Stability of heat kernel estimates for symmetric jump processes on metric measure spaces

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Abstract

In this paper, we consider symmetric jump processes of mixed-type on metric measure spaces under general volume doubling condition, and establish stability of two-sided heat kernel estimates and heat kernel upper bounds. We obtain their stable equivalent characterizations in terms of the jumping kernels, variants of cut-off Sobolev inequalities, and the Faber-Krahn inequalities. In particular, we establish stability of heat kernel estimates for α-stable-like processes even with $\alpha \geq 2$ when the underlying spaces have walk dimensions larger than 2, which has been one of the major open problems in this area.

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1 Introduction and Main Results

1.1 Setting

Let (M, d) be a locally compact separable metric space, and let μ be a positive Radon measure on M with full support. We will refer to such a triple (M, d, μ) as a *metric measure space*, and denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(M; \mu)$. Throughout the paper, we assume for simplicity that $\mu(M) = \infty$. We would emphasize that in this paper we do not assume M to be connected nor (M, d) to be geodesic.

We consider a regular *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(M; \mu)$. By the Beurling-Deny formula, such form can be decomposed into three terms — the strongly local term, the pure-jump term and the killing term (see [FOT, Theorem 4.5.2]). Throughout this paper, we consider the form that consists of the pure-jump term only; namely there exists a symmetric Radon measure $J(\cdot, \cdot)$ on $M \times M \setminus \text{diag}$, where diag denotes the diagonal set $\{(x, x) : x \in M\}$, such that

$$\mathcal{E}(f, g) = \int_{M \times M \setminus \text{diag}} (f(x) - f(y)(g(x) - g(y))\, J(dx, dy), \quad f, g \in \mathcal{F}. \tag{1.1}$$

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Since $(\mathcal{E},\mathcal{F})$ is regular, each function $f\in\mathcal{F}$ admits a quasi-continuous version $\widetilde{f}$ on M (see [FOT, Theorem 2.1.3]). Throughout the paper, we will abuse notation and take the quasi-continuous version of f without writing $\tilde{f}$. Let $\mathcal{L}$ be the (negative definite) L^2-*generator* of $(\mathcal{E},\mathcal{F})$ on $L^2(M;\mu)$; this is, $\mathcal{L}$ is the self-adjoint operator in $L^2(M;\mu)$ such that

$$\mathcal{E}(f,g)=-\langle \mathcal{L}f,g\rangle \quad \text{for all } f\in\mathcal{D}(\mathcal{L}) \text{ and } g\in\mathcal{F}.$$

Let $\{P_t\}_{t\geq 0}$ be the associated *semigroup*. Associated with the regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(M;\mu)$ is an μ-symmetric *Hunt process* $X=\{X_t,t\geq 0;\mathbb{P}^x,x\in M\setminus\mathcal{N}\}$. Here $\mathcal{N}$ is a properly exceptional set for $(\mathcal{E},\mathcal{F})$ in the sense that $\mu(\mathcal{N})=0$ and $\mathbb{P}^x(X_t\in\mathcal{N}$ for some $t>0)=0$ for all $x\in M\setminus\mathcal{N}$. This Hunt process is unique up to a properly exceptional set — see [FOT, Theorem 4.2.8]. We fix X and $\mathcal{N}$, and write $M_0=M\setminus\mathcal{N}$. While the semigroup $\{P_t\}_{t\geq 0}$ associated with $\mathcal{E}$ is defined on $L^2(M;\mu)$, a more precise version with better regularity properties can be obtained, if we set, for any bounded Borel measurable function f on M,

$$P_tf(x)=\mathbb{E}^x f(X_t),\quad x\in M_0.$$

The *heat kernel* associated with the semigroup $\{P_t\}_{t\geq 0}$ (if it exists) is a measurable function $p(t,x,y): M_0\times M_0\to(0,\infty)$ for every $t>0$, such that

$$\mathbb{E}^x f(X_t)=P_tf(x)=\int p(t,x,y)f(y)\,\mu(dy),\quad x\in M_0, f\in L^\infty(M;\mu), \tag{1.2}$$

$$p(t,x,y)=p(t,y,x)\quad \text{for all } t>0,\ x,y\in M_0, \tag{1.3}$$

$$p(s+t,x,z)=\int p(s,x,y)p(t,y,z)\,\mu(dy)\quad \text{for all } s>0,t>0,\ x,z\in M_0. \tag{1.4}$$

While (1.2) only determines $p(t,x,\cdot)$ μ-a.e., using the Chapman-Kolmogorov equation (1.4) one can regularize $p(t,x,y)$ so that (1.2)–(1.4) hold for every point in M_0. See [BBCK, Theorem 3.1] and [GT, Section 2.2] for details. We call $p(t,x,y)$ the *heat kernel* on the *metric measure Dirichlet space* (or *MMD space*) $(M,d,\mu,\mathcal{E})$. By (1.2), sometime we also call $p(t,x,y)$ the *transition density function* with respect to the measure μ for the process X. Note that in some arguments of our paper, we can extend (without further mention) $p(t,x,y)$ to all x, $y\in M$ by setting $p(t,x,y)=0$ if x or y is outside M_0. The existence of the heat kernel allows to extend the definition of P_tf to all measurable functions f by choosing a Borel measurable version of f and noticing that the integral (1.2) does not change if function f is changed on a set of measure zero.

Denote the ball centered at x with radius r by $B(x,r)$ and $\mu(B(x,r))$ by $V(x,r)$. When the metric measure space M is an Alhfors d-regular set on $\mathbb{R}^n$ with $d\in(0,n]$ (that is, $V(x,r)\asymp r^d$ for $r\in(0,1]$), and the Radon measure $J(dx,dy)=J(x,y)\,\mu(dx)\,\mu(dy)$ for some non-negative symmetric function $J(x,y)$ such that

$$J(x,y)\asymp d(x,y)^{-(d+\alpha)},\quad x,y\in M \tag{1.5}$$

for some $0<\alpha<2$, it is established in [CK1] that the corresponding Markov process X has infinite lifetime, and has a jointly Hölder continuous transition density function $p(t,x,y)$ with respect to the measure μ, which enjoys the following two-sided estimate

$$p(t,x,y)\asymp t^{-d/\alpha}\wedge\frac{t}{d(x,y)^{d+\alpha}} \tag{1.6}$$

for any $(t,x,y) \in (0,1] \times M \times M$. Here for two positive functions f, g, notation $f \asymp g$ means f/g is bounded between two positive constants, and $a \wedge b := \min\{a,b\}$. Moreover, if M is a global d-set; that is, if $V(x,r) \asymp r^d$ holds for all $r > 0$, then the estimate (1.6) holds for all $(t,x,y) \in (0,\infty) \times M \times M$. We call the above Hunt process X an α-stable-like process on M. Note that when $M = \mathbb{R}^d$ and $J(x,y) = c|x-y|^{-(d+\alpha)}$ for some constants $\alpha \in (0,2)$ and $c > 0$, X is a rotationally symmetric α-stable Lévy process on $\mathbb{R}^d$. The estimate (1.6) can be regarded as the jump process counterpart of the celebrated Aronson estimates for diffusions. Since $J(x,y)$ is the weak limit of $p(t,x,y)/t$ as $t \to 0$, heat kernel estimate (1.6) implies (1.5). Hence the results from [CK1] give a stable characterization for α-stable-like heat kernel estimates when $\alpha \in (0,2)$ and the metric measure space M is a d-set for some constant $d > 0$. This result has later been extended to mixed stable-like processes [CK2] and to diffusions with jumps [CK3], with some growth condition on the rate function ϕ such as

$$\int_0^r \frac{s}{\phi(s)}\,ds \le c\frac{r^2}{\phi(r)} \quad \text{for } r > 0. \tag{1.7}$$

For α-stable-like processes where $\phi(r) = r^\alpha$, condition (1.7) corresponds exactly to $0 < \alpha < 2$. Some of the key methods used in [CK1] were inspired by a previous work [BL] on random walks on integer lattice $\mathbb{Z}^d$.

The notion of d-set arises in the theory of function spaces and in fractal geometry. Geometrically, self-similar sets are typical examples of d-sets. There are many self-similar fractals on which there exist fractal diffusions with walk dimension $d_w > 2$ (that is, diffusion processes with scaling relation $\textit{time} \approx \textit{space}^{d_w}$). This is the case, for example, for the Sierpinski gasket in $\mathbb{R}^n$ ($n \ge 2$) which is a d-set with $d = \log(n+1)/\log 2$ and has walk dimension $d_w = \log(n+3)/\log 2$, and for the Sierpinski carpet in $\mathbb{R}^n$ ($n \ge 2$) which is a d-set with $d = \log(3^n - 1)/\log 3$ and has walk dimension $d_w > 2$; see [B]. A direct calculation shows (see [BSS, Sto]) that the β-subordination of the fractal diffusions on these fractals are jump processes whose Dirichlet forms $(\mathcal{E}, \mathcal{F})$ are of the form given above with $\alpha = \beta d_w$ and their transition density functions have two-sided estimate (1.6). Note that as $\beta \in (0,1)$, $\alpha \in (0, d_w)$ so α can be larger than 2. When $\alpha > 2$, the approach in [CK1] ceases to work as it is hopeless to construct good cut-off functions a priori in this case. A long standing open problem in the field is whether estimate (1.6) holds for generic jump processes with jumping kernel of the form (1.5) for any $\alpha \in (0, d_w)$. A related open question is to find a characterization for heat kernel estimate (1.6) that is stable under "rough isometries". Do they hold on general metric measure spaces with volume doubling (VD) and reverse volume doubling (RVD) properties (see Definition 1.1 below for these two terminologies)? These are the questions we will address in this paper.

For diffusions on manifolds with walk dimension 2, a remarkable fundamental result obtained independently by Grigor'yan [Gr2] and Saloff-Coste [Sa] asserts that the following are equivalent: (i) Aronson-type Gaussian bounds for heat kernel, (ii) parabolic Harnack equality, and (iii) VD and Poincaré inequality. This result is then extended to strongly local Dirichlet forms on metric measure spaces in [BM, St1, St2] and to graphs in [De]. For diffusions on fractals with walk dimension larger than 2, the above equivalence still holds but one needs to replace (iii) by (iii') VD, Poincaré inequality and a cut-off Sobolev inequality; see [BB2, BBK1, AB]. For heat kernel estimates of symmetric jump processes in general metric measure spaces, as mentioned above, when $\alpha \in (0,2)$ and the metric measure space M is a d-set, characterizations of α-stable-like heat kernel estimates were obtained in [CK1] which are stable under rough isometries; see [CK2, CK3] for further extensions. For the equivalent characterizations of heat kernel estimates for symmetric jump processes analogous to the situation when $\alpha \ge 2$, there are some efforts such as [BGK1, Theorem 1.2] and [GHL2, Theorem 2.3] but none of these characterizations are stable under rough isometries. In [BGK1, Theorem 0.3], assuming that $(\mathcal{E}, \mathcal{F})$ is conservative, $V(x,r) \leq cr^d$ for some constant $d > 0$ and that $p(t,x,x) \leq ct^{-d/\alpha}$ for any $x \in M$ and $t > 0$, an equivalent characterization for the heat kernel upper bound estimate in (1.6) is given in terms of certain exit time estimates. Under the assumption that $(\mathcal{E}, \mathcal{F})$ is conservative, the Radon measure $J(dx, dy) = J(x,y)\,\mu(dx)\,\mu(dy)$ for some non-negative symmetric function $J(x,y)$, and $V(x,r) \leq cr^d$ for some constant $d > 0$, it is shown in [GHL2] that heat kernel upper bound estimate in (1.6) holds if and only if $p(t,x,x) \leq c_1 t^{d/\alpha}$, $J(x,y) \leq c_2 d(x,y)^{-(d+\alpha)}$, and the following survival estimate holds: there are constants $\delta, \varepsilon \in (0,1)$ so that $\mathbb{P}^x(\tau_{B(x,r)} \leq t) \leq \varepsilon$ for all $x \in M$, $r > 0$ and $t^{1/\alpha} \leq \delta r$. In both [BGK1, GHL2], α can be larger than 2. We note that when $\alpha < 2$, further equivalent characterizations of heat kernel estimates are given for jump processes on graphs [BBK2, Theorem 1.5], some of which are stable under rough isometries. Also, when the Dirichlet form of the jump process is parabolic (namely the capacity of any non-empty compact subset of M is positive [GHL2, Definition 6.3], which is equivalent to that every singleton has positive capacity), an equivalent characterization of heat kernel estimates is given in [GHL2, Theorem 6.17], which is stable under rough isometries.

1.2 Heat kernel

In this paper, we are concerned with both upper bound and two-sided estimates on $p(t,x,y)$ for mixed stable-like processes on general metric measure spaces including α-stable-like processes with $\alpha \geq 2$. To state our results precisely, we need a number of definitions.

Definition 1.1. (i) We say that (M,d,μ) satisfies the *volume doubling property* (VD) if there exists a constant $C_\mu \geq 1$ such that for all $x \in M$ and $r > 0$,

$$V(x,2r) \leq C_\mu V(x,r). \tag{1.8}$$

(ii) We say that (M,d,μ) satisfies the *reverse volume doubling property* (RVD) if there exist constants $d_1 > 0$, $c_\mu > 0$ such that for all $x \in M$ and $0 < r \leq R$,

$$\frac{V(x,R)}{V(x,r)} \geq c_\mu \Big(\frac{R}{r}\Big)^{d_1}. \tag{1.9}$$

VD condition (1.8) is equivalent to the existence of $d_2 > 0$ and $\widetilde{C}_\mu > 0$ so that

$$\frac{V(x,R)}{V(x,r)} \leq \widetilde{C}_\mu \Big(\frac{R}{r}\Big)^{d_2} \quad \text{for all } x \in M \text{ and } 0 < r \leq R, \tag{1.10}$$

while RVD condition (1.9) is equivalent to the existence of $l_\mu > 1$ and $\widetilde{c}_\mu > 1$ so that

$$V(x, l_\mu r) \geq \widetilde{c}_\mu V(x,r) \quad \text{for all } x \in M \text{ and } r > 0. \tag{1.11}$$

Since μ has full support on M, we have $\mu(B(x,r)) > 0$ for every $x \in M$ and $r > 0$. Under VD condition, we have from (1.10) that for all $x \in M$ and $0 < r \leq R$,

$$\frac{V(x,R)}{V(y,r)} \leq \frac{V(y, d(x,y)+R)}{V(y,r)} \leq \widetilde{C}_\mu \Big(\frac{d(x,y)+R}{r}\Big)^{d_2}. \tag{1.12}$$

On the other hand, under RVD, we have from (1.11) that

$$\mu\big(B(x_0, l_\mu r) \setminus B(x_0, r)\big) > 0 \quad \text{for each } x_0 \in M \text{ and } r > 0.$$

It is known that VD implies RVD if M is connected and unbounded. See, for example [GH, Proposition 5.1 and Corollary 5.3].

Let $\mathbb{R}_+ := [0, \infty)$, and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing continuous function with $\phi(0) = 0$, $\phi(1) = 1$ and satisfying that there exist constants $c_1, c_2 > 0$ and $\beta_2 \geq \beta_1 > 0$ such that

$$c_1 \Big(\frac{R}{r}\Big)^{\beta_1} \leq \frac{\phi(R)}{\phi(r)} \ \leq \ c_2 \Big(\frac{R}{r}\Big)^{\beta_2} \quad \text{for all } 0 < r \leq R. \tag{1.13}$$

Note that (1.13) is equivalent to the existence of constants $c_3, l_0 > 1$ such that

$$c_3^{-1} \phi(r) \leq \phi(l_0 r) \leq c_3 \, \phi(r) \quad \text{for all } r > 0.$$

Definition 1.2. We say J_ϕ holds if there exists a non-negative symmetric function $J(x, y)$ so that for $\mu \times \mu$-almost all $x, y \in M$,

$$J(dx, dy) = J(x, y) \, \mu(dx) \, \mu(dy), \tag{1.14}$$

and

$$\frac{c_1}{V(x, d(x, y))\phi(d(x, y))} \leq J(x, y) \leq \frac{c_2}{V(x, d(x, y))\phi(d(x, y))} \tag{1.15}$$

We say that $\mathrm{J}_{\phi,\leq}$ (resp. $\mathrm{J}_{\phi,\geq}$) if (1.14) holds and the upper bound (resp. lower bound) in (1.15) holds.

Remark 1.3. (i) Since changing the value of $J(x, y)$ on a subset of $M \times M$ having zero $\mu \times \mu$-measure does not affect the definition of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M; \mu)$, without loss of generality, we may and do assume that in condition J_ϕ ($\mathrm{J}_{\phi,\geq}$ and $\mathrm{J}_{\phi,\leq}$, respectively) that (1.15) (and the corresponding inequality) holds for every $x, y \in M$. In addition, by the symmetry of $J(\cdot, \cdot)$, we may and do assume that $J(x, y) = J(y, x)$ for all $x, y \in M$.

(ii) Note that, under VD, for every $\lambda > 0$, there are constants $0 < c_1 < c_2$ so that for every $r > 0$,

$$c_1 V(y, r) \leq V(x, r) \leq c_2 V(y, r) \quad \text{for } x, y \in M \text{ with } d(x, y) \leq \lambda r. \tag{1.16}$$

Indeed, by (1.12), we have for every $r > 0$ and $x, y \in M$ with $d(x, y) \leq \lambda r$,

$$\widetilde{C}_\mu^{-1} (1 + \lambda)^{-d_2} \leq \frac{V(x, r)}{V(y, r)} \leq \widetilde{C}_\mu (1 + \lambda)^{d_2}.$$

Taking $\lambda = 1$ and $r = d(x, y)$ in (1.16) shows that, under VD the bounds in condition (1.15) are consistent with the symmetry of $J(x, y)$.

Definition 1.4. Let $U \subset V$ be open sets of M with $U \subset \overline{U} \subset V$. We say a non-negative bounded measurable function φ is a *cut-off function for* $U \subset V$, if $\varphi = 1$ on U, $\varphi = 0$ on V^c and $0 \leq \varphi \leq 1$ on M.

For $f, g \in \mathcal{F}$, we define the carré du-Champ operator $\Gamma(f, g)$ for the non-local Dirichlet form $(\mathcal{E}, \mathcal{F})$ by

$$\Gamma(f, g)(dx) = \int_{y \in M} (f(x) - f(y))(g(x) - g(y)) \, J(dx, dy).$$

Clearly $\mathcal{E}(f, g) = \Gamma(f, g)(M)$.

Let $\mathcal{F}_b = \mathcal{F} \cap L^\infty(M,\mu)$. It can be verified (see [CKS, Lemma 3.5 and Theorem 3.7]) that for any $f \in \mathcal{F}_b$, $\Gamma(f,f)$ is the unique Borel measure (called the *energy measure*) on M satisfying

$$\int_M g\, d\Gamma(f,f) = \mathcal{E}(f,fg) - \frac{1}{2}\mathcal{E}(f^2,g), \quad f,g \in \mathcal{F}_b.$$

Note that the following chain rule holds: for $f,g,h \in \mathcal{F}_b$,

$$\int_M d\Gamma(fg,h) = \int_M f\, d\Gamma(g,h) + \int_M g\, d\Gamma(f,h).$$

Indeed, this can be easily seen by the following equality

$$f(x)g(x) - f(y)g(y) = f(x)(g(x)-g(y)) + g(y)(f(x)-f(y)), \quad x,y \in M.$$

We now introduce a condition that controls the energy of cut-off functions.

Definition 1.5. Let ϕ be an increasing function on $\mathbb{R}_+$.

(i) **(Condition** $\mathrm{CSJ}(\phi)$**)** We say that condition $\mathrm{CSJ}(\phi)$ holds if there exist constants $C_0 \in (0,1]$ and $C_1, C_2 > 0$ such that for every $0 < r \le R$, almost all $x_0 \in M$ and any $f \in \mathcal{F}$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0,R) \subset B(x_0,R+r)$ so that the following holds:

$$\begin{aligned}\int_{B(x_0,R+(1+C_0)r)} f^2\, d\Gamma(\varphi,\varphi) \le & C_1 \int_{U\times U^*} (f(x)-f(y))^2\, J(dx,dy) \\ & + \frac{C_2}{\phi(r)} \int_{B(x_0,R+(1+C_0)r)} f^2\, d\mu,\end{aligned} \tag{1.17}$$

where $U = B(x_0,R+r) \setminus B(x_0,R)$ and $U^* = B(x_0,R+(1+C_0)r) \setminus B(x_0,R-C_0r)$.

(ii) **(Condition** $\mathrm{SCSJ}(\phi)$**)** We say that condition $\mathrm{SCSJ}(\phi)$ holds if there exist constants $C_0 \in (0,1]$ and $C_1, C_2 > 0$ such that for every $0 < r \le R$ and almost all $x_0 \in M$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0,R) \subset B(x_0,R+r)$ so that (1.17) holds for any $f \in \mathcal{F}$.

Clearly $\mathrm{SCSJ}(\phi) \Longrightarrow \mathrm{CSJ}(\phi)$.

Remark 1.6. (i) $\mathrm{SCSJ}(\phi)$ is a modification of $\mathrm{CSA}(\phi)$ that was introduced in [AB] for strongly local Dirichlet forms as a weaker version of the so called cut-off Sobolev inequality $\mathrm{CS}(\phi)$ in [BB2, BBK1]. For strongly local Dirichlet forms the inequality corresponding to $\mathrm{CSJ}(\phi)$ is called generalized capacity condition in [GHL3]. As we will see in Theorem 1.15 below, $\mathrm{SCSJ}(\phi)$ and $\mathrm{CSJ}(\phi)$ are equivalent under $\mathrm{FK}(\phi)$ (see Definition 1.8 below) and $\mathrm{J}_{\phi,\le}$.

(ii) The main difference between $\mathrm{CSJ}(\phi)$ here and $\mathrm{CSA}(\phi)$ in [AB] is that the integrals in the left hand side and in the second term of the right hand side of the inequality (1.17) are over $B(x,R+(1+C_0)r)$ (containing U^*) instead of over U for [AB]. Note that the integral over U^c is zero in the left hand side of (1.17) for the case of strongly local Dirichlet forms. As we see in the arguments of the stability of heat kernel estimates for jump processes, it is important to fatten the annulus and integrate over U^* rather than over U. Another difference from $\mathrm{CSA}(\phi)$ is that in [AB] the first term of the right hand side is $\frac{1}{8}\int_U \varphi^2\, d\Gamma(f,f)$. However, we will prove in Proposition 2.4 that $\mathrm{CSJ}(\phi)$ implies the stronger inequality $\mathrm{CSJ}(\phi)_+$ under some regular conditions VD, (1.13) and $\mathrm{J}_{\phi,\le}$. See [AB, Lemma 5.1] for the case of strongly local Dirichlet forms.

(iii) As will be proved in Proposition 2.3 (iv), under VD and (1.13), if (1.17) holds for some $C_0 > 0$, then it holds for all $C_0' \ge C_0$ (with possibly different $C_2 > 0$).

(iv) By the definition above, it is clear that if $\phi_1 \le \phi_2$, then CSJ(ϕ_2) implies CSJ(ϕ_1).

Remark 1.7. Under VD, (1.13) and $\mathrm{J}_{\phi,\le}$, SCSJ(ϕ) always holds if $\beta_2 < 2$, where β_2 is the exponent in (1.13). In particular, SCSJ(ϕ) holds for $\phi(r) = r^\alpha$ with $\alpha < 2$. Indeed, for any fixed $x_0 \in M$ and $r, R > 0$, we choose a non-negative cut-off function $\varphi(x) = h(d(x_0, x))$, where $h \in C^1([0,\infty))$ such that $0 \le h \le 1$, $h(s) = 1$ for all $s \le R$, $h(s) = 0$ for $s \ge R + r$ and $h'(s) \le 2/r$ for all $s \ge 0$. Then, by $\mathrm{J}_{\phi,\le}$, for almost every $x \in M$,

$$\begin{aligned}
\frac{d\Gamma(\varphi,\varphi)}{d\mu}(x) &= \int (\varphi(x) - \varphi(y))^2 J(x,y)\,\mu(dy) \\
&\le \int_{\{d(x,y)\ge r\}} J(x,y)\,\mu(dy) + \frac{4}{r^2}\int_{\{d(x,y)\le r\}} d(x,y)^2 J(x,y)\,\mu(dy) \\
&\le \int_{\{d(x,y)\ge r\}} J(x,y)\,\mu(dy) + \frac{4}{r^2}\sum_{i=0}^{\infty}\int_{\{2^{-i-1}r<d(x,y)\le 2^{-i}r\}} d(x,y)^2 J(x,y)\,\mu(dy) \\
&\le \frac{c_1}{\phi(r)} + \frac{c_1}{r^2}\sum_{i=0}^{\infty}\frac{V(x,2^{-i}r)2^{-2i}r^2}{V(x,2^{-i-1}r)\phi(2^{-i-1}r)} \\
&\le \frac{c_1}{\phi(r)} + \frac{c_2}{\phi(r)}\sum_{i=0}^{\infty} 2^{-i(2-\beta_2)} \le \frac{c_3}{\phi(r)},
\end{aligned}$$

where in the third inequality we have used Lemma 2.1 below, and the forth inequality is due to VD and (1.13). Thus (1.17) holds.

We next introduce the Faber-Krahn inequality, see [GT, Section 3.3] for more details. For $\lambda > 0$, we define

$$\mathcal{E}_\lambda(f,g) = \mathcal{E}(f,g) + \lambda\int_M f(x)g(x)\,\mu(dx) \quad \text{for } f,g \in \mathcal{F}.$$

For any open set $D \subset M$, $\mathcal{F}_D$ is defined to be the $\mathcal{E}_1$-closure in $\mathcal{F}$ of $\mathcal{F} \cap C_c(D)$. Define

$$\lambda_1(D) = \inf\{\mathcal{E}(f,f) : f \in \mathcal{F}_D \text{ with } \|f\|_2 = 1\}, \tag{1.18}$$

the bottom of the Dirichlet spectrum of $-\mathcal{L}$ on D.

Definition 1.8. The MMD space $(M, d, \mu, \mathcal{E})$ satisfies the *Faber-Krahn inequality* FK(ϕ), if there exist positive constants C and ν such that for any ball $B(x,r)$ and any open set $D \subset B(x,r)$,

$$\lambda_1(D) \ge \frac{C}{\phi(r)}(V(x,r)/\mu(D))^\nu. \tag{1.19}$$

We remark that since $V(x,r) \ge \mu(D)$ for $D \subset B(x,r)$, if (1.19) holds for some $\nu = \nu_0 > 0$, it holds for every $\nu \in (0, \nu_0)$. So without loss of generality, we may and do assume $0 < \nu < 1$.

Recall that $X = \{X_t\}$ is the Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M;\mu)$ with proper exceptional set $\mathcal{N}$, and $M_0 := M \setminus \mathcal{N}$. For a set $A \subset M$, define the exit time $\tau_A = \inf\{t > 0 : X_t \in A^c\}$.

Definition 1.9. We say that E_ϕ holds if there is a constant $c_1 > 1$ such that for all $r > 0$ and all $x \in M_0$,

$$c_1^{-1}\phi(r) \le \mathbb{E}^x[\tau_{B(x,r)}] \le c_1\phi(r).$$

We say that $\mathrm{E}_{\phi,\le}$ (resp. $\mathrm{E}_{\phi,\ge}$) holds if the upper bound (resp. lower bound) in the inequality above holds.

Under (1.13), it is easy to see that $\mathrm{E}_{\phi,\ge}$ and $\mathrm{E}_{\phi,\le}$ imply the following statements respectively:

$$\begin{aligned} &\mathbb{E}^y[\tau_{B(x,r)}] \ge c_2\phi(r) \quad \text{for all } x \in M,\ y \in B(x,r/2)\cap M_0,\ r>0; \\ &\mathbb{E}^y[\tau_{B(x,r)}] \le c_3\phi(r) \quad \text{for all } x \in M,\ y \in M_0,\ r>0. \end{aligned}$$

Indeed, for $y \in B(x,r/2)\cap M_0$, we have $\mathbb{E}^y[\tau_{B(x,r)}] \ge \mathbb{E}^y[\tau_{B(y,r/2)}] \ge c_1^{-1}\phi(r/2) \ge c_2\phi(r)$. Similarly, for $y \in B(x,r)\cap M_0$, we have $\mathbb{E}^y[\tau_{B(x,r)}] \le \mathbb{E}^y[\tau_{B(y,2r)}] \le c_1\phi(2r) \le c_3\phi(r)$ (and $\mathbb{E}^y[\tau_{B(x,r)}] = 0$ for $y \in M_0 \setminus B(x,r)$).

Definition 1.10. We say $\mathrm{EP}_{\phi,\le}$ holds if there is a constant $c > 0$ such that for all $r, t > 0$ and all $x \in M_0$,

$$\mathbb{P}^x(\tau_{B(x,r)} \le t) \le \frac{ct}{\phi(r)}.$$

We say $\mathrm{EP}_{\phi,\le,\varepsilon}$ holds, if there exist constants $\varepsilon, \delta \in (0,1)$ such that for any ball $B = B(x_0,r)$ with radius $r > 0$,

$$\mathbb{P}^x(\tau_B \le \delta\phi(r)) \le \varepsilon \quad \text{for all } x \in B(x_0, r/4)\cap M_0.$$

It is clear that $\mathrm{EP}_{\phi,\le}$ implies $\mathrm{EP}_{\phi,\le,\varepsilon}$. We will prove in Lemma 4.16 below that under (1.13), E_ϕ implies $\mathrm{EP}_{\phi,\le,\varepsilon}$.

Definition 1.11. (i) We say that $\mathrm{HK}(\phi)$ holds if there exists a heat kernel $p(t,x,y)$ of the semigroup $\{P_t\}$ associated with $(\mathcal{E},\mathcal{F})$, which has the following estimates for all $t > 0$ and all $x, y \in M_0$,

$$\begin{aligned} &c_1\Big(\frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))}\Big) \\ &\le\ p(t,x,y) \le c_2\Big(\frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))}\Big), \end{aligned} \tag{1.20}$$

where $c_1, c_2 > 0$ are constants independent of $x, y \in M_0$ and $t > 0$. Here the inverse function of the strictly increasing function $t \mapsto \phi(t)$ is denoted by $\phi^{-1}(t)$.

(ii) We say $\mathrm{UHK}(\phi)$ (resp. $\mathrm{LHK}(\phi)$) holds if the upper bound (resp. the lower bound) in (1.20) holds.

(iii) We say $\mathrm{UHKD}(\phi)$ holds if there is a constant $c > 0$ such that for all $t > 0$ and all $x \in M_0$,

$$p(t,x,x) \le \frac{c}{V(x,\phi^{-1}(t))}.$$

Remark 1.12. We have three remarks about this definition.

(i) First, note that under VD

$$\frac{1}{V(y,\phi^{-1}(t))}\wedge\frac{t}{V(y,d(x,y))\phi(d(x,y))}\asymp\frac{1}{V(x,\phi^{-1}(t))}\wedge\frac{t}{V(x,d(x,y))\phi(d(x,y))}. \quad (1.21)$$

Therefore we can replace $V(x,d(x,y))$ by $V(y,d(x,y))$ in (1.20) by modifying the values of c_1 and c_2. This is because

$$\frac{1}{V(x,\phi^{-1}(t))}\le\frac{t}{V(x,d(x,y))\phi(d(x,y))}$$

if and only if $d(x,y)\le\phi^{-1}(t)$, and by (1.12),

$$\tilde{C_\mu}^{-1}\Big(1+\frac{d(x,y)}{\phi^{-1}(t)}\Big)^{-d_2}\le\frac{V(x,\phi^{-1}(t))}{V(y,\phi^{-1}(t))}\le\tilde{C_\mu}\Big(1+\frac{d(x,y)}{\phi^{-1}(t)}\Big)^{d_2}.$$

This together with (1.16) yields (1.21).

(ii) By the Cauchy-Schwarz inequality, one can easily see that UHKD(ϕ) is equivalent to the existence of $c_1>0$ so that

$$p(t,x,y)\le\frac{c_1}{\sqrt{V(x,\phi^{-1}(t))V(y,\phi^{-1}(t))}}\quad\text{for } x,y\in M_0 \text{ and } t>0.$$

Consequently, by Remark 1.3(ii), under VD, UHKD(ϕ) implies that for every $c_1>0$ there is a constant $c_2>0$ so that

$$p(t,x,y)\le\frac{c_2}{V(x,\phi^{-1}(t))}\quad\text{for } x,y\in M_0 \text{ with } d(x,y)\le c_1\phi^{-1}(t).$$

(iii) It will be implied by Theorem 1.13 and Lemma 5.6 below that if VD, (1.13) and HK(ϕ) hold, then the heat kernel $p(t,x,y)$ is Hölder continuous on (x,y) for every $t>0$, and so (1.20) holds for all $x,y\in M$.

In the following, we say $(\mathcal{E},\mathcal{F})$ *is conservative* if its associated Hunt process X has infinite lifetime. This is equivalent to $P_t1=1$ a.e. on M_0 for every $t>0$. It follows from Proposition 3.1(ii) that LHK(ϕ) implies that $(\mathcal{E},\mathcal{F})$ is conservative. We can now state the stability of the heat kernel estimates HK(ϕ). The following is the main result of this paper.

Theorem 1.13. *Assume that the metric measure space (M,d,μ) satisfies* VD *and* RVD*, and ϕ satisfies* (1.13)*. Then the following are equivalent:*
(1) HK(ϕ).
(2) J_ϕ *and* E_ϕ.
(3) J_ϕ *and* SCSJ(ϕ).
(4) J_ϕ *and* CSJ(ϕ).

Remark 1.14. (i) When ϕ satisfies (1.13) with $\beta_2<2$, by Remark 1.7, SCSJ(ϕ) holds and so in this case we have by Theorem 1.13 that HK(ϕ) $\iff$ J_ϕ. Thus Theorem 1.13 not only recovers but also extends the main results in [CK1, CK2] except for the cases where $J(x,y)$ decays exponentially when $d(x,y)$ is large, in the sense that the underlying spaces here are general metric measure spaces satisfying VD and RVD.

(ii) A new point of Theorem 1.13 is that it gives us the stability of heat kernel estimates for general symmetric jump processes of mixed-type, including α-stable-like processes with $\alpha \geq 2$, on general metric measure spaces when the underlying spaces have walk dimension larger than 2. In particular, if (M, d, μ) is a metric measure space on which there is an anomalous diffusion with walk dimension $d_w > 2$ such as Sierpinski gaskets or carpets, one can deduce from the subordinate anomalous diffusion the two-sided heat kernel estimates of any symmetric jump processes with jumping kernel $J(x, y)$ of α-stable type or mixed stable type; see Section 6 for details. This in particular answers a long standing problem in the field.

In the process of establishing Theorem 1.13, we also obtain the following characterizations for UHK(ϕ).

Theorem 1.15. *Assume that the metric measure space* (M, d, μ) *satisfies* VD *and* RVD*, and* ϕ *satisfies* (1.13)*. Then the following are equivalent:*
(1) UHK(ϕ) *and* $(\mathcal{E}, \mathcal{F})$ *is conservative.*
(2) UHKD(ϕ)*,* $\mathrm{J}_{\phi,\leq}$ *and* E_ϕ*.*
(3) FK(ϕ)*,* $\mathrm{J}_{\phi,\leq}$ *and* SCSJ(ϕ)*.*
(4) FK(ϕ)*,* $\mathrm{J}_{\phi,\leq}$ *and* CSJ(ϕ)*.*

We point out that UHK(ϕ) alone does not imply the conservativeness of the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$. For example, censored (also called resurrected) α-stable processes in upper half spaces with $\alpha \in (1, 2)$ enjoy UHK(ϕ) with $\phi(r) = r^\alpha$ but have finite lifetime; see [CT, Theorem 1.2]. We also note that RVD are only used in the proofs of UHKD(ϕ) $\Longrightarrow$ FK(ϕ) and $\mathrm{J}_{\phi,\geq} \Longrightarrow$ FK(ϕ).

