$ and $B=B(x_0,r)$,

$$p^B(t,x,y) \ge \frac{c_1}{V(x_0,\phi^{-1}(t))}, \quad x,y\in B(x_0,\varepsilon\phi^{-1}(t))\cap M_0.$$

(ii) We say that the UJS holds if there is a symmetric function $J(x,y)$ so that $J(x,dy)=J(x,y)\,\mu(dy)$, and there is a constant $c>0$ such that for μ-a.e. $x,y\in M$ with $x\ne y$,

$$J(x,y) \le \frac{c}{V(x,r)}\int_{B(x,r)} J(z,y)\,\mu(dz) \quad \text{for every } 0<r\le d(x,y)/2.$$

(iii) We say that the (*weak*) *Poincaré inequality* (PI(ϕ)) holds if there exist constants $c>0$ and $\kappa\ge1$ such that for any ball $B_r=B(x,r)$ with $x\in M$ and for any $f\in\mathscr{F}_b$,

$$\int_{B_r}(f-\overline{f}_{B_r})^2\,d\mu \le c\phi(r)\int_{B_{\kappa r}\times B_{\kappa r}}(f(y)-f(x))^2\,J(dx,dy),$$

where $\overline{f}_{B_r}=\frac{1}{\mu(B_r)}\int_{B_r} f\,d\mu$ is the average value of f on B_r.

The following is the main result of [15].

Theorem 5. *Suppose that the metric measure space* (M,d,μ) *satisfies* VD *and* RVD*, and* ϕ *satisfies* (9)*. Then the following are equivalent:*
(1) PHI(ϕ).
(2) $\mathrm{PHI}^+(\phi)$.
(3) UHK(ϕ)*,* NDL(ϕ) *and* UJS.

(4) $\mathrm{NDL}(\phi)$ *and* UJS.
(5) $\mathrm{PHR}(\phi)$, $\mathrm{E}_{\phi,\leq}$ *and* UJS.
(6) EHR, E_ϕ *and* UJS.
(7) $\mathrm{PI}(\phi)$, $\mathrm{J}_{\phi,\leq}$, $\mathrm{CSJ}(\phi)$ *and* UJS.

We remark that any of the conditions above implies the conservativeness of the process X. As a corollary of Theorem 2 and Theorem 5 (noting that J_ϕ implies UJS), we have the following.

Corollary 1. *Suppose that the metric measure space* (M,d,μ) *satisfies* VD *and* RVD*, and* ϕ *satisfies* (9)*. Then*

$$\mathrm{HK}(\phi) \Longleftrightarrow \mathrm{PHI}(\phi) + \mathrm{J}_{\phi,\geq}.$$

Unlike the diffusion case (3), heat kernel estimates and parabolic Harnack inequalities are no longer equivalent for discontinuous Markov processes.

3 L^p-mean value inequality

In this section, we establish L^p-mean value inequality for every $p \in (0,2]$ for symmetric jump processes. See [8, 9, 25] for the recent study on elliptic Harnack inequalities and mean value inequalities of fractional Laplacian operators.

Definition 9. Let D be an open subset of M. A function f is said to be locally in $\mathscr{F}_D$, denoted as $f \in \mathscr{F}_D^{loc}$, if for every relatively compact subset U of D, there is a function $g \in \mathscr{F}_D$ such that $f = g$ m-a.e. on U. We say that a nearly Borel measurable function u on M is *$\mathscr{E}$-subharmonic* (resp. *$\mathscr{E}$-harmonic, $\mathscr{E}$-superharmonic*) in D if $u \in \mathscr{F}_D^{loc}$ that is locally bounded, and satisfies

$$\int_{U\times V^c} |u(y)|J(dx,dy) < \infty$$

for any relatively compact open sets U and V of M with $\bar{U} \subset V \subset \bar{V} \subset D$, and

$$\mathscr{E}(u,\varphi) \leq 0 \quad (\text{resp. } = 0, \geq 0)$$

for any $0 \leq \varphi \in \mathscr{F}_D$.

