Minicourse at the Department of Mathematics Kyoto University

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Quenched invariance principle for symmetric diffusions in a degenerate random environment

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Based on a joint work with Alberto Chiarini

INTRODUCTION

LECTURE 1. The homogenization problem of diffusions in divergence form, and the construction of the process using Dirichlet forms.

LECTURE 2. Harmonic coordinates and corrector, martingale CLT and the ergodicity of the environment process viewed from the diffusion.

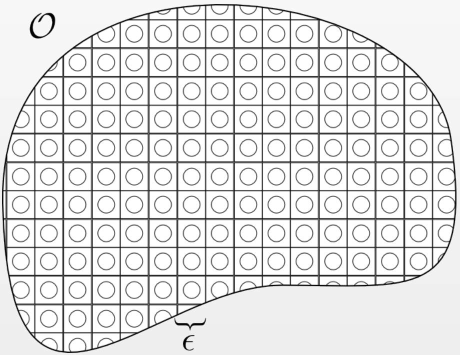
LECTURE 3. The sublinearity of the corrector via Moser iteration.

LECTURE 4. The local limit theorem and the parabolic Harnack inequality.

Motivation

Consider a conductor occupying a region $\mathcal{O} \subset \mathbb{R}^d$ and suppose that the conducting material is *inhomogeneous*. For example:

- composite materials (glasses, metal alloys, conductors);
- crystals, materials with impurities, materials with holes;



Motivation

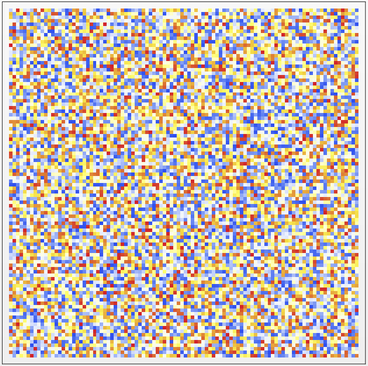
Question.

Given that the variations of the conductivity are on a very small scale,

- Can we describe the macroscopic properties of the material?
- Can we give a satisfactory approximation of the motion of a particle living in the inhomogeneous medium?

Homogenization is the art of extracting homogeneous effective parameters from disordered heterogeneous media.

Lecture 1. Diffusions in divergence form and construction of the process.



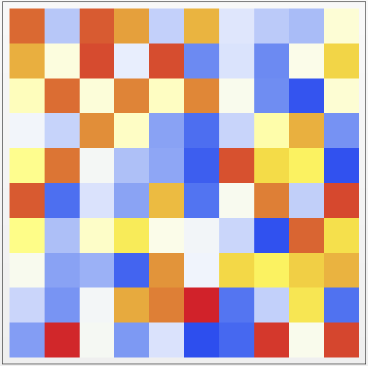


Figure: a possible realization of a random field.

Sampling a diffusion in Random Environment

To sample a *diffusion in Random Environment* there are two steps:

- First Step: we sample the Random Environment ω from a probability space $(\Omega, \mathcal{G}, \mu)$.
- Second Step: we sample the diffusion X according to a law $\mathbb{P}^\omega$ on $C([0,\infty];\mathbb{R}^d)$ depending on the environment ω. The diffusion is associated to

$$L^\omega u(x) = \operatorname{div}(a^\omega(x)\nabla u(x)).$$

The goal. Understanding the long-time behaviour of X_t under $\mathbb{P}^\omega$,

$$\epsilon X_{t/\epsilon^2} \longrightarrow \boxed{?} \quad \text{as } \epsilon \to 0.$$

The Random Environment

A stationary and ergodic Random Environment is a probability space $(\Omega, \mathcal{G}, \mu)$ equipped with a measurable group of shifts $\{\tau_x\}_{x\in\mathbb{R}^d}$ such that

- (group) $\tau_{x+y} = \tau_x \circ \tau_y$ and $\tau_0 = id_\Omega$.
- (stationarity) $\mu \circ \tau_x^{-1} = \mu$ for all $x \in \mathbb{R}^d$.
- (ergodicity) if $\tau_x A = A$ for all $x \in \mathbb{R}^d \implies \mu(A) \in \{0, 1\}$.

Given $v : \Omega \to \mathbb{R}$ we set

$$v^\omega(x) := v(\tau_x\omega), \quad x \in \mathbb{R}^d, \omega \in \Omega,$$

to be the random field associated to v. $(v^{\tau_y\omega}(x) = v^\omega(x + y))$.

Ergodic theorem: if $\mathbb{E}_\mu[|v|^s] < \infty$, then μ-a.a.

$$\|v^\omega\|_{s,B} := \left(\frac{1}{|B|}\int_B |v^\omega(x)|^s \, dx\right)^{1/s} \longrightarrow \mathbb{E}_\mu[|v|^s]^{1/s}, \qquad |B| \uparrow +\infty.$$

The symmetric diffusion

Fix an ergodic and stationary random environment $(\Omega, \mathcal{G}, \mu)$ and $d \geq 2$. Consider the operator in divergence form

$$L^\omega u(x) = \mathrm{div}(a^\omega(x) \nabla u(x))$$

where $x \mapsto a^\omega(x)$ is a random field associated to $a : \Omega \to \mathbb{R}^{d \times d}$ which is symmetric and satisfies:

A.1 there exist $\lambda, \Lambda : \Omega \to [0, +\infty]$ such that

$$\lambda(\omega)|\xi|^2 \leq a(\omega)\xi \cdot \xi \leq \Lambda(\omega)|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \ \mu\text{-a.s.}$$

A.2 there exist $p, q \in [1, +\infty]$ such that $1/p + 1/q < 2/d$ and

$$\mathbb{E}_\mu[\lambda^{-q}] < +\infty, \quad \mathbb{E}_\mu[\Lambda^p] < +\infty.$$

A.3 $x \mapsto 1/\lambda^\omega(x)$ and $x \mapsto \Lambda^\omega(x)$ belong to $L^\infty_{loc}(\mathbb{R}^d)$ μ-a.s.