We emphasize again that in our main results above, the underlying metric measure space (M, d, μ) is only assumed to satisfy the general VD and RVD. Neither uniform VD nor uniform RVD property is assumed. We do not assume M to be connected nor (M, d) to be geodesic.

As mentioned earlier, parabolic Harnack inequality is equivalent to the two-sided Aronson type heat kernel estimates for diffusion processes. In a subsequent paper [CKW], we study stability of parabolic Harnack inequality for symmetric jump processes on metric measure spaces.

Definition 1.16. (i) We say that a Borel measurable function $u(t, x)$ on $[0, \infty) \times M$ is *parabolic* (or *caloric*) on $D = (a, b) \times B(x_0, r)$ for the process X if there is a properly exceptional set $\mathcal{N}_u$ associated with the process X so that for every relatively compact open subset U of D, $u(t, x) = \mathbb{E}^{(t,x)} u(Z_{\tau_U})$ for every $(t, x) \in U \cap ([0, \infty) \times (M \backslash \mathcal{N}_u))$.

(ii) We say that the *parabolic Harnack inequality* (PHI(ϕ)) holds for the process X, if there exist constants $0 < C_1 < C_2 < C_3 < C_4$, $C_5 > 1$ and $C_6 > 0$ such that for every $x_0 \in M$, $t_0 \geq 0$, $R > 0$ and for every non-negative function $u = u(t, x)$ on $[0, \infty) \times M$ that is parabolic on cylinder $Q(t_0, x_0, \phi(C_4 R), C_5 R) := (t_0, t_0 + \phi(C_4 R)) \times B(x_0, C_5 R)$,

$$\operatorname{ess\,sup}_{Q_-} u \leq C_6 \operatorname{ess\,inf}_{Q_+} u, \tag{1.22}$$

where $Q_- := (t_0 + \phi(C_1 R), t_0 + \phi(C_2 R)) \times B(x_0, R)$ and $Q_+ := (t_0 + \phi(C_3 R), t_0 + \phi(C_4 R)) \times B(x_0, R)$.

We note that the above PHI(ϕ) is called a weak parabolic Harnack inequality in [BGK2], in the sense that (1.22) holds for some $C_1, \cdots, C_5$. It is called a parabolic Harnack inequality in

[BGK2] if (1.22) holds for any choice of positive constants $C_1, \cdots, C_5$ with $C_6 = C_6(C_1, \dots, C_5) < \infty$. Since our underlying metric measure space may not be geodesic, one can not expect to deduce parabolic Harnack inequality from weak parabolic Harnack inequality.

As a consequence of Theorem 1.13 and various equivalent characterizations of parabolic Harnack inequality established in [CKW], we have the following.

Theorem 1.17. *Suppose that the metric measure space* (M, d, μ) *satisfies* VD *and* RVD*, and* ϕ *satisfies* (1.13)*. Then*

$$\mathrm{HK}(\phi) \Longleftrightarrow \mathrm{PHI}(\phi) + \mathrm{J}_{\phi,\geq}.$$

Thus for symmetric jump processes, parabolic Harnack inequality $\mathrm{PHI}(\phi)$ is strictly weaker than $\mathrm{HK}(\phi)$. This fact was proved for symmetric jump processes on graphs with $V(x,r) \asymp r^d$, $\phi(r) = r^\alpha$ for some $d \geq 1$ and $\alpha \in (0,2)$ in [BBK2, Theorem 1.5].

Some of the main results of this paper were presented at the 38th Conference on Stochastic Processes and their Applications held at the University of Oxford, UK from July 13-17, 2015 and at the International Conference on Stochastic Analysis and Related Topics held at Wuhan University, China from August 3-8, 2015. While we were at the final stage of finalizing this paper, we received a copy of [MS1, MS2] from M. Murugan. Stability of discrete-time long range random walks of stable-like jumps on infinite connected locally finite graphs is studied in [MS2]. Their results are quite similar to ours when specialized to the case of $\phi(r) = r^\alpha$ but the techniques and the settings are somewhat different. They work on discrete-time random walks on infinite connected locally finite graphs equipped with graph distance, while we work on continuous-time symmetric jump processes on general metric measure space and with much more general jumping mechanisms. Moreover, it is assumed in [MS2] that there is a constant $c \geq 1$ so that $c^{-1} \leq \mu(\{x\}) \leq c$ for every $x \in M$ and the d-set condition that there are constants $C \geq 1$ and $d_f > 0$ so that $C^{-1} r^{d_f} \leq V(x,r) \leq C r^{d_f}$ for every $x \in M$ and $r \geq 1$, while we only assume general VD and RVD. Technically, their approach is to generalize the so-called Davies' method (to obtain the off-diagonal heat kernel upper bound from the on-diagonal upper bound) to be applicable when $\alpha > 2$ under the assumption of cut-off Sobolev inequalities. Quite recently, we also learned from A. Grigor'yan [GHH] that they are also working on the same topic of this paper on metric measure spaces with the d-set condition and the conservativeness assumption on $(\mathcal{E}, \mathcal{F})$. Their results are also quite similar to ours, again specialized to the case of $\phi(r) = r^\alpha$, but the techniques are also somewhat different. Their approach [GHH] is to deduce a kind of weak Harnack inequalities first from J_ϕ and $\mathrm{CSJ}(\phi)$, which they call generalized capacity condition. They then obtain uniform Hölder continuity of harmonic functions, which plays the key role for them to obtain the near-diagonal lower heat kernel bound that corresponds to (3.2). As we see below, our approach is different from theirs. We emphasize here that in this paper we do not assume a priori that $(\mathcal{E}, \mathcal{F})$ is conservative.

The rest of the paper is organized as follows. In the next section, we present some preliminary results about $\mathrm{J}_{\phi,\leq}$ and $\mathrm{CSJ}(\phi)$. In particular, in Proposition 2.4 we show that the leading constant in $\mathrm{CSJ}(\phi)$ is self-improving. Sections 3, 4 and 5 are devoted to the proofs of (1) $\Longrightarrow$ (3), (4) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) in Theorems 1.13 and 1.15, respectively. Among them, Section 4 is the most difficult part, where in Subsection 4.2 we establish the Caccioppoli inequality and the L^p-mean value inequality for subharmonic functions associated with symmetric jump processes, and in Subsection 4.4 Meyer's decomposition is realized for jump processes in the VD setting. Both subsections are of interest in their own. In Section 6, some examples are given to illustrate the applications of our results, and a counterexample is also given to indicate that

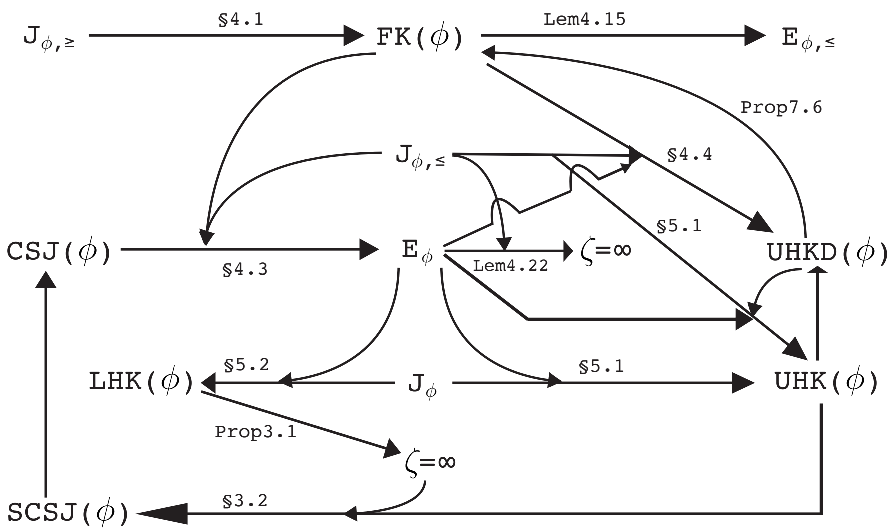

Figure 1: diagram

CSJ(ϕ) is necessary for HK(ϕ) in general setting. For reader's convenience, some known facts used in this paper are streamlined and collected in Subsections 7.1-7.4 of the Appendix. In connection with the implication of (3) $\Longrightarrow$ (1) in Theorem 1.15, we show in Subsection 7.5 that SCSJ(ϕ) + $\mathrm{J}_{\phi,\leq} \Longrightarrow (\mathcal{E},\mathcal{F})$ is conservative; in other words FK(ϕ) is not needed for establishing the conservativeness of $(\mathcal{E},\mathcal{F})$. We remark that, in order to increase the readability of the paper, we have tried to make the paper as self-contained as possible. Figure 1 illustrates implications of various conditions and flow of our proofs.

Throughout this paper, we will use c, with or without subscripts, to denote strictly positive finite constants whose values are insignificant and may change from line to line. For $p \in [1,\infty]$, we will use $\|f\|_p$ to denote the L^p-norm in $L^p(M;\mu)$. For $B = B(x_0, r)$ and $a > 0$, we use aB to denote the ball $B(x_0, ar)$.

2 Preliminaries

For basic properties and definitions related to Dirichlet forms, such as the relation between regular Dirichlet forms and Hunt processes, associated semigroups, resolvents, capacity and quasi-continuity, we refer the reader to [CF, FOT].

We begin with the following estimate, which is essentially given in [CK2, Lemma 2.1].

Lemma 2.1. *Assume that* VD *and* (1.13) *hold. Then there exists a constant* $c_0 > 0$ *such that*

$$\int_{B(x,r)^c} \frac{1}{V(x,d(x,y))\,\phi(d(x,y))}\,\mu(dy) \leq \frac{c_0}{\phi(r)} \quad \textit{for every } x \in M \textit{ and } r > 0. \tag{2.1}$$

Thus if, in addition, $\mathrm{J}_{\phi,\leq}$ *holds, then there exists a constant* $c_1 > 0$ *such that*

$$\int_{B(x,r)^c} J(x,y)\,\mu(dy) \leq \frac{c_1}{\phi(r)} \quad \textit{for every } x \in M \textit{ and } r > 0.$$

Proof. For completeness, we present a proof here. By $\mathrm{J}_{\phi,\le}$ and VD, we have for every $x \in M$ and $r > 0$,

$$\begin{aligned}&\int_{B(x,r)^c} \frac{1}{V(x,d(x,y))\,\phi(d(x,y))}\,\mu(dy)\\&=\sum_{i=0}^{\infty}\int_{B(x,2^{i+1}r)\setminus B(x,2^ir)} \frac{1}{V(x,d(x,y))\,\phi(d(x,y))}\,\mu(dy)\\&\le \sum_{i=0}^{\infty}\frac{1}{V(x,2^ir)\,\phi(2^ir)}V(x,2^{i+1}r)\\&\le c_2\sum_{i=0}^{\infty}\frac{1}{\phi(2^ir)}\le \frac{c_3}{\phi(r)}\sum_{i=0}^{\infty}2^{-i\beta_1}\le \frac{c_4}{\phi(r)},\end{aligned}$$

where the lower bound in (1.13) is used in the second to the last inequality. □

Fix $\rho > 0$ and define a bilinear form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ by

$$\mathcal{E}^{(\rho)}(u,v) = \int (u(x)-u(y))(v(x)-v(y))\mathbf{1}_{\{d(x,y)\le\rho\}}\,J(dx,dy). \tag{2.2}$$

Clearly, the form $\mathcal{E}^{(\rho)}(u,v)$ is well defined for $u,v \in \mathcal{F}$, and $\mathcal{E}^{(\rho)}(u,u) \le \mathcal{E}(u,u)$ for all $u \in \mathcal{F}$. Assume that VD, (1.13) and $\mathrm{J}_{\phi,\le}$ hold. Then we have by Lemma 2.1 that for all $u \in \mathcal{F}$,

$$\begin{aligned}\mathcal{E}(u,u)-\mathcal{E}^{(\rho)}(u,u) &= \int (u(x)-u(y))^2\mathbf{1}_{\{d(x,y)>\rho\}}\,J(dx,dy)\\&\le 4\int_M u^2(x)\,\mu(dx)\int_{B(x,\rho)^c} J(x,y)\,\mu(dy) \le \frac{c_0\|u\|_2^2}{\phi(\rho)}.\end{aligned}\tag{2.3}$$

Thus $\mathcal{E}_1(u,u)$ is equivalent to $\mathcal{E}_1^{(\rho)}(u,u) := \mathcal{E}^{(\rho)}(u,u) + \|u\|_2^2$ for every $u \in \mathcal{F}$. Hence $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M;\mu)$. Throughout this paper, we call $(\mathcal{E}^{(\rho)}, \mathcal{F})$ ρ-truncated Dirichlet form. The Hunt process associated with $(\mathcal{E}^{(\rho)}, \mathcal{F})$ can be identified in distribution with the Hunt process of the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ by removing those jumps of size larger than ρ.

Assume that $\mathrm{J}_{\phi,\le}$ holds, and in particular (1.14) holds. Define $J(x,dy) = J(x,y)\,\mu(dy)$. Let $J^{(\rho)}(dx,dy) = \mathbf{1}_{\{d(x,y)\le\rho\}}J(dx,dy)$, $J^{(\rho)}(x,dy) = \mathbf{1}_{\{d(x,y)\le\rho\}}J(x,dy)$, and $\Gamma^{(\rho)}(f,g)$ be the carré du-Champ operator of the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$; namely,

$$\mathcal{E}^{(\rho)}(f,g) = \int_M \mu(dx)\int_M (f(x)-f(y))(g(x)-g(y))\,J^{(\rho)}(x,dy) =: \int_M d\Gamma^{(\rho)}(f,g).$$

We now define variants of $\mathrm{CSJ}(\phi)$.

Definition 2.2. Let ϕ be an increasing function on $\mathbb{R}_+$ with $\phi(0) = 0$, and $C_0 \in (0,1]$. For any $x_0 \in M$ and $0 < r \le R$, set $U = B(x_0, R+r)\setminus B(x_0,R)$, $U^* = B(x_0, R+(1+C_0)r)\setminus B(x_0, R-C_0r)$ and $U^{*\prime} = B(x_0, R+2r)\setminus B(x_0, R-r)$.

(i) We say that condition $\mathrm{CSJ}^{(\rho)}(\phi)$ holds if the following holds for all $\rho > 0$: there exist constants $C_0 \in (0,1]$ and $C_1, C_2 > 0$ such that for every $0 < r \le R$, almost all $x_0 \in M$

and any $f \in \mathcal{F}$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0,R) \subset B(x_0,R+r)$ so that the following holds for all $\rho > 0$:

$$\begin{aligned}\int_{B(x_0,R+(1+C_0)r)} f^2\, d\Gamma^{(\rho)}(\varphi,\varphi) \le & C_1 \int_{U\times U^*} (f(x)-f(y))^2\, J^{(\rho)}(dx,dy) \\ & + \frac{C_2}{\phi(r\wedge\rho)} \int_{B(x_0,R+(1+C_0)r)} f^2\, d\mu.\end{aligned} \tag{2.4}$$

(ii) We say that condition CSAJ(ϕ) holds if there exist constants $C_0 \in (0,1]$ and $C_1, C_2 > 0$ such that for every $0 < r \le R$, almost all $x_0 \in M$ and any $f \in \mathcal{F}$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0,R) \subset B(x_0,R+r)$ so that the following holds for all $\rho > 0$:

$$\int_{U^*} f^2\, d\Gamma(\varphi,\varphi) \le C_1 \int_{U\times U^*} (f(x)-f(y))^2\, J(dx,dy) + \frac{C_2}{\phi(r)} \int_{U^*} f^2\, d\mu. \tag{2.5}$$

(iii) We say that condition CSAJ$^{(\rho)}$(ϕ) holds if the following holds for all $\rho > 0$: there exist constants $C_0 \in (0,1]$ and $C_1, C_2 > 0$ such that for every $0 < r \le R$, almost all $x_0 \in M$ and any $f \in \mathcal{F}$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0,R) \subset B(x_0,R+r)$ so that the following holds for all $\rho > 0$:

$$\int_{U^*} f^2\, d\Gamma^{(\rho)}(\varphi,\varphi) \le c_1 \int_{U\times U^*} (f(x)-f(y))^2\, J^{(\rho)}(dx,dy) + \frac{C_2}{\phi(r\wedge\rho)} \int_{U^*} f^2\, d\mu.$$

(iv) We say that condition CSJ$^{(\rho)}$(ϕ)$_+$ holds if the following holds for all $\rho > 0$: for any $\varepsilon > 0$, there exists a constant $c_1(\varepsilon) > 0$ such that for every $0 < r \le R$, almost all $x_0 \in M$ and any $f \in \mathcal{F}$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0,R) \subset B(x_0,R+r)$ so that the following holds for all $\rho > 0$:

$$\begin{aligned}\int_{B(x_0,R+2r)} f^2\, d\Gamma^{(\rho)}(\varphi,\varphi) \le & \varepsilon \int_{U\times U^{*\prime}} \varphi^2(x)(f(x)-f(y))^2\, J^{(\rho)}(dx,dy) \\ & + \frac{c_1(\varepsilon)}{\phi(r\wedge\rho)} \int_{B(x_0,R+2r)} f^2\, d\mu.\end{aligned} \tag{2.6}$$

(v) We say that condition CSAJ$^{(\rho)}$(ϕ)$_+$ holds if the following holds for all $\rho > 0$: for any $\varepsilon > 0$, there exists a constant $c_1(\varepsilon) > 0$ such that for every $0 < r \le R$, almost all $x_0 \in M$ and any $f \in \mathcal{F}$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0,R) \subset B(x_0,R+r)$ so that the following holds for all $\rho > 0$:

$$\int_{U^{*\prime}} f^2\, d\Gamma^{(\rho)}(\varphi,\varphi) \le \varepsilon \int_{U\times U^{*\prime}} \varphi^2(x)\,(f(x)-f(y))^2\, J^{(\rho)}(dx,dy) + \frac{c_1(\varepsilon)}{\phi(r\wedge\rho)} \int_{U^{*\prime}} f^2\, d\mu.$$

For open subsets A and B of M with $A \subset B$, and for any $\rho > 0$, define

$$\mathrm{Cap}^{(\rho)}(A,B) = \inf\{\mathcal{E}^{(\rho)}(\varphi,\varphi) : \varphi \in \mathcal{F},\ \varphi|_A = 1,\ \varphi|_{B^c} = 0\}.$$

Proposition 2.3. *Let ϕ be an increasing function on $\mathbb{R}_+$. Assume that* VD, (1.13) *and* $\mathrm{J}_{\phi,\le}$ *hold. The following hold.*

(1) CSJ(ϕ) *is equivalent to* CSJ$^{(\rho)}$(ϕ).

(2) CSJ(ϕ) *is implied by* CSAJ(ϕ).

(3) CSAJ(ϕ) *is equivalent to* CSAJ$^{(\rho)}(\phi)$.

(4) *If* CSJ$^{(\rho)}(\phi)$ *(resp.* CSAJ$^{(\rho)}(\phi)$*) holds for some* $C_0 > 0$*, then for any* $C_0' \geq C_0$*, there exist constants* $C_1, C_2 > 0$ *(where* C_2 *depends on* C_0'*) such that* CSJ$^{(\rho)}(\phi)$ *(resp.* CSAJ$^{(\rho)}(\phi)$*) holds.*

(5) *If* CSJ(ϕ) *holds, then there is a constant* $c_0 > 0$ *such that for every* $0 < r \leq R$*,* $\rho > 0$ *and almost all* $x \in M$*,*

$$\mathrm{Cap}^{(\rho)}(B(x,R), B(x,R+r)) \leq c_0 \frac{V(x,R+r)}{\phi(r\wedge\rho)}.$$

In particular, we have

$$\mathrm{Cap}(B(x,R), B(x,R+r)) \leq c_0 \frac{V(x,R+r)}{\phi(r)}. \tag{2.7}$$

Proof. (1) Letting $\rho \to \infty$, we see that (2.4) implies (1.17). Now, assume that (1.17) holds. Then for any $x_0 \in M$, $\rho > 0$ and $f \in \mathcal{F}$,

$$\begin{aligned}
&\int_{B(x_0,R+(1+C_0)r)} f^2\,d\Gamma^{(\rho)}(\varphi,\varphi)\\
\leq\ &\int_{B(x_0,R+(1+C_0)r)} f^2\,d\Gamma(\varphi,\varphi)\\
\leq\ & C_1\int_{U\times U^*}(f(x)-f(y))^2\,J(dx,dy) + \frac{C_2}{\phi(r)}\int_{B(x_0,R+(1+C_0)r)} f^2\,d\mu\\
\leq\ & C_1\int_{U\times U^*}(f(x)-f(y))^2\,J^{(\rho)}(dx,dy) + 2C_1\int_{U\times U^*}(f^2(x)+f^2(y))\mathbf{1}_{\{d(x,y)>\rho\}}\,J(dx,dy)\\
&+\frac{C_2}{\phi(r)}\int_{B(x_0,R+(1+C_0)r)} f^2\,d\mu\\
\leq\ & C_1\int_{U\times U^*}(f(x)-f(y))^2\,J^{(\rho)}(dx,dy) + \frac{C_3}{\phi(r\wedge\rho)}\int_{B(x_0,R+(1+C_0)r)} f^2\,d\mu,
\end{aligned}$$

where Lemma 2.1 is used in the last inequality.

(2) Fix $x_0 \in M$. Let $\varphi \in \mathcal{F}_b$ be a cut-off function for $B(x_0,R) \subset B(x_0,R+r)$. Since $\varphi(x) = 1$ on $x \in B(x_0,R)$, we have for $f \in \mathcal{F}$,

$$\begin{aligned}
\int_{B(x_0,R-C_0r)} f^2\,d\Gamma(\varphi,\varphi) &= \int_{B(x_0,R-C_0r)} f^2(x)\,\mu(dx)\int_M (1-\varphi(y))^2 J(x,y)\,\mu(dy)\\
&\leq \int_{B(x_0,R-C_0r)} f^2(x)\,\mu(dx)\int_{B(x_0,R)^c} J(x,y)\,\mu(dy)\\
&\leq \int_{B(x_0,R-C_0r)} f^2(x)\,\mu(dx)\int_{B(x,C_0r)^c} J(x,y)\,\mu(dy)\\
&\leq \frac{c_1}{\phi(C_0r)}\int_{B(x_0,R-C_0r)} f^2\,d\mu
\end{aligned}$$

$$\leq \frac{c_2}{\phi(r)} \int_{B(x_0,R-C_0r)} f^2\, d\mu,$$

where we used Lemma 2.1 and (1.13) in the last two inequalities. This together with (2.5) gives us the desired conclusion.

(3) This can be proved in the same way as (1).

(4) This is easy. Indeed, for $C_0' \geq C_0$, set $D_1 = B(x_0, R+(1+C_0')r) \setminus B(x_0, R+(1+C_0)r)$ and $D_2 = B(x_0, R-C_0r) \setminus B(x_0, R-C_0'r)$, where we set $B(x_0, R-C_0'r) = \emptyset$ for $C_0' > R/r$. Let $\varphi \in \mathcal{F}_b$ be a cut-off function for $B(x_0, R) \subset B(x_0, R+r)$. Then for any $f \in \mathcal{F}$ and $\rho > 0$,

$$\begin{aligned}\int_{D_1} f^2\, d\Gamma^{(\rho)}(\varphi,\varphi) &= \int_{D_1} f^2(x)\,\mu(dx) \int_{B(x_0,R+r)} \varphi^2(y) J^{(\rho)}(x,y)\,\mu(dy)\\ &\leq \int_{D_1} f^2(x)\,\mu(dx) \int_{B(x,C_0r)^c} J(x,y)\,\mu(dy)\\ &\leq \frac{c_1}{\phi(r)} \int_{D_1} f^2\, d\mu,\end{aligned}$$

where Lemma 2.1 and (1.13) are used in the last inequality. Similarly, for any $f \in \mathcal{F}$ and $\rho > 0$,

$$\int_{D_2} f^2\, d\Gamma^{(\rho)}(\varphi,\varphi) \leq \frac{c_2}{\phi(r)} \int_{D_2} f^2\, d\mu.$$

From both inequalities above we can get the desired assertion for $C_0' \geq C_0$.

(5) In view of (1) and (4), $\mathrm{CSJ}^{(\rho)}(\phi)$ holds for every $\rho > 0$ and we can and do take $C_0 = 1$ in (1.17). Fix $x_0 \in M$ and write $B_s := B(x_0, s)$ for $s \geq 0$. Let $f \in \mathcal{F}$ such that $f|_{B_{R+2r}} = 1$ and $f|_{B^c_{R+3r}} = 0$. For any $\rho > 0$, let $\varphi \in \mathcal{F}_b$ be the cut-off function for $B_R \subset B_{R+r}$ associated with f in $\mathrm{CSJ}^{(\rho)}(\phi)$. Then

$$\begin{aligned}\mathrm{Cap}^{(\rho)}(B_R, B_{R+r}) &\leq \int_{B_{R+2r}} d\Gamma^{(\rho)}(\varphi,\varphi) + \int_{B^c_{R+2r}} d\Gamma^{(\rho)}(\varphi,\varphi)\\ &= \int_{B_{R+2r}} f^2\, d\Gamma^{(\rho)}(\varphi,\varphi) + \int_{B^c_{R+2r}} d\Gamma^{(\rho)}(\varphi,\varphi)\\ &\leq c_1 \int_{(B_{R+r}\setminus B_R)\times(B_{R+2r}\setminus B_{R-r})} (f(x)-f(y))^2\, J^{(\rho)}(dx,dy)\\ &\quad + \frac{c_2}{\phi(r\wedge\rho)} \int_{B_{R+2r}} f^2\, d\mu + \int_{B^c_{R+2r}} \mu(dx) \int_{B_{R+r}} \varphi^2(y) J(x,y)\,\mu(dy)\\ &\leq \frac{c_2\mu(B_{R+2r})}{\phi(r\wedge\rho)} + \frac{c_3\mu(B_{R+r})}{\phi(r)}\\ &\leq \frac{c_4\mu(B_{R+r})}{\phi(r\wedge\rho)},\end{aligned}$$

where we used $\mathrm{CSJ}^{(\rho)}(\phi)$ in the second inequality and Lemma 2.1 in the third inequality.

Now let f_ρ be the potential whose $\mathcal{E}^{(\rho)}$-norm gives the capacity. Then the Cesàro mean of a subsequence of f_ρ converges in $\mathcal{E}_1$-norm, say to f, and $\mathcal{E}(f,f)$ is no less than the capacity corresponding to $\rho = \infty$. So (2.7) is proved. □

We next show that the leading constant in $\mathrm{CSJ}^{(\rho)}(\phi)$ (resp. $\mathrm{CSAJ}^{(\rho)}(\phi)$) is self-improving in the following sense.

Proposition 2.4. *Suppose that* VD, (1.13) *and* $\mathrm{J}_{\phi,\le}$ *hold. Then the following hold.*

(1) $\mathrm{CSJ}^{(\rho)}(\phi)$ *is equivalent to* $\mathrm{CSJ}^{(\rho)}(\phi)_+$.

(2) $\mathrm{CSAJ}^{(\rho)}(\phi)$ *is equivalent to* $\mathrm{CSAJ}^{(\rho)}(\phi)_+$.

Proof. We only prove (1), since (2) can be verified similarly. It is clear that $\mathrm{CSJ}^{(\rho)}(\phi)_+$ implies that $\mathrm{CSJ}^{(\rho)}(\phi)$. Below, we assume that $\mathrm{CSJ}(\phi)$ holds.

Fix $x_0 \in M$, $0 < r \le R$ and $f \in \mathcal{F}$. For $s > 0$, set $B_s = B(x_0, s)$. The goal is to construct a cut-off function $\varphi \in \mathcal{F}_b$ for $B_R \subset B_{R+r}$ which satisfies (2.6).

For $\lambda > 0$ which is determined later, let

$$s_n = c_0 r e^{-n\lambda/(2\beta_2)},$$

where $c_0 := c_0(\lambda)$ is chosen so that $\sum_{n=1}^{\infty} s_n = r$ and β_2 is given in (1.13). Set $r_0 = 0$ and

$$r_n = \sum_{k=1}^{n} s_k, \quad n \ge 1.$$

Clearly, $R < R + r_1 < R + r_2 < \cdots < R + r$. For any $n \ge 0$, define $U_n := B_{R+r_{n+1}} \setminus B_{R+r_n}$, and $U_n^* = B_{R+r_{n+1}+s_{n+1}} \setminus B_{R+r_n-s_{n+1}}$. By $\mathrm{CSJ}^{(\rho)}(\phi)$ (with $C_0 = 1$; see Proposition 2.3 (4)), there exists a cut-off function φ_n for $B_{R+r_n} \subset B_{R+r_{n+1}}$ such that

$$\begin{aligned}\int_{B_{R+r_{n+1}+s_{n+1}}} f^2\, d\Gamma^{(\rho)}(\varphi_n, \varphi_n) \le & C_1 \int_{U_n \times U_n^*} (f(x) - f(y))^2\, J^{(\rho)}(dx, dy) \\ & + \frac{C_2}{\phi(s_{n+1} \wedge \rho)} \int_{B_{R+r_{n+1}+s_{n+1}}} f^2\, d\mu.\end{aligned} \tag{2.8}$$

Let $b_n = e^{-n\lambda}$ and define

$$\varphi = \sum_{n=1}^{\infty} (b_{n-1} - b_n)\varphi_n. \tag{2.9}$$

Then φ is a cut-off function for $B_R \subset B_{R+r}$, because $\varphi = 1$ on B_R and $\varphi = 0$ on B_{R+r}^c. On U_n we have $\varphi = (b_{n-1} - b_n)\varphi_n + b_n$, so that $b_n \le \varphi \le b_{n-1}$ on U_n. In particular, on U_n

$$b_{n-1} - b_n \le \frac{\varphi(b_{n-1} - b_n)}{b_n} = (e^{\lambda} - 1)\varphi. \tag{2.10}$$

Below, we verify that the function φ defined by (2.9) satisfies (2.6) and $\varphi \in \mathcal{F}_b$. For this, we will make a non-trivial and substantial modification of the proof of [AB, Lemma 5.1]. Set

$$F_{n,m}(x, y) = f^2(x)(\varphi_n(x) - \varphi_n(y))(\varphi_m(x) - \varphi_m(y))$$

for any $n, m \ge 1$. Then

$$\int_{B_{R+2r}} f^2\, d\Gamma^{(\rho)}(\varphi, \varphi) = \int_{B_{R+2r}} f^2(x) \int_M \Big(\sum_{n=1}^{\infty} (b_{n-1} - b_n)(\varphi_n(x) - \varphi_n(y))\Big)^2 J^{(\rho)}(dx, dy)$$

$$\begin{aligned}
&\le \int_{B_{R+2r}} \int_M \Big[2\sum_{n=1}^{\infty}\sum_{m=1}^{n-2} (b_{n-1}-b_n)(b_{m-1}-b_m) F_{n,m}(x,y) \\
&\qquad + 2\sum_{n=2}^{\infty} (b_{n-1}-b_n)(b_{n-2}-b_{n-1}) F_{n,n-1}(x,y) \\
&\qquad + \sum_{n=1}^{\infty} (b_{n-1}-b_n)^2 F_{n,n}(x,y) \Big] J^{(\rho)}(dx,dy) \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For $n \ge m+2$, since $F_{n,m}(x,y) = 0$ for $x, y \in B_{R+r_n}$ or $x, y \notin B_{R+r_{m+1}}$, we can deduce that $F_{n,m}(x,y) \neq 0$ only if $x \in B_{R+r_{m+1}}, y \notin B_{R+r_n}$ or $x \notin B_{R+r_n}, y \in B_{R+r_{m+1}}$. Since $|F_{n,m}(x,y)| \le f^2(x)$, using Lemma 2.1, we have

$$\begin{aligned}
&\int_{B_{R+2r}} \int_M F_{n,m}(x,y)\, J^{(\rho)}(dx,dy) \\
&\qquad = \int_{B_{R+2r}\cap B_{R+r_{m+1}}} \int_{B^c_{R+r_n}} \cdots + \int_{B_{R+2r}\cap B^c_{R+r_n}} \int_{B_{R+r_{m+1}}} \cdots \\
&\qquad \le \frac{c}{\phi(\sum_{k=m+2}^{n} s_k)} \int_{B_{R+2r}} f^2(x)\,\mu(dx) \\
&\qquad \le \frac{c}{\phi(s_{m+2})} \int_{B_{R+2r}} f^2(x)\,\mu(dx).
\end{aligned} \tag{2.11}$$

Note that, according to (1.13), we have

$$\frac{\phi(r)}{\phi(s_{k+2})} \le c' \Big(\frac{r}{c_0(\lambda) r e^{-(k+2)\lambda/(2\beta_2)}} \Big)^{\beta_2} = c' \frac{e^{\lambda} e^{k\lambda/2}}{c_0(\lambda)^{\beta_2}} = \frac{c' e^{\lambda} (e^{\lambda}-1)^{1/2}}{c_0(\lambda)^{\beta_2} (b_{k-1}-b_k)^{1/2}}.$$

Therefore,

$$(b_{k-1}-b_k)^{1/2} \phi(s_{k+2})^{-1} \le c_1(\lambda) \phi(r)^{-1}. \tag{2.12}$$

This together with (2.11) implies

$$\begin{aligned}
I_1 &\le 2\sum_{n=1}^{\infty}\sum_{m=1}^{n-2} (b_{n-1}-b_n)(b_{m-1}-b_m) \frac{c}{\phi(s_{m+2})} \int_{B_{R+2r}} f^2(x)\,\mu(dx) \\
&\le \sum_{n=1}^{\infty}\sum_{m=1}^{n-2} (b_{n-1}-b_n)(b_{m-1}-b_m)^{1/2} \frac{c_2(\lambda)}{\phi(r)} \int_{B_{R+2r}} f^2(x)\,\mu(dx) \\
&\le \frac{c_3(\lambda)}{\phi(r)} \int_{B_{R+2r}} f^2(x)\,\mu(dx),
\end{aligned}$$

because $\sum_{m=1}^{\infty} (b_{m-1}-b_m)^{1/2} = c_4(\lambda)$ and $\sum_{n=1}^{\infty} (b_{n-1}-b_n) = 1$. For I_2, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
I_2 &\le 2\sum_{n=2}^{\infty} \Big(\int_{B_{R+2r}} \int_M (b_{n-1}-b_n)^2 F_{n,n}(x,y)^2\, J^{(\rho)}(dx,dy) \Big)^{1/2} \\
&\qquad \times \Big(\int_{B_{R+2r}} \int_M (b_{n-2}-b_{n-1})^2 F_{n-1,n-1}(x,y)^2\, J^{(\rho)}(dx,dy) \Big)^{1/2}
\end{aligned}$$

$$\le 2\, I_3,$$

where we used $2(ab)^{1/2} \le a+b$ for $a, b \ge 0$ in the last inequality. For I_3,

$$\begin{aligned}
&\int_{B_{R+2r}} \int_M F_{n,n}(x,y)\, J^{(\rho)}(dx,dy) \\
&= \Big(\int_{B_{R+r_{n+1}+s_{n+1}}} \int_M + \int_{B_{R+2r} \setminus B_{R+r_{n+1}+s_{n+1}}} \int_M \Big) F_{n,n}(x,y)\, J^{(\rho)}(dx,dy) \\
&\le \int_{B_{R+r_{n+1}+s_{n+1}}} \int_M F_{n,n}(x,y)\, J^{(\rho)}(dx,dy) + \frac{c}{\phi(s_{n+1})} \int_{B_{R+2r}} f^2(x)\, \mu(dx) \\
&\le C_1 \int_{U_n \times U_n^*} (f(x)-f(y))^2\, J^{(\rho)}(dx,dy) + \frac{c + C_2}{\phi(s_{n+1} \wedge \rho)} \int_{B_{R+2r}} f^2(x)\, \mu(dx),
\end{aligned}$$

where we used Lemma 2.1 in the second line and (2.8) in the last line. Using (2.10) and (2.12), and noting that $s_{k+1} \ge s_{k+2}$ and $\sum_{m=1}^\infty (b_{m-1}-b_m)^{3/2} + \sum_{m=1}^\infty (b_{m-1}-b_m)^2 = c_5(\lambda)$, we have

$$I_3 \le C_3 (e^\lambda - 1)^2 \int_{U \times U^{*\prime}} \varphi^2(x)(f(x)-f(y))^2\, J^{(\rho)}(dx,dy) + \frac{c_6(\lambda)}{\phi(r \wedge \rho)} \int_{B_{R+2r}} f^2(x)\, \mu(dx),$$

where we used the facts that $\{U_n\}$ are disjoint, $\bigcup_n U_n = U$, and $U_n^* \subset U^{*\prime}$ for all $n \ge 1$. For any $\varepsilon > 0$, we now choose λ so that $3C_3(e^\lambda - 1)^2 = \varepsilon$, and obtain (2.6).