The following is established in [10, Theorem 2.11 and Lemma 2.3] first for harmonic functions, and then extended in [16, Theorem 2.9] to subharmonic functions.

Theorem 6. *Let D be an open subset of M, and let u be a bounded function. Then u is $\mathscr{E}$-harmonic* (*resp. $\mathscr{E}$-subharmonic*) *in D if and only if u is harmonic* (*resp. subharmonic*) *in D.*

Following [9, 14], we define the *nonlocal tail* $\mathrm{Tail}(u;x_0,r)$ of a Borel measurable function u on M in the complement of the ball $B(x_0,r)$ by

$$\mathrm{Tail}\,(u;x_0,r) := \phi(r)\int_{B(x_0,r)^c} \frac{|u(z)|}{V(x_0,d(x_0,z))\phi(d(x_0,z))}\,\mu(dz).$$

For simplicity, we denote $B(x_0,r)$ by $B_r(x_0)$. The following L^2-mean value inequality has been obtained in [14, Proposition 4.10].

Proposition 1. **(L^2-mean value inequality)** *Assume* VD, (9), FK(ϕ), CSJ(ϕ) *and* $\mathrm{J}_{\phi,\le}$ *hold. For any $x_0 \in M$ and $r > 0$, let u be a bounded $\mathscr{E}$-subharmonic in $B_r(x_0)$. Then there is a constant $c_0 > 0$ independent of x_0 and r so that*

$$\operatorname{ess\,sup}_{B_{r/2}(x_0)} u \le c_0 \left[\left(\frac{1}{V(x_0,r)}\int_{B_r(x_0)} u^2\,d\mu\right)^{1/2} + \mathrm{Tail}\,(u;x_0,r/2)\right]. \tag{18}$$

Using Proposition 1, we can establish the following L^p-mean value inequality for every $p \in (0,2)$ for bounded $\mathscr{E}$-subharmonic functions.

Theorem 7. **(L^p-mean value inequality with $p \in (0,2)$)** *Assume that* VD, (9), FK(ϕ), CSJ(ϕ) *and* $\mathrm{J}_{\phi,\le}$ *hold. For any $x_0 \in M$ and $r > 0$, let u be bounded and $\mathscr{E}$-subharmonic in $B_r(x_0)$ such that $u \ge 0$ on $B_r(x_0)$. Then for any $\sigma \in (0,1)$ and $p \in (0,2)$,*

$$\begin{aligned}\operatorname{ess\,sup}_{B_{\sigma r}(x_0)} u \le & \frac{c_0}{(1-\sigma)^{2(d_2+\beta_2-\beta_1)/p}} \\ & \times \left[\left(\frac{1}{V(x_0,r)}\int_{B_r(x_0)} |u|^p\,d\mu\right)^{1/p} + \mathrm{Tail}\,(u;x_0,r/2)\right],\end{aligned} \tag{19}$$

where β_1,β_2 are the constants in (9)*, d_2 is the exponent in* (8) *from* VD*, and $c_0 > 0$ is a constant independent of x_0, σ and r.*

Proof. To prove (19), it suffices to consider the case when $\sigma \ge 1/2$. In this case, for any $\sigma \le t < s \le 1$ and $z \in B_{tr}(x_0)$, applying Proposition 1 with $B_{(s-t)r}(z)$ playing the role of $B_r(x_0)$, we get that

$$u(z) \le c_1 \left[\frac{1}{(s-t)^{d_2/2}}\left(\frac{1}{V(x_0,sr)}\int_{B_{sr}(x_0)} u^2\,d\mu\right)^{1/2} + \mathrm{Tail}\,(u;z,(s-t)r/2)\right],$$

where we have used the facts that $B_{(s-t)r}(z) \subset B_{sr}(x_0)$ for any $z \in B_{tr}(x_0)$, and

$$\frac{V(x_0,sr)}{V(z,(s-t)r)} \le c'\left(1+\frac{d(x_0,z)+sr}{(s-t)r}\right)^{d_2} \le c''\left(1+\frac{tr+sr}{(s-t)r}\right)^{d_2} \le \frac{c'''}{(s-t)^{d_2}},$$

thanks to VD and (9).