Next, we prove that $\varphi \in \mathcal{F}_b$. Let $\varphi^{(i)} = \sum_{n=1}^i (b_{n-1}-b_n)\varphi_n$ for $i \ge 1$. It is clear that $\varphi^{(i)} \in \mathcal{F}_b$ and $\varphi^{(i)} \to \varphi$ as $i \to \infty$. So in order to prove $\varphi \in \mathcal{F}_b$, it suffices to verify that

$$\lim_{i,j \to \infty} \mathcal{E}(\varphi^{(i)} - \varphi^{(j)}, \varphi^{(i)} - \varphi^{(j)}) = 0. \tag{2.13}$$

Indeed, for any $i > j$, we can follow the arguments above and obtain that

$$\begin{aligned}
&\int_{B_{R+2r}} d\Gamma(\varphi^{(i)} - \varphi^{(j)}, \varphi^{(i)} - \varphi^{(j)}) \\
&\le e^{-j\lambda} \left(c_7(\lambda) \int_{U \times U^{*\prime}} (f(x)-f(y))^2\, J(dx,dy) + \frac{c_8(\lambda)}{\phi(r)} \int_{B_{R+2r}} f^2(x)\, \mu(dx) \right).
\end{aligned}$$

On the other hand, by Lemma 2.1 and the fact that $\operatorname{supp}(\varphi^{(i)} - \varphi^{(j)}) \subset B_{R+r}$,

$$\begin{aligned}
\int_{B_{R+2r}^c} d\Gamma(\varphi^{(i)} - \varphi^{(j)}, \varphi^{(i)} - \varphi^{(j)}) &\le \left(\sum_{n=j+1}^i (b_{n-1}-b_n) \right)^2 \int_{B_{R+2r}^c} \int_{B_{R+r}} J(x,y)\, \mu(dy)\, \mu(dx) \\
&\le e^{-j\lambda} \frac{c_9(\lambda)}{\phi(r)} \mu(B_{R+r}).
\end{aligned}$$

Combining with both inequalities above, we can get that (2.13) holds true. □

As a direct consequence of Proposition 2.3(1) and Proposition 2.4(1), we have the following corollary.

Corollary 2.5. *Suppose that* VD*,* (1.13)*,* $J_{\phi,\le}$ *and* CSJ(ϕ) *hold. Then there exists a constant* $c_1 > 0$ *such that for every* $0 < r \le R$*, almost all* $x_0 \in M$ *and any* $f \in \mathcal{F}$*, there exists a cut-off function* $\varphi \in \mathcal{F}_b$ *for* $B(x_0, R) \subset B(x_0, R + r)$ *so that the following holds for all* $\rho \in (0, \infty]$*:*

$$\begin{aligned}\int_{B(x_0,R+2r)} f^2\, d\Gamma^{(\rho)}(\varphi,\varphi) \le \frac{1}{8} \int_{U\times U^{*\prime}} \varphi^2(x)(f(x)-f(y))^2\, J^{(\rho)}(dx, dy) \\ + \frac{c_1}{\phi(r\wedge\rho)} \int_{B(x_0,R+2r)} f^2\, d\mu,\end{aligned} \tag{2.14}$$

where $U = B(x_0, R + r) \setminus B(x_0, R)$ *and* $U^{*\prime} = B(x_0, R + 2r) \setminus B(x_0, R - r)$*.*

Remark 2.6. According to all the arguments above, we can easily obtain that Propositions 2.3, 2.4 and Corollary 2.5 with small modifications (i.e. the cut-off function $\varphi \in \mathcal{F}_b$ can be chosen to be independent of $f \in \mathcal{F}$) hold for SCSJ(ϕ).

We close this subsection by the following statement.

Lemma 2.7. *Assume that* VD*,* (1.13) UHK(ϕ) *hold and* $(\mathcal{E},\mathcal{F})$ *is conservative. Then* $EP_{\phi,\le}$ *holds.*

Proof. We first verify that there is a constant $c_1 > 0$ such that for each $t, r > 0$ and for almost all $x \in M$,

$$\int_{B(x,r)^c} p(t, x, y)\, \mu(dy) \le \frac{c_1 t}{\phi(r)}.$$

Indeed, we only need to consider the case that $\phi(r) > t$; otherwise, the inequality above holds trivially with $c_1 = 1$. According to UHK(ϕ), VD and (1.13), for any $t, r > 0$ with $\phi(r) > t$ and almost all $x \in M$,

$$\begin{aligned}\int_{B(x,r)^c} p(t, x, y)\, \mu(dy) &= \sum_{i=0}^{\infty} \int_{B(x,2^{i+1}r)\setminus B(x,2^i r)} p(t, x, y)\, \mu(dy) \\ &\le \sum_{i=0}^{\infty} \frac{c_2 t V(x, 2^{i+1}r)}{V(x, 2^i r)\phi(2^i r)} \le \frac{c_3 t}{\phi(r)} \sum_{i=0}^{\infty} 2^{-i\beta_1} \le \frac{c_4 t}{\phi(r)}.\end{aligned}$$

Now, since $(\mathcal{E},\mathcal{F})$ is conservative, by the strong Markov property, for any each $t, r > 0$ and for almost all $x \in M$,

$$\begin{aligned}\mathbb{P}^x(\tau_{B(x,r)} \le t) &= \mathbb{P}^x(\tau_{B(x,r)} \le t, X_{2t} \in B(x, r/2)^c) + \mathbb{P}^x(\tau_{B(x,r)} \le t, X_{2t} \in B(x, r/2)) \\ &\le \mathbb{P}^x(X_{2t} \in B(x, r/2)^c) + \sup_{z\notin B(x,r)^c, s\le t} \mathbb{P}^z(X_{2t-s} \in B(z, r/2)^c) \\ &\le \frac{c_5 t}{\phi(r)},\end{aligned}$$

which yields $EP_{\phi,\le}$. (Note that the conservativeness of $(\mathcal{E},\mathcal{F})$ is used in the equality above. Indeed, without the conservativeness, there must be an extra term $\mathbb{P}^x(\tau_{B(x,r)} \le t, \zeta \le 2t)$ in the right hand side of the above equality, where ζ is the lifetime of X.) □

3 Implications of heat kernel estimates

In this section, we will prove (1) $\Longrightarrow$ (3) in Theorems 1.13 and 1.15. We point out that, under VD, RVD and (1.13), UHK(ϕ) $\Longrightarrow$ FK(ϕ) is given in Proposition 7.6 in the Appendix.

3.1 UHK(ϕ) + $(\mathcal{E}, \mathcal{F})$ is conservative $\Longrightarrow$ $\mathrm{J}_{\phi,\leq}$, and HK(ϕ) $\Longrightarrow$ J_ϕ

We first show the following, where, for future reference, it is formulated for a general Hunt process Y that admits no killings inside.

Proposition 3.1. *Suppose that $Y = \{Y_t, t \geq 0, \mathbb{P}^x, x \in E\}$ is an arbitrary Hunt process on a locally compact separable metric space E that admits no killings inside E. Denote its lifetime by ζ.*

(i) *If there is a constant $c_0 > 0$ so that*

$$\mathbb{P}^x(\zeta = \infty) \geq c_0 \quad \textit{for every } x \in E, \tag{3.1}$$

then $\mathbb{P}^x(\zeta = \infty) = 1$ for every $x \in E$.

(ii) *Suppose that* VD *holds, the heat kernel $p(t, x, y)$ of the process Y exists, and there exist constants $\varepsilon \in (0, 1)$ and $c_1 > 0$ such that for any $x \in E$ and $t > 0$,*

$$p(t, x, y) \geq \frac{c_1}{V(x, \phi^{-1}(t))} \quad \textit{for } y \in B(x, \varepsilon\phi^{-1}(t)), \tag{3.2}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing continuous function with $\phi(0) = 0$. Then $\mathbb{P}^x(\zeta = \infty) = 1$ for every $x \in E$. In particular, LHK(ϕ) *implies $\zeta = \infty$ a.s.*

Proof. (i) Let $\{\mathcal{F}_t^Y; t \geq 0\}$ be the minimal augmented filtration generated by the Hunt process Y, and set $u(x) := \mathbb{P}^x(\zeta = \infty)$. Then we have $u(x) \geq c_0 > 0$ for $x \in E$. Note that

$$u(Y_t) = \mathbf{1}_{\{\zeta > t\}} u(Y_t) = \mathbb{E}^x \left[\mathbf{1}_{\{\zeta = \infty\}} | \mathcal{F}_t^Y \right]$$

is a bounded martingale with $\lim_{t\to\infty} u(Y_t) = \mathbf{1}_{\{\zeta=\infty\}}$. Let $\{K_j; j \geq 1\}$ be an increasing sequence of compact sets so that $\cup_{j=1}^\infty K_j = E$ and define $\tau_j = \inf\{t \geq 0 : Y_t \notin K_j\}$. Since the Hunt process Y admits no killings inside E, we have $\tau_j < \zeta$ a.s. for every $j \geq 1$. Clearly $\lim_{j\to\infty} \tau_j = \zeta$. By the optional stopping theorem, we have for $x \in E$,

$$\begin{aligned} u(x) &= \lim_{j\to\infty} \mathbb{E}^x u(Y_{\tau_j}) = \mathbb{E}^x \left[\lim_{j\to\infty} u(Y_{\tau_j}) \right] = \mathbb{E}^x \left[\lim_{j\to\infty} u(Y_{\tau_j}) \mathbf{1}_{\{\zeta<\infty\}} + \lim_{t\to\infty} u(Y_t) \mathbf{1}_{\{\zeta=\infty\}} \right] \\ &\geq c_0 \mathbb{P}^x(\zeta < \infty) + \mathbb{P}^x(\zeta = \infty) = c_0 \mathbb{P}^x(\zeta < \infty) + u(x). \end{aligned}$$

It follows that $\mathbb{P}^x(\zeta < \infty) = 0$ for every $x \in E$.

(ii) By (3.2) and the equivalent characterization (1.10) of VD, we have for every $x \in E$ and $t > 0$,

$$\mathbb{P}^x(\zeta > t) \geq \int_{B(x,\varepsilon\phi^{-1}(t))} p(t, x, y)\, \mu(dy) \geq \int_{B(x,\varepsilon\phi^{-1}(t))} \frac{c_1}{V(x, \phi^{-1}(t))}\, \mu(dy) \geq c_2 > 0.$$

Passing $t \to \infty$, we get $\mathbb{P}^x(\zeta = \infty) \geq c_2$ for every $x \in E$. The conclusion now follows immediately from (i). □

Remark 3.2. (i) The condition that Y admits no killings inside E is needed for Proposition 3.1 to hold. That is, condition (3.1) alone does not guarantee Y is conservative. Here is a counterexample. Let Y be the process obtained from a Brownian motion $W = \{W_t\}$ in $\mathbb{R}^3$ killed according to the potential $q(x) := \mathbf{1}_{B(0,1)}(x)$. That is, for $f \geq 0$ on $\mathbb{R}^3$,

$$\mathbb{E}^x[f(Y_t)] = \mathbb{E}^x \left[f(W_t) \exp\left(-\int_0^t \mathbf{1}_{B(0,1)}(W_s)\, ds \right) \right]. \tag{3.3}$$

Denote by ζ the lifetime of Y. We claim that (3.1) holds for Y. Indeed, for three-dimensional Brownian motion W, we have

$$\inf_{x\in\mathbb{R}^3:|x|\geq 2} \mathbb{P}^x(\sigma^W_{B(0,1)} = \infty) = 1 - \sup_{x\in\mathbb{R}^3:|x|\geq 2} \mathbb{P}^x(\sigma^W_{B(0,1)} < \infty) = 1 - \sup_{x\in\mathbb{R}^3:|x|\geq 2} \frac{1}{|x|} = \frac{1}{2},$$

where $\sigma^W_{B(0,1)} = \inf\{t \geq 0 : W_t \in B(0,1)\}$. Clearly for $x \in B(0,2)^c$,

$$\mathbb{P}^x(\zeta = \infty) \geq \mathbb{P}^x(\sigma^W_{B(0,1)} = \infty) \geq \frac{1}{2}. \tag{3.4}$$

On the other hand, if we use $p(t,x,y)$ and $p^0(t,x,y)$ to denote the transition density function of Y and W with respect to the Lebesgue measure on $\mathbb{R}^3$ respectively, then we have by (3.3) that

$$e^{-t}p^0(t,x,y) \leq p(t,x,y) \leq p^0(t,x,y) \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^3.$$

Hence there is a constant $c_1 \in (0,1)$ so that

$$\mathbb{P}^x\left(Y_1 \in \mathbb{R}^3 \setminus B(0,2)\right) \geq c_1 \quad \text{for every } x \in B(0,1).$$

Using the Markov property of Y at time 1, we have from (3.4) that $\mathbb{P}^x(\zeta = \infty) \geq c_1/2$ for every $x \in B(0,1)$. This establishes (3.1) with $c_0 = c_1/2$. However $\mathbb{P}^x(\zeta < \infty) > 0$ for every $x \in \mathbb{R}^3$.

(ii) In the setting of this paper, X is the symmetric Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ given by (1.1) that has no killing term. So X always admits no killings inside M.

The next proposition in particular shows that UHK(ϕ) implies (1.14).

Proposition 3.3. *Under* VD *and* (1.13),

$$\mathrm{UHK}(\phi) \text{ and } (\mathcal{E}, \mathcal{F}) \text{ is conservative} \Longrightarrow \mathrm{J}_{\phi,\leq},$$

and

$$\mathrm{HK}(\phi) \Longrightarrow \mathrm{J}_\phi.$$

Proof. The proof is easy and standard, and we only consider $\mathrm{HK}(\phi) \Longrightarrow \mathrm{J}_\phi$ for simplicity. Consider the form $\mathcal{E}^{(t)}(f,g) := \langle f - P_t f, g\rangle/t$. Since $(\mathcal{E}, \mathcal{F})$ is conservative by Proposition 3.1(ii), we can write

$$\mathcal{E}^{(t)}(f,g) = \frac{1}{2t}\int_M\int_M (f(x) - f(y))(g(x) - g(y))p(t,x,y)\,\mu(dx)\,\mu(dy).$$

It is well known that $\lim_{t\to 0} \mathcal{E}^{(t)}(f,g) = \mathcal{E}(f,g)$ for all $f, g \in \mathcal{F}$. Let A, B be disjoint compact sets, and take $f, g \in \mathcal{F}$ such that $\operatorname{supp} f \subset A$ and $\operatorname{supp} g \subset B$. Then

$$\mathcal{E}^{(t)}(f,g) = -\frac{1}{t}\int_A\int_B f(x)g(y)p(t,x,y)\,\mu(dy)\,\mu(dx) \xrightarrow{t\to 0} -\int_A\int_B f(x)g(y)\,J(dx,dy).$$

Using HK(ϕ), we obtain

$$\int_A\int_B f(x)g(y)\,J(dx,dy) \asymp \int_A\int_B \frac{f(x)g(y)}{V(x,d(x,y))\phi(d(x,y))}\,\mu(dy)\,\mu(dx),$$

for all $f, g \in \mathcal{F}$ such that $\operatorname{supp} f \subset A$ and $\operatorname{supp} g \subset B$. Since A, B are arbitrary disjoint compact sets, it follows that $J(dx,dy)$ is absolutely continuous w.r.t. $\mu(dx)\,\mu(dy)$, and J_ϕ holds. □

3.2 UHK(ϕ) and $(\mathcal{E},\mathcal{F})$ is conservative $\Longrightarrow$ SCSJ(ϕ)

In this subsection, we give the proof that UHK(ϕ) and the conservativeness of $(\mathcal{E},\mathcal{F})$ imply SCSJ(ϕ). For $D \subset M$ and $\lambda > 0$, define

$$G_\lambda^D f(x) = \mathbb{E}^x \int_0^{\tau_D} e^{-\lambda t} f(X_t)\,dt, \quad x \in M_0.$$

Lemma 3.4. *Suppose that* VD, (1.13) *and* UHK(ϕ) *hold, and* $(\mathcal{E},\mathcal{F})$ *is conservative. Let* $x_0 \in M$, $0 < r \le R$, *and define*

$$\begin{aligned} D_0 &= B(x_0, R + 9r/10) \setminus \overline{B}(x_0, R + r/10), \\ D_1 &= B(x_0, R + 4r/5) \setminus \overline{B}(x_0, R + r/5), \\ D_2 &= B(x_0, R + 3r/5) \setminus \overline{B}(x_0, R + 2r/5). \end{aligned}$$

Let $\lambda = \phi(r)^{-1}$, *and set* $h = G_\lambda^{D_0}\mathbf{1}_{D_1}$. *Then* $h \in \mathcal{F}_{D_0}$ *and* $h(x) \le \phi(r)$ *for all* $x \in M_0$. *Moreover, there exists a constant* $c_1 > 0$, *independent of* x_0, r *and* R, *so that* $h(x) \ge c_1\phi(r)$ *for all* $x \in D_2 \cap M_0$.

Proof. That $h \in \mathcal{F}_{D_0}$ follows by [FOT, Theorem 4.4.1]. The definition of h implies that $h(x) = 0$ for $x \notin \overline{D}_0$, and the upper bound on h is elementary, since $h \le G_\lambda^M \mathbf{1} = \lambda^{-1} = \phi(r)$.

By Lemma 2.7, we can choose a constant $\delta_{1/2} > 0$ such that for all $r > 0$ and all $x \in M_0$,

$$\mathbb{P}^x(\tau_{B(x,r)} \le \delta_{1/2}\phi(r)) \le \frac{1}{2}.$$

For any $x \in D_2 \cap M_0$, $B_1 := B(x, r/5) \subset D_1$. Hence

$$\begin{aligned} h(x) &= \mathbb{E}^x \int_0^{\tau_{D_0}} e^{-\lambda t}\mathbf{1}_{D_1}(X_t)\,dt \\ &\ge \mathbb{E}^x\left[\int_0^{\tau_{B_1}} e^{-\lambda t}\mathbf{1}_{B_1}(X_t)\,dt;\ \tau_{B_1} > \delta_{1/2}\phi(r/5)\right] \\ &\ge \mathbb{P}^x(\tau_{B_1} > \delta_{1/2}\phi(r/5))\left[\int_0^{\delta_{1/2}\phi(r/5)} e^{-\lambda t}\,dt\right] \ge c_1\phi(r), \end{aligned}$$

where we used (1.13) in the last inequality. □

We also need the following property for non-local Dirichlet forms.

Lemma 3.5. *For each $f, g \in \mathcal{F}_b$, $\eta > 0$ and any subset $D \subset M$,*

$$\begin{aligned}
&(1-\eta^{-1}) \int_{D\times D} f^2(x)(g(x)-g(y))^2\, J(dx,dy) \\
&\le \int_{D\times D} (g(x)f^2(x)-g(y)f^2(y))(g(x)-g(y))\, J(dx,dy) \\
&\quad + \eta \int_{D\times D} g^2(x)(f(x)-f(y))^2\, J(dx,dy)
\end{aligned} \tag{3.5}$$

Proof. For any $f, g \in \mathcal{F}_b$, we can easily get that

$$\begin{aligned}
&\int_{D\times D} f^2(x)(g(x)-g(y))^2\, J(dx,dy) \\
&= \int_{D\times D} (g(x)f^2(x)-g(y)f^2(y))(g(x)-g(y))\, J(dx,dy) \\
&\quad - \frac{1}{2} \int_{D\times D} (f^2(x)-f^2(y))(g^2(x)-g^2(y))\, J(dx,dy).
\end{aligned} \tag{3.6}$$

Then according to the Cauchy-Schwarz inequality, for any $\eta > 0$,

$$\begin{aligned}
&\Big| \int_{D\times D} (f^2(x)-f^2(y))(g^2(x)-g^2(y))\, J(dx,dy) \Big| \\
&\le \left(\int_{D\times D} \eta(g(x)+g(y))^2(f(x)-f(y))^2\; J(dx,dy) \right)^{1/2} \\
&\quad \times \left(\int_{D\times D} \eta^{-1}(f(x)+f(y))^2(g(x)-g(y))^2\, J(dx,dy) \right)^{1/2} \\
&\le \left(\int_{D\times D} 4\eta g^2(x)(f(x)-f(y))^2\, J(dx,dy) \right)^{1/2} \\
&\quad \times \left(\int_{D\times D} 4\eta^{-1} f^2(x)(g(x)-g(y))^2\, J(dx,dy) \right)^{1/2} \\
&\le 2\eta \int_{D\times D} g^2(x)(f(x)-f(y))^2\, J(dx,dy) \\
&\quad + 2\eta^{-1} \int_{D\times D} f^2(x)(g(x)-g(y))^2\, J(dx,dy),
\end{aligned}$$

where we have used the fact $ab \le \frac{1}{2}(a^2+b^2)$ for all $a, b \ge 0$ in the last inequality. Plugging this into (3.6), we obtain (3.5). □

Proposition 3.6. *Suppose that* VD*,* (1.13) *and* UHK(ϕ) *hold, and* $(\mathcal{E}, \mathcal{F})$ *is conservative. Then* SCSJ(ϕ) *holds.*

Proof. By the dominated convergence theorem, we only need to verify that SCSJ(ϕ) holds for any $f \in \mathcal{F}_b$. For any $x_0 \in M$ and $s > 0$, let $B_s = B(x_0, s)$. For $0 < r \le R$, let $U = B_{R+r} \setminus B_R$

and $U^* = B_{R+3r/2} \setminus B_{R-r/2}$. Let D_i be those as in Lemma 3.4, and $\lambda = \phi(\lambda)^{-1}$. For $x \in M_0$, set

$$g(x) = \frac{G_\lambda^{D_0} \mathbf{1}_{D_1}(x)}{c^* \phi(r)},$$
$$\varphi(x) = \begin{cases} 1 \wedge g(x) & \text{if } x \in B_{R+r/2}^c \cap M_0, \\ 1 & \text{if } x \in B_{R+r/2} \cap M_0, \end{cases}$$

where c^* is the constant c_1 in Lemma 3.4. Then by Lemma 3.4, $\varphi = 0$ on B_{R+r}^c, and $\varphi = 1$ on B_R.

We first claim

$$\int_{U^*} f^2 \, d\Gamma(\varphi, \varphi) \le \int_{U^*} f^2 \, d\Gamma(g, g) + \frac{c_1}{\phi(r)} \int_{U^*} f^2 \, d\mu, \quad f \in \mathcal{F}_b. \tag{3.7}$$

Indeed, by decomposing the regions of integrals, we have

$$\begin{aligned} \int_{U^*} f^2 \, d\Gamma(\varphi, \varphi) &= \int_{B_{R+r/2} \setminus B_{R-r/2}} \int_{B_{R+r} \setminus B_{R+r/2}} + \int_{B_{R+3r/2} \setminus B_{R+r/2}} \int_{B_{R+r/2}} \\ &\quad + \int_{B_{R+3r/2} \setminus B_{R+r/2}} \int_{B_{R+r} \setminus B_{R+r/2}} + \int_{B_{R+3r/2} \setminus B_{R-r/2}} \int_{B_{R+r}^c} \\ &=: I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where the first integral of each term in the right hand side is with respect to x. Here we used the fact

$$\int_{B_{R+r/2} \setminus B_{R-r/2}} f^2(x) \, \mu(dx) \int_{B_{R+r/2}} (\varphi(x) - \varphi(y))^2 J(x, y) \, \mu(dy) = 0,$$

because $\varphi(x) = \varphi(y) = 1$ when $x, y \in B_{R+r/2}$. By Lemma 2.1 and (1.13), we have

$$\begin{aligned} I_1 &= \int_{B_{R+r/2} \setminus B_{R-r/2}} f^2(x) \, \mu(dx) \int_{B_{R+r} \setminus B_{R+3r/5}} (1 - \varphi(y))^2 J(x, y) \, \mu(dy) \\ &\le \frac{c_1}{\phi(r/10)} \int_{B_{R+r/2} \setminus B_{R-r/2}} f^2 \, d\mu \le \frac{c_2}{\phi(r)} \int_{B_{R+r/2} \setminus B_{R-r/2}} f^2 \, d\mu. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_{B_{R+3r/2} \setminus B_{R+3r/5}} f^2(x) (\varphi(x) - 1)^2 \, \mu(dx) \int_{B_{R+r/2}} J(x, y) \, \mu(dy) \\ &\le \frac{c_3}{\phi(r)} \int_{B_{R+3r/2} \setminus B_{R+3r/5}} f^2 \, d\mu, \\ I_4 &= \int_{B_{R+9r/10} \setminus B_{R-r/2}} f^2(x) \varphi^2(x) \, \mu(dx) \int_{B_{R+r}^c} J(x, y) \, \mu(dy) \\ &\le \frac{c_4}{\phi(r)} \int_{B_{R+9r/10} \setminus B_{R-r/2}} f^2 \, d\mu. \end{aligned}$$

Finally, we have

$$I_3 = \int_{B_{R+3r/2} \setminus B_{R+r/2}} f^2(x) \, \mu(dx) \int_{B_{R+r} \setminus B_{R+r/2}} (\varphi(x) - \varphi(y))^2 J(x, y) \, \mu(dy)$$

$$
\begin{aligned}
&\le \int_{B_{R+3r/2}\setminus B_{R+r/2}} f^2(x)\,\mu(dx) \int_{B_{R+r}\setminus B_{R+r/2}} (g(x)-g(y))^2 J(x,y)\,\mu(dy)\\
&\le \int_{U^*} f^2\, d\Gamma(g,g),
\end{aligned}
$$

so that (3.7) is proved.

Next, using Lemma 2.1 and (3.5) with $\eta = 2$, we have for any $f \in \mathcal{F}_b$,

$$
\begin{aligned}
\int_{U^*} f^2\, d\Gamma(g,g) &\le \int_{U^*\times U^*} f^2(x)(g(x)-g(y))^2\, J(dx,dy) + \int_{U^*\times U^{*c}} f^2(x) g^2(x)\, J(dx,dy)\\
&\le 2\int_{U^*\times U^*} (f^2(x)g(x)-f^2(y)g(y))(g(x)-g(y))\, J(dx,dy) \qquad (3.8)\\
&\quad + 4\int_{U^*\times U^*} g^2(x)(f(x)-f(y))^2\, J(dx,dy) + \frac{c_5}{\phi(r)}\int_U f^2\, d\mu,
\end{aligned}
$$

where in the last inequality we have used the fact that g is zero outside U.

With $\lambda = \phi(r)^{-1}$, we have for any $f \in \mathcal{F}_b$,

$$
\begin{aligned}
&\int_{U^*\times U^*} (f^2(x)g(x)-f^2(y)g(y))(g(x)-g(y))\, J(dx,dy)\\
&\le \int_{(U^*\times U^*)\cup(U^{*c}\times U^*)\cup(U^*\times U^{*c})} (f^2(x)g(x)-f^2(y)g(y))(g(x)-g(y))\, J(dx,dy)\\
&= \int_M d\Gamma(f^2 g, g) = \mathcal{E}(f^2 g, g) \le \mathcal{E}_\lambda(f^2 g, g) \qquad (3.9)\\
&= (c^*\phi(r))^{-1}\mathcal{E}_\lambda(f^2 g, G_\lambda^{D_0}\mathbf{1}_{D_1})\\
&= (c^*\phi(r))^{-1}\langle f^2 g, \mathbf{1}_{D_1}\rangle\\
&\le (c^*\phi(r))^{-1}\int_U f^2 g\, d\mu.
\end{aligned}
$$

Here we used [FOT, Theorem 4.4.1] and the fact that $f^2 g \in \mathcal{F}_{D_0}$ to obtain the third equality. Plugging (3.9) into (3.8), and using the facts that $g \le c_6$ and g is zero outside U, we obtain

$$
\begin{aligned}
\int_{U^*} f^2\, d\Gamma(g,g) &\le 4\int_{U^*\times U^*} g^2(x)(f(x)-f(y))^2\, J(dx,dy) + \frac{2}{c^*\phi(r)}\int_U f^2 g\, d\mu + \frac{c_5}{\phi(r)}\int_U f^2\, d\mu\\
&\le 4c_6^2 \int_{U\times U^*} (f(x)-f(y))^2\, J(dx,dy) + \Big(\frac{2c_6}{c^*} + c_5\Big)\frac{1}{\phi(r)}\int_U f^2\, d\mu.
\end{aligned}
$$

This and (3.7) imply CSAJ(ϕ) for any $f \in \mathcal{F}_b$ with the strong form (i.e. the cut-off function is independent of $f \in \mathcal{F}_b$) with $C_0 = \frac{1}{2}$. Therefore, the desired assertion follows from Proposition 2.3(2) and Remark 2.6. □

As mentioned in the beginning of this section, UHK(ϕ) implies FK(ϕ) by Proposition 7.6 under VD, RVD and (1.13). This completes the proof of (1) $\Longrightarrow$ (3) part in Theorems 1.13 and 1.15. Note also that (3) $\Longrightarrow$ (4) part in Theorems 1.13 and 1.15 holds trivially.

4 Implications of CSJ(ϕ) and $\mathrm{J}_{\phi,\ge}$

In this section, we will prove (4) $\Longrightarrow$ (2) in Theorems 1.13 and 1.15.

4.1 $\mathrm{J}_{\phi,\geq} \Longrightarrow \mathrm{FK}(\phi)$

We first prove that under VD and (1.13), $\mathrm{J}_{\phi,\geq}$ implies the local Nash inequality introduced by Kigami ([Ki]). Note that for the uniform VD case, the following lemma was proved in [CK2, Theorem 3.1]. The proof below is similar to that of [CK2, Theorem 3.1].

Lemma 4.1. *Under* VD, (1.13) *and* $\mathrm{J}_{\phi,\geq}$, *there is a constant* $c_0 > 0$ *such that for any* $s > 0$,

$$\|u\|_2^2 \le c_0\Big(\frac{\|u\|_1^2}{\inf_{z\in \operatorname{supp} u} V(z,s)} + \phi(s)\mathcal{E}(u,u)\Big), \quad \forall u \in \mathcal{F}\cap L^1(M;\mu).$$

Proof. For any $u \in \mathcal{F}\cap L^1(M;\mu)$ and $s>0$, define

$$u_s(x) := \frac{1}{V(x,s)}\int_{B(x,s)} u(z)\,\mu(dz) \quad \text{for } x\in M.$$

For $A\subset M$ and $s>0$, denote $A^s := \{z\in M : d(z,A) < s\}$. Using (1.12), we have

$$\|u_s\|_\infty \le \frac{c_1\|u\|_1}{\inf_{z_0\in(\operatorname{supp} u)^s} V(z_0,s)} \le \frac{c_1'\|u\|_1}{\inf_{z\in\operatorname{supp} u} V(z,2s)} \le \frac{c_1'\|u\|_1}{\inf_{z\in\operatorname{supp} u} V(z,s)}$$

and

$$\begin{aligned}\|u_s\|_1 &\le \int_{(\operatorname{supp} u)^s} \frac{1}{V(x,s)}\,\mu(dx)\int_{B(x,s)} |u(z)|\,\mu(dz)\\ &= \int_{\operatorname{supp} u} |u(z)|\,\mu(dz)\int_{(\operatorname{supp} u)^s\cap B(z,s)} \frac{1}{V(x,s)}\,\mu(dx) \le c_2\|u\|_1.\end{aligned}$$

In particular,

$$\|u_s\|_2^2 \le \|u_s\|_\infty\|u_s\|_1 \le \frac{c_3\|u\|_1^2}{\inf_{z\in\operatorname{supp} u} V(z,s)}.$$

Therefore, for $u\in\mathcal{F}\cap L^1(M;\mu)$, by $\mathrm{J}_{\phi,\geq}$,

$$\begin{aligned}\|u\|_2^2 &\le 2\|u-u_s\|_2^2 + 2\|u_s\|_2^2\\ &\le 2\int_M \left(\frac{1}{V(x,s)}\int_{B(x,s)} (u(x)-u(y))^2\mu(dy)\right)\mu(dx) + \frac{2c_3\|u\|_1^2}{\inf_{z\in\operatorname{supp} u} V(z,s)}\\ &\le c_4\int_M \left(\frac{1}{V(x,s)}\int_{B(x,s)} (u(x)-u(y))^2 J(x,y)\,\phi(s)V(x,s))\,\mu(dy)\right)\mu(dx)\\ &\quad + \frac{2c_3\|u\|_1^2}{\inf_{z\in\operatorname{supp} u} V(z,s)}\\ &\le c_5\,\phi(s)\int_M\int_{B(x,s)} (u(x)-u(y))^2 J(x,y)\,\mu(dy)\,\mu(dx)\\ &\quad + \frac{2c_3\|u\|_1^2}{\inf_{z\in\operatorname{supp} u} V(z,s)}\\ &\le c_6\left(\phi(s)\mathcal{E}(u,u) + \frac{\|u\|_1^2}{\inf_{z\in\operatorname{supp} u} V(z,s)}\right).\end{aligned}$$

We thus obtain the desired inequality. □

We then conclude by Proposition 7.4 that $\mathrm{J}_{\phi,\geq} \Longrightarrow \mathrm{FK}(\phi)$ under VD, RVD and (1.13).

By Proposition 7.7 in Appendix (see also [BBCK, Theorem 3.1] and [GT, Section 2.2]), it follows that there is a proper exceptional set $\mathcal{N}$ so that the Hunt process $\{X_t\}$ has a transition density function $p(t,x,y)$ for every $x, y \in M \setminus \mathcal{N}$.

4.2 Caccioppoli and L^1-mean value inequalities

In this subsection, we establish mean value inequalities for subharmonic functions. Though in this paper, we only need mean value inequalities for the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$, we choose to first establish these inequalities for subharmonic functions of the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ and then indicate how these proofs can be modified to establish similar inequalities for subharmonic functions of the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$. There are several reasons for doing so: (i) the mean value inequalities for the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ will be used as one of the key tools in the study of the stability of parabolic Harnack inequality in our subsequent paper [CKW]; (ii) since the proofs share many common parts and ideas in the truncated and non-truncated settings, it is more efficient to do it in this way; (iii) although they share many common ideas in these two settings, there are also some differences; see the paragraph proceeding the statement of Proposition 4.11, by putting together in one place clearly reveals differences and difficulties in the setting of jump processes as for the diffusion case.

We first need to introduce the analytic characterization of subharmonic functions and to extend the definition of bilinear form $\mathcal{E}$. Let D be an open subset of M. Recall that a function f is said to be locally in $\mathcal{F}_D$, denoted as $f \in \mathcal{F}_D^{loc}$, if for every relatively compact subset U of D, there is a function $g \in \mathcal{F}_D$ such that $f = g$ m-a.e. on U.

The next lemma is proved in [C, Lemma 2.6].

Lemma 4.2. *Let D be an open subset of M. Suppose u is a function in $\mathcal{F}_D^{loc}$ that is locally bounded on D and satisfies that*

$$\int_{U \times V^c} |u(y)|\, J(dx, dy) < \infty \tag{4.1}$$

for any relatively compact open sets U and V of M with $\bar{U} \subset V \subset \bar{V} \subset D$. Then for every $v \in C_c(D) \cap \mathcal{F}$, the expression

$$\int (u(x) - u(y))(v(x) - v(y))\, J(dx, dy)$$

is well defined and finite; it will still be denoted as $\mathcal{E}(u, v)$.

As noted in [C, (2.3)], since $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M; \mu)$, for any relatively compact open sets U and V with $\bar{U} \subset V$, there is a function $\psi \in \mathcal{F} \cap C_c(M)$ such that $\psi = 1$ on U and $\psi = 0$ on V. Consequently,

$$\int_{U \times V^c} J(dx, dy) = \int_{U \times V^c} (\psi(x) - \psi(y))^2\, J(dx, dy) \leq \mathcal{E}(\psi, \psi) < \infty,$$

so each bounded function u satisfies (4.1).

Definition 4.3. Let D be an open subset of M.

(i) We say that a nearly Borel measurable function u on M is *$\mathcal{E}$-subharmonic* (resp. *$\mathcal{E}$-harmonic, $\mathcal{E}$-superharmonic*) in D if $u \in \mathcal{F}_D^{loc}$, satisfies condition (4.1) and

$$\mathcal{E}(u,\varphi) \le 0 \quad (\text{resp. } = 0, \ge 0)$$

for any $0 \le \varphi \in \mathcal{F}_D$.

(ii) A nearly Borel measurable function u on M is said to be *subharmonic* (resp. *harmonic, superharmonic*) in D (with respect to the process X) if for any relatively compact subset $U \subset D$, $t \mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable submartingale (resp. martingale, supermartingale) under $\mathbb{P}^x$ for q.e. $x \in U$.

The following result is established in [C, Theorem 2.11 and Lemma 2.3] first for harmonic functions, and then extended in [ChK, Theorem 2.9] to subharmonic functions.

Theorem 4.4. *Let D be an open subset of M, and let u be a bounded function. Then u is $\mathcal{E}$-harmonic (resp. $\mathcal{E}$-subharmonic) in D if and only if u is harmonic (resp. subharmonic) in D.*

To establish the Caccioppoli inequality, we also need the following definition.

Definition 4.5. For a Borel measurable function u on M, we define its *nonlocal tail* in the ball $B(x_0,r)$ by

$$\mathrm{Tail}\,(u;x_0,r) = \phi(r)\int_{B(x_0,r)^c} \frac{|u(z)|}{V(x_0,d(x_0,z))\phi(d(x_0,z))}\,\mu(dz). \tag{4.2}$$

Suppose that VD and (1.13) hold. Observe that in view of (2.1), $\mathrm{Tail}\,(u;x_0,r)$ is finite if u is bounded. Note also that $\mathrm{Tail}\,(u;x_0,r)$ is finite by the Hölder inequality and (2.1) whenever $u \in L^p(M;\mu)$ for any $p \in [1,\infty)$ and $r>0$. As mentioned in [CKP], the key-point in the present nonlocal setting is how to manage the nonlocal tail.

We first show that CSJ(ϕ) enables us to prove a Caccioppoli inequality for $\mathcal{E}$-subharmonic functions. Note that the Caccioppoli inequality below is different from that in [CKP, Lemma 1.4], since our argument is heavily based on CSJ(ϕ).

Lemma 4.6. (Caccioppoli inequality) *For $x_0 \in M$ and $s>0$, let $B_s = B(x_0,s)$. Suppose that* VD, (1.13), CSJ(ϕ) *and* $\mathrm{J}_{\phi,\le}$ *hold. For $0<r<R$, let u be an $\mathcal{E}$-subharmonic function on B_{R+r} for the Dirichlet form $(\mathcal{E},\mathcal{F})$, and $v=(u-\theta)^+$ for $\theta \ge 0$. Also, let φ be the cut-off function for $B_{R-r} \subset B_R$ associated with v in* CSJ(ϕ). *Then there exists a constant $c>0$ independent of x_0,R,r and θ such that*

$$\int_{B_{R+r}} d\Gamma(v\varphi,v\varphi) \le \frac{c}{\phi(r)}\left[1+\frac{1}{\theta}\left(1+\frac{R}{r}\right)^{d_2+\beta_2-\beta_1}\mathrm{Tail}\,(u;x_0,R+r)\right]\int_{B_{R+r}} u^2\,d\mu. \tag{4.3}$$

Proof. Since u is $\mathcal{E}$-subharmonic on B_{R+r} for the Dirichlet form $(\mathcal{E},\mathcal{F})$ and $\varphi^2 v \in \mathcal{F}_{B_R}$,

$$\begin{aligned}0 \ge \mathcal{E}(u,\varphi^2 v) &= \int_{B_{R+r}\times B_{R+r}} (u(x)-u(y))(\varphi^2(x)v(x)-\varphi^2(y)v(y))\,J(dx,dy)\\ &\quad + 2\int_{B_{R+r}\times B_{R+r}^c} (u(x)-u(y))\varphi^2(x)v(x)\,J(dx,dy)\\ &=: I_1 + 2I_2.\end{aligned} \tag{4.4}$$

For I_1, we may and do assume without loss of generality that $u(x) \geq u(y)$; otherwise just exchange the roles of x and y below. We have

$$\begin{aligned}
&(u(x)-u(y))(\varphi^2(x)v(x)-\varphi^2(y)v(y))\\
&=(u(x)-u(y))\varphi^2(x)(v(x)-v(y))+(u(x)-u(y))(\varphi^2(x)-\varphi^2(y))v(y)\\
&\geq \varphi^2(x)(v(x)-v(y))^2+(v(x)-v(y))(\varphi^2(x)-\varphi^2(y))v(y)\\
&\geq \varphi^2(x)(v(x)-v(y))^2-\frac{1}{8}(\varphi(x)+\varphi(y))^2(v(x)-v(y))^2-2v^2(y)(\varphi(x)-\varphi(y))^2\\
&\geq \frac{3}{4}\varphi^2(x)(v(x)-v(y))^2-\frac{1}{4}\varphi^2(y)(v(x)-v(y))^2-2v^2(y)(\varphi(x)-\varphi(y))^2.
\end{aligned}$$

where the first inequality follows from the facts that for any $x, y \in M$, $u(x)-u(y) \geq v(x)-v(y)$ and $(u(x)-u(y))v(y) = (v(x)-v(y))v(y)$, while in the second and third equalities we used the facts that $ab \geq -\frac{1}{8}a^2 - 2b^2$ and $(a+b)^2 \leq 2a^2+2b^2$, respectively, for all $a, b \in \mathbb{R}$. This together with the symmetry of $J(dx, dy)$ yields that

$$I_1 \geq \frac{1}{2}\int_{B_{R+r}\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J(dx,dy) - 2\int_{B_{R+r}\times B_{R+r}} v^2(x)(\varphi(x)-\varphi(y))^2\, J(dx,dy).$$

For I_2, note that

$$(u(x)-u(y))\varphi^2(x)v(x) = ((u(x)-\theta)-(u(y)-\theta))\varphi^2(x)v(x) \geq (v(x)-v(y))\varphi^2(x)v(x) \geq -v(x)v(y).$$

Note also that $v \leq vu/\theta \leq u^2/\theta$. Hence we have

$$\begin{aligned}
I_2 &= \int_{B_R\times B_{R+r}^c} (u(x)-u(y))\varphi^2(x)v(x)\, J(dx,dy)\\
&\geq -\int_{B_R} v\, d\mu \left[\sup_{x\in B_R}\int_{B_{R+r}^c} v(y)\, J(x,dy)\right]\\
&\geq -\frac{1}{\theta}\int_{B_R} u^2\, d\mu \left[\sup_{x\in B_R}\int_{B_{R+r}^c} v(y)\, J(x,dy)\right]\\
&\geq -\frac{c_1}{\theta\phi(r)}\left[\left(1+\frac{R}{r}\right)^{d_2+\beta_2-\beta_1}\phi(R+r)\int_{B_{R+r}^c}\frac{|u(y)|}{V(x_0,d(x_0,y))\phi(d(x_0,y))}\,\mu(dy)\right]\int_{B_R} u^2\, d\mu\\
&= -\frac{c_1}{\theta\phi(r)}\left[\left(1+\frac{R}{r}\right)^{d_2+\beta_2-\beta_1}\mathrm{Tail}\,(u;x_0,R+r)\right]\int_{B_R} u^2\, d\mu,
\end{aligned}$$

where the last inequality follows from the fact that $v \leq |u|$, $\mathrm{J}_{\phi,\leq}$ as well as (1.12) and (1.13) which imply that for any $x \in B_R$ and $y \in B_{R+r}^c$,

$$\frac{V(x_0,d(x_0,y))\phi(d(x_0,y))}{V(x,d(x,y))\phi(d(x,y))} \leq c'\Big(1+\frac{d(x_0,x)}{d(x,y)}\Big)^{d_2+\beta_2} \leq c''\Big(1+\frac{R}{r}\Big)^{d_2+\beta_2}$$

and

$$\frac{\phi(r)}{\phi(R+r)} \leq c'''\Big(1+\frac{R}{r}\Big)^{-\beta_1}.$$

Putting the estimates for I_1 and I_2 above into (4.4), we arrive at

$$\begin{aligned}
0 \le 4 & \int_{B_{R+r}\times B_{R+r}} v^2(x)(\varphi(x)-\varphi(y))^2\, J(dx,dy) \\
& - \int_{B_R\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J(dx,dy) \\
& + \frac{c_2}{\theta\phi(r)}\left[\left(1+\frac{R}{r}\right)^{d_2+\beta_2-\beta_1} \operatorname{Tail}(u;x_0,R+r)\right]\int_{B_R} u^2\, d\mu \\
\le 4 & \int_{B_{R+r}} v^2\, d\Gamma(\varphi,\varphi) - \int_{B_R\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J(dx,dy) \\
& + \frac{c_2}{\theta\phi(r)}\left[\left(1+\frac{R}{r}\right)^{d_2+\beta_2-\beta_1} \operatorname{Tail}(u;x_0,R+r)\right]\int_{B_R} u^2\, d\mu.
\end{aligned} \tag{4.5}$$

On the other hand, using the inequality $(a+b)^2 \le 2(a^2+b^2)$ for all $a,b\in\mathbb{R}$ and Lemma 2.1, we have

$$\begin{aligned}
& \int_{B_{R+r}} d\Gamma(v\varphi, v\varphi) \\
& = \int_{B_{R+r}\times M} (v(x)\varphi(x)-v(y)\varphi(y))^2\, J(dx,dy) \\
& \le \int_{B_{R+r}\times B_{R+r}} \big(v(x)(\varphi(x)-\varphi(y))+\varphi(y)(v(x)-v(y))\big)^2\, J(dx,dy) \\
& \quad + \int_{B_R} v^2(x)\varphi^2(x)\int_{B^c_{R+r}} J(dx,dy) \\
& \le 2\Big[\int_{B_{R+r}\times B_{R+r}} v^2(x)(\varphi(x)-\varphi(y))^2\, J(dx,dy) \\
& \quad + \int_{B_{R+r}\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J(dx,dy)\Big] + \frac{c_3}{\phi(r)}\int_{B_R} v^2\, d\mu \\
& \le 2\int_{B_{R+r}} v^2\, d\Gamma(\varphi,\varphi) + 2\int_{B_R\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J(dx,dy) + \frac{c_3}{\phi(r)}\int_{B_R} u^2\, d\mu.
\end{aligned} \tag{4.6}$$

Combining (4.5) with (4.6), we have for $a>0$,

$$\begin{aligned}
& a\int_{B_{R+r}} d\Gamma(v\varphi, v\varphi) \\
& \le (2a+4)\int_{B_{R+r}} v^2\, d\Gamma(\varphi,\varphi) + (2a-1)\int_{B_R\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J(dx,dy) \\
& \quad + \frac{c_4(1+a)}{\phi(r)}\left[1+\frac{1}{\theta}\left(1+\frac{R}{r}\right)^{d_2+\beta_2-\beta_1} \operatorname{Tail}(u;x_0,R+r)\right]\int_{B_R} u^2\, d\mu.
\end{aligned} \tag{4.7}$$

Next by using (2.14) for v with $\rho=\infty$, we have

$$\int_{B_{R+r}} v^2\, d\Gamma(\varphi,\varphi) \le \frac{1}{8}\int_{B_R\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J(dx,dy) + \frac{c_0}{\phi(r)}\int_{B_{R+r}} v^2\, d\mu. \tag{4.8}$$

Plugging this into (4.7) with $a = 2/9$ (so that $(4+2a)/8 + (2a-1) = 0$), we obtain

$$\frac{2}{9}\int_{B_{R+r}} d\Gamma(v\varphi, v\varphi) \le \frac{c_5}{\phi(r)}\left[1 + \frac{1}{\theta}\left(1+\frac{R}{r}\right)^{d_2+\beta_2-\beta_1} \mathrm{Tail}\,(u; x_0, R+r)\right]\int_{B_{R+r}} u^2\, d\mu,$$

which proves the desired assertion. □

Remark 4.7. In order to obtain (4.3) we need that the constant in the first term on the right hand side of (2.14) was less than 1/4. On the other hand, we note that (4.8) is weaker than (2.14) yielded by CSJ(ϕ), which can strengthen the first term in the right hand side of (4.8) into

$$\frac{1}{8}\int_{U\times U^*} \varphi^2(x)(v(x)-v(y))^2\, J(dx, dy)$$

with $U = B_R \setminus B_{R-r}$ and $U^* = B_{R+r} \setminus B_{R-2r}$.

The key step in the proof of the mean value inequality is the following comparison over balls. For a ball $B = B(x_0, r) \subset M$ and a function w on B, write

$$I(w, B) = \int_B w^2\, d\mu.$$

The following lemma can be proved similarly to that of [AB, Lemma 3.5] (see also [Gr1, Lemma 3.2]) with very minor corrections due to B_{R+r} instead of B_R. For completeness, we give the proof below.

Lemma 4.8. *For $x_0 \in M$ and $s > 0$, let $B_s = B(x_0, s)$. Suppose* VD, (1.13), FK(ϕ), CSJ(ϕ) *and* $\mathrm{J}_{\phi,\le}$ *hold. For $R, r_1, r_2 > 0$ with $r_1 \in [\frac{1}{2}R, R]$ and $r_1 + r_2 \le R$, let u be an $\mathcal{E}$-subharmonic function on B_R for the Dirichlet form $(\mathcal{E}, \mathcal{F})$, and $v = (u-\theta)_+$ for some $\theta > 0$. Set $I_0 = I(u, B_{r_1+r_2})$ and $I_1 = I(v, B_{r_1})$. We have*

$$I_1 \le \frac{c_1}{\theta^{2\nu} V(x_0, R)^\nu} I_0^{1+\nu}\left(1+\frac{r_1}{r_2}\right)^{\beta_2}\left[1 + \left(1+\frac{r_1}{r_2}\right)^{d_2+\beta_2-\beta_1}\frac{\mathrm{Tail}\,(u; x_0, R/2)}{\theta}\right], \tag{4.9}$$

where ν is the constant appearing in the FK(ϕ) *inequality* (1.19), *d_2 is the constant in* (1.10) *from* VD, *and c_1 is a constant independent of θ, x_0, R, r_1 and r_2.*

Proof. Set

$$D = \{x \in B_{r_1+r_2/2} : u(x) > \theta\}.$$

Let φ be a cut-off function for $B_{r_1} \subset B_{r_1+r_2/2}$ associated with v in CSJ(ϕ).

As in [Gr1] the proof uses the following five inequalities:

$$\int_{B_{r_1+r_2/2}} u^2\, d\mu \le I_0, \tag{4.10}$$

$$\int_{B_{r_1+r_2}} d\Gamma(v\varphi, v\varphi) \le \frac{c_0}{\phi(r_2)}\left[1 + \frac{1}{\theta}\left(1+\frac{r_1}{r_2}\right)^{d_2+\beta_2-\beta_1} \mathrm{Tail}\,(u; x_0, R/2)\right] I_0, \tag{4.11}$$

$$2\int_D d\Gamma(v\varphi, v\varphi) \ge \lambda_1(D)\int_D v^2\varphi^2\, d\mu, \tag{4.12}$$

$$\lambda_1(D) \geq C\mu(B_{r_1+r_2})^{\nu}\phi(r_1+r_2)^{-1}\mu(D)^{-\nu}, \tag{4.13}$$

$$\mu(D) \leq \theta^{-2}\int_{B_{r_1+r_2/2}} u^2\, d\mu. \tag{4.14}$$

Of these, (4.10) holds trivially. The inequality (4.11) follows immediately from (4.3) since, by VD and (1.13),

$$\mathrm{Tail}\,(u; x_0, r_1+r_2) \leq c_1 \mathrm{Tail}\,(u; x_0, R/2).$$

Inequality (4.12) is immediate from the variational definition (1.18) of $\lambda_1(D)$ and the facts that $v\varphi \in \mathcal{F}_D$ and

$$2\int_D d\Gamma(v\varphi, v\varphi) \geq \mathcal{E}(v\varphi, v\varphi).$$

Indeed, since $v\varphi = 0$ on D^c, we have

$$\begin{aligned}
\mathcal{E}(v\varphi, v\varphi) =& \left(\int_{D\times D} + \int_{D\times D^c} + \int_{D^c\times D} + \int_{D^c\times D^c}\right)\big(v(x)\varphi(x) - v(y)\varphi(y)\big)^2 J(dx, dy) \\
=& \left(\int_{D\times D} + \int_{D\times D^c} + \int_{D^c\times D}\right)\big(v(x)\varphi(x) - v(y)\varphi(y)\big)^2 J(dx, dy) \\
\leq& \left(\int_{D\times M} + \int_{M\times D}\right)\big(v(x)\varphi(x) - v(y)\varphi(y)\big)^2 J(dx, dy) \\
=& 2\int_{D\times M} \big(v(x)\varphi(x) - v(y)\varphi(y)\big)^2 J(dx, dy) \\
=& 2\int_D d\Gamma(v\varphi, v\varphi),
\end{aligned}$$

where the third equality follows from the symmetry of $J(dx, dy)$. (4.13) follows from the Faber-Krahn inequality (1.19), VD and (1.13). (4.14) is just Markov's inequality.

Putting (4.10) into (4.14), we get

$$\mu(D) \leq I_0/\theta^2. \tag{4.15}$$

By VD, (1.13), (4.12), (4.13) and (4.15), we have

$$\begin{aligned}
\int_D d\Gamma(v\varphi, v\varphi) &\geq \frac{C\mu(B_{r_1+r_2})^{\nu}}{\phi(r_1+r_2)\mu(D)^{\nu}}\int_D v^2\varphi^2\, d\mu \\
&= \frac{C\mu(B_{r_1+r_2})^{\nu}}{\phi(r_1+r_2)\mu(D)^{\nu}}\int_{B_{r_1+r_2/2}} v^2\varphi^2\, d\mu \\
&\geq \frac{C'V(x_0,R)^{\nu}\theta^{2\nu}}{\phi(r_1)I_0^{\nu}}\int_{B_{r_1+r_2/2}} v^2\varphi^2\, d\mu \\
&\geq \frac{C''V(x_0,R)^{\nu}\theta^{2\nu}}{\phi(r_1)I_0^{\nu}}\int_{B_{r_1}} v^2\, d\mu \\
&= \frac{C''V(x_0,R)^{\nu}\theta^{2\nu}}{\phi(r_1)I_0^{\nu}} I_1,
\end{aligned}$$

where in the last inequality we used the fact $\varphi = 1$ on B_{r_1}. Combining the inequality above with (4.11) and (1.13), we obtain the desired estimate (4.9). □

We need the following elementary iteration lemma, see, e.g., [Giu, Lemma 7.1].

Lemma 4.9. *Let $\beta > 0$ and let $\{A_j\}$ be a sequence of real positive numbers such that*

$$A_{j+1} \le c_0 b^j A_j^{1+\beta}$$

with $c_0 > 0$ and $b > 1$. If

$$A_0 \le c_0^{-1/\beta} b^{-1/\beta^2},$$

then we have

$$A_j \le b^{-j/\beta} A_0, \tag{4.16}$$

which in particular yields $\lim_{j\to\infty} A_j = 0$.

Proof. We proceed by induction. The inequality (4.16) is obviously true for $j = 0$. Assume now that holds for j. We have

$$A_{j+1} \le c_0 b^j b^{-j(1+\beta)/\beta} A_0^{1+\beta} = (c_0 b^{1/\beta} A_0^{\beta}) b^{-(j+1)/\beta} A_0 \le b^{-(j+1)/\beta} A_0,$$

so (4.16) holds for $j + 1$. □

Proposition 4.10. (L^2-mean value inequality) *Let $x_0 \in M$ and $R > 0$. Assume* VD, (1.13), FK(ϕ), CSJ(ϕ) *and* $\mathrm{J}_{\phi,\le}$ *hold, and let u be a bounded $\mathcal{E}$-subharmonic in $B(x_0, R)$. Then for any $\delta > 0$,*

$$\operatorname{ess\,sup}_{B(x_0,R/2)} u \le c_1 \left[\left(\frac{(1+\delta^{-1})^{1/\nu}}{V(x_0,R)} \int_{B(x_0,R)} u^2\, d\mu \right)^{1/2} + \delta \mathrm{Tail}\,(u; x_0, R/2) \right], \tag{4.17}$$

where ν is the constant appearing in the FK(ϕ) *inequality* (1.19), *and $c_1 > 0$ is a constant independent of x_0, R and δ.*

In particular, there is a constant $c > 0$ independent of x_0 and R so that

$$\operatorname{ess\,sup}_{B(x_0,R/2)} u \le c \left[\left(\frac{1}{V(x_0,R)} \int_{B(x_0,R)} u^2\, d\mu \right)^{1/2} + \mathrm{Tail}\,(u; x_0, R/2) \right]. \tag{4.18}$$

Proof. We first set up some notations. For $i \ge 0$ and $\theta > 0$, let $r_i = \frac{1}{2}(1 + 2^{-i})R$ and $\theta_i = (1 - 2^{-i})\theta$. For any $x_0 \in M$ and $s > 0$, let $B_s = B(x_0, s)$. Define

$$I_i = \int_{B_{r_i}} (u - \theta_i)_+^2\, d\mu, \quad i \ge 0.$$

By [ChK, Corollary 2.10(iv)], for any $i \ge 0$, $(u - \theta_i)_+$ is an $\mathcal{E}$-subharmonic function for the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on B_R. Then, thanks to Lemma 4.8, by (4.9) applied to the function $(u - \theta_i)$ in $B_{r_{i+1}} \subset B_{r_i}$,

$$\begin{aligned}
I_{i+1} &= \int_{B_{r_{i+1}}} (u - \theta_{i+1})_+^2\, d\mu = \int_{B_{r_{i+1}}} \left[(u - \theta_i) - (\theta_{i+1} - \theta_i) \right]_+^2 d\mu \\
&\le \frac{c_2}{(\theta_{i+1} - \theta_i)^{2\nu} V(x_0,R)^\nu} I_i^{1+\nu} \left(\frac{r_i}{r_i - r_{i+1}} \right)^{\beta_2} \\
&\quad \times \left[1 + \left(\frac{r_i}{r_i - r_{i+1}} \right)^{d_2+\beta_2-\beta_1} \frac{\mathrm{Tail}\,(u; x_0, R/2)}{(\theta_{i+1} - \theta_i)} \right] \\
&\le \frac{c_3 2^{(1+2\nu+d_2+2\beta_2-\beta_1)i}}{\theta^{2\nu} V(x_0,R)^\nu} I_i^{1+\nu} \left[1 + \frac{\mathrm{Tail}\,(u; x_0, R/2)}{\theta} \right].
\end{aligned}$$

In the following, we take

$$\theta = \delta \mathrm{Tail}\,(u; x_0, R/2) + \sqrt{c_* \frac{I_0}{V(x_0,R)}}, \quad \delta > 0,$$

where $c_* = [(1+\delta^{-1})c_3]^{1/\nu} 2^{(1+2\nu+d_2+2\beta_2-\beta_1)/\nu^2}$. It is easy to see that

$$I_0 \le \left[\frac{c_3}{\theta^{2\nu} V(x_0,R)^\nu} \Big(1 + \frac{\mathrm{Tail}\,(u; x_0, R/2)}{\theta} \Big) \right]^{-1/\nu} 2^{-(1+2\nu+d_2+2\beta_2-\beta_1)/\nu^2}.$$

Then by Lemma 4.9, we have $I_i \to 0$ as $i \to \infty$. Hence

$$\int_{B_{R/2}} (u-\theta)_+^2 \, d\mu \le \inf_i I_i = 0,$$

which implies that

$$\text{ess sup}_{B_{R/2}} u \le \theta \le c_4 \left[\left(\frac{(1+\delta^{-1})^{1/\nu} I_0}{V(x_0,R)} \right)^{1/2} + \delta \mathrm{Tail}\,(u; x_0, R/2) \right].$$

This proves (4.17). □

In the following, we consider L^2 and L^1 mean value inequalities for $\mathcal{E}$-subharmonic functions for truncated Dirichlet forms. In the truncated situation we can no longer use the nonlocal tail of subharmonic functions, and the remedy is to enlarge the integral regions in the right hand side of the mean value inequalities. These mean value inequalities will be used in the next subsection to consider the stability of heat kernel.

Proposition 4.11. **(L^2-mean value inequality for ρ-truncated Dirichlet forms)** *Assume* VD*,* (1.13)*,* FK(ϕ)*,* CSJ(ϕ) *and* $J_{\phi,\le}$ *hold. There are positive constants $c_1, c_2 > 0$ so that for $x_0 \in M$, $\rho, R > 0$, and for any bounded $\mathcal{E}^{(\rho)}$-subharmonic function u on $B(x_0,R)$ for the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$, we have*

$$\text{ess sup}_{B(x_0,R/2)} u^2 \le \frac{c_1}{V(x_0,R)} \Big(1+\frac{\rho}{R}\Big)^{d_2/\nu} \Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu} \int_{B(x_0,R+\rho)} u^2 \, d\mu. \tag{4.19}$$

Here, ν is the constant in FK(ϕ)*, d_2 and β_2 are the exponents in* (1.10) *from* VD *and* (1.13) *respectively.*

Proof. The proof is mainly based on that of Proposition 4.10. For simplicity, we only present the main different steps, and the details are left to the interested readers.

First, we apply the argument in the proof of Lemma 4.6 to the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$. In this truncated setting, we estimate the term I_2 in (4.4) as follows.

$$\begin{aligned} I_2 &= \int_{B_R \times B_{R+r}^c} (u(x)-u(y)) \varphi^2(x) v(x) \, J^{(\rho)}(dx,dy) \\ &\ge - \int_{B_R} v(x) \, \mu(dx) \left[\sup_{x \in B_R} \int_{B_{R+r}^c} v(y) J^{(\rho)}(x,y) \, \mu(dy) \right] \end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{1}{\theta}\int_{B_R} u^2(x)\,\mu(dx)\left[\sup_{x\in B_R}\int_{B_{R+r}^c} v(y)J^{(\rho)}(x,y)\,\mu(dy)\right]\\
&\geq -\frac{1}{\theta}\int_{B_R} u^2(x)\,\mu(dx)\left[\frac{c_1}{\phi(r)}\Big(\sup_{x\in B_R}\frac{1}{V(x,r)}\Big)\int_{B_{R+\rho}} v(y)\,\mu(dy)\right]\\
&\geq -\frac{c_2}{\phi(r)}\bigg[\Big(\frac{R+\rho}{r}\Big)^{d_2}\frac{1}{\theta V(x_0,R+\rho)}\int_{B_{R+\rho}}|u|(y)\,\mu(dy)\bigg]\int_{B_R} u^2(x)\,\mu(dx),
\end{aligned}
$$

where in the second and third inequality we have used the fact that $v \leq vu/\theta \leq u^2/\theta$ and the condition $\mathrm{J}_{\phi,\leq}$ respectively, while the last inequality follows from that for any $x \in B_R$,

$$
\frac{V(x,r)}{V(x_0,R+\rho)} \geq \frac{V(x,r)}{V(x,2R+\rho)} \geq c'\Big(\frac{R+\rho}{r}\Big)^{-d_2},
$$

thanks to VD.

On the other hand, we do the upper estimate for $\int_{B_{R+r}} d\Gamma(v\varphi, v\varphi)$ just as (4.6), but using ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ instead. Indeed, we have

$$
\begin{aligned}
\int_{B_{R+r}} d\Gamma(v\varphi, v\varphi) \leq &\int_{B_{R+r}\times M} (v(x)\varphi(x)-v(y)\varphi(y))^2\, J^{(\rho)}(dx,dy)\\
&+2\int_{B_R} v^2(x)\varphi^2(x)\int_{d(x,y)\geq\rho} J(dx,dy)\\
&+2\int_{M} v^2(y)\varphi^2(y)\int_{d(x,y)\geq\rho} J(dx,dy)\\
\leq &\int_{B_{R+r}\times B_{R+r}} \big(v(x)(\varphi(x)-\varphi(y))+\varphi(y)(v(x)-v(y))\big)^2\, J^{(\rho)}(dx,dy)\\
&+\int_{B_R} v^2(x)\varphi^2(x)\int_{B_{R+r}^c} J^{(\rho)}(dx,dy)+\frac{c_1'}{\phi(\rho)}\int_{B_R} v^2\,d\mu\\
\leq &2\Big(\int_{B_{R+r}\times B_{R+r}} v^2(x)(\varphi(x)-\varphi(y))^2\, J^{(\rho)}(dx,dy)\\
&+\int_{B_{R+r}\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J^{(\rho)}(dx,dy)\Big)+\frac{c_1''}{\phi(\rho\wedge r)}\int_{B_R} v^2\,d\mu\\
\leq &2\int_{B_{R+r}} v^2\, d\Gamma^{(\rho)}(\varphi,\varphi)+2\int_{B_R\times B_{R+r}} \varphi^2(x)(v(x)-v(y))^2\, J^{(\rho)}(dx,dy)\\
&+\frac{c_2''}{\phi(r)}\left(1+\frac{r}{\rho}\right)^{\beta_2}\int_{B_R} u^2\,d\mu.
\end{aligned}
$$

Having both two estimates above at hand, one can change (4.3) in Lemma 4.6 into

$$
\int_{B_{R+r}} d\Gamma(v\varphi, v\varphi) \leq \frac{c}{\phi(r)}\left[1+\left(1+\frac{r}{\rho}\right)^{\beta_2}+\left(\frac{R+\rho}{r}\right)^{d_2}\frac{1}{\theta V(x_0,R+\rho)}\int_{B_{R+\rho}} u\,d\mu\right]\int_{B_{R+r}} u^2\,d\mu,
$$

where $c>0$ is a constant independent of x_0, R, r, ρ and θ. This in turn gives us the following conclusion instead of (4.9) in Lemma 4.8:

$$
I_1 \leq \frac{c_1}{\theta^{2\nu}V(x_0,R)^{\nu}} I_0^{1+\nu}\left(\frac{r_1}{r_2}\right)^{\beta_2}\left[1+\left(1+\frac{r_2}{\rho}\right)^{\beta_2}+\left(\frac{r_1+\rho}{r_2}\right)^{d_2}\frac{1}{\theta V(x_0,R+\rho)}\int_{B_{R+\rho}}|u|\,d\mu\right].
$$

Finally, following the argument of Proposition 4.10, we can obtain that for any bounded $\mathcal{E}^{(\rho)}$-subharmonic function u associated with the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ on $B(x_0, R)$, it holds

$$\begin{aligned} \text{ess sup}_{B(x_0,R/2)} u^2 \le c_0 \Bigg[& \left(\frac{1}{V(x_0, R+\rho)} \int_{B(x_0,R+\rho)} u \, d\mu \right)^2 \\ & + \left(1 + \frac{\rho}{R}\right)^{d_2/\nu} \left(1 + \frac{R}{\rho}\right)^{\beta_2/\nu} \frac{1}{V(x_0,R)} \int_{B(x_0,R)} u^2 \, d\mu \Bigg], \end{aligned} \tag{4.20}$$

where ν is the constant in FK(ϕ), d_2 and β_2 are the constants in VD and (1.13) respectively, and $c_0 > 0$ is a constant independent of x_0, ρ and R. Hence, the desired assertion (4.19) immediately follows from (4.20). □

As a consequence of Proposition 4.11, we have the following L^1-mean value inequality for truncated Dirichlet forms.

Corollary 4.12. **(L^1-mean value inequality for ρ-truncated Dirichlet forms)** *Assume* VD, (1.13), FK(ϕ), CSJ(ϕ) *and* $\mathrm{J}_{\phi,\le}$ *hold. There are positive constants $c_1, c_2 > 0$ so that for $x_0 \in M$, $\rho, R > 0$, and for any nonnegative, bounded and $\mathcal{E}^{(\rho)}$-subharmonic function u on $B(x_0, R)$ for the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$, we have*

$$\text{ess sup}_{B(x_0,R/2)} u \le \frac{c_2}{V(x_0,R)} \left(1 + \frac{\rho}{R}\right)^{d_2/\nu} \left(1 + \frac{R}{\rho}\right)^{\beta_2/\nu} \int_{B(x_0,R+\rho)} u \, d\mu. \tag{4.21}$$

Here, ν is the constant in FK(ϕ), *d_2 and β_2 are the exponents in* (1.10) *from* VD *and* (1.13) *respectively.*

Proof. Fix $x_0 \in M$ and $R > 0$. For any $s > 0$, let $B_s = B(x_0, s)$. For $n \ge 0$, let $r_n = R2^{-n-1}$. Note that $\{r_n\}$ is decreasing such that $r_0 = R/2$ and $r_\infty = 0$, and $\{B_{r_n}\}$ is decreasing and $\{B_{R-r_n}\}$ is increasing such that $B_0 = B_{R-r_0} = B(x_0, R/2)$ and $B_{R-r_\infty} = B(x_0, R)$. Take arbitrary point $\xi \in B_{R-r_{n-1}}$; then since $r_n = r_{n-1}/2$, we have $B(\xi, r_n) \subset B_{R-r_n}$. Applying (4.20) with $x_0 = \xi$ and $R = r_n$, we have

$$\begin{aligned} u(\xi)^2 \le c_1 \Bigg[& \left(\frac{1}{V(\xi, r_n+\rho)} \int_{B(\xi,r_n+\rho)} u \, d\mu \right)^2 \\ & + \left(1 + \frac{\rho}{r_n}\right)^{d_2/\nu} \left(1 + \frac{r_n}{\rho}\right)^{\beta_2/\nu} \frac{1}{V(\xi,r_n)} \int_{B(\xi,r_n)} u^2 \, d\mu \Bigg], \end{aligned} \tag{4.22}$$

where $c_1 > 0$ does not depend on ξ, r_n and ρ.

In the following, let

$$M_n = \text{ess sup}_{B_{R-r_n}} u \quad \text{and} \quad A = \frac{1}{V(x_0,R)} \int_{B(x_0,R+\rho)} u \, d\mu.$$

Since $B(\xi, r_n) \subset B_{R-r_n}$, we have

$$\int_{B(\xi,r_n)} u^2 \, d\mu \le M_n V(x_0, R) A.$$

Note that, by VD,

$$\frac{V(x_0,R)}{V(\xi,r_n+\rho)} \le \frac{V(x_0,R+\rho)}{V(\xi,r_n+\rho)} \le c'\left(1+\frac{d(x_0,\xi)+R+\rho}{r_n+\rho}\right)^{d_2} \le c''2^{nd_2}$$

and

$$\frac{V(x_0,R)}{V(\xi,r_n)} \le c'''2^{nd_2}.$$

Plugging these estimates into (4.22), we have

$$u(\xi)^2 \le c_2 2^{2nd_2}A^2 + c_3\Big(1+\frac{\rho}{R}\Big)^{d_2/\nu}\Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu} M_n A 2^{nd_2(1+1/\nu)}.$$

Since ξ is an arbitrary point in $B_{R-r_{n-1}}$, we obtain

$$M_{n-1}^2 \le c_4\Big(1+\frac{\rho}{R}\Big)^{d_2/\nu}\Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu}(A+e^{nb(1/\nu-1)}M_n)e^{2nb}A, \tag{4.23}$$

where $b = d_2\log 2$.