Next, by splitting the integration domain of the integral in $\mathrm{Tail}\,(u;z,(s-t)r/2)$ into the sets $B_{r/2}(x_0)\backslash B_{(s-t)r/2}(z)$ and $M\backslash(B_{r/2}(x_0)\cup B_{(s-t)r/2}(z))$, we get that

$$\mathrm{Tail}\,(u;z,(s-t)r/2)$$

$$
\begin{aligned}
&=\phi((s-t)r/2)\int_{B_{r/2}(x_0)\setminus B_{(s-t)r/2}(z)}\frac{|u(y)|}{V(z,d(z,y))\phi(d(z,y))}\,\mu(dy)\\
&\quad+\phi((s-t)r/2)\int_{M\setminus(B_{r/2}(x_0)\cup B_{(s-t)r/2}(z))}\frac{|u(y)|}{V(z,d(z,y))\phi(d(z,y))}\,\mu(dy)\\
&\le\int_{B_{r/2}(x_0)\setminus B_{(s-t)r/2}(z)}\frac{|u(y)|}{V(z,d(z,y))}\,\mu(dy)\\
&\quad+\phi((s-t)r/2)\int_{M\setminus(B_{r/2}(x_0)\cup B_{(s-t)r/2}(z))}\frac{|u(y)|}{V(z,d(z,y))\phi(d(z,y))}\,\mu(dy)\\
&\le\frac{c_1}{(s-t)^{d_2}}\frac{1}{V(x_0,r/2)}\int_{B_{r/2}(x_0)}|u|\,d\mu+\frac{c_2}{(s-t)^{d_2+\beta_2-\beta_1}}\mathrm{Tail}\,(u;x_0,r/2)\\
&\le\frac{c_3}{(s-t)^{d_2+\beta_2-\beta_1}}\left[\frac{1}{V(x_0,sr)}\int_{B_{sr}(x_0)}|u|\,d\mu+\mathrm{Tail}\,(u;x_0,r/2)\right]\\
&\le\frac{c_3}{(s-t)^{d_2+\beta_2-\beta_1}}\left[\left(\frac{1}{V(x_0,sr)}\int_{B_{sr}(x_0)}u^2\,d\mu\right)^{1/2}+\mathrm{Tail}\,(u;x_0,r/2)\right],
\end{aligned}
$$

where in the second inequality we have used the following two facts that for any $z\in B_{tr}(x_0)$ and $y\in B_{r/2}(x_0)\setminus B_{(s-t)r/2}(z)$,

$$
\frac{V(x_0,r/2)}{V(z,d(z,y))}\le c_4\left(1+\frac{d(x_0,z)+r/2}{d(z,y)}\right)^{d_2}\le\frac{c_5}{(s-t)^{d_2}};
$$

for $z\in B_{tr}(x_0)$ and $y\notin B_{r/2}(x_0)\cup B_{(s-t)r/2}(z)$,

$$
\frac{V(x_0,d(x_0,y))\phi(d(x_0,y))}{V(z,d(z,y))\phi(d(z,y))}\le\frac{c_6}{(s-t)^{d_2+\beta_2}}
$$

and

$$
\frac{\phi((s-t)r/2)}{\phi(r/2)}\le c_7(s-t)^{\beta_1},
$$

due to VD and (9) again.