Our goal is to prove

$$M_0 \le c_0\Big(1+\frac{\rho}{R}\Big)^{d_2/\nu}\Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu}A$$

for some constant $c_0>0$ independent of x_0, R and ρ. If $M_0\le A$, then we are done, and so we only need to consider the case $M_0>A$. Then $A<M_0\le e^{nb(1/\nu-1)}M_n$ for all $n\ge 0$, because $\{M_n\}$ is increasing and without loss of generality we may and do assume that $\nu<1$. Therefore, (4.23) implies

$$M_{n-1}^2 \le 2c_4\Big(1+\frac{\rho}{R}\Big)^{d_2/\nu}\Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu}e^{nb(1+1/\nu)}M_nA.$$

From here we can argue similarly to [CG, p. 689-690]. By iterating the inequality above, we have

$$M_0^{2^n} \le \exp\left(b(1+1/\nu)\sum_{i=1}^n i2^{n-i}\right)\left[2c_4\Big(1+\frac{\rho}{R}\Big)^{d_2/\nu}\Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu}A\right]^{1+2+2^2+\cdots 2^{n-1}} M_n.$$

So

$$\begin{aligned} M_0 \le & c_5\left[2c_4\Big(1+\frac{\rho}{R}\Big)^{d_2/\nu}\Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu}A\right]^{1-2^{-n}} M_n^{2^{-n}} \\ \le & c_6\left[\Big(1+\frac{\rho}{R}\Big)^{d_2/\nu}\Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu}\right]A(M_n/A)^{2^{-n}}. \end{aligned}$$

Since u is bounded in B_R, $M_n\le c_7$ for all $n\ge 0$ and some constant $c_7>0$, so we have $\lim_{n\to\infty}(M_n/A)^{2^{-n}}=1$. We thus obtain

$$M_0 \le c_6\Big(1+\frac{\rho}{R}\Big)^{d_2/\nu}\Big(1+\frac{R}{\rho}\Big)^{\beta_2/\nu}A.$$

The proof is complete. □

4.3 $\mathrm{FK}(\phi) + \mathrm{J}_{\phi,\le} + \mathrm{CSJ}(\phi) \Longrightarrow \mathrm{E}_\phi$

The main result of this subsection is as follows.

Proposition 4.13. *Assume* VD, (1.13), $\mathrm{FK}(\phi)$, $\mathrm{J}_{\phi,\le}$ *and* $\mathrm{CSJ}(\phi)$ *hold. Then* E_ϕ *holds.*

In order to prove this, we first show that

Lemma 4.14. *Assume that* VD, (1.13) *and* $\mathrm{FK}(\phi)$ *hold. Then* $\mathrm{E}_{\phi,\le}$ *holds.*

Proof. By Proposition 7.3, under VD and (1.13), $\mathrm{FK}(\phi)$ implies that there is a constant $C > 0$ such that for any ball $B := B(x, r)$ with $x \in M$ and $r > 0$,

$$\operatorname{ess\,sup}_{x',y' \in B} p^B(t, x', y') \le \frac{C}{V(x, r)} \left(\frac{\phi(r)}{t}\right)^{1/\nu},$$

where ν is the constant in $\mathrm{FK}(\phi)$. Then for any $T \in (0, \infty)$ and all $x \in M_0$,

$$\begin{aligned}
\mathbb{E}^x \tau_B &= \int_0^\infty P_t^B \mathbf{1}_B(x)\,dt = \int_0^T P_t^B \mathbf{1}_B(x)\,dt + \int_T^\infty P_t^B \mathbf{1}_B(x)\,dt \\
&\le T + \int_T^\infty \int_B p^B(t, x, y)\,\mu(dy)\,dt \\
&\le T + C \int_T^\infty \left(\frac{\phi(r)}{t}\right)^{1/\nu} dt \le T + C_1 \phi(r)^{1/\nu} T^{1-1/\nu},
\end{aligned}$$

where in the last inequality we have used the fact that the constant ν in $\mathrm{FK}(\phi)$ can be assumed that $\nu \in (0, 1)$. Setting $T = \phi(r)$, we conclude that $\mathbb{E}^x \tau_B \le C_2 \phi(r)$. This proves $\mathrm{E}_{\phi,\le}$. □

Let $\{X_t^{(\rho)}\}$ be the Hunt process associated with the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$. For $\lambda > 0$, let ξ_λ be an exponential distributed random variable with mean $1/\lambda$, which is independent of the ρ-truncated process $\{X_t^{(\rho)}\}$.

Lemma 4.15. *Assume that* VD, (1.13), $\mathrm{FK}(\phi)$, $\mathrm{J}_{\phi,\le}$ *and* $\mathrm{CSJ}(\phi)$ *hold. Then for any* $c_0 \in (0, 1)$, *there exists a constant* $c_1 > 0$ *such that for all* $R > 0$ *and all* $x \in M_0$,

$$\mathbb{E}^x \left[\tau_{B(x,R)}^{(c_0 R)} \wedge \xi_{\phi(R)^{-1}}\right] \ge c_1 \phi(R).$$

Proof. For fixed $c_0 \in (0, 1)$ and $R > 0$, set $\rho = c_0 R$. Set $B = B(x, R)$, $\lambda = 1/\phi(R)$ and $u_\lambda(x) = \mathbb{E}^x\big(\tau_B^{(\rho)} \wedge \xi_\lambda\big)$ for $x \in M_0$; here and in the following we make some abuse of notation and use $\mathbb{E}$ for the expectation of the product measure of the truncated process $\{X_t^{(\rho)}\}$ and ξ_λ. Then for all $x \in M_0$,

$$u_\lambda(x) = \mathbb{E}^x \left[\int_0^{\tau_B^{(\rho)} \wedge \xi_\lambda} \mathbf{1}(X_t^{(\rho)})\,dt\right] = \mathbb{E}^x \left[\int_0^{\tau_B^{(\rho)}} e^{-\lambda t} \mathbf{1}(X_t^{(\rho)})\,dt\right] = G_\lambda^{(\rho),B} \mathbf{1}(x),$$

where $G_\lambda^{(\rho),B}$ is the λ-order resolvent for the truncated process $\{X_t^{(\rho)}\}$ killed on exiting B. Clearly u_λ is bounded and is in $\mathcal{F}_B^{(\rho)}$. Moreover, $u_\lambda(X^{(\rho)}_{t \wedge \tau_B^{(\rho)}})$ is a bounded supermartingale under $\mathbb{P}^x$ for every $x \in B \cap M_0$.

Set $u_{\lambda,\varepsilon} = u_\lambda + \varepsilon$ for any $\varepsilon > 0$. Since $t \mapsto u_{\lambda,\varepsilon}(X^{(\rho)}_{t\wedge\tau^{(\rho)}_B})$ is a bounded supermartingale under $\mathbb{P}^x$ for every $x \in B \cap M_0$, we have by Theorem 4.4 that $u_{\lambda,\varepsilon} \in \mathcal{F}^{(\rho),loc}_B$ and is $\mathcal{E}^{(\rho)}$-superharmonic in B. By $\mathrm{J}_{\phi,\le}$, CSJ(ϕ) and Proposition 2.3(5), we can choose a non-negative cut-off function $\varphi \in \mathcal{F}^{(\rho)}_B$ for $\frac{1+c_0}{2}B \subset B$ such that

$$\mathcal{E}^{(\rho)}(\varphi,\varphi) \le \frac{c_1\mu(B)}{\phi(R)}$$

and so

$$\mathcal{E}^{(\rho)}_\lambda(\varphi,\varphi) = \mathcal{E}^{(\rho)}(\varphi,\varphi) + \lambda\langle\varphi,\varphi\rangle \le \frac{c_1\mu(B)}{\phi(R)} + \lambda\mu(B) \le \frac{c_2\mu(B)}{\phi(R)}.$$

Furthermore, choose a continuous function g on $[0,\infty)$ such that $g(0) = 0$, $g(t) = \varepsilon^2 t^{-1}$ for $t \ge \varepsilon$ and $|g(t) - g(t')| \le |t - t'|$ for all $t, t' \ge 0$. According to [FOT, Theorem 1.4.2 (v) and (iii)], $u^{-1}_{\lambda,\varepsilon} = \varepsilon^{-2} g(u_{\lambda,\varepsilon}) \in \mathcal{F}^{(\rho),loc}_B$ and $u^{-1}_{\lambda,\varepsilon}\varphi^2 \in \mathcal{F}^{(\rho)}_B$. Thus, it follows from the fact

$$(u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y))(u_{\lambda,\varepsilon}(x)^{-1}\varphi^2(x) - u_{\lambda,\varepsilon}(y)^{-1}\varphi^2(y)) \le (\varphi(x) - \varphi(y))^2$$

that

$$\mathcal{E}^{(\rho)}_\lambda(u_{\lambda,\varepsilon}, u^{-1}_{\lambda,\varepsilon}\varphi^2) = \mathcal{E}^{(\rho)}(u_{\lambda,\varepsilon}, u^{-1}_{\lambda,\varepsilon}\varphi^2) + \lambda\langle u_{\lambda,\varepsilon}, u^{-1}_{\lambda,\varepsilon}\varphi^2\rangle \le \mathcal{E}^{(\rho)}(\varphi,\varphi) + \lambda\langle\varphi,\varphi\rangle = \mathcal{E}^{(\rho)}_\lambda(\varphi,\varphi).$$

Therefore,

$$\mathcal{E}^{(\rho)}_\lambda(u_{\lambda,\varepsilon}, u^{-1}_{\lambda,\varepsilon}\varphi^2) \le \frac{c_2\mu(B)}{\phi(R)}.$$

On the other hand, noticing again that $u^{-1}_{\lambda,\varepsilon}\varphi^2 \in \mathcal{F}^{(\rho)}_B$,

$$\begin{aligned}
\mathcal{E}^{(\rho)}_\lambda(u_{\lambda,\varepsilon}, u^{-1}_{\lambda,\varepsilon}\varphi^2) &= \varepsilon\mathcal{E}^{(\rho)}_\lambda(1, u^{-1}_{\lambda,\varepsilon}\varphi^2) + \mathcal{E}^{(\rho)}_\lambda(u_\lambda, u^{-1}_{\lambda,\varepsilon}\varphi^2)\\
&= \varepsilon\lambda\langle 1, u^{-1}_{\lambda,\varepsilon}\varphi^2\rangle + \langle 1, u^{-1}_{\lambda,\varepsilon}\varphi^2\rangle\\
&\ge \langle 1, u^{-1}_{\lambda,\varepsilon}\varphi^2\rangle \ge \int_{\frac{1+c_0}{2}B} u^{-1}_{\lambda,\varepsilon}\, d\mu,
\end{aligned}$$

and so

$$\int_{\frac{1+c_0}{2}B} u^{-1}_{\lambda,\varepsilon}\, d\mu \le \frac{c_2\mu(B)}{\phi(R)}.$$

Since $u_{\lambda,\varepsilon} \ge \varepsilon$ is $\mathcal{E}^{(\rho)}$-superharmonic in B with respect to the truncated process $\{X^{(\rho)}_t\}$, it follows that $u^{-1}_{\lambda,\varepsilon}(X^{(\rho)}_{t\wedge\tau^{(\rho)}_B})$ is a bounded $\mathbb{P}^x$-submartingale for every $x \in B \cap M_0$. Thus in view of Theorem 4.4, $u^{-1}_{\lambda,\varepsilon}$ is $\mathcal{E}^{(\rho)}$-subharmonic in B. Applying the L^1-mean value inequality (4.21) to $u^{-1}_{\lambda,\varepsilon}$ on $\frac{1-c_0}{2}B$, we get that

$$\operatorname{ess\,sup}_{\frac{1-c_0}{4}B} u^{-1}_{\lambda,\varepsilon} \le \frac{c_3}{\mu(B)} \int_{\frac{1+c_0}{2}B} u^{-1}_{\lambda,\varepsilon}\, d\mu \le \frac{c_4}{\phi(R)}.$$

Whence, $\operatorname{ess\,inf}_{\frac{1-c_0}{4}B} u_{\lambda,\varepsilon} \ge c_5\phi(R)$. Letting $\varepsilon \to 0$, we get $\operatorname{ess\,inf}_{\frac{1-c_0}{4}B} u_\lambda \ge c_5\phi(R)$. This yields the desired estimate. □

The next lemma is standard.

Lemma 4.16. *If* E_ϕ *holds, then for all* $x \in M_0$ *and* $r, t > 0$,

$$\mathbb{P}^x(\tau_{B(x,r)} \le t) \le 1 - \frac{c_1\phi(r)}{\phi(2r)} + \frac{c_2 t}{\phi(2r)}. \tag{4.24}$$

In particular, if (1.13) *and* E_ϕ *hold, then* $\mathrm{EP}_{\phi,\le,\varepsilon}$ *holds, i.e. for any ball* $B := B(x_0, r)$ *with* $x_0 \in M$ *and radius* $r > 0$*, there are constants* $\delta, \varepsilon \in (0,1)$ *such that*

$$\mathbb{P}^x(\tau_B \le t) \le \varepsilon \quad \text{for all } x \in B(x_0, r/4) \cap M_0 \tag{4.25}$$

provided that $t \le \delta\phi(r)$.

Proof. Suppose that there are constants $c_2 \ge c_1 > 0$ such that for all $x \in M_0$ and $r > 0$,

$$c_1\phi(r) \le \mathbb{E}^x \tau_{B(x,r)} \le c_2\phi(r).$$

Since for any $t > 0$, $\tau_{B(x,r)} \le t + (\tau_{B(x,r)} - t)\mathbf{1}_{\{\tau_{B(x,r)} \ge t\}}$, we have by the Markov property

$$\begin{aligned}\mathbb{E}^x \tau_{B(x,r)} &\le t + \mathbb{E}^x\Big[\mathbf{1}_{\{\tau_{B(x,r)} > t\}}\mathbb{E}^{X_t}[\tau_{B(x,r)} - t]\Big] \le t + \mathbb{P}^x(\tau_{B(x,r)} > t) \sup_{z \in B(x,r)} \mathbb{E}^z \tau_{B(x,r)} \\ &\le t + \mathbb{P}^x(\tau_{B(x,r)} > t) \sup_{z \in B(x,r)} \mathbb{E}^z \tau_{B(z,2r)} \le t + c_2\mathbb{P}^x(\tau_{B(x,r)} > t)\phi(2r).\end{aligned}$$

Then for all $x \in M_0$, $c_1\phi(r) \le \mathbb{E}^x\tau_{B(x,r)} \le t + c_2\mathbb{P}^x(\tau_{B(x,r)} > t)\phi(2r)$, proving (4.24). Since

$$\mathbb{P}^x(\tau_{B(x_0,r)} \le t) \le \mathbb{P}^x(\tau_{B(x,3r/4)} \le t), \quad x \in B(x_0, r/4) \cap M_0.$$

inequality (4.25) follows from (4.24) and (1.13). □

Lemma 4.17. *Assume that* VD, (1.13), FK(ϕ), $\mathrm{J}_{\phi,\le}$ *and* CSJ(ϕ) *hold. Then there exists a constant* $c_1 > 0$ *such that for all* $x \in M_0$ *and all* $R > 0$,

$$\mathbb{E}^x \tau_{B(x,R)} \ge c_1\phi(R).$$

Proof. Let $B = B(x, R)$, $\rho = cR$ for some $c \in (0,1)$ and $\lambda = 1/\phi(R)$. Recall that ξ_λ is an exponential distributed random variable with mean $1/\lambda$, which is independent of the ρ-truncated process $\{X_t^{(\rho)}\}$. Since it is clear that for all $x \in M_0$,

$$\mathbb{E}^x\Big[\tau_B^{(\rho)} \wedge \xi_\lambda\Big] \le \mathbb{E}^x\xi_\lambda = \phi(R),$$

using Lemma 4.15, we have

$$\mathbb{E}^x\Big[\tau_B^{(\rho)} \wedge \xi_\lambda\Big] \asymp \phi(R).$$

So by an argument similar to that of Lemma 4.16, we have for all $x \in M_0$,

$$\mathbb{P}^x\big(\tau_B^{(\rho)} \wedge \xi_\lambda \le t\big) \le 1 - c_1 + c_2 t/\phi(R).$$

In particular, choosing $c_3 > 0$ small enough, we have

$$\mathbb{P}^x(\tau_B^{(\rho)} \geq c_3\phi(R)) \geq \mathbb{P}^x\big(\tau_B^{(\rho)} \wedge \xi_\lambda \geq c_3\phi(R)\big) \geq c_4 > 0.$$

Next, let T_ρ be the first time that the size of jump bigger than ρ occurs for the original process $\{X_t\}$, and let $\{X_t^{(\rho)}\}$ be the truncated process associated with $\{X_t\}$. Then, as in the proof of [BGK1, Lemma 3.1(a)], we have

$$\mathbb{P}(T_\rho > t|\mathcal{F}_\infty^{X^{(\rho)}}) = \exp\left(-\int_0^t \mathcal{J}(X_s^{(\rho)})\,ds\right) \geq e^{-c_5 t/\phi(\rho)},$$

where

$$\mathcal{J}(x) := \int_{B(x,\rho)^c} J(x,y)\,\mu(dy) \leq c_5/\phi(\rho),$$

thanks to Lemma 2.1. So

$$\mathbb{P}(T_\rho > c_3\phi(R)|\mathcal{F}_\infty^{X^{(\rho)}}) \geq c_6.$$

This implies

$$\mathbb{P}^x\big(\tau_B^{(\rho)} \wedge T_\rho > c_3\phi(R)\big) = \mathbb{E}^x\left[\mathbf{1}_{\{\tau_B^{(\rho)} \geq c_3\phi(R)\}}\mathbb{E}^x\left[\mathbf{1}_{\{T_\rho > c_3\phi(R)\}}|\mathcal{F}_\infty^{X^{(\rho)}}\right]\right] \geq c_4 c_6 > 0.$$

Note that $\tau_B \geq \tau_B^{(\rho)} \wedge T_\rho$. (In fact, if $\tau_B^{(\rho)} < T_\rho$, then $\tau_B = \tau_B^{(\rho)}$; if $\tau_B^{(\rho)} \geq T_\rho$, then, by the fact that the truncated process $\{X_t^{(\rho)}\}$ coincides with the original $\{X_t\}$ till T_ρ, we also have $\tau_B \geq T_\rho$.) We obtain

$$\mathbb{P}^x(\tau_B > c_3\phi(R)) \geq c_4 c_6,$$

and so the desired estimate holds. □

4.4 $\mathrm{FK}(\phi) + \mathrm{E}_\phi + \mathrm{J}_{\phi,\leq} \Longrightarrow \mathrm{UHKD}(\phi)$

If $V(x,r) \asymp r^d$ for each $r > 0$ and $x \in M$ with some constant $d > 0$, then $\mathrm{FK}(\phi) \Longrightarrow \mathrm{UHKD}(\phi)$ is well-known; e.g. see the remark in the proof of [GT, Theorem 4.2]. However, in non-uniform VD settings, it is highly non-trivial to establish the on-diagonal upper bound estimate $\mathrm{UHKD}(\phi)$ from $\mathrm{FK}(\phi)$. Below, we will adopt the truncating argument and significantly modify the iteration techniques in [Ki, Proof of Theorem 2.9] and [GH, Lemma 5.6]. Without further mention, throughout the proof we will assume that μ and ϕ satisfy VD and (1.13), respectively.

Recall that for $\rho > 0$, $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is the ρ-truncated Dirichlet form defined as in (2.2). It is clear that for any function $f, g \in \mathcal{F}$ with $\mathrm{dist}(\mathrm{supp}\, f, \mathrm{supp}\, g) > \rho$, $\mathcal{E}^{(\rho)}(f,g) = 0$. For any non-negative open set $D \subset M$, denote by $\{P_t^D\}$ and $\{Q_t^{(\rho),D}\}$ the semigroups of $(\mathcal{E}, \mathcal{F}_D)$ and $(\mathcal{E}^{(\rho)}, \mathcal{F}_D)$, respectively. We write $\{Q_t^{(\rho),M}\}$ as $\{Q_t^{(\rho)}\}$ for simplicity.

We next give the following preliminary heat kernel estimate.

Lemma 4.18. *Suppose that* VD, (1.13), $\mathrm{FK}(\phi)$ *and* $\mathrm{J}_{\phi,\leq}$ *hold. For any ball $B = B(x,r)$ with $x \in M$ and $r > 0$, the semigroup $\{Q_t^{(\rho),B}\}$ possesses the heat kernel $q^{(\rho),B}(t,x,y)$, which satisfies that there exist constants $C, c_0, \nu > 0$ (independent of ρ) such that for all $t > 0$ and $x', y' \in B \cap M_0$,*

$$q^{(\rho),B}(t,x',y') \leq \frac{C}{V(x,r)}\left(\frac{\phi(r)}{t}\right)^{1/\nu} \exp\left(\frac{c_0 t}{\phi(\rho)}\right).$$

Proof. First, by Proposition 7.3, FK(ϕ) implies that there exist constants $C_1, \nu > 0$ such that for any ball $B = B(x, r)$,

$$\frac{V(x,r)^\nu}{\phi(r)} \|u\|_2^{2+2\nu} \le C_1 \mathcal{E}(u,u) \|u\|_1^{2\nu}, \quad \forall u \in \mathcal{F}_B.$$

According to (2.3), there is a constant $c_0 > 0$ such that

$$\frac{V(x,r)^\nu}{\phi(r)} \|u\|_2^{2+2\nu} \|u\|_1^{-2\nu} \le C_1 \Big(\mathcal{E}^{(\rho)}(u,u) + \frac{c_0 \|u\|_2^2}{\phi(\rho)} \Big) =: C_1 \mathcal{E}^{(\rho)}_{c_0/\phi(\rho)}(u,u), \quad \forall u \in \mathcal{F}_B^{(\rho)}.$$

According to Proposition 7.3 again (to the Dirichelt form $\mathcal{E}^{(\rho)}_{c_0/\phi(\rho)}$), this yields the required assertion. □

Let $\{X_t^{(\rho)}\}$ be the Hunt process associated with the Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$. For any subset open set D, let $\tau_D^{(\rho)}$ be the first exit time from D by the Hunt process $\{X_t^{(\rho)}\}$.

Lemma 4.19. *Suppose that* VD, (1.13), E_ϕ *and* $\mathrm{J}_{\phi,\le}$ *hold. Then there are constants* $c_1, c_2 > 0$ *such that for any* $r, t, \rho > 0$,

$$\mathbb{P}^x(\tau^{(\rho)}_{B(x,r)} \le t) \le 1 - c_1 + \frac{c_2 t}{\phi(2r) \wedge \phi(\rho)}, \quad x \in M_0.$$

Proof. First, by (1.13), E_ϕ and Lemma 4.16, we know that for all $x \in M_0$ and $r, t > 0$,

$$\mathbb{P}^x(\tau_{B(x,r)} \le t) \le 1 - c_1 + \frac{c_2 t}{\phi(2r)}.$$

Denote by $B = B(x, r)$ for $x \in M$ and $r > 0$. According to Lemma 7.8, for all $t > 0$ and all $x \in M_0$,

$$P_t^B \mathbf{1}_B(x) \le Q_t^{(\rho),B} \mathbf{1}_B(x) + \frac{c_3 t}{\phi(\rho)}. \tag{4.26}$$

Combining both estimates above with the facts that

$$1 - P_t^B \mathbf{1}_B(x) = \mathbb{P}^x(\tau_B \le t), \quad 1 - Q_t^{(\rho),B} \mathbf{1}_B(x) = \mathbb{P}^x(\tau_B^{(\rho)} \le t),$$

we prove the desired assertion. □

Lemma 4.20. *Suppose that* VD, (1.13), E_ϕ *and* $\mathrm{J}_{\phi,\le}$ *hold. Then there are constants* $\varepsilon \in (0,1)$ *and* $c > 0$ *such that for any* $r, \lambda, \rho > 0$ *with* $\lambda \ge \frac{c}{\phi(r \wedge \rho)}$,

$$\mathbb{E}^x[e^{-\lambda \tau^{(\rho)}_{B(x,r)}}] \le 1 - \varepsilon, \quad x \in M_0.$$

Proof. Denote by $B = B(x, r)$. Using Lemma 4.19, we have for any $t > 0$ and all $x \in M_0$,

$$\begin{aligned} \mathbb{E}^x\left[e^{-\lambda \tau_B^{(\rho)}}\right] &= \mathbb{E}^x\left[e^{-\lambda \tau_B^{(\rho)}} \mathbf{1}_{\{\tau_B^{(\rho)} < t\}}\right] + \mathbb{E}^x\left[e^{-\lambda \tau_B^{(\rho)}} \mathbf{1}_{\{\tau_B^{(\rho)} \ge t\}}\right] \\ &\le \mathbb{P}^x(\tau_B^{(\rho)} < t) + e^{-\lambda t} \le 1 - c_1 + \frac{c_2 t}{\phi(2r) \wedge \phi(\rho)} + e^{-\lambda t}. \end{aligned}$$

Next, set $\varepsilon = \frac{c_1}{4} > 0$. Taking $t = c_3 \phi(r \wedge \rho)$ for some $c_3 > 0$ such that $\frac{c_2 t}{\phi(2r)} + \frac{c_2 t}{\phi(\rho)} \le 2\varepsilon$, and $\lambda > 0$ such that $e^{-\lambda t} \le \varepsilon$ in the inequality above, we obtain the desired assertion. □

The following lemma furthermore improves the estimate established in Lemma 4.20.

Lemma 4.21. *Suppose that* VD*,* (1.13)*,* E_ϕ *and* $J_{\phi,\le}$ *hold. Then there exist constants* $C, c_0 > 0$ *such that for all* $x \in M_0$ *and* $R, \rho > 0$

$$\mathbb{E}^x\left[e^{-\frac{c}{\phi(\rho)}\tau^{(\rho)}_{B(x,R)}}\right] \le C\exp\left(-c_0 R/\rho\right), \tag{4.27}$$

where $c > 0$ *is the constant in Lemma* 4.20*. In particular,* $(\mathcal{E}, \mathcal{F})$ *is conservative.*

Proof. We only need to consider the case that $\rho \in (0, R/2)$, since the conclusion holds trivially when $\rho \ge R/2$. For simplicity, we drop the superscript ρ from $\tau^{(\rho)}$. For any $z \in M_0$ and $R > 0$, set $\tau = \tau_{B(z,R)}$. For any fixed $0 < r < \frac{R}{2}$, set $n = \left[\frac{R}{2r}\right]$. Let $u(x) = \mathbb{E}^x[e^{-\lambda\tau}]$ for $x \in M_0$, and $m_k = \|u\|_{L^\infty(\overline{B(z,kr)};\mu)}$, $k = 1, 2, \cdots, n$. For any $0 < \varepsilon' < \varepsilon$ where ε is the constant for Lemma 4.20, we can choose $x_k \in \overline{B(z, kr)} \cap M_0$ such that $(1-\varepsilon')m_k \le u(x_k) \le m_k$. For any $k \le n-1$, $B(x_k, r) \subset B(z, (k+1)r) \subset B(z, R)$.

Next, we consider the following function in $B(x_k, r) \cap M_0$:

$$v_k(x) = \mathbb{E}^x[e^{-\lambda\tau_k}],$$

where $\tau_k = \tau_{B(x_k,r)}$. Recall that $\{X^{(\rho)}_t\}$ is the Hunt process associated with the semigroup $\{Q^{(\rho)}_t\}$. By the strong Markov property, for any $x \in B(x_k, r) \cap M_0$,

$$\begin{aligned} u(x) = \mathbb{E}^x[e^{-\lambda\tau}] &= \mathbb{E}^x\left[e^{-\lambda\tau_k}e^{-\lambda(\tau-\tau_k)}\right] \\ &= \mathbb{E}^x\left[e^{-\lambda\tau_k}\mathbb{E}^{X^{(\rho)}_{\tau_k}}(e^{-\lambda\tau})\right] = \mathbb{E}^x\left[e^{-\lambda\tau_k}u(X^{(\rho)}_{\tau_k})\right] \\ &\le \mathbb{E}^x\left[e^{-\lambda\tau_k}\right]\|u\|_{L^\infty(\overline{B(x_k,r+\rho)};\mu)} = v_k(x)\|u\|_{L^\infty(\overline{B(x_k,r+\rho)};\mu)}, \end{aligned}$$

where we have used the fact that $X^{(\rho)}_{\tau_k} \in \overline{B(x_k, r+\rho)}$ in the inequality above. It follows that for any $0 < \rho \le r$,

$$u(x_k) \le v_k(x_k)\|u\|_{L^\infty(\overline{B(x_k,r+\rho)};\mu)} \le v_k(x_k)m_{k+2},$$

whence

$$(1-\varepsilon')m_k \le v_k(x_k)m_{k+2}.$$

According to Lemma 4.20, if $\lambda \ge \frac{c}{\phi(\rho)}$ and $0 < \rho \le r$ (here c is the constant in Lemma 4.20), then

$$(1-\varepsilon')m_k \le (1-\varepsilon)m_{k+2},$$

whence it follows by iteration that

$$u(z) \le m_1 \le \left(\frac{1-\epsilon}{1-\varepsilon'}\right)^{n-1} m_{2n-1} \le C\exp\left(-c_0\frac{R}{r}\right),$$

where in the last inequality we have used that $n \ge \frac{R}{2r} - 1$, $m_{2n-1} \le 1$ and $c_0 := \frac{1}{2}\log\frac{1-\varepsilon'}{1-\varepsilon}$. This completes the proof of (4.27).

To see that this implies that $(\mathcal{E}, \mathcal{F})$ is conservative, take $R \to \infty$ in (4.27). Then one has $\mathbb{E}^x\left(e^{-\frac{c}{\phi(\rho)}\zeta^{(\rho)}}\right) = 0$ for all $x \in M_0$, where $\zeta^{(\rho)}$ is the lifetime of $\{X^{(\rho)}_t\}$. So we conclude $\zeta^{(\rho)} = \infty$ a.s. This together with Lemma 2.1 implies that $(\mathcal{E}, \mathcal{F})$ is conservative. Indeed, the process

$\{X_t\}$ can be obtained from $\{X_t^{(\rho)}\}$ through Meyer's construction as discussed in Section 7.2, and therefore the conservativeness of $(\mathcal{E}, \mathcal{F})$ follows immediately from that of $(\mathcal{E}^{(\rho)}, \mathcal{F})$ corresponding to the process $\{X_t^{(\rho)}\}$. □

Since $\mathrm{J}_{\phi,\geq}$ implies FK(ϕ) under an additional assumption RVD (see Subsection 4.1) and FK(ϕ) + $\mathrm{J}_{\phi,\leq}$ + CSJ(ϕ) imply E_ϕ (see Subsection 4.3), together with the above lemma, we see that each of Theorem 1.13 (2), (3), (4) and Theorem 1.15 (2), (3), (4) implies the conservativeness of $(\mathcal{E}, \mathcal{F})$.

As a direct consequence of Lemma 4.21, we have the following corollary.

Corollary 4.22. *Suppose that* VD, (1.13), E_ϕ *and* $\mathrm{J}_{\phi,\leq}$ *hold. Then there exist constants* $C, c_1, c_2 > 0$ *such that for any* $R, \rho > 0$ *and for all* $x \in M_0$,

$$\mathbb{P}^x(\tau^{(\rho)}_{B(x,R)} \leq t) \leq C \exp\left(-c_1 \frac{R}{\rho} + c_2 \frac{t}{\phi(\rho)}\right). \tag{4.28}$$

In particular, for any $\varepsilon > 0$*, there is a constant* $c_0 > 0$ *such that for any ball* $B = B(x, R)$ *with* $x \in M_0$ *and* $R > 0$*, and any* $\rho > 0$ *with* $\phi(\rho) \geq t$ *and* $R \geq c_0\rho$,

$$\mathbb{P}^z(\tau^{(\rho)}_B \leq t) \leq \varepsilon \quad \text{for all } z \in B(x, R/2) \cap M_0.$$

Proof. Denote by $B = B(x, R)$ for $x \in M$ and $R > 0$. Using Lemma 4.21, we obtain that, for any $t, \rho > 0$ and all $x \in M_0$,

$$\begin{aligned}\mathbb{P}^x(\tau^{(\rho)}_B \leq t) =& \mathbb{P}^x(e^{-\frac{c}{\phi(\rho)}\tau^{(\rho)}_B} \geq e^{-c\frac{t}{\phi(\rho)}}) \leq e^{c\frac{t}{\phi(\rho)}} \mathbb{E}^x(e^{-\frac{c}{\phi(\rho)}\tau^{(\rho)}_B}) \\ \leq& C \exp\left(-c_1 \frac{R}{\rho} + c\frac{t}{\phi(\rho)}\right).\end{aligned}$$

This proves the first assertion. The second assertion immediately follows from the first one and the fact that $\mathbb{P}^z(\tau^{(\rho)}_B \leq t) \leq \mathbb{P}^z(\tau^{(\rho)}_{B(z,R/2)} \leq t)$ for all $z \in B(x, R/2) \cap M_0$. □

Given the above control of the exit time, we now aim to prove UHKD(ϕ). As the first step, we obtain the on-diagonal upper bound for the heat kernel of $\{Q^{(\rho)}_t\}$. The proof is a non-trivial modification of [GH, Lemma 5.6]. For any open subset D of M and any $\rho > 0$, we define $D_\rho = \{x \in M : d(x, D) < \rho\}$. Recall that, for $B = B(x_0, r)$ and $a > 0$, we use aB to denote the ball $B(x_0, ar)$

Proposition 4.23. *Suppose that* VD, (1.13), FK(ϕ), E_ϕ *and* $\mathrm{J}_{\phi,\leq}$ *hold. Then the semigroup* $\{Q^{(\rho)}_t\}$ *possesses the heat kernel* $q^{(\rho)}(t, x, y)$*, and there are two constants* C, $c > 0$ *such that for any* $x \in M$ *and* $\rho, t > 0$ *with* $\phi(\rho) \geq ct$,

$$\operatorname{ess\,sup}_{x', y' \in B(x,\rho)} q^{(\rho)}(t, x', y') \leq \frac{C}{V(x, \rho)} \left(\frac{\phi(\rho)}{t}\right)^{1/\nu}. \tag{4.29}$$

Proof. Fix $x_0 \in M$. For any $t > 0$, $R > r + \rho$ and $r \geq \rho$, set $U = B(x_0, r)$ and $D = B(x_0, R)$. Then $\frac{1}{4}U_\rho \subset \frac{1}{2}U$. By Corollary 4.22, for any $\varepsilon \in (0, 1)$ (which is assumed to be chosen small enough), there is a constant $c_0 := c_0(\varepsilon) > 1$ large enough such that for all $\phi(\rho) \geq t$ and $r \geq c_0\rho$,

$$\operatorname{ess\,sup}_{x \in \frac{1}{4}U_\rho}(1 - Q^{(\rho),U}_t \mathbf{1}_U(x)) \leq \operatorname{ess\,sup}_{x \in \frac{1}{2}U}(1 - Q^{(\rho),U}_t \mathbf{1}_U(x))$$

$$=\text{ess sup}_{x\in \frac{1}{2}U}\mathbb{P}^x(\tau_U^{(\rho)}\le t)\le \varepsilon.$$

Then by (7.2) in Lemma 7.9 with $V=\frac{1}{4}U_\rho$, we have for any $t,s>0$, $\phi(\rho)\ge t$ and $r\ge c_0\rho$,

$$\begin{aligned}\text{ess sup}_{x,y\in\frac{1}{4}U_\rho} q^{(\rho),D}(t+s,x,y) &\le \text{ess sup}_{x,y\in U} q^{(\rho),U}(t,x,y)+\varepsilon\,\text{ess sup}_{x,y\in U_\rho} q^{(\rho),D}(s,x,y)\\ &\le \text{ess sup}_{x,y\in U_\rho} q^{(\rho),U_\rho}(t,x,y)+\varepsilon\,\text{ess sup}_{x,y\in U_\rho} q^{(\rho),D}(s,x,y).\end{aligned}$$

Furthermore, due to Lemma 4.18, there exist constants c_1, $\nu>0$ (independent of c_0) such that for any $r,\rho,t>0$ with $\phi(\rho)\ge t$ and $r\ge c_0\rho$,

$$\text{ess sup}_{x,y\in U_\rho} q^{(\rho),U_\rho}(t,x,y)\le \frac{c_1}{V(x_0,r)}\left(\frac{\phi(r)}{t}\right)^{1/\nu} := Q_t(r).$$

According to both inequalities above, we obtain that for any $t,s>0$, $R>r+\rho$, $\phi(\rho)\ge t$ and $r\ge c_0\rho$,

$$\text{ess sup}_{x,y\in\frac{1}{4}U_\rho} q^{(\rho),D}(t+s,x,y)\le Q_t(r)+\varepsilon\,\text{ess sup}_{x,y\in U_\rho} q^{(\rho),D}(s,x,y). \tag{4.30}$$

Now, for fixed $t>0$, let $\phi(\rho)\ge t$ and

$$t_k=\frac{1}{2}(1+2^{-k})t,\quad r_k=4^kc_0\rho-\rho,\quad B_k=B(x_0,r_k+\rho)$$

for $k\ge 0$. In particular, $t_0=t$, $r_0=(c_0-1)\rho$ and $B_0=B(x_0,c_0\rho)$.

Applying (4.30) with $r=r_{k+1}$, $s=t_{k+1}$ and $t+s=t_k$ yielding that

$$\text{ess sup}_{x,y\in B_k} q^{(\rho),D}(t_k,x,y)\le Q_{2^{-(k+2)}t}(r_{k+1})+\varepsilon\,\text{ess sup}_{x,y\in B_{k+1}} q^{(\rho),D}(t_{k+1},x,y), \tag{4.31}$$

where we have used the facts that $\phi(\rho)\ge t\ge t_k$ and $r_k\ge c_0\rho$ for all $k\ge0$. Note that, by (1.13),

$$\begin{aligned}Q_{2^{-(k+2)}t}(r_{k+1}) &= \frac{c_1}{V(x_0,r_{k+1})}\left(\frac{\phi(r_{k+1})}{2^{-(k+2)}t}\right)^{1/\nu}\\ &\le \frac{c_1}{V(x_0,r_k)}\left(\frac{\phi(r_k)}{2^{-(k+1)}t}\right)^{1/\nu}2^{1/\nu}c'\left(\frac{r_{k+1}}{r_k}\right)^{\beta_2/\nu}\\ &\le LQ_{2^{-(k+1)}t}(r_k),\end{aligned}$$

where L is a constant independent of c_0 and x_0. Without loss of generality, we may and do assume that ε is small enough and $L\ge 2^{1/\nu}$ such that $\varepsilon L\le\frac{1}{2}$. By this inequality, we can get that

$$Q_{2^{-(k+2)}t}(r_{k+1})\le LQ_{2^{-(k+1)}t}(r_k)\le L^2Q_{2^{-k}t}(r_{k-1})\le\cdots\le L^{k+2}Q_t(r_0).$$

Hence, it follows from (4.31) that

$$\text{ess sup}_{x,y\in B_k} q^{(\rho),D}(t_k,x,y)\le L^{k+2}Q_t(r_0)+\varepsilon\,\text{ess sup}_{x,y\in B_{k+1}} q^{(\rho),D}(t_{k+1},x,y),$$

which gives by iteration that for any positive integer n,

$$\begin{aligned}\text{ess sup}_{x,y\in B_0} q^{(\rho),D}(t_0,x,y) \le& L^2(1+L\varepsilon+(L\varepsilon)^2+\cdots)Q_t(r_0)\\ &+\varepsilon^n\,\text{ess sup}_{x,y\in B_n} q^{(\rho),D}(t_n,x,y)\\ \le& 2L^2Q_t(r_0)+\varepsilon^n\,\text{ess sup}_{x,y\in B_n} q^{(\rho),D}(t_n,x,y),\end{aligned} \tag{4.32}$$

as long as $B_n \subset D$.

By Lemma 4.18, VD and (1.13), there exists a constant $L_1 > 0$ (also independent of c_0) such that

$$\text{ess sup}_{x,y\in B_n} q^{(\rho),B_n}(t_n, x, y) \le c'' Q_{t_n}(r_n) \le c''' L_1^n Q_t(r_0).$$

Again, without loss of generality, we may and do assume that $L_1 \le L$ and so $0 < \varepsilon L_1 \le \frac{1}{2}$; otherwise, we replace L with $L + L_1$ below. In particular,

$$\lim_{n\to\infty} \varepsilon^n \,\text{ess sup}_{x,y\in B_n} q^{(\rho),B_n}(t_n, x, y) \le c''' Q_t(r_0) \lim_{n\to\infty} (\varepsilon L_1)^n = 0.$$

Putting both estimates above into (4.32) with $D = B_n$, we find that

$$\limsup_{n\to\infty} \text{ess sup}_{x,y\in B_0} q^{(\rho),B_n}(t, x, y) \le 2L^2 Q_t((c_0 - 1)\rho). \tag{4.33}$$

Having (4.33) at hand, we can follow the argument of [GH, Lemma 5.6] to complete the proof, see [GH, p. 540]. Indeed, the sequence $\{q^{(\rho),B_n}(t, \cdot, \cdot)\}$ increases as $n \to \infty$ and converges almost everywhere on $M \times M$ to a non-negative measurable function $q^{(\rho)}(t, \cdot, \cdot)$; see [GT, Theorem 2.12 (b) and (c)]. The function $q^{(\rho)}(t, \cdot, \cdot)$ is finite almost everywhere since

$$\int_{B_n} q^{(\rho),B_n}(t, x, y)\, \mu(dy) \le 1.$$

For any non-negative function $f \in L^2(M; \mu)$, we have by the monotone convergence theorem,

$$\lim_{n\to\infty} \int_{B_n} q^{(\rho),B_n}(t, x, y) f(y)\, \mu(dy) = \int q^{(\rho)}(t, x, y) f(y)\, \mu(dy).$$

On the other hand,

$$\lim_{n\to\infty} \int_{B_n} q^{(\rho),B_n}(t, x, y) f(y)\, \mu(dy) = \lim_{n\to\infty} Q_t^{(\rho),B_n} f(x) = Q_t^{(\rho)} f(x),$$

see [GT, Theorem 2.12(c)] again. Hence, $q^{(\rho)}(t, x, y)$ is the heat kernel of $\{Q_t^{(\rho)}\}$. Thus it follows from (4.33) that there exist constants C, $c > 0$ (independent of ρ) such that (4.29) holds for all $x_0 \in M$, $t > 0$ and $\phi(\rho) \ge ct$. □

For any $\rho > 0$ and $x, y \in M$, set

$$J_\rho(x, y) := J(x, y)\mathbf{1}_{\{d(x,y)>\rho\}}.$$

Using the Meyer's decomposition and Lemma 7.2(i), we have the following estimate

$$p(t, x, y) \le q^{(\rho)}(t, x, y) + \mathbb{E}^x\Big[\int_0^t \int_M J_\rho(Y_s, z) p_{t-s}(z, y)\, \mu(dz)\, ds\Big], \quad x, y \in M_0. \tag{4.34}$$

The following is a key proposition.

Proposition 4.24. *Suppose that* VD, (1.13), E_ϕ *and* $\mathrm{J}_{\phi,\le}$ *hold. Then there exists a constant* $c_1 > 0$ *such that the following estimate holds for all* $t, \rho > 0$ *and all* $x \in M_0$,

$$\mathbb{E}^x \left[\int_0^t \int_M J_\rho(X_s^{(\rho)}, z) p(t - s, z, y)\, \mu(dz)\right] \le \frac{c_1 t}{V(x, \rho)\phi(\rho)} \exp\Big(c_1 \frac{t}{\phi(\rho)}\Big).$$

Proof. By $\mathrm{J}_{\phi,\le}$, $J_\rho(x,y) \le \frac{c_1}{V(x,\rho)\phi(\rho)}$ for all $x, y \in M$. By the fact that $p(t,z,y) = p(t,y,z)$, for all $x \in M_0$,

$$\begin{aligned}
&\mathbb{E}^x\left[\int_0^t \int_M J_\rho(X_s^{(\rho)}, z) p(t-s,z,y)\,\mu(dz)\right]\\
&\le c_1 \mathbb{E}^x\left[\int_0^t \frac{1}{V(X_s^{(\rho)},\rho)\phi(\rho)}\,ds\right]\\
&= c_1 \sum_{k=1}^\infty \mathbb{E}^x\left[\int_0^t \frac{1}{V(X_s^{(\rho)},\rho)\phi(\rho)}\,ds; \tau^{(\rho)}_{B(x,k\rho)} \ge t > \tau^{(\rho)}_{B(x,(k-1)\rho)}\right]\\
&=: c_1 \sum_{k=1}^\infty I_k.
\end{aligned}$$

If $t \le \tau^{(\rho)}_{B(x,k\rho)}$, then $d(X_s^{(\rho)}, x) \le k\rho$ for all $s \le t$. This along with VD yields that for all $k \ge 1$,

$$\begin{aligned}
\frac{1}{V(X_s^{(\rho)},\rho)\phi(\rho)} \le& \frac{c_2 k^{d_2}}{V(X_s^{(\rho)},2k\rho)\phi(\rho)} \le \frac{c_2 k^{d_2}}{\inf_{d(z,x)\le k\rho} V(z,2k\rho)\phi(\rho)}\\
\le& \frac{c_2 k^{d_2}}{V(x,k\rho)\phi(\rho)} \le \frac{c_2 k^{d_2}}{V(x,\rho)\phi(\rho)}.
\end{aligned}$$

In particular, we have

$$I_1 \le \frac{c_2}{V(x,\rho)\phi(\rho)}.$$

Thus, by Corollary 4.22, for all $k \ge 2$,

$$\begin{aligned}
I_k \le& \frac{c_3 t k^{d_2}}{V(x,\rho)\phi(\rho)} \mathbb{P}^x(\tau^{(\rho)}_{B(x,(k-1)\rho)} < t)\\
\le& \frac{c_4 t}{V(x,\rho)\phi(\rho)} e^{c_5 \frac{t}{\phi(\rho)}} k^{d_2} e^{-c_6 k} \le \frac{c_4 t}{V(x,\rho)\phi(\rho)} e^{c_5 \frac{t}{\phi(\rho)}} e^{-c_7 k}.
\end{aligned}$$

This yields the desired assertion. □

Given all the above estimates, we can obtain the main theorem in this subsection.

Theorem 4.25. *Suppose that* VD*,* (1.13)*,* FK(ϕ)*,* E_ϕ *and* $\mathrm{J}_{\phi,\le}$ *hold. Then* UHKD(ϕ) *is satisfied, i.e. there is a constant $c > 0$ such that for all $x \in M_0$ and $t > 0$,*

$$p(t,x,x) \le \frac{c}{V(x,\phi^{-1}(t))}.$$

Proof. For each $t > 0$, set $\rho = \phi^{-1}(ct)$, where $c > 0$ is the constant in Proposition 4.23. Then by Proposition 4.23, for all $x \in M_0$,

$$q^{(\rho)}(t,x,x) \le \frac{c_1}{V(x,\phi^{-1}(t))}.$$

Using this, (4.34) and Proposition 4.24, for all $x \in M_0$, we have

$$p(t,x,x) \le q^{(\rho)}(t,x,x) + \frac{c_2 t}{V(x,\rho)\phi(\rho)} \exp\left(c_2 \frac{t}{\phi(\rho)}\right) \le \frac{c_3}{V(x,\phi^{-1}(t))},$$

thanks to $\phi(\rho) = ct$, VD and (1.13). □

5 Consequences of condition J_ϕ and mean exit time condition E_ϕ

In this section, we will first prove (2) $\Longrightarrow$ (1) in Theorem 1.15 and then prove (2) $\Longrightarrow$ (1) in Theorem 1.13. Without any mention, throughout the proof we will assume that μ and ϕ satisfy VD, RVD and (1.13) respectively. (Indeed, RVD is only used in the proof of $\mathrm{J}_{\phi,\geq} \Longrightarrow \mathrm{FK}(\phi)$.) We note that (2) implies the conservativeness of $(\mathcal{E},\mathcal{F})$ due to Lemma 4.21.

Recall again that, for any $\rho>0$, $(\mathcal{E}^{(\rho)},\mathcal{F})$ defined in (2.2) denotes the ρ-truncated Dirichlet form obtained by ρ-truncation for the jump density of the original Dirichlet form $(\mathcal{E},\mathcal{F})$. Let $\{X_t^{(\rho)}\}$ be the Hunt process associated with the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F})$. For any open subset $D\subset M$, let $\tau_D^{(\rho)}$ be the first exit time of the process $\{X_t^{(\rho)}\}$. For any open subset $D\subset M$ and $\rho>0$, set $D_\rho=\{x\in M: d(x,D)<\rho\}$.

5.1 $\mathrm{UHKD}(\phi)+\mathrm{J}_{\phi,\leq}+\mathrm{E}_\phi \Longrightarrow \mathrm{UHK}(\phi)$, $\mathrm{J}_\phi+\mathrm{E}_\phi \Longrightarrow \mathrm{UHK}(\phi)$

We begin with the following improved statement for $\mathrm{UHKD}(\phi)$.

Lemma 5.1. *Under* VD *and* (1.13)*, if* $\mathrm{UHKD}(\phi)$*,* $\mathrm{J}_{\phi,\leq}$ *and* E_ϕ *hold, then there is a constant* $c>0$ *such that for any* $t>0$ *and all* $x,y\in M_0$*,*

$$p(t,x,y)\leq c\left(\frac{1}{V(x,\phi^{-1}(t))}\wedge\frac{1}{V(y,\phi^{-1}(t))}\right).$$

Proof. First, using the first conclusion in Lemma 7.2(ii), Lemma 2.1 and $\mathrm{UHKD}(\phi)$, we can easily see that the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F})$ has the heat kernel $q^{(\rho)}(t,x,y)$, and

$$q^{(\rho)}(t,x,x)\leq p(t,x,x)\exp\Big(c_1\frac{t}{\phi(\rho)}\Big)\leq\frac{c_2}{V(x,\phi^{-1}(t))}\exp\Big(c_1\frac{t}{\phi(\rho)}\Big),$$

for all $t>0$ and all $x\in M_0$, where $c_1,c_2>0$ are independent of ρ. Then by the symmetry of $q^{(\rho)}(t,x,y)$ and the Cauchy-Schwarz inequality, for all $t>0$ and all $x,y\in M_0$,

$$q^{(\rho)}(t,x,y)\leq\sqrt{q^{(\rho)}(t,x,x)q^{(\rho)}(t,y,y)}\leq\frac{c_2}{\sqrt{V(x,\phi^{-1}(t))V(y,\phi^{-1}(t))}}\exp\Big(c_1\frac{t}{\phi(\rho)}\Big).\tag{5.1}$$

Second, let U and V be two open subsets of M such that U_ρ and V_ρ are precompact, and $U\cap V=\emptyset$. According to Lemma 7.10, for any $t>0$ and all $x\in U\cap M_0$ and $y\in V\cap M_0$,

$$\begin{aligned}q^{(\rho)}(2t,x,y)\leq&\,\mathbb{P}^x(\tau_U^{(\rho)}\leq t)\operatorname{ess\,sup}_{t\leq t'\leq 2t}\|q^{(\rho)}(t',\cdot,y)\|_{L^\infty(U_\rho,\mu)}\\&+\mathbb{P}^y(\tau_V^{(\rho)}\leq t)\operatorname{ess\,sup}_{t\leq t'\leq 2t}\|q^{(\rho)}(t',\cdot,x)\|_{L^\infty(V_\rho;\mu)}\\\leq&\Big(\mathbb{P}^x(\tau_U^{(\rho)}\leq t)+\mathbb{P}^y(\tau_V^{(\rho)}\leq t)\Big)\operatorname{ess\,sup}_{x'\in U_\rho,y'\in V_\rho,t\leq t'\leq 2t}q^{(\rho)}(t',x',y').\end{aligned}$$

Then taking $U=B(x,r)$ and $V=B(y,r)$ with $r=\frac14 d(x,y)$ in the inequality above, and using Corollary 4.22 and (5.1), we find that for any $t,\rho>0$ and all $x,y\in M_0$,

$$\begin{aligned}q^{(\rho)}(2t,x,y)\leq&\,c_3\exp\Big(-c_4\frac{r}{\rho}+c_5\frac{t}{\phi(\rho)}\Big)\operatorname{ess\,sup}_{x'\in B(x,r+\rho),y'\in B(y,r+\rho)}\frac{1}{\sqrt{V(x',\phi^{-1}(t))V(y',\phi^{-1}(t))}}\\\leq&\frac{c_6}{V(x,\phi^{-1}(t))}\left(1+\frac{r+\rho}{\phi^{-1}(t)}\right)^{d_2}\exp\Big(-c_4\frac{r}{\rho}+c_5\frac{t}{\phi(\rho)}\Big).\end{aligned}$$

This along with (4.34) and Proposition 4.24 yields that for any $t, \rho > 0$ and all $x, y \in M_0$,

$$p(t,x,y) \le c_7 \left[\frac{1}{V(x,\phi^{-1}(t))} \left(1 + \frac{r+\rho}{\phi^{-1}(t)}\right)^{d_2} \exp\Big(-c_4 \frac{r}{\rho}\Big) + \frac{t}{V(x,\rho)\phi(\rho)} \right] \exp\Big(c_8 \frac{t}{\phi(\rho)}\Big).$$

Taking $\rho = c_9 \phi^{-1}(t)$ with some constant $c_9 > 0$ in the inequality above and using the fact that the function $f(r) = (1+r)^{d_2} e^{-r}$ is bounded on $[0, \infty)$, we furthermore get that for all $t > 0$ and all $x, y \in M_0$,

$$p(t,x,y) \le \frac{c_{10}}{V(x,\phi^{-1}(t))},$$

which in turn gives us the desired assertion by the symmetry of $p(t, x, y)$. $\square$

Lemma 5.2. *Under* VD *and* (1.13), *if* UHKD(ϕ), $\mathrm{J}_{\phi,\le}$ *and* E_ϕ *hold, then the* ρ*-truncated Dirichlet form* $(\mathcal{E}^{(\rho)}, \mathcal{F})$ *has the heat kernel* $q^{(\rho)}(t, x, y)$*, and it satisfies that for any* $t > 0$ *and all* $x, y \in M_0$,

$$q^{(\rho)}(t,x,y) \le c_1 \left(\frac{1}{V(x,\phi^{-1}(t))} + \frac{1}{V(y,\phi^{-1}(t))} \right) \exp\Big(c_2 \frac{t}{\phi(\rho)} - c_3 \frac{d(x,y)}{\rho}\Big),$$

where c_1, c_2, c_3 *are positive constants independent of* ρ.

Consequently, for any $t > 0$ *and all* $x, y \in M_0$,

$$q^{(\rho)}(t,x,y) \le \frac{c_4}{V(x,\phi^{-1}(t))} \left(1 + \frac{d(x,y)}{\phi^{-1}(t)}\right)^{d_2} \exp\Big(c_2 \frac{t}{\phi(\rho)} - c_3 \frac{d(x,y)}{\rho}\Big).$$

Proof. (i) The existence of $q^{(\rho)}(t, x, y)$ has been mentioned in the proof of Lemma 5.1. Furthermore, according to Lemma 7.2(2), Lemma 2.1 and Lemma 5.1, there exist $c_1, c_2 > 0$ such that for all $t > 0$ and all $x, y \in M_0$,

$$q^{(\rho)}(t,x,y) \le c_1 \left(\frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{1}{V(y,\phi^{-1}(t))} \right) \exp\Big(c_2 \frac{t}{\phi(\rho)}\Big). \tag{5.2}$$

Therefore, in order to prove the desired assertion, below we only need to consider the case that $d(x, y) \ge 2\rho$.

By Corollary 4.22, for any ball $B(x, r)$, $t > 0$ and all $z \in B(x, \rho) \cap M_0$ with $r > \rho$,

$$\begin{aligned} Q_t^{(\rho)} \mathbf{1}_{B(x,r)^c}(z) &\le \mathbb{P}^z(\tau^{(\rho)}_{B(x,r)} \le t) \le \mathbb{P}^z(\tau^{(\rho)}_{B(z,r-\rho)} \le t) \\ &\le c_3 \exp\left(-c_4 \frac{r}{\rho} + c_3 \frac{t}{\phi(\rho)}\right), \end{aligned} \tag{5.3}$$

where $c_3, c_4 > 0$ are independent of ρ.

(ii) Fix $x_0, y_0 \in M$ and $t > 0$. Set $r = \frac{1}{2} d(x_0, y_0)$. By the semigroup property, we have that

$$\begin{aligned} q^{(\rho)}(2t,x,y) &= \int_M q^{(\rho)}(t,x,z) q^{(\rho)}(t,z,y)\,\mu(dz) \\ &\le \int_{B(x_0,r)^c} q^{(\rho)}(t,x,z) q^{(\rho)}(t,z,y)\,\mu(dz) + \int_{B(y_0,r)^c} q^{(\rho)}(t,x,z) q^{(\rho)}(t,z,y)\,\mu(dz). \end{aligned}$$

Using (5.2) and (5.3), we obtain that

$$\begin{aligned}\int_{B(x_0,r)^c} q^{(\rho)}(t,x,z)q^{(\rho)}(t,z,y)\,\mu(dz) &\le \frac{c_1}{V(y,\phi^{-1}(t))}\exp\Big(c_2\frac{t}{\phi(\rho)}\Big)\int_{B(x_0,r)^c} q^{(\rho)}(t,x,z)\,\mu(dz)\\ &\le \frac{c_5}{V(y,\phi^{-1}(t))}\exp\Big(c_5\frac{t}{\phi(\rho)}-c_4\frac{r}{\rho}\Big)\end{aligned}$$

for μ-almost all $x \in B(x_0,\rho)$ and $y \in M$. Similarly, by the symmetry of $q^{(\rho)}(t,z,y)$,

$$\begin{aligned}\int_{B(y_0,r)^c} q^{(\rho)}(t,x,z)q^{(\rho)}(t,z,y)\,\mu(dz) &\le \frac{c_1}{V(x,\phi^{-1}(t))}\exp\Big(c_2\frac{t}{\phi(\rho)}\Big)\int_{B(y_0,r)^c} q^{(\rho)}(t,z,y)\,\mu(dz)\\ &= \frac{c_1}{V(x,\phi^{-1}(t))}\exp\Big(c_2\frac{t}{\phi(\rho)}\Big)\int_{B(y_0,r)^c} q^{(\rho)}(t,y,z)\,\mu(dz)\\ &\le \frac{c_6}{V(x,\phi^{-1}(t))}\exp\Big(c_6\frac{t}{\phi(\rho)}-c_4\frac{r}{\rho}\Big)\end{aligned}$$

for μ-almost all $y \in B(x_0,\rho)$ and $x \in M$. Hence, since x_0 and y_0 are arbitrary, we get the first required assertion. Then the second one immediately follows from the first one and VD. □

Now, we can prove the following main result.

Proposition 5.3. *Under* VD *and* (1.13), *if* UHKD(ϕ), $\mathrm{J}_{\phi,\le}$ *and* E_ϕ *hold, then we have* UHK(ϕ).

Proof. (i) We first prove that there are $N \in \mathbb{N}$ with $N > (\beta_1 + d_2)/\beta_1$ and $C_0 \ge 1$ such that for each $t, r > 0$ and all $x \in M_0$,

$$\int_{B(x,r)^c} p(t,x,y)\,\mu(dy) \le C_0\left(\frac{\phi^{-1}(t)}{r}\right)^{\theta}, \tag{5.4}$$

where $\theta = \beta_1 - (\beta_1 + d_2)/N$, and d_2 and β_1 are constants from VD and (1.13) respectively. Indeed, we only need to consider the case that $r > \phi^{-1}(t)$. For any $\rho, t > 0$ and all $x, y \in M_0$, by (4.34) and Proposition 4.24, we have

$$p(t,x,y) \le q^{(\rho)}(t,x,y) + \frac{c_1 t}{V(x,\rho)\phi(\rho)}\exp\left(\frac{c_2 t}{\phi(\rho)}\right),$$

where $c_1, c_2 > 0$ are constants independent of ρ. Now, for fixed large $N \in \mathbb{N}$ (which will be specified later), define

$$\rho_n = 2^{n\alpha} r^{1-1/N}\phi^{-1}(t)^{1/N}, \quad n \in \mathbb{N},$$

where $\alpha \in (d_2/(d_2+\beta_1) \vee 1/2, 1)$. Since $r > \phi^{-1}(t)$ and $2\alpha \ge 1$, we have

$$\phi^{-1}(t) \le \rho_n \le 2^n r, \quad \frac{2^n r}{\rho_n} \le \frac{\rho_n}{\phi^{-1}(t)}. \tag{5.5}$$

In particular, by (1.13), $t/\phi(\rho_n) \le c_3$. Plugging these into Lemma 5.2, we have that there are constants $c_4, c_5 > 0$ such that for every $t > 0$ and all $x, y \in M_0$ with $2^n r \le d(x,y) \le 2^{n+1} r$,

$$q^{(\rho_n)}(t,x,y) \le \frac{c_4}{V(x,\phi^{-1}(t))}\left(\frac{2^n r}{\phi^{-1}(t)}\right)^{d_2}\exp\left(-\frac{c_5 2^n r}{\rho_n}\right).$$

Thus, there is a constant $c_6 > 0$ such that for every $t > 0$ and all $x \in M_0$,

$$\begin{aligned}
\int_{B(x,r)^c} p(t,x,y)\,\mu(dy) =& \sum_{n=0}^{\infty} \int_{B(x,2^{n+1}r)\setminus B(x,2^n r)} p(t,x,y)\,\mu(dy) \\
\le& \sum_{n=0}^{\infty} \frac{c_6}{V(x,\phi^{-1}(t))} \left(\frac{2^n r}{\phi^{-1}(t)}\right)^{d_2} \exp\left(-\frac{c_5 2^n r}{\rho_n}\right) V(x,2^n r) \\
&+ \sum_{n=0}^{\infty} \frac{c_6 t V(x,2^n r)}{V(x,\rho_n)\phi(\rho_n)} \\
=:& I_1 + I_2.
\end{aligned}$$

We first estimate I_2. Take N large enough so that $\beta_1 - (\beta_1 + d_2)/N > 0$. Then using VD, (1.13) and (5.5), we have

$$\begin{aligned}
I_2 \le& c_7 \sum_{n=0}^{\infty} \Big(\frac{\phi^{-1}(t)}{\rho_n}\Big)^{\beta_1} \Big(\frac{2^n r}{\rho_n}\Big)^{d_2} \\
=& c_7 \Big(\frac{\phi^{-1}(t)}{r}\Big)^{\beta_1 - (\beta_1+d_2)/N} \sum_{n=0}^{\infty} 2^{n(d_2 - \alpha(d_2+\beta_1))} \\
\le& c_8 \Big(\frac{\phi^{-1}(t)}{r}\Big)^{\beta_1 - (\beta_1+d_2)/N},
\end{aligned}$$

where in the last inequality we used the fact $d_2 - \alpha(d_2 + \beta_1) < 0$ due to the choice of α. We next estimate I_1. Note that for each $K \in \mathbb{N}$, there exists a constant $c_K > 0$ such that $e^{-x} \le c_K x^{-K}$ for all $x \ge 1$. Now choose K large enough so that $K/N > 2d_2 + \beta_1$ and $(1-\alpha)K > 2d_2$. Then using VD, (1.13) and (5.5) again, we have

$$\begin{aligned}
I_1 \le& \sum_{n=0}^{\infty} \frac{c_{9,K}}{V(x,\phi^{-1}(t))} \Big(\frac{2^n r}{\phi^{-1}(t)}\Big)^{d_2} \Big(\frac{\rho_n}{2^n r}\Big)^{K} V(x,2^n r) \\
\le& c_{10,K} \sum_{n=0}^{\infty} \Big(\frac{2^n r}{\phi^{-1}(t)}\Big)^{2d_2} \Big(\frac{\phi^{-1}(t)^{1/N}}{2^{n(1-\alpha)} r^{1/N}}\Big)^{K} \\
=& c_{10,K} \Big(\frac{\phi^{-1}(t)}{r}\Big)^{K/N - 2d_2} \sum_{n=0}^{\infty} 2^{n(2d_2 - (1-\alpha)K)} \\
\le& c_{11,K} \Big(\frac{\phi^{-1}(t)}{r}\Big)^{K/N - 2d_2} \le c_{11,K} \Big(\frac{\phi^{-1}(t)}{r}\Big)^{\beta_1}.
\end{aligned}$$

Combining with all estimations above, we obtain the desired estimate (5.4).

(ii) For any ball B with radius r, by (5.4), there is a constant $c_1 > 0$ such that

$$1 - P_t^B \mathbf{1}_B(x) = \mathbb{P}^x(\tau_B \le t) \le c_1 \left(\frac{r}{\phi^{-1}(t)}\right)^{-\theta} \quad \text{all } x \in \frac{1}{4}B \cap M_0, \tag{5.6}$$

e.g. see the proof of Lemma 2.7. (Note that due to Lemma 4.21, $(\mathcal{E}, \mathcal{F})$ is conservative.)

Combining (5.6) with (4.26), we find that

$$1 - Q_t^{(\rho),B} \mathbf{1}_B(x) \le c_2 \left[\left(\frac{r}{\phi^{-1}(t)}\right)^{-\theta} + \frac{t}{\phi(\rho)}\right] \quad \text{for all } x \in \frac{1}{4}B \cap M_0, \tag{5.7}$$

where $Q_t^{(\rho),B}$ is the semigroup for the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}_B)$, and the constant c_2 is independent of ρ.

Next, we prove the following improvement of estimate in Lemma 5.2: for all $t > 0$, $k \geq 1$, and all $x_0, y_0 \in M$ with $d(x_0, y_0) > 4k\rho$,

$$q^{(\rho)}(t,x,y) \leq c_3(k)\left(\frac{1}{V(x,\phi^{-1}(t))} + \frac{1}{V(y,\phi^{-1}(t))}\right)\exp\left(c_4\frac{t}{\phi(\rho)}\right)\left(1 + \frac{\rho}{\phi^{-1}(t)}\right)^{-(k-1)\theta} \tag{5.8}$$

for almost all $x \in B(x_0,\rho)$ and $y \in B(y_0,\rho)$. By (5.2), it suffices to consider the case that $\rho \geq \phi^{-1}(t)$. Indeed, fix $k \geq 1$, $t > 0$ and $x_0, y_0 \in M_0$. Set $r = \frac{1}{2}d(x_0,y_0) > 2k\rho$. By (5.7) and Lemma 7.11,

$$Q_t^{(\rho)}\mathbf{1}_{B(x_0,r)^c}(x) \leq c_5(k)\left[\left(\frac{\rho}{\phi^{-1}(t)}\right)^{-\theta} + \frac{t}{\phi(\rho)}\right]^{k-1} \quad \text{for almost all } x \in B(x_0,\rho).$$

It is easy to see that

$$\left(\frac{\rho}{\phi^{-1}(t)}\right)^{-\theta} \geq c_3\frac{t}{\phi(\rho)} \quad \text{for all } \rho > \phi^{-1}(t),$$

(here c_3 is the constant in (1.13)) and so for almost all $x \in B(x_0,\rho)$,

$$Q_t^{(\rho)}\mathbf{1}_{B(x_0,r)^c}(x) \leq c_6(k)\left(\frac{\rho}{\phi^{-1}(t)}\right)^{-(k-1)\theta}.$$

Then using (5.2) and the estimate above, we can follow part (ii) in the proof of Lemma 5.2 to obtain (5.8).

(iii) Finally we prove the desired upper bound for $p(t,x,y)$. For any fixed $x_0, y_0 \in M$, let $r = \frac{1}{2}d(x_0,y_0)$. We only need to show that

$$p(t,x,y) \leq \frac{C}{V(x,\phi^{-1}(t))}\left(1 \wedge \frac{V(x,\phi^{-1}(t))t}{V(x,r)\phi(r)}\right)$$

for all $t > 0$, $0 < \rho < r$ small enough and almost all $x \in B(y_0,\rho)$ and $y \in B(x_0,\rho)$. As before, by Lemma 5.1, without loss of generality we may and do assume that $r/\phi^{-1}(t)$ is large enough. Take $k = 1 + [(2d_2 + \beta_2)/\theta]$ and $\rho = r/(8k)$. Using (4.34), Proposition 4.24 and (5.8), we obtain

$$\begin{aligned}
p(t,x,y) &\leq \frac{c_7(k)}{V(x,\phi^{-1}(t))}\left(1 + \frac{d(x,y)}{\phi^{-1}(t)}\right)^{d_2}\left(\frac{\rho}{\phi^{-1}(t)} + 1\right)^{-(k-1)\theta} + \frac{c_0't}{V(x,\rho)\phi(\rho)} \\
&\leq c_8(k)\left[\frac{1}{V(x,\phi^{-1}(t))}\left(\frac{r}{\phi^{-1}(t)}\right)^{-(k-1)\theta+d_2} + \frac{t}{V(x,r)\phi(r)}\right] \\
&\leq \frac{c_9(k)t}{V(x,r)\phi(r)}
\end{aligned}$$

for all $t > 0$, and almost all $x \in B(x_0,\rho)$ and $y \in B(y_0,\rho)$. The proof is complete. □

$\mathrm{J}_{\phi,\geq} \Longrightarrow \mathrm{FK}(\phi)$ has been proved in Subsection 4.1 by the additional assumption RVD, and $\mathrm{FK}(\phi) + \mathrm{E}_\phi + \mathrm{J}_{\phi,\leq} \Longrightarrow \mathrm{UHKD}(\phi)$ has been proved in Subsection 4.4. Combining these with Proposition 5.3, we also obtain $\mathrm{J}_\phi + \mathrm{E}_\phi \Longrightarrow \mathrm{UHK}(\phi)$.

5.2 $\mathrm{J}_\phi + \mathrm{E}_\phi \Longrightarrow \mathrm{LHK}(\phi)$

Proposition 5.4. *If* VD, (1.13), E_ϕ *and* J_ϕ *hold, then we have* $\mathrm{LHK}(\phi)$.

Proof. The proof is split into two steps, and the first one is concerned with the near-diagonal lower bound estimate.