Combining both estimates above, we find that for any $1/2\le t\le s\le 1$,

$$
\mathrm{ess\,sup}_{B_{tr}(x_0)}u\le\frac{c_8}{(s-t)^{d_2+\beta_2-\beta_1}}\left[\left(\frac{1}{V(x_0,sr)}\int_{B_{sr}(x_0)}u^2\,d\mu\right)^{1/2}+\mathrm{Tail}\,(u;x_0,r/2)\right].
$$

Recall that $u\ge 0$ on $B_r(x_0)$. By VD and the standard Young inequality with exponents $2/(2-p)$ and $2/p$ for $0<p<2$, we know that for any $1/2\le t\le s\le 1$,

$$
\begin{aligned}
&(s-t)^{d_2+\beta_2-\beta_1}\left(\frac{1}{V(x_0,sr)}\int_{B_{sr}(x_0)}u^2\,d\mu\right)^{1/2}\\
&\le c_9(\mathrm{ess\,sup}_{B_{sr}(x_0)}u)^{(2-p)/2}\frac{1}{(s-t)^{d_2+\beta_2-\beta_1}}\left(\frac{1}{V(x_0,r)}\int_{B_r(x_0)}|u|^p\,d\mu\right)^{1/2}
\end{aligned}
$$

$$\leq \frac{1}{2}\text{ess sup}_{B_{sr}(x_0)}u + \frac{c_{10}}{(s-t)^{2(d_2+\beta_2-\beta_1)/p}}\left(\frac{1}{V(x_0,r)}\int_{B_r(x_0)}|u|^p\,d\mu\right)^{1/p}.$$

Thus, we have for any $0<p<2$ and $1/2\leq t\leq s\leq 1$,

$$\begin{aligned}\text{ess sup}_{B_{tr}(x_0)}u &\leq \frac{1}{2}\text{ess sup}_{B_{sr}(x_0)}u\\ &+\frac{c_{11}}{(s-t)^{2(d_2+\beta_2-\beta_1)/p}}\left[\left(\frac{1}{V(x_0,r)}\int_{B_r(x_0)}|u|^p\,d\mu\right)^{1/p}+\text{Tail}\,(u;x_0,r/2)\right].\end{aligned}$$

Therefore, the desired assertion (19) now follows from Lemma 1 below. □

The following lemma is taken from [19, Lemma 1.1], which is used in the proof of Theorem 7.

Lemma 1. *Let $f(t)$ be a non-negative bounded function defined for $0\leq T_0\leq t\leq T_1$. Suppose that for $T_0\leq t\leq s\leq T_1$ we have*

$$f(t)\leq A(s-t)^{-\alpha}+B+\theta f(s),$$

where A,B,α,θ are non-negative constants, and $\theta<1$. Then there exists a constant c depending only on α and θ such that for every $T_0\leq r\leq R\leq T_1$, we have

$$f(r)\leq c\Big(A(R-r)^{-\alpha}+B\Big).$$

Proof. Consider the sequence $\{t_i; i\geq 0\}$ defined by $t_0=r$ and $t_{i+1}=t_i+(1-\delta)\delta^i(R-r)$ with $\delta\in(0,1)$. By iteration

$$f(t_0)\leq\theta^k f(t_k)+\left(\frac{A}{(1-\delta)^\alpha}(R-r)^{-\alpha}+B\right)\sum_{i=0}^{k-1}\theta^i\delta^{-i\alpha}.$$

We now choose δ such that $\delta^{-\alpha}\theta<1$ and let $k\to\infty$, getting the desired assertion holds with $c=(1-\delta)^{-\alpha}(1-\theta\delta^{-\alpha})^{-1}$. □

References

1. S. Andres and M.T. Barlow. Energy inequalities for cutoff-functions and some applications. *J. Reine Angew. Math.* **699** (2015), 183–215.
2. D.G. Aronson. Bounds on the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.* **73** (1967), 890–1896.
3. M.T. Barlow and R.F. Bass. Stability of parabolic Harnack inequalities. *Trans. Amer. Math. Soc.* **356** (2003), 1501–1533.
4. M.T. Barlow, R.F. Bass and T. Kumagai. Stability of parabolic Harnack inequalities on metric measure spaces. *J. Math. Soc. Japan* **58** (2006), 485–519.
5. M.T. Barlow, R.F. Bass and T. Kumagai. Parabolic Harnack inequality and heat kernel estimates for random walks with long range jumps. *Math. Z.* **261** (2009), 297–320.

6. M.T. Barlow, A. Grigor'yan and T. Kumagai. On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan* **64** (2012), 1091–1146.
7. A. Björn and J. Björn. *Nonlinear Potential Theory on Metric Spaces*. EMS Tracts in Mathematics, vol. **17**, European Math. Soc., Zurich 2011.
8. A.D. Castro, T. Kuusi and G. Palatucci. Nonlocal Harnack inequalities. *J. Funct. Anal.* **267** (2014), 1807–1836.
9. A.D. Castro, T. Kuusi and G. Palatucci. Local behavior of fractional p-minimizers. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* **33** (2016), 1279–1299.
10. Z.-Q. Chen. On notions of harmonicity. *Proc. Amer. Math. Soc.* **137** (2009), 3497–3510.
11. Z.-Q. Chen and M. Fukushima. *Symmetric Markov Processes, Time Change, and Boundary Theory*. Princeton Univ. Press, Princeton 2012.
12. Z.-Q. Chen and T. Kumagai. Heat kernel estimates for stable-like processes on d-sets. *Stochastic Process Appl.* **108** (2003), 27–62.
13. Z.-Q. Chen and T. Kumagai. Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Relat. Fields* **140** (2008), 277–317.
14. Z.-Q. Chen, T. Kumagai and J. Wang. Stability of heat kernel estimates for symmetric non-local Dirichlet forms. Preprint 2016, available at `arXiv:1604.04035`
15. Z.-Q. Chen, T. Kumagai and J. Wang. Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms. Preprint 2016, available at `arXiv:1609.07594`
16. Z.-Q. Chen and K. Kuwae. On subhamonicity for symmetric Markov processes. *J. Math. Soc. Japan* **64** (2012), 1181-1209.
17. T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Math. Iberoamericana* **15** (1999), 181–232.
18. M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes*. de Gruyter, Berlin, 2nd rev. and ext. ed., 2011.
19. M. Giaquinta and E. Giusti. On the regularity of the minima of variational integrals. *Acta Math.* **148** (1982), 31–46.
20. D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp.
21. A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. (in Russian) *Matem. Sbornik.* **182** (1991), 55–87. (English transl.) *Math. USSR Sbornik* **72** (1992), 47–77.
22. A. Grigor'yan and J. Hu. Upper bounds of heat kernels on doubling spaces. *Mosco Math. J.* **14** (2014), 505–563.
23. A. Grigor'yan, E. Hu and J. Hu. Two-sided estimates of heat kernels of jump type Dirichlet forms. Preprint 2016, available at `https://www.math.uni-bielefeld.de/~grigor/gcap.pdf`.
24. A. Grigor'yan, J. Hu and K.-S. Lau. Generalized capacity, Harnack inequality and heat kernels on metric spaces. *J. Math. Soc. Japan* **67** (2015), 1485–1549.
25. T. Kuusi, G. Mingione and Y. Sire. Nonlocal equations with measure data. *Commun. Math. Phys.* **337** (2015), 1317–1368.
26. P. Li and S.-T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156** (1986) no. 3-4, 153–201.
27. Z.-M. Ma and M. Röckner. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Springer-Verlag, New York, 1992.
28. J. Moser. On Harnack's inequality for parabolic differential equations. *Comm. Pure Appl. Math.* **17** (1964), 101–134.
29. M. Murugan and L. Saloff-Coste. Heat kernel estimates for anomalous heavy-tailed random walks. Preprint 2015, available at `arXiv:1512.02361`.
30. L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Inter. Math. Res. Notices* **2** (1992), 27–38.
31. L. Saloff-Coste. *Aspects of Sobolev-type Inequalities.* Lond. Math. Soc. Lect. Notes, vol. **289**, Cambridge Univerisity Press, Cambridge 2002.
32. K.-T. Sturm. Analysis on local Dirichlet spaces -III. The parabolic Harnack inequality. *J. Math. Pures Appl.* **75** (1996), 273–297.
33. A. Torchinsky. *Real-variable Methods in Harmonic Analysis.* Academic Press, London 1986.