(i) The argument for the near-diagonal lower bound estimate is standard; we present it here for the sake of completeness. It follows from E_ϕ and Lemma 4.16 that there exist constants $c_0 \geq 1$ and $c_1 \in (0,1)$ so that for all $x \in M_0$ and $t, r > 0$ with $r \geq c_0\phi^{-1}(t)$,

$$\int_{B(x,r)^c} p(t,x,y)\,\mu(dy) \leq \mathbb{P}^x(\tau_{B(x,r)} \leq t) \leq c_1.$$

This and the conservativeness of $(\mathcal{E},\mathcal{F})$(which is due to Lemma 4.21) imply that

$$\int_{B(x,c_0\phi^{-1}(t))} p(t,x,y)\,\mu(dy) \geq 1 - c_1.$$

By the semigroup property and the Cauchy-Schwarz inequality, we get for all $x \in M_0$

$$\begin{aligned} p(2t,x,x) = \int_M p(t,x,y)^2\,\mu(dy) &\geq \frac{1}{V(x,c_0\phi^{-1}(t))}\Big(\int_{B(x,c_0\phi^{-1}(t))} p(t,x,y)\,\mu(dy)\Big)^2 \\ &\geq \frac{c_2}{V(x,\phi^{-1}(t))}. \end{aligned} \tag{5.9}$$

Furthermore, by (5.10) below, we can take $\delta > 0$ small enough and find that for almost all $y \in B(x,\delta\phi^{-1}(t))$,

$$p(2t,x,y) \geq p(2t,x,x) - \frac{c_3}{V(x,\phi^{-1}(t))}\delta^\theta \geq \frac{c_4}{V(x,\phi^{-1}(t))}.$$

This proves that there are constants $\delta_1, c_5 > 0$ such that for all $t > 0$, almost all $x \in M$ and $y \in B(x,\delta_1\phi^{-1}(t))$,

$$p(t,x,y) \geq \frac{c_5}{V(x,\phi^{-1}(t))}.$$

(ii) The argument below is motivated by [CZ, Section 4.4]. According to the result in Subsection 5.1, Lemma 4.21 and Lemma 2.7, $\mathrm{UHK}(\phi)$ and so $\mathrm{EP}_{\phi,\leq}$ holds, i.e. for all $x \in M_0$ and $t, r > 0$,

$$\mathbb{P}^x(\tau_{B(x,r)} \leq t) \leq c_6 t/\phi(r).$$

In particular, there are $a \in (0,1/2)$ and $\delta_2 \in (0,\delta_1)$ (independent of t) such that for all $t > 0$, $\delta_1\phi^{-1}((1-a)t) \geq \delta_2\phi^{-1}(t)$, and for all $x \in M_0$ and $t > 0$,

$$\mathbb{P}^x(\tau_{B(x,2\delta_2\phi^{-1}(t)/3)} \leq at) \leq 1/2.$$

For $A \subset M$, let

$$\sigma_A = \inf\{t > 0 : X_t \in A\}.$$

Now, for all $x \in M_0$ and $y \in M$ with $d(x,y) \geq \delta_1\phi^{-1}(t)$,

$$\begin{aligned} \mathbb{P}^x(X_{at} \in\, & B(y,\delta_1\phi^{-1}((1-a)t))) \\ &\geq \mathbb{P}^x(X_{at} \in B(y,\delta_2\phi^{-1}(t))) \end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{P}^x\left(\sigma_{B(y,\delta_2\phi^{-1}(t)/3)} \leq at; \sup_{s\in[\sigma_{B(y,\delta_2\phi^{-1}(t)/3)},at]} d(X_s, X_{\sigma_{B(y,\delta_2\phi^{-1}(t)/3)}}) \leq 2\delta_2\phi^{-1}(t)/3\right)\\
&\geq \mathbb{P}^x(\sigma_{B(y,\delta_2\phi^{-1}(t)/3)} \leq at) \inf_{z\in B(y,\delta_2\phi^{-1}(t)/3)} \mathbb{P}^z(\tau_{B(z,2\delta_2\phi^{-1}(t)/3)} > at)\\
&\geq \frac{1}{2}\mathbb{P}^x(\sigma_{B(y,\delta_2\phi^{-1}(t)/3)} \leq at)\\
&\geq \frac{1}{2}\mathbb{P}^x\left(X_{(at)\wedge\tau_{B(x,2\delta_2\phi^{-1}(t)/3)}} \in B(y,\delta_2\phi^{-1}(t)/3)\right).
\end{aligned}
$$

For any $x, y \in M$ with $d(x,y) \geq \delta_1\phi^{-1}(t) \geq \delta_2\phi^{-1}(t)$, $B(y,\delta_2\phi^{-1}(t)/3) \subset B(x, 2\delta_2\phi^{-1}(t)/3)^c$. Then by $J_{\phi,\geq}$ and Lemma 7.1, for all $x \in M_0$,

$$
\begin{aligned}
&\mathbb{P}^x\left(X_{(at)\wedge\tau_{B(x,2\delta_2\phi^{-1}(t)/3)}} \in B(y,\delta_2\phi^{-1}(t)/3)\right)\\
&= \mathbb{E}^x\left[\sum_{s\leq(at)\wedge\tau_{B(x,2\delta_2\phi^{-1}(t)/3)}} \mathbf{1}_{\{X_s\in B(y,\delta_2\phi^{-1}(t)/3)\}}\right]\\
&\geq \mathbb{E}^x\left[\int_0^{(at)\wedge\tau_{B(x,2\delta_2\phi^{-1}(t)/3)}} ds \int_{B(y,\delta_2\phi^{-1}(t)/3)} J(X_s,u)\,\mu(du)\right]\\
&\geq c_7\mathbb{E}^x\left[\int_0^{(at)\wedge\tau_{B(x,2\delta_2\phi^{-1}(t)/3)}} ds \int_{B(y,\delta_2\phi^{-1}(t)/3)} \frac{1}{V(u,d(X_s,u))\phi(d(X_s,u))}\,\mu(du)\right]\\
&\geq c_8\mathbb{E}^x\left[(at)\wedge\tau_{B(x,2\delta_2\phi^{-1}(t)/3)}\right] V(y,\delta_2\phi^{-1}(t)/3)\frac{1}{V(y,d(x,y))\phi(d(x,y))}\\
&\geq c_8 at\mathbb{P}^x\left[\tau_{B(x,2\delta_2\phi^{-1}(t)/3)} \geq at\right] V(y,\delta_2\phi^{-1}(t)/3)\frac{1}{V(y,d(x,y))\phi(d(x,y))}\\
&\geq \frac{c_9 tV(y,\phi^{-1}(t))}{V(x,d(x,y))\phi(d(x,y))},
\end{aligned}
$$

where in the third inequality we have used the fact that

$$
d(X_s,u) \leq d(X_s,x) + d(x,y) + d(y,u) \leq d(x,y) + \delta_2\phi^{-1}(t) \leq 2d(x,y).
$$

Therefore, for almost all $x, y \in M$ with $d(x,y) \geq \delta_1\phi^{-1}(t)$,

$$
\begin{aligned}
p(t,x,y) &\geq \int_{B(y,\delta_1\phi^{-1}(t))} p(at,x,z)p((1-a)t,z,y)\,\mu(dz)\\
&\geq \inf_{z\in B(y,\delta_1\phi^{-1}((1-a)t))} p((1-a)t,z,y)\int_{B(y,\delta_1\phi^{-1}((1-a)t))} p(at,x,z)\,\mu(dz)\\
&\geq \frac{c_{10}}{V(y,\phi^{-1}(t))}\cdot\frac{c_9 tV(y,\phi^{-1}(t))}{V(x,d(x,y))\phi(d(x,y))}\\
&= \frac{c_{11}t}{V(x,d(x,y))\phi(d(x,y))}.
\end{aligned}
$$

The proof is complete. □

Remark 5.5. We emphasis that the on-diagonal lower bound estimate (5.9) is based on E_ϕ only.

The following lemma has been used in the proof above.

Lemma 5.6. *Under* VD*,* (1.13)*,* J_ϕ *and* E_ϕ*, the heat kernel* $p(t,x,y)$ *is Hölder continuous with respect to* (x,y)*. More explicitly, there exist constants* $\theta \in (0,1)$ *and* $c_3 > 0$ *such that for all* $t > 0$ *and* $x,y,z \in M$*,*

$$|p(t,x,y) - p(t,x,z)| \le \frac{c_3}{V(x,\phi^{-1}(t))} \left(\frac{d(y,z)}{\phi^{-1}(t)}\right)^\theta. \tag{5.10}$$

Proof. The proof is essentially the same as that of [CK1, Theorem 4.14], and we should highlight a few different steps. Let $Z := \{V_s, X_s\}_{s\ge 0}$ be a space-time process where $V_s = V_0 - s$. The filtration generated by Z satisfying the usual conditions will be denoted by $\{\widetilde{\mathcal{F}}_s; s \ge 0\}$. The law of the space-time process $s \mapsto Z_s$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. For every open subset D of $[0,\infty) \times M$, define $\tau_D = \inf\{s > 0 : Z_s \notin D\}$ and $\sigma_D = \inf\{t > 0 : Z_t \in D\}$.

According to Subsection 5.1, $J_\phi + E_\phi$ imply UHK(ϕ). Then by Lemma 2.7, $EP_{\phi,\le}$ holds, i.e. there is a constant $c_0 \in (0,1)$ such that for all $x \in M_0$,

$$\mathbb{P}^{(0,x)}(\tau_{B(x,r)} \le c_0\phi(r)) \le 1/2. \tag{5.11}$$

Let $Q(t,x,r) = [t, t + c_0\phi(r)] \times B(x,r)$. Then, following the argument of [CK2, Lemma 6.2] and using the Lévy system for the process $\{X_t\}$ (see Lemma 7.1), we can obtain that there is a constant $c_1 > 0$ such that for all $x \in M_0$, $t, r > 0$ and any compact subset $A \subset Q(t,x,r)$

$$\mathbb{P}^{(t,x)}(\sigma_A < \tau_{Q(t,x,r)}) \ge c_1 \frac{m \otimes \mu(A)}{V(x,r)\phi(r)}, \tag{5.12}$$

where $m \otimes \mu$ is a product measure of the Lebesgue measure m on $\mathbb{R}_+$ and μ on M. Note that unlike [CK2, Lemma 6.2], here (5.12) is satisfied for all $r > 0$ not only $r \in (0,1]$, which is due to the fact (5.11) holds for all $r > 0$.

Also by the Lévy system of the process $\{X_t\}$ (see Lemma 7.1), we find that there is a constant $c_2 > 0$ such that for all $x \in M_0$, $t, r > 0$ and $s \ge 2r$,

$$\begin{aligned}
\mathbb{P}^{(t,x)}(X_{\tau_{Q(t,x,r)}} \notin B(x,s)) &= \mathbb{E}^{(t,x)} \int_0^{\tau_{Q(t,x,r)}} \int_{B(x,s)^c} J(X_v,u)\,\mu(du)\,dv \\
&\le \mathbb{E}^{(t,x)} \int_0^{\tau_{Q(t,x,r)}} \int_{B(X_v,s/2)^c} J(X_v,u)\,\mu(du)\,dv \\
&\le c_2 \frac{\phi(r)}{\phi(s)},
\end{aligned} \tag{5.13}$$

where in the last inequality we have used Lemma 2.1 and E_ϕ.

Having (5.12) and (5.13) at hand, one can follow the argument of [CK1, Theorem 4.14] to get that the Hölder continuity of bounded parabolic functions, and so the desired assertion for the heat kernel $p(t,x,y)$. □

Remark 5.7. The proof above is based on (5.11), (5.12) and (5.13). According to Lemma 4.16, (5.11) is a consequence of E_ϕ; while, from the argument above, (5.13) can be deduced from $J_{\phi,\le}$ and $E_{\phi,\le}$. (5.12) is the so called Krylov type estimate, which is a key to yield the Hölder continuity of bounded parabolic functions, and where $J_{\phi,\ge}$ is used.

6 Applications and Example

6.1 Applications

We first give examples of ϕ such that condition (1.13) is satisfied (see [CK2, Example 2.3]).

Example 6.1. (1) Assume that there exist $0 < \beta_1 \le \beta_2 < \infty$ and a probability measure ν on $[\beta_1, \beta_2]$ such that

$$\phi(r) = \int_{\beta_1}^{\beta_2} r^{\beta}\, \nu(d\beta), \quad r \ge 0.$$

Then (1.13) is satisfied. Clearly, ϕ is a continuous strictly increasing function with $\phi(0) = 0$. Note that some additional restriction of the range of β_2 should be imposed for the corresponding Dirichlet form to be regular. (For instance, $\beta_2 < 2$ when $M = \mathbb{R}^n$.) In this case,

$$J(x, y) \asymp \frac{1}{V(x, d(x, y)) \int_{\beta_1}^{\beta_2} d(x, y)^{\beta}\, \nu(d\beta)}, \quad x, y \in M. \tag{6.1}$$

When $\beta_1 = \beta_2 = \beta$ (i.e. $\nu(\{\beta\}) = 1$), a symmetric jump process whose jump density is comparable to (6.1) is called a symmetric β-stable like process.

(2) Similarly, consider the following increasing function

$$\phi(r) = \left(\int_{\beta_1}^{\beta_2} r^{-\beta}\, \nu(d\beta) \right)^{-1} \quad \text{for} \ \ r > 0, \quad \phi(0) = 0,$$

where ν is a finite measure on $[\beta_1, \beta_2] \subset (0, \infty)$. Then (1.13) is satisfied. Again, ϕ is a continuous strictly increasing function, and some additional restriction of the range of β_2 should be imposed for the corresponding Dirichlet form to be regular. In this case,

$$J(x, y) \asymp \frac{1}{V(x, d(x, y))} \int_{\beta_1}^{\beta_2} \frac{1}{d(x, y)^{\beta}}\, \nu(d\beta), \quad x, y \in M.$$

A particular case is when ν is a discrete measure. For example, when $\nu(A) = \sum_{i=1}^{N} \delta_{\alpha_i}(A)$ for some $\alpha_i \in (0, \infty)$ with $1 \le i \le N$ and $N \ge 1$,

$$J(x, y) \asymp \sum_{i=1}^{N} \frac{1}{V(x, d(x, y)) d(x, y)^{\alpha_i}}.$$

We now give an important class of examples where β, β_2 in (1.13) could be strictly larger than 2, and then discuss the stability of heat kernel estimates.

The first class of examples are given as subordinations of diffusion processes on fractals. First, let us define the Sierpinski carpet as a typical example of fractals. Set $E_0 = [0, 1]^n$. For any $l \in \mathbb{N}$ with $l \ge 2$, let

$$\mathcal{Q} = \Big\{ \Pi_{i=1}^{n} [(k_i - 1)/l, k_i/l] : 1 \le k_i \le l,\ k_i \in \mathbb{N},\ 1 \le i \le n \Big\}.$$

For any $l \le N \le l^n$, let F_i $(1 \le i \le N)$ be orientation preserving affine maps of E_0 onto some element of $\mathcal{Q}$. (Without loss of generality, let $F_1(x) = l^{-1}x$ for $x \in E_0$ and assume that the sets $F_i(E_0)$ are distinct.) Set $I = \{1, \dots, N\}$ and $E_1 = \cup_{i \in I} F_i(E_0)$. Then there exists a unique non-empty compact set $\hat{M} \subset E_0$ such that $\hat{M} = \cup_{i \in I} F_i(\hat{M})$. $\hat{M}$ is called a Sierpinski carpet if the following hold:

(SC1) (Symmetry) E_1 is preserved by all the isometries of the unit cube E_0.

(SC2) (Connectedness) E_1 is connected.

(SC3) (Non-diagonality) Let B be a cube in E_0 which is the union of 2^d distinct elements of $\mathcal{Q}$. (So B has side length $2l^{-1}$.) If $\mathrm{Int}(E_1 \cap B) \neq \emptyset$, then it is connected.

(SC4) (Borders included property) E_1 contains the set $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_d = 0\}$.

Note that Sierpinski carpets are infinitely ramified in the sense that $\hat{M}$ can not be disconnected by removing a finite number of points. Let

$$E_k := \bigcup_{i_1,\cdots,i_k \in I} F_{i_1} \circ \cdots \circ F_{i_k}(E_0), \quad M_{\mathrm{pre}} := \bigcup_{k \geq 0} l^k E_k \ \text{ and } \ M := \bigcup_{k \geq 0} l^k \hat{M}.$$

M_{pre} is called a pre-carpet, and M is called an unbounded carpet. Both Hausdorff dimensions of $\hat{M}$ and M with respect to the Euclidean metric are $d = \log N / \log l$. Let μ be the (normalized) Hausdorff measure on M. The following has been proved in [BB1]:

There exists a μ-symmetric conservative diffusion on M that has a symmetric jointly continuous transition density $\{q(t,x,y) : t > 0, x, y \in M\}$ with the following estimates for all $t > 0, x, y \in M$:

$$c_1 t^{-\alpha/\beta_*} \exp\left(- c_2 \Big(\frac{|x-y|^{\beta_*}}{t} \Big)^{\frac{1}{\beta_*-1}} \right) \leq q(t,x,y) \leq c_3 t^{-\alpha/\beta_*} \exp\left(- c_4 \Big(\frac{|x-y|^{\beta_*}}{t} \Big)^{\frac{1}{\beta_*-1}} \right), \tag{6.2}$$

where $0 < \alpha \leq n$ and $\beta_* \geq 2$. In fact, it is known that there exist μ-symmetric diffusion processes with the above heat kernel estimates on various fractals including the Sierpinski gaskets and nested fractals, and typically $\beta_* > 2$. For example, for the two-dimensional Sierpinski gasket, $\alpha = \log 3/\log 2$ and $\beta_* = \log 5/\log 2$ (see [B, K2] for details).

Next, let us consider a more general situation. Let (M, d, μ) be a metric measure space as in the setting of this paper that satisfies VD and RVD. Assume that there exists a μ-symmetric diffusion process $\{Z_t\}$ that admits no killings inside M, and has a symmetric and jointly continuous transition density $\{q(t,x,y) : t > 0, x, y \in M\}$ with the following estimates for all $t > 0, x, y \in M$:

$$\frac{c_1}{V(x,\Psi^{-1}(t))} \exp\Big(- c_2 \Big(\frac{\Psi(d(x,y))}{t} \Big)^{\gamma_1} \Big) \leq q(t,x,y) \leq \frac{c_3}{V(x,\Psi^{-1}(t))} \exp\Big(- c_4 \Big(\frac{\Psi(d(x,y))}{t} \Big)^{\gamma_2} \Big), \tag{6.3}$$

where $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing continuous function with $\Psi(0) = 0$, $\Psi(1) = 1$ and satisfying (1.13). The lower bound in (6.3) implies that

$$q(t,x,y) \geq \frac{c_1 e^{-c_2}}{V(x,\Psi^{-1}(t))} \quad \text{for } d(x,y) \leq \Psi^{-1}(t)$$

and so we conclude by Proposition 3.1(ii) that the process $\{Z_t\}$ has infinite lifetime.

Clearly (6.2) is a special case of (6.3) with $V(x,r) \asymp r^\alpha$, $\Psi(s) = s^{\beta_*}$ and $\gamma_1 = \gamma_2 = 1/(\beta_* - 1)$. A typical example that the local and global structures of Ψ differ is a so called fractal-like manifold. It is a 2-dimensional Riemannian manifold whose global structure is like that of the fractal. For example, one can construct it from $M_{\rm pre}$ by changing each bond to a cylinder and smoothing the connection to make it a manifold. One can naturally construct a Brownian motion on the surfaces of cylinders. Using the stability of heat kernel estimates like (6.3) (see for instance [BBK1] for details), one can show that any divergence operator $\mathcal{L} = \sum_{i,j=1}^2 \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ in local coordinates on such manifolds that satisfies the uniform elliptic condition obeys (6.3) with $\Psi(s) = s^2 + s^{\beta_*}$.

We now subordinate the diffusion $\{Z_t\}$ whose heat kernel enjoys (6.3). Let $\{\xi_t\}$ be a subordinator that is independent of $\{Z_t\}$; namely, it is an increasing Lévy process on $\mathbb{R}_+$. Let $\bar\phi$ be the Laplace exponent of the subordinator, i.e.

$$\mathbb{E}[\exp(-\lambda \xi_t)] = \exp(-t\bar\phi(\lambda)), \quad \lambda, t > 0.$$

It is known that $\bar\phi$ is a Bernstein function, i.e. it is a C^∞ function on $\mathbb{R}_+$ and $(-1)^n D^n \bar\phi \le 0$ for all $n \ge 0$. See for instance [SSV] for the general theory of subordinations. See also [BSS, K1, Sto] for subordinations on fractals. By the general theory, there exist $a, b \ge 0$ and a measure μ on $\mathbb{R}_+$ satisfying $\int_0^\infty (1 \wedge t)\,\mu(dt) < \infty$ such that

$$\bar\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt). \tag{6.4}$$

Below, we assume that $\bar\phi$ is a complete Bernstein function; namely, the measure $\mu(dt)$ has a completely monotone density $\mu(t)$, i.e. $(-1)^n D^n \mu \ge 0$ for all $n \ge 0$. Assume further that $\bar\phi$ satisfies (1.13) with different β_1, β_2 from those for Ψ, and that furthermore $\beta_1, \beta_2 \in (0,1)$. Then $a = b = 0$ in (6.4) and one can obtain $\mu(t) \asymp \bar\phi(1/t)/t$ (see [KSV, Theorem 2.2]).

The process $\{X_t\}$ defined by $X_t = Z_{\xi_t}$ for any $t \ge 0$ is called a subordinate process. Let $\{\eta_t(u) : t > 0, u \ge 0\}$ be the distribution density of $\{\xi_t\}$. It is known (see for instance [BSS, Sto]) that the Lévy density $J(\cdot,\cdot)$ and the heat kernel $p(t,\cdot,\cdot)$ of X are given by

$$J(x,y) = \int_0^\infty q(u,x,y)\mu(u)\,du, \tag{6.5}$$

$$p(t,x,y) = \int_0^\infty q(u,x,y)\eta_t(u)\,du \quad \text{for all} \quad t > 0,\ x,y \in M. \tag{6.6}$$

Define

$$\phi(r) = \frac{1}{\bar\phi(1/\Psi(r))}. \tag{6.7}$$

Then ϕ also satisfies (1.13) (with different β_1, β_2 from those for $\bar\phi$ and Ψ). From now on, we discuss whether $p(t,\cdot,\cdot)$ satisfies HK(ϕ) or not. The most classical case is when (M, d, μ) is the Euclidean space $\mathbb{R}^d$ equipped with the Lebesgue measure μ, $\{Z_t\}$ is Brownian motion on $\mathbb{R}^d$ (and so $\beta_* = 2$ and $\gamma_1 = \gamma_2 = 1$), and $\bar\phi(t) = t^{\alpha/2}$ with $0 < \alpha < 2$. In this case $\{\xi_t\}$ is an $\alpha/2$-stable subordinator and the corresponding subordinate process is the rotationally symmetric α-stable process on $\mathbb{R}^d$. For a diffusion on a fractal whose heat kernel enjoys (6.2) for some $\beta_* > 2$, it is proved in [BSS, Theorem 3.1] that $p(t,\cdot,\cdot)$ satisfies HK(ϕ) with $\phi(r) = r^{\beta_*\alpha/2}$ when $\bar\phi(t) = t^{\alpha/2}$. (Note that $\beta_*\alpha/2 > 2$ when $\alpha > 4/\beta_*$.) The proof uses (6.6) and some estimates of $\eta_t(u)$ such as

$$\eta_t(u) \le c_5 t u^{-1-\alpha/2}, \quad t, u > 0.$$

Now let us consider the case $\Psi(s) = s^{\beta_{*,1}} + s^{\beta_{*,2}}$ with $2 \le \beta_{*,1} \le \beta_{*,2}$ (e.g. the fractal-like manifold is a special case in that $\beta_{*,1} = 2$), and $\bar{\phi}(t) = t^{\alpha_1/2} + t^{\alpha_2/2}$ for some $0 < \alpha_1 \le \alpha_2 < 2$. For this case, $\{\xi_t\}$ is a sum of independent $\alpha_1/2$- and $\alpha_2/2$-subordinators, so the distribution density $\eta_t(u)$ is a convolution of their distribution densities. Hence we have

$$\eta_t(u) \le c_6 t/(u^{1+\alpha_1/2} \wedge u^{1+\alpha_2/2}). \tag{6.8}$$

By elementary but tedious computations (along similar lines as in the proof of [BSS, Theorem 3.1]), one can deduce that $p(t,\cdot,\cdot)$ satisfies HK(ϕ) with

$$\phi(r) = r^{\alpha_2\beta_{*,1}/2} 1_{\{r\le 1\}} + r^{\alpha_1\beta_{*,2}/2} 1_{\{r>1\}}, \tag{6.9}$$

which is (up to constant multiplicative) the same as (6.7). In fact, the computation by using (6.6) also requires various estimates of $\eta_t(u)$, which are in general rather complicated. An alternative way is to prove first J_ϕ by using (6.5), which is easier since we have $\mu(t) \asymp \bar{\phi}(1/t)/t$. Then we can obtain

$$p(t,x,y) \le \frac{c_7 t}{V(x, d(x,y))\phi(d(x,y))}$$

by plugging (6.8) into (6.6). Integrating this, we have $\mathbb{P}^x(X_t \notin B(x,r)) \le c_8 t/\phi(r)$ for all $x \in M$ and $r,t > 0$. Note that since the diffusion process $\{Z_t\}$ has infinite lifetime, so does the subordinated process $\{X_t\}$. Then, following the argument of Lemma 2.7, we can get that $\mathbb{P}^x(\tau_{B(x,r)} \le t) \le c_9 t/\phi(r)$ for all $x \in M$ and $r,t > 0$. Consequently, by taking $\varepsilon > 0$ sufficiently small, we have

$$\mathbb{P}^x(\tau_{B(x,r)} \ge \phi(\varepsilon r)) = 1 - \mathbb{P}^x(\tau_{B(x,r)} < \phi(\varepsilon r)) \ge 1 - \frac{c_9\phi(\varepsilon r)}{\phi(r)} \ge c_{10} > 0,$$

which implies $\mathrm{E}_{\phi,\ge}$. Under VD and RVD, J_ϕ implies $\mathrm{E}_{\phi,\le}$ (which is due to Section 4.1 and Lemma 4.14). Therefore, by Theorem 1.13, we conclude that $p(t,\cdot,\cdot)$ satisfies HK(ϕ).

The above argument shows that HK(ϕ) is satisfied for the subordinated process $\{X_t\}$ when $\bar{\phi}(t) = t^{\alpha_1/2} + t^{\alpha_2/2}$. It follows from our stability theorem, Theorem 1.13, that for any symmetric pure jump process on the above mentioned space whose jumping kernel enjoys J_ϕ with ϕ given by (6.9), it enjoys the two-sided heat kernel estimates HK(ϕ).

The stability results we discuss above are new in general, especially for high dimensional Sierpinski carpets. However, if we restrict the framework so that (roughly) $\alpha < \beta_*$ in (6.2) (which is the case for diffusions on the Sierpinski gaskets, for instance), then the stability for the heat kernel was already established in [GHL2]. See [GHL2, Examples 6.16 and 6.20] for related examples.

6.2 Counterexample

In this subsection, we show that J_ϕ does not imply HK(ϕ) through the following counterexample.

Example 6.2. **(J_ϕ does not imply** HK(ϕ).**)** In [BBK2, CK1], it is proved in the setting of graphs or d-sets that J_ϕ is equivalent to HK(ϕ), when $V(x,r) \asymp r^d$ and $\phi(r) = r^\alpha$ with $0 < \alpha < 2$. Here, we give an example that this is not the case in general.

Let $M = \mathbb{R}^d$, $\phi(r) = r^\alpha + r^\beta$ with $0 < \alpha < 2 < \beta$, and

$$J(x,y) \asymp \frac{1}{|x-y|^d \phi(|x-y|)}, \quad x, y \in \mathbb{R}^d.$$

Note that $\phi(r) \asymp r^{\alpha}$ if $r \le 1$, and $\phi(r) \asymp r^{\beta}$ if $r \ge 1$. This example clearly satisfies J_ϕ. We first prove the following

$$p(t,x,y) \le \begin{cases} c_1 t^{-d/\alpha}, & t \in (0,1], \\ c_2 t^{-d/2}, & t \in [1,\infty). \end{cases} \tag{6.10}$$

Indeed, for the truncated process $\{X_t^{(1)}\}$ with

$$J_0(x,y) = J(x,y)\mathbf{1}_{\{|x-y|\le 1\}} \asymp \frac{1}{|x-y|^d \phi(|x-y|)} \mathbf{1}_{\{|x-y|\le 1\}},$$

it is proved in [CKK, Proposition 2.2] that (6.10) holds. Since (6.10) is equivalent to

$$\theta(\|u\|_2^2) \le c_3\, \mathcal{E}(u,u) \quad \text{for every } u \in \mathcal{F} \text{ with } \|u\|_1 = 1, \tag{6.11}$$

where $\theta(r) = r^{1+\alpha/d} \vee r^{1+2/d}$ (see for instance [Cou, theorem II.5]), it follows from the fact $J_0(x,y) \preceq J(x,y)$ that (6.11) and so (6.10) hold for the original process $\{X_t\}$. So if we take $t = c_4(r^\alpha \vee r^2)$ for $c_4 > 0$ large enough, then for all x, $x_0 \in \mathbb{R}^d$ and $r > 0$,

$$\mathbb{P}^x(X_t \in B(x_0,r)) = \int_{B(x_0,r)} p(t,x,z)\, dz \le c_5(t^{-d/\alpha} \vee t^{-d/2}) r^d \le \tfrac{1}{2}.$$

This implies $\mathbb{P}^x(\tau_{B(x_0,r)} > t) \le \frac{1}{2}$. Using the strong Markov property of X, we have for all $x, x_0 \in \mathbb{R}^d$, $\mathbb{P}^x(\tau_{B(x_0,r)} > kt) \le 2^{-k}$ and so $\mathbb{E}^x \tau_{B(x_0,r)} \le c_6 t = c_4 c_6 (r^\alpha \vee r^2)$. Thus E_ϕ fails, and so $\mathrm{HK}(\phi)$ does not hold either.

7 Appendix

7.1 The Lévy system formula

The following formula is used many times in this paper. See, for example [CK2, Appendix A] for the proof.

Lemma 7.1. *Let f be a non-negative measurable function on $\mathbb{R}_+ \times M \times M$ that vanishes along the diagonal. Then for every $t \ge 0$, $x \in M_0$ and stopping time T (with respect to the filtration of $\{X_t\}$),*

$$\mathbb{E}^x \left[\sum_{s \le T} f(s, X_{s-}, X_s) \right] = \mathbb{E}^x \left[\int_0^T \int_M f(s, X_s, y)\, J(X_s, dy)\, ds \right].$$

7.2 Meyer's decomposition

We use the following construction of Meyer [Me] for jump processes. Assume that $J(x,y) = J'(x,y) + J''(x,y)$ for any $x, y \in M$, and that there exists a constant $C > 0$ such that

$$\mathcal{J}(x) = \int J''(x,y)\, \mu(dy) \le C \quad \text{for all } x \in M.$$

Note that, by Lemma 2.1 the assumption above holds when VD, (1.13) and $\mathrm{J}_{\phi,\le}$ are satisfied. Let $\{Y_t\}$ be a process corresponding to the jumping kernel $J'(x,y)$. Then we can construct a process $\{X_t\}$ corresponding to the jumping kernel $J(x, y)$ by the following procedure. Let ξ_i, $i \geq 1$, be i.i.d. exponential random variables of parameter 1 independent of $\{Y_t\}$. Set

$$H_t = \int_0^t \mathcal{J}(Y_s)\, ds, \quad T_1 = \inf\{t \geq 0 : H_t \geq \xi_1\} \quad \text{and} \quad Q(x, y) = \frac{J''(x, y)}{\mathcal{J}(x)}.$$

We remark that $\{Y_t\}$ is a.s. continuous at T_1. We let $X_t = Y_t$ for $0 \leq t < T_1$, and then define X_{T_1} with law $Q(X_{T_1-}, y)\, \mu(dy) = Q(Y_{T_1}, y)\, \mu(dy)$. The construction now proceeds in the same way from the new space-time starting point (T_1, X_{T_1}). Since $\mathcal{J}(x)$ is bounded, there can be a.s. only finitely many extra jumps added in any bounded time interval. In [Me] it is proved that the resulting process corresponds to the jumping kernel $J(x, y)$.

In the following, we assume that both $\{X_t\}$ and $\{Y_t\}$ have transition densities. Denote by $p^X(t, x, y)$ and $p^Y(t, x, y)$ the transition density of $\{X_t\}$ and $\{Y_t\}$, respectively. The relation below between $p^X(t, x, y)$ and $p^Y(t, x, y)$ has been shown in [BGK1, Lemma 3.1 and (3.5)] and [BBCK, Lemma 3.6].

Lemma 7.2. *For almost all $x, y \in M$, we have*

(1)
$$p^X(t, x, y) \leq p^Y(t, x, y) + \mathbb{E}^x \int_0^t ds \int J''(Y_s, z)\, p^X(t - s, z, y)\, \mu(dz).$$

(2) *Let $A \in \sigma(Y_t, 0 < t < \infty)$. Then for almost all $x \in M$,*

$$\mathbb{P}^x(A) \leq e^{t\, \|\mathcal{J}\|_\infty} \mathbb{P}^x(A \cap \{X_s = Y_s \text{ for all } 0 \leq s \leq t\}). \tag{7.1}$$

In particular,

$$p^Y(t, x, y) \leq p^X(t, x, y) e^{t\, \|\mathcal{J}\|_\infty}.$$

Note that, by (7.1), if the process $\{X_t\}$ has transition density functions, so does $\{Y_t\}$.

7.3 Some results related to FK(ϕ).

The following is a general equivalence of FK(ϕ) for regular Dirichlet forms.

Proposition 7.3. *Assume that* VD *and* (1.13) *hold. Then the following are equivalent.*

(1) FK(ϕ).

(2) Nash$(\phi)_B$*; namely, there exist constants $C_1, \nu > 0$ such that for each $x \in M$ and $r > 0$,*

$$\frac{V(x, r)^\nu}{\phi(r)} \|u\|_2^{2+2\nu} \leq C_1 \mathcal{E}(u, u) \|u\|_1^{2\nu}, \quad u \in \mathcal{F}_{B(x,r)}.$$

(3) *There exist constants $C_1, \nu > 0$ such that for any ball $B = B(x, r)$, the Dirichlet heat kernel $p^B(t, \cdot, \cdot)$ exists and satisfies that*

$$\operatorname{ess\,sup}_{y,z \in B} p^B(t, y, z) \leq \frac{C_1}{V(x, r)} \Big(\frac{\phi(r)}{t}\Big)^{1/\nu}, \quad t > 0.$$

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) can be proved similarly to [GH, Lemmas 5.4 and 5.5] by choosing $a = CV(x,r)^\nu/\phi(r)$ in the paper.

(3) $\Longrightarrow$ (1) can be proved similarly to the approach of [GH, p. 553]. Note that [GH] discusses the case $\phi(r) = r^\beta$, but the generalization to ϕ is easy by using (1.13). $\square$

Under VD and RVD, we have further statements for FK(ϕ).

Proposition 7.4. *Assume that* VD*,* RVD *and* (1.13) *hold. Consider the following inequalities:*

(1) FK(ϕ).

(2) *There exist constants* $c_1, \nu > 0$ *such that for each* $x \in M$ *and* $r > 0$*,*

$$\|u\|_2^{2+2\nu} \le \frac{c_1}{V(x,r)^\nu}\|u\|_1^{2\nu}\Big(\|u\|_2^2 + \phi(r)\mathcal{E}(u,u)\Big), \quad u \in \mathcal{F}_{B(x,r)}.$$

(3) $\mathrm{Nash}(\phi)_{\mathrm{loc}}$*; namely, there exists a constant* $c_2 > 0$ *such that for each* $s > 0$*,*

$$\|u\|_2^2 \le c_2\Big(\frac{\|u\|_1^2}{\inf_{z\in \operatorname{supp} u} V(z,s)} + \phi(s)\mathcal{E}(u,u)\Big), \quad u \in \mathcal{F}\cap L^1(M;\mu).$$

We have (1) $\Longleftrightarrow$ (2) $\Longleftarrow$ (3)*.*

Proof. (1) $\Longleftrightarrow$ $\mathrm{Nash}(\phi)_B$ is in Proposition 7.3. (2) $\Longleftrightarrow$ $\mathrm{Nash}(\phi)_B$ is given in [BCS, Proposition 3.4.1] (they are proved for the case $\phi(t) = t^2$ but the modifications are easy), while (3) $\Longrightarrow$ (2) is given in [BCS, Proposition 3.1.4]. We note that in all the proofs above RVD is used only in (2) $\Longrightarrow$ $\mathrm{Nash}(\phi)_B$, and (2) $\Longleftarrow$ $\mathrm{Nash}(\phi)_B$ holds trivially. We thus obtain the desired results. $\square$

We now define the weak Poincaré inequality which will be used in the forthcoming paper [CKW].

Definition 7.5. We say that the *weak Poincaré inequality* (PI(ϕ)) holds if there exist constants $C > 0$ and $\kappa \ge 1$ such that for any ball $B_r = B(x,r)$ with $x \in M$ and for any $f \in \mathcal{F}_b$,

$$\int_{B_r} (f - \overline{f}_{B_r})^2\, d\mu \le C\phi(r)\int_{B_{\kappa r}\times B_{\kappa r}} (f(y)-f(x))^2\, J(dx,dy),$$

where $\overline{f}_{B_r} = \frac{1}{\mu(B_r)}\int_{B_r} f\, d\mu$ is the average value of f on B_r.

Proposition 7.6. *Assume that* VD *and* (1.13) *hold. Then either* PI(ϕ) *or* UHKD(ϕ) *implies* $\mathrm{Nash}(\phi)_{\mathrm{loc}}$*. Consequently, if* VD*,* RVD *and* (1.13) *are satisfied, then either* PI(ϕ) *or* UHKD(ϕ) *implies* FK(ϕ)*.*

Proof. (i) When $\phi(t) = t^2$, this fact that PI(ϕ) $\Longrightarrow$ $\mathrm{Nash}(\phi)_{\mathrm{loc}}$ is well-known; see for example [Sa, Theorem 2.1]. Generalization to this setting is a line by line modification. Then the second assertion follows from Proposition 7.4.

(ii) That UHKD(ϕ) implies $\mathrm{Nash}(\phi)_{\mathrm{loc}}$ can be proved similarly to [Ki, Corollary 2.4]. (We note that in [Ki, Corollary 2.4] it is proved for the case $\phi(t) = t^\beta$, but the modifications are easy.) One also can prove this similarly to the approach of [GH, p. 551–552]. Note that [GH] discusses the case $\phi(r) = r^\beta$, but the generalization to ϕ is also easy. $\square$

Proposition 7.7. *Under* VD *and* (1.13), FK(ϕ) *implies that the semigroup* $\{P_t\}$ *is locally ultracontractive, which in turn yields that*

(1) *there exists a properly exceptional set* $\mathcal{N} \subset M$ *such that, for any open subset* $D \subset M$*, the semigroup* $\{P_t^D\}$ *possesses the heat kernel* $p^D(t, x, y)$ *with domain* $D \setminus \mathcal{N} \times D \setminus \mathcal{N}$*.*

(2) *Let* $\varphi(x, y) : M_0 \times M_0 \to [0, \infty]$ *be a upper semi-continuous function such that for some open set* $D \subset M$ *and for some* $t > 0$*,*

$$p^D(t, x, y) \le \varphi(x, y)$$

for almost all $x, y \in D$*. Then the inequality above holds for all* $x, y \in D \setminus \mathcal{N}$*.*

Proof. The statement of Proposition 7.3 tells us that, under VD, (1.13) and FK(ϕ), there exist constants C_1, $\nu > 0$ such that for any ball $B = B(x, r)$ with $x \in M$ and $r > 0$, and any $t > 0$,

$$\|P_t^B\|_{L^1(B;\mu)\to L^\infty(B;\mu)} \le \frac{C_\nu}{V(x, r)} \left(\frac{\phi(r)}{t}\right)^{1/\nu}.$$

Therefore, the semigroup $\{P_t\}$ is locally ultracontractive. The other assertions follow from [BBCK, Theorem 6.1] and [GT, Theorem 2.12].

7.4 Some results related to the (Dirichlet) heat kernel

Recall that for any $\rho > 0$, $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is the ρ-truncated Dirichlet form, which is obtained by ρ-truncation for the jump density of the original Dirichlet form $(\mathcal{E}, \mathcal{F})$, i.e.

$$\mathcal{E}^{(\rho)}(f, g) = \int (f(x) - f(y))(g(x) - g(y))\mathbf{1}_{\{d(x,y)\le\rho\}}\, J(dx, dy).$$

As mentioned in Section 2, if VD, (1.13) and $\mathrm{J}_{\phi,\le}$ hold, then $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M; \mu)$. Let $\{X_t^{(\rho)}\}$ be the process associated with $(\mathcal{E}^{(\rho)}, \mathcal{F})$. For any non-negative open set $D \subset M$, as before we denote by $\{P_t^D\}$ and $\{Q_t^{(\rho),D}\}$ the semigroups of $(\mathcal{E}, \mathcal{F}_D)$ and $(\mathcal{E}^{(\rho)}, \mathcal{F}_D)$, respectively. (We write $\{Q_t^{(\rho),M}\}$ as $\{Q_t^{(\rho)}\}$ for simplicity.) Most of results in this subsection have been proved in [GHL2]. To be self-contained, we present new proofs by making full use of the probabilistic ideas.

The following lemma was proved in [GHL2, Proposition 4.6].

Lemma 7.8. *Suppose that* VD, (1.13) *and* $\mathrm{J}_{\phi,\le}$ *hold. Let* D *be the open subset of* M*. Then there exists a constant* $c > 0$ *such that for any* $t > 0$*, almost all* $x \in D$ *and any non-negative* $f \in L^2(D; \mu) \cap L^\infty(D; \mu)$*,*

$$|P_t^D f(x) - Q_t^{(\rho),D} f(x)| \le c\|f\|_\infty \frac{t}{\phi(\rho)}.$$

Proof. Note that $P_t^D f(x) = \mathbb{E}^x(f(X_t)\mathbf{1}_{\{\tau_D > t\}})$ and $Q_t^{(\rho),D} f(x) = \mathbb{E}^x(f(X_t^{(\rho)})\mathbf{1}_{\{\tau_D^{(\rho)} > t\}})$. Let

$$T_\rho = \inf\big\{t > 0 : d(X_t, X_{t-}) > \rho\big\}.$$

It is clear that $X_t = X_t^{(\rho)}$ for all $t < T_\rho$. Thus by [BGK1, Lemma 3.1(a)],

$$
\begin{aligned}
|P_t^D f(x) - Q_t^{(\rho),D} f(x)| &\le \left|\mathbb{E}^x(f(X_t)\mathbf{1}_{\{T_\rho \le t < \tau_D\}})\right| + \left|\mathbb{E}^x(f(X_t^{(\rho)})\mathbf{1}_{\{T_\rho \le t < \tau_D^{(\rho)}\}})\right| \\
&\le 2\|f\|_\infty \, \mathbb{P}^x(T_\rho \le t) \\
&\le 2\|f\|_\infty \left(1 - \exp\left(-t \operatorname{ess\,sup}_{z\in M} \int_{B(z,\rho)^c} J(z,y)\,\mu(dy)\right)\right) \\
&\le 2\|f\|_\infty t \operatorname{ess\,sup}_{z\in M} \int_{B(z,\rho)^c} J(z,y)\,\mu(dy),
\end{aligned}
$$

where the inequality $1 - e^{-r} \le r$ for all $r > 0$ was used in the last inequality. The desired conclusion now follows from Lemma 2.1. □

We need the following comparison of heat kernels in different domains.

Lemma 7.9. *Let V, U and D be open subsets of M such that $U_\rho := \{z \in M : d(z,U) < \rho\}$ is precompact, $V \subset U$ and $U_\rho \subset D$. Then for all t, $s > 0$,*

$$
\begin{aligned}
\operatorname{ess\,sup}_{x,y\in V} q^{(\rho),D}(t+s,x,y) \le& \operatorname{ess\,sup}_{x,y\in U} q^{(\rho),U}(t,x,y) \\
&+ \operatorname{ess\,sup}_{x\in V} \mathbb{P}^x(\tau_U^{(\rho)} \le t) \operatorname{ess\,sup}_{x,y\in U_\rho} q^{(\rho),D}(s,x,y).
\end{aligned}
\tag{7.2}
$$

Proof. For simplicity, in the proof $\{X_t^{(\rho)}\}$ denotes the subprocess of $\{X_t^{(\rho)}\}$ on exiting D. For any fixed x, $y \in V$, one can choose $r > 0$ small enough such that $B(x,r) \subset V$ and $B(y,r) \subset V$. Let $0 \le f, g \in L^1(D;\mu)$ be such that $\operatorname{supp} f \subset B(x,r)$ and $\operatorname{supp} g \subset B(y,r)$. Then we have

$$
\begin{aligned}
&\mathbb{E}\left[f(X_0^{(\rho)})g(X_{t+s}^{(\rho)})\right] \\
&= \mathbb{E}\left[f(X_0^{(\rho)})g(X_{t+s}^{(\rho)}) : \tau_U^{(\rho)} > t\right] + \mathbb{E}\left[f(X_0^{(\rho)})g(X_{t+s}^{(\rho)}) : \tau_U^{(\rho)} \le t\right] \\
&= \mathbb{E}\left[f(X_0^{(\rho)})\mathbf{1}_{\{\tau_U^{(\rho)} > t\}} \mathbb{E}^{X_t^{(\rho)}} g(X_s^{(\rho)})\right] + \mathbb{E}\left[f(X_0^{(\rho)})\mathbf{1}_{\{\tau_U^{(\rho)} \le t\}} \mathbb{E}^{X^{(\rho)}_{\tau_U^{(\rho)}}} g(X^{(\rho)}_{t+s-\tau_U^{(\rho)}})\right] \\
&= \mathbb{E}\left[f(X_0^{(\rho)})\mathbf{1}_{\{\tau_U^{(\rho)} > t\}} Q_s g(X_t^{(\rho)})\right] + \mathbb{E}\left[f(X_0^{(\rho)})\mathbf{1}_{\{\tau_U^{(\rho)} \le t\}} \mathbb{E}^{X^{(\rho)}_{\tau_U^{(\rho)}}} g(X^{(\rho)}_{t+s-\tau_U^{(\rho)}})\right] \\
&= \mathbb{E}\left[f(X_0^{(\rho)}) Q_t^{(\rho),U}(Q_s g)(X_0^{(\rho)})\right] + \mathbb{E}\left[f(X_0^{(\rho)})\mathbf{1}_{\{\tau_U^{(\rho)} \le t\}} \mathbb{E}^{X^{(\rho)}_{\tau_U^{(\rho)}}} g(X^{(\rho)}_{t+s-\tau_U^{(\rho)}})\right] \\
&\le \|f\|_{L^1(D;\mu)} \|Q_s g\|_{L^1(D;\mu)} \operatorname{ess\,sup}_{x',y'\in U} q^{(\rho),U}(t,x',y') \\
&\quad + \|f\|_{L^1(D;\mu)} \operatorname{ess\,sup}_{x'\in V} \mathbb{P}^{x'}(\tau_U^{(\rho)} \le t) \|g\|_{L^1(D;\mu)} \operatorname{ess\,sup}_{x',y'\in U^\rho, s\le t'\le t+s} q^{(\rho),D}(t',x',y') \\
&\le \|f\|_{L^1(D;\mu)} \|g\|_{L^1(D;\mu)} \operatorname{ess\,sup}_{x',y'\in U} q^{(\rho),U}(t,x',y') \\
&\quad + \|f\|_{L^1(D;\mu)} \operatorname{ess\,sup}_{x'\in V} \mathbb{P}^{x'}(\tau_U^{(\rho)} \le t) \|g\|_{L^1(D;\mu)} \operatorname{ess\,sup}_{x',y'\in U^\rho, s\le t'\le t+s} q^{(\rho),D}(t',x',y'),
\end{aligned}
$$

where we have used the strong Markov property and the fact that $X^{(\rho)}_{\tau_U^{(\rho)}} \in U^{(\rho)}$ in the first inequality.

Furthermore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}q^{(\rho),D}(t,x',y') &= \int q^{(\rho),D}(t/2,x',z)q^{(\rho),D}(t/2,z,y')\,\mu(dz)\\ &\le \sqrt{\int \left(q^{(\rho),D}(t/2,x',z)\right)^2\,\mu(dz)}\sqrt{\int \left(q^{(\rho),D}(t/2,y',z)\right)^2\,\mu(dz)}\\ &= \sqrt{q^{(\rho),D}(t,x',x')}\sqrt{q^{(\rho),D}(t,y',y')},\end{aligned}$$

and so

$$\text{ess sup}_{x',y'\in U^\rho} q^{(\rho),D}(t,x',y') = \text{ess sup}_{x'\in U^\rho} q^{(\rho),D}(t,x',x').$$

Therefore,

$$\text{ess sup}_{x'\in U^\rho} q^{(\rho),D}(t,x',x') = \sup_{\|f\|_{L^1(U_\rho;\mu)}\le 1} \langle Q_t^{(\rho),D} f, f\rangle = \sup_{\|f\|_{L^1(U_\rho;\mu)}\le 1} \langle Q_{t/2}^{(\rho),D} f, Q_{t/2}^{(\rho),D} f\rangle,$$

which implies that the function $s \mapsto \text{ess sup}_{x',y'\in U^\rho} q^{(\rho),D}(s,x',y')$ is decreasing, i.e.

$$\text{ess sup}_{x',y'\in U^\rho, s\le t'\le t+s} q^{(\rho),D}(t',x',y') = \text{ess sup}_{x',y'\in U^\rho} q^{(\rho),D}(s,x',y').$$

Hence,

$$\begin{aligned}\frac{\mathbb{E}\left[f(X_0^{(\rho)})g(X_{t+s}^{(\rho)})\right]}{\|f\|_{L^1(D;\mu)}\|g\|_{L^1(D;\mu)}} &\le \text{ess sup}_{x',y'\in U} q^{(\rho),U}(t,x',y')\\ &\quad + \text{ess sup}_{x'\in V} \mathbb{P}^{x'}(\tau_U^{(\rho)} \le t)\text{ess sup}_{x',y'\in U^\rho} q^{(\rho),D}(s,x',y').\end{aligned}$$

Taking the esssup with respect to f, g and letting $r \to 0$, we can get

$$\begin{aligned}q^{(\rho),D}(t+s,x,y) \le& \text{ess sup}_{x',y'\in U} q^{(\rho),U}(t,x',y')\\ &+ \text{ess sup}_{x\in V} \mathbb{P}^x(\tau_U^{(\rho)} \le t)\,\text{ess sup}_{x',y'\in U_\rho} q^{(\rho),D}(s,x',y')\end{aligned}$$

proving the desired assertion. □

The following lemma gives us the way to get heat kernel estimates in term of the exit time and the on-diagonal heat kernel estimates, e.g. see [GHL1, Theorem 5.1 and (5.13)].

Lemma 7.10. *Let U and V be open subsets of M such that $U \cap V = \emptyset$. For any $t, s > 0$ and almost all $x \in U$ and $y \in V$,*

$$\begin{aligned}q^{(\rho)}(t+s,x,y) \le& \mathbb{P}^x(\tau_U^{(\rho)} \le t)\,\text{ess sup}_{s\le t'\le t+s} \|q^{(\rho)}(t',\cdot,y)\|_{L^\infty(U_\rho;\mu)}\\ &+ \mathbb{P}^y(\tau_V^{(\rho)} \le s)\,\text{ess sup}_{t\le t'\le s+t} \|q^{(\rho)}(t',\cdot,x)\|_{L^\infty(V_\rho;\mu)}.\end{aligned}$$

Proof. For any fixed $x \in U$ and $y \in V$, choose $0 < r < \frac{1}{2}d(x,y)$, and let $f = \mathbf{1}_{B(x,r)}$ and $g = \mathbf{1}_{B(y,r)}$. Then by the time reversal property of the symmetric process $\{X_t^{(\rho)}\}$ and the strong Markov property,

$$\mathbb{E}\left[f(X_0^{(\rho)})g(X_{t+s}^{(\rho)})\right] = \mathbb{E}\left[f(X_0^{(\rho)})g(X_{t+s}^{(\rho)}); \tau_U^{(\rho)} < t\right] + \mathbb{E}\left[f(X_0^{(\rho)})g(X_{t+s}^{(\rho)}); \tau_U^{(\rho)} \ge t\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[f(X_0^{(\rho)})g(X_{t+s}^{(\rho)}); \tau_U^{(\rho)} < t\right] + \mathbb{E}\left[g(X_0^{(\rho)})f(X_{t+s}^{(\rho)}); \tau_V^{(\rho)} < s\right] \\
&\le \mathbb{E}\big[f(X_0^{(\rho)})\mathbf{1}_{\{\tau_U^{(\rho)} \le t\}}\mathbb{E}^{X_{\tau_U}^{(\rho)}} g(X^{(\rho)}_{t+s-\tau_U^{(\rho)}})\big] \\
&\quad + \mathbb{E}\left[g(X_0^{(\rho)})\mathbf{1}_{\{\tau_V^{(\rho)} \le s\}}\mathbb{E}^{X_{\tau_V}^{(\rho)}} f(X^{(\rho)}_{t+s-\tau_V^{(\rho)}})\right] \\
&\le \mathbb{E}\left[f(X_0^{(\rho)})\mathbf{1}_{\{\tau_U^{(\rho)} \le t\}}\right] \operatorname{ess\,sup}_{z\in U_\rho, s\le t'\le t+s}\mathbb{E}^z g\big[X_{t'}^{(\rho)}\big] \\
&\quad + \mathbb{E}\left[g(X_0^{(\rho)})\mathbf{1}_{\{\tau_V^{(\rho)} \le s\}}\right] \operatorname{ess\,sup}_{z\in V_\rho, t\le t'\le t+s}\mathbb{E}^z f\big[X_{t'}^{(\rho)}\big].
\end{aligned}
$$

Dividing both sides with $\mu(B(x,r))\mu(B(y,r))$ and letting $r \to 0$, we can obtain the desired estimate. □

The following result was proved in [GHL2, Theorem 3.1].

Lemma 7.11. *Assume that for any ball B with radius $r > 0$ and any $t > 0$,*

$$\mathbb{P}^z(\tau_B^{(\rho)} \le t) \le \psi(r,t) \quad \textit{for almost all } z \in \frac{1}{4}B,$$

where $\psi(r,\cdot)$ is a non-decreasing function for all $r > 0$. Then for any ball $B(x,r)$, $t > 0$ and any integer $k \ge 1$,

$$Q_t^{(\rho)}\mathbf{1}_{B(x,k(r+\rho))^c}(z) \le \psi(r,t)^k \quad \textit{for almost all } z \in B(x,r/4). \tag{7.3}$$

Consequently, for any ball $B(x,R)$ with $R > \rho$, $t > 0$ and any integer $k \ge 1$,

$$Q_t^{(\rho)}\mathbf{1}_{B(x,kR)^c}(z) \le \psi(R-\rho,t)^{k-1} \quad \textit{for almost all } z \in B(x,R).$$

Proof. We prove (7.3) by induction in k. Indeed, for $k = 1$,

$$Q_t^{(\rho)}\mathbf{1}_{B(x,r+\rho)^c}(z) \le \mathbb{P}^z(\tau_{B(x,r)}^{(\rho)} \le t) \le \psi(r,t) \quad \text{for almost all } z \in B(x,r/4).$$

For the inductive step from k to $k+1$, we use the strong Markov property and get that for almost all $z \in B(x,r/4)$,

$$
\begin{aligned}
Q_t^{(\rho)}\mathbf{1}_{B(x,(k+1)(r+\rho))^c}(z) &= \mathbb{E}^z\left[\mathbf{1}_{\{\tau_{B(x,r)}^{(\rho)} < t\}}\mathbb{P}^{X^{(\rho)}_{\tau^{(\rho)}_{B(x,r)}}}\Big(X^{(\rho)}_{t-\tau^{(\rho)}_{B(x,r)}} \notin B(x,(k+1)(r+\rho))\Big)\right] \\
&\le \mathbb{P}^z(\tau_{B(x,r)}^{(\rho)} < t) \operatorname{ess\,sup}_{y\in B(x,r+\rho), s\le t} Q_s^{(\rho)}\mathbf{1}_{B(y,k(r+\rho))^c}(y) \\
&\le \psi(r,t)^{k+1}.
\end{aligned}
$$

Here, in the first inequality above we have used the facts that $X^{(\rho)}_{\tau^{(\rho)}_{B(x,r)}} \in B(x,r+\rho)$, and for $z \notin B(x,(k+1)(r+\rho))$ and $y \in B(x,r+\rho)$, it holds $d(z,y) \ge d(x,z) - d(y,x) \ge k(r+\rho)$. The last inequality above follows from the assumption that $\psi(r,\cdot)$ is a non-decreasing function for all $r > 0$. This proves (7.3).

Finally, let $r = R - \rho > 0$. Then by (7.3), for any $y \in B(x,R)$ and $k \ge 1$,

$$Q_t^{(\rho)}\mathbf{1}_{B(x,(k+1)R)^c}(z) \le Q_t^{(\rho)}\mathbf{1}_{B(y,kR)^c}(z) \le \phi(R-\rho,t)^k \quad \text{for almost all } z \in B(y,r/4).$$

Covering $B(x,R)$ by a countable family of balls like $B(y,r/4)$ with $y \in B(x,R)$ and renaming k to $k-1$, we prove the second assertion. □

7.5 $\mathrm{SCSJ}(\phi) + \mathrm{J}_{\phi,\le} \Longrightarrow (\mathcal{E},\mathcal{F})$ is conservative

We will prove the following statement in this subsection of the Appendix. Although this theorem is not used in the main body of the paper, we include it here since it indicates that $\mathrm{FK}(\phi)$ is not required to deduce the conservativeness. See the paragraph after the statement of Theorem 1.15 for related discussions.

Theorem 7.12. *Assume that* VD *and* (1.13) *hold. Then,*

$$\mathrm{SCSJ}(\phi) + \mathrm{J}_{\phi,\le} \Longrightarrow (\mathcal{E},\mathcal{F}) \textit{ is conservative.}$$

Under VD, (1.13) and $\mathrm{J}_{\phi,\le}$, in view of Lemma 2.1 and Meyer's construction of adding and removing jumps in Subsection 7.2, $(\mathcal{E},\mathcal{F})$ is conservative if and only if so is $(\mathcal{E}^{(\rho)},\mathcal{F})$ for some (and hence for any) $\rho > 0$. Therefore, to prove the conservativeness of $(\mathcal{E},\mathcal{F})$, it suffices to establish it for $(\mathcal{E}^{(\rho)},\mathcal{F})$ for some $\rho > 0$. Our proof is based on Davies' method [Da], similar to what is done in [AB, Section 6] for diffusion processes.

We first give some notations. Fix $x_0 \in M$ and $r > 0$, let $B_r = B(x_0, r)$. Suppose $\mathrm{SCSJ}(\phi)$ holds. Let φ_n be the associated cut-off function for $B_{n\rho} \subset B_{(n+1)\rho}$ in $\mathrm{SCSJ}(\phi)$, and $\{a_n; n \ge -1\}$ an increasing sequence with $a_{-1} = a_0 \ge 0$. Set

$$\widetilde{\varphi} = a_0 + \sum_{n=0}^{\infty} (a_{n+1} - a_n)(1 - \varphi_n). \tag{7.4}$$

Note that

$$\widetilde{\varphi} = a_0 + \sum_{k=0}^{n-1} (a_{k+1} - a_k)(1 - \varphi_k) \le a_n \quad \text{on } B_{n\rho}, \tag{7.5}$$

and for $0 \le j < n$,

$$\tilde{\varphi} \ge a_0 + \sum_{k=0}^{j-1} (a_{k+1} - a_k)(1 - \varphi_k) = a_j \quad \text{on } M \setminus B_{j\rho}. \tag{7.6}$$

We have the following statement.

Lemma 7.13. *Assume that* VD, (1.13), $\mathrm{J}_{\phi,\le}$ *and* $\mathrm{CSJ}(\phi)$ *hold. Then for any* $f \in \mathcal{F}_b$,

$$\int_M f^2 \, d\Gamma^{(\rho)}(\widetilde{\varphi}, \widetilde{\varphi}) \le A_0 \left(\frac{1}{8} \int_M \tilde{\varphi}^2 \, d\Gamma^{(\rho)}(f,f) + \frac{C_0}{\phi(\rho)} \int_M \tilde{\varphi}^2 f^2 \, d\mu \right), \tag{7.7}$$

where

$$A_0 := \sup_{n \ge 0} \left(\frac{a_{n+1} - a_n}{a_{n-1}} \right)^2. \tag{7.8}$$

Proof. By considering $f\varphi_n$ in place of f and then taking $n \to \infty$ if needed, we may assume without loss of generality that $f \in \mathcal{F}_b$ has compact support. Thus in view of (7.5), the right hand side of (7.7) is finite. Let $U_n = B_{(n+1)\rho} \setminus B_{n\rho}$ and $U_n^* = B_{(n+2)\rho} \setminus B_{(n-1)\rho}$. Note that

$$\Gamma^{(\rho)}(1 - \varphi_n, 1 - \varphi_m) = \Gamma^{(\rho)}(\varphi_n, \varphi_m) = 0$$

for any $m \geq n+3$, and $\Gamma^{(\rho)}(1-\varphi_n, 1-\varphi_n) = \Gamma^{(\rho)}(\varphi_n, \varphi_n) = 0$ outside U_n^*. Then using the Cauchy-Schwarz inequality, CSJ(ϕ) and Proposition 2.4(2) (with $\varepsilon = \frac{1}{48}$ in $\mathrm{CSAJ}_+^{(\rho)}$), we have

$$\begin{aligned}
\int_M f^2\, d\Gamma^{(\rho)}(\widetilde{\varphi}, \widetilde{\varphi}) &\leq 2\sum_{n=0}^{\infty}\sum_{n\leq m}(a_{n+1}-a_n)(a_{m+1}-a_m)\int_M f^2\, d\Gamma^{(\rho)}(\varphi_n, \varphi_m) \\
&= 2\sum_{n=0}^{\infty}\sum_{n\leq m\leq n+2}(a_{n+1}-a_n)(a_{m+1}-a_m)\int_M f^2\, d\Gamma^{(\rho)}(\varphi_n, \varphi_m) \\
&= \sum_{n=0}^{\infty}\sum_{n\leq m\leq n+2}(a_{n+1}-a_n)^2\int_M f^2\, d\Gamma^{(\rho)}(\varphi_n, \varphi_n) \\
&\quad + \sum_{n=0}^{\infty}\sum_{n\leq m\leq n+2}(a_{m+1}-a_m)^2\int_M f^2\, d\Gamma^{(\rho)}(\varphi_m, \varphi_m) \\
&\leq 6\sum_{n=0}^{\infty}(a_{n+1}-a_n)^2\int_M f^2\, d\Gamma^{(\rho)}(\varphi_n, \varphi_n) \\
&= 6\sum_{n=0}^{\infty}(a_{n+1}-a_n)^2\int_{U_n^*} f^2\, d\Gamma^{(\rho)}(\varphi_n, \varphi_n) \\
&\leq \sum_{n=0}^{\infty}(a_{n+1}-a_n)^2\left(\frac{1}{8}\int_{U_n} d\Gamma^{(\rho)}(f,f) + \frac{c_1}{\phi(\rho)}\int_{U_n^*} f^2\, d\mu\right) \\
&\leq \sum_{n=0}^{\infty}\left(\frac{a_{n+1}-a_n}{a_{n-1}}\right)^2\left(\frac{1}{8}\int_{U_n}\tilde{\varphi}^2\, d\Gamma^{(\rho)}(f,f) + \frac{c_1}{\phi(\rho)}\int_{U_n^*}\tilde{\varphi}^2 f^2\, d\mu\right),
\end{aligned}$$

where in the last inequality we have used the fact that $a_{n-1} \leq \tilde{\varphi} \leq a_{n+2}$ on U_n^* from (7.5) and (7.6). The proof is complete. □

We also need the following lemma.

Lemma 7.14. *Assume that* VD, (1.13), $\mathrm{J}_{\phi,\leq}$ *and* SCSJ(ϕ) *hold. Let* $\tilde{\varphi}$ *and* A_0 *be as in* (7.4) *and* (7.8), *respectively. Suppose that* $A_0 \leq 1$. *Let* f *have compact support, and set* $u(t) = Q_t^{(\rho)} f$. *Then, we have*

$$\int_0^t ds \int_M \tilde{\varphi}^2\, d\Gamma^{(\rho)}(u(s), u(s)) \leq 2\|f\tilde{\varphi}\|_2^2 \exp\left(\frac{4C_0 t}{\phi(\rho)}\right). \tag{7.9}$$

Proof. Let $(a_n)_{n\geq -1}$ and φ_n as above. For any $N \geq 1$, set

$$\tilde{\varphi}_{0,N} = a_0 + \sum_{n=0}^{N}(a_{n+1}-a_n)(1-\varphi_n)$$

and

$$h_N(t) = \|u(t)\tilde{\varphi}_{0,N}\|_2^2.$$

We write $u(t,x) = Q_t^{(\rho)} f(x)$. Since $u(t)$ and $\widetilde{\varphi}_{0,N}^2 u(t) \in \mathcal{F}$,

$$h_N'(t) = -\,2\mathcal{E}^{(\rho)}(u(t), \tilde{\varphi}_{0,N}^2 u(t))$$

$$
\begin{aligned}
&= -2\int_{M\times M}(u(t,x)-u(t,y))(\tilde\varphi_{0,N}^2(x)u(t,x)-\tilde\varphi_{0,N}^2(y)u(t,y))\,J^{(\rho)}(dx,dy)\\
&= -2\int_{M\times M}(u(t,x)-u(t,y))^2\tilde\varphi_{0,N}^2(x)\,J^{(\rho)}(dx,dy)\\
&\quad -2\int_{M\times M}(\tilde\varphi_{0,N}^2(x)-\tilde\varphi_{0,N}^2(y))u(t,y)(u(t,x)-u(t,y))\,J^{(\rho)}(dx,dy)\\
&\le -2\int_{M\times M}(u(t,x)-u(t,y))^2\tilde\varphi_{0,N}^2(x)\,J^{(\rho)}(dx,dy)\\
&\quad +\frac14\int_{M\times M}(\tilde\varphi_{0,N}(x)+\tilde\varphi_{0,N}(y))^2(u(t,x)-u(t,y))^2\,J^{(\rho)}(dx,dy)\\
&\quad +4\int_{M\times M}u(t,y)^2(\tilde\varphi_{0,N}(x)-\tilde\varphi_{0,N}(y))^2\,J^{(\rho)}(dx,dy)\\
&\le -2\int_{M\times M}(u(t,x)-u(t,y))^2\tilde\varphi_{0,N}^2(x)\,J^{(\rho)}(dx,dy)\\
&\quad +\frac12\int_{M\times M}(\tilde\varphi_{0,N}^2(x)+\tilde\varphi_{0,N}^2(y))(u(t,x)-u(t,y))^2\,J^{(\rho)}(dx,dy)\\
&\quad +4\int_{M\times M}u(t,y)^2(\tilde\varphi_{0,N}(x)-\tilde\varphi_{0,N}(y))^2\,J^{(\rho)}(dx,dy)\\
&\le -\int_{M\times M}(u(t,x)-u(t,y))^2\tilde\varphi_{0,N}^2(x)\,J^{(\rho)}(dx,dy)\\
&\quad +4\int_{M\times M}u(t,x)^2(\varphi_{0,N}(x)-\varphi_{0,N}(y))^2\,J^{(\rho)}(dx,dy)\\
&= -\int_{M\times M}\tilde\varphi_{0,N}^2\,d\Gamma^{(\rho)}(u(t),u(t))+4\int_{M\times M}u(t)\,d\Gamma^{(\rho)}(\varphi_{0,N},\varphi_{0,N}),
\end{aligned}
$$

where in the first inequality we used the fact that $2ab \le \frac{a^2}{4}+4b^2$ for all $a,b\in\mathbb{R}$, and in the last inequality

$$\varphi_{0,N} := \sum_{n=0}^{N}(a_{n+1}-a_n)\varphi_n = -\tilde\varphi_{0,N}+a_{N+1}.$$

So by (the proof of) Lemma 7.13 and the assumption $A_0\le 1$,

$$h_N'(t) \le -\frac12\int_M \tilde\varphi_{0,N}^2\,d\Gamma^{(\rho)}(u(t),u(t))+\frac{4C_0}{\phi(\rho)}h_N(t). \tag{7.10}$$

In particular,

$$h_N' \le \frac{4C_0}{\phi(\rho)}h_N$$

and hence

$$h_N(t)\le h_N(0)\exp\left(\frac{4C_0t}{\phi(\rho)}\right)=\|f\tilde\varphi_{0,N}\|_2^2\exp\left(\frac{4C_0t}{\phi(\rho)}\right).$$

Using the inequality above and integrating (7.10), we obtain

$$h_N(t)-h_N(0)+\frac12\int_0^t ds\int_M \tilde\varphi_{0,N}^2\,d\Gamma^{(\rho)}(u(s),u(s))\le \|f\tilde\varphi_{0,N}\|_2^2(e^{4C_0t/\phi(\rho)}-1).$$

Since $h_N(0)=\|f\tilde\varphi_{0,N}\|_2^2$, letting $N\to\infty$ gives us the desired assertion. □

Proof of Theorem 7.12. We mainly follow the argument of [Da, Theorem 7] and make use of Lemma 7.14 above. Let $f \geq 0$ be a bounded function with compact support and let $u(t) = Q_t^{(\rho)} f$. As mentioned in the remark below Theorem 7.12, it is sufficient to verify that $Q_t^{(\rho)}\mathbf{1} = 1$ μ-a.e for every $t > 0$. Since $\int_M Q_t^{(\rho)} f\, d\mu = \int_M f\, Q_t^{(\rho)}\mathbf{1}\, d\mu$, it reduces to show that

$$\int_M f\, d\mu \leq \int_M u(t)\, d\mu \tag{7.11}$$

for some $t > 0$.

For any $n \geq 0$, let $a_n = s^n$ with $s > 1$ such that $s(s-1) \leq 1$, and set $a_{-1} = 1$. In particular, with A_0 defined by (7.8), we have $A_0 = s^2(1-s)^2 \leq 1$. Let φ_n and $\tilde{\varphi}$ be defined as in the paragraph containing (7.4). Set $U_n^* = B_{(n+2)\rho} \setminus B_{(n-1)\rho}$. Then for $t \in (0, 1]$, by the Cauchy-Schwarz inequality and Lemma 7.14, for any $t \in (0, 1)$,

$$\begin{aligned}
\langle f, \varphi_n \rangle - \langle u(t), \varphi_n \rangle &= -\int_0^t \frac{d}{ds} \langle u(s), \varphi_n \rangle\, ds \\
&= \int_0^t ds \int_M \Gamma^{(\rho)}(u(s), \varphi_n) \\
&= \int_0^t ds \int_M \tilde{\varphi} \cdot \tilde{\varphi}^{-1}\, d\Gamma^{(\rho)}(u(s), \varphi_n) \\
&\leq \left(\int_0^t ds \int_M \tilde{\varphi}^2\, d\Gamma^{(\rho)}(u(s), u(s)) \right)^{1/2} \left(\int_0^t ds \int_M \tilde{\varphi}^{-2}\, d\Gamma^{(\rho)}(\varphi_n, \varphi_n) \right)^{1/2} \\
&\leq \sqrt{2} \| f \tilde{\varphi} \|_2 e^{2C_0 t/\phi(\rho)} (\sup_{U_n^*} \tilde{\varphi}^{-1}) \left(\int_{U_n^*} \Gamma^{(\rho)}(\varphi_n, \varphi_n) \right)^{1/2},
\end{aligned}$$

where in the last inequality we used again the fact that $\Gamma^{(\rho)}(\varphi_n, \varphi_n) = 0$ outside U_n^*. Note that on U_n^*, we have from (7.5) and (7.6) that $a_{n-1} \leq \tilde{\varphi} \leq a_{n+2}$ and so $\sup_{U_n^*} \tilde{\varphi}^{-1} \leq a_{n-1}^{-1}$. On the other hand, using SCSJ(ϕ) with $f \in \mathcal{F} \cap C_c(M)$ such that $f|_{B_{(n+2)\rho}} = 1$, we find that

$$\int_{U_n^*} \Gamma^{(\rho)}(\varphi_n, \varphi_n) \leq \frac{c_1}{\phi(\rho)} \mu(U_n^*).$$

Combining all all the conclusions above, we get

$$\langle f, \varphi_n \rangle - \langle u(t), \varphi_n \rangle \leq \sqrt{2}\, \| f \tilde{\varphi} \|_2 \exp\left(\frac{2C_0 t}{\phi(\rho)} + \frac{1}{2} \log\left(\frac{c_1}{\phi(\rho)} \mu(U_n^*) \right) - \log a_{n-1} \right).$$

Noting that due to VD, $\mu(U_n^*) \leq \mu(B_{(n+2)\rho}) \leq c_2(\rho) n^{d_2}$ for any $n \geq 0$, and $a_n = s^n$ for $s > 1$, one can easily see that the right hand side of the inequality above converges to 0 when $n \to \infty$. Since

$$\int_M u(t)\, d\mu = \lim_{n \to \infty} \int_M u(t) \varphi_n\, d\mu \quad \text{and} \quad \int_M f\, d\mu = \lim_{n \to \infty} \int_M f \varphi_n\, d\mu,$$

we get (7.11) and the conservativeness of $(\mathcal{E}, \mathcal{F})$. □

Remark 7.15. By using the arguments above, one can study the stochastic completeness in terms of SCSJ(ϕ) for jump processes in general settings, namely to obtain some sufficient condition for the stochastic completeness without VD assumption. See [AB, Theorem 1.16 and Section 7] for related discussions about diffusions.

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