Notation: for $r \geq 1$, $B \subset \mathbb{R}^d$

$$\|f\|_{r,B} := \left(\frac{1}{|B|} \int_B |f(x)|^r \, dx \right)^{\frac{1}{r}}.$$

By the Ergodic Theorem, for all $x \in \mathbb{R}^d$

$$\lim_{R\to\infty} \|1/\lambda^\omega\|_{q,B(x,R)} = \mathbb{E}_\mu[\lambda^{-q}]^{1/q} < \infty,$$
$$\lim_{R\to\infty} \|\Lambda^\omega\|_{p,B(x,R)} = \mathbb{E}_\mu[\Lambda^{p}]^{1/p} < \infty.$$

Remark: λ^ω and Λ^ω are not necessarily **volume doubling** and

$$\sup_{R>0} \sup_{x\in\mathbb{R}^d} \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} \lambda^\omega(x) \, dx \right) \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} (\lambda^\omega(x))^{-\frac{1}{p-1}} \, dx \right)^{p-1}$$

is possibly **infinite** $\Rightarrow$ Not in a Muckenhaupt class (Similar for Λ^ω).

Problem: $x \mapsto a^\omega(x)$ is **not** smooth $\Rightarrow L^\omega = \mathrm{div}(a^\omega \nabla \cdot)$ is **not** well defined.

Solution: we rather consider the Dirichlet form on $L^2(\mathbb{R}^d, dx)$

$$\mathcal{E}^\omega(u, v) := \int_{\mathbb{R}^d} a^\omega \nabla u \cdot \nabla v \, dx$$

with $u, v \in \mathcal{F}^\omega$, being $\mathcal{F}^\omega$ the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to $\mathcal{E}^\omega(\cdot, \cdot) + (\cdot, \cdot)_{L^2}$ in $L^2(\mathbb{R}^d, dx)$.

Fukushima [FOT94, Theorem 7.2.2] $\Rightarrow$ there exist a reversible diffusion $(X_t, \mathbb{P}_x^\omega)$, $x \in \mathbb{R}^d$, which is uniquely determined up to the ambiguity of a *properly exceptional set* $\mathcal{N}^\omega \subset \mathbb{R}^d$, possibly dependent on $\omega \in \Omega$.

We use A.3 to remove the ambiguity of the exceptional set $\mathcal{N}^\omega$.

Quenched Invariance Principle

Theorem (Chiarini, Deuschel 2014)

Let A.1, A.2 and A.3 be satisfied and $(X_t, \mathbb{P}_x^\omega)$, $x \in \mathbb{R}^d$ *be the minimal diffusion process associated to* $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ *on* $L^2(\mathbb{R}^d, dx)$.
Then, for μ*-almost all* ω*, in distribution under* $\mathbb{P}_0^\omega$ *on* $C([0,\infty), \mathbb{R}^d)$

$$\epsilon X_{\cdot/\epsilon^2} \xrightarrow{d} \mathbf{D}^{1/2} W_\cdot \quad \epsilon \to 0,$$

where W *is a standard d-dimensional Brownian motion and* $\mathbf{D}$ *is a deterministic non-degenerate covariance matrix.*

Related works

The problem of quenched Invariance Principle for diffusions has a long history:

- Papanicolau, Varadhan (1979), Kozlov (1979), bounded and smooth coefficients.
- Osada (1983), bounded measurable coefficients.
- Fannjiang, Komorowski (1997), smooth coefficients, antisymmetric part unbounded.
- Lejay (2001), bounded, measurable coefficients, divergence form operator with lower order terms.
- Ba, Mathieu (2015), periodic environment, measurable coefficients, only the first moment is needed.

Further contributions: Hairer, Landim, Olla, Pardoux, Piatnitski, Snitzman, Zhikov ...

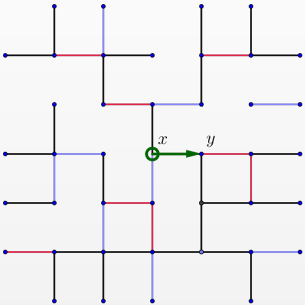
Related works – the discrete counterpart

The Random Conductance Model

Put random weights on the edges of the Euclidian lattice, $c^{\omega}_{xy} \in [0, \infty)$ according to a stationary and ergodic law $\mathbb{P}$. Look at the continuous time Markov chain X_t with jump rates

$$P^{\omega}_{xy} = \frac{c^{\omega}_{xy}}{\sum_{z \sim x} c^{\omega}_{xz}}.$$

What about the long time behavior of X_t?



Many contributions:

Andres, Barlow, Biskup, Deuschel, Hambly, Kumagai, Mathieu, Piatniski, Slowik...

The strategy

For the proof of the invariance principle we use the decomposition

$$X_t = \text{martingale} \; + \; \text{corrector}.$$

There are two steps.

1. Find a function $y : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ which is $\mathcal{E}^\omega$-harmonic for μ-a.a ω. Then $M_t := y(X_t, \omega)$ is a $\mathbb{P}_0^\omega$-martingale and one shows

$$\epsilon M_{\cdot/\epsilon^2} \xrightarrow{d} \mathbf{D}^{1/2} W_\cdot \quad \mu\text{-a.s.}$$

   through an application of the CLT for martingales (Helland 1982).
2. Prove the sublinearity of the corrector $\chi : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, $\chi(x,\omega) := x - y(x,\omega)$ on balls, that is,

$$\lim_{\epsilon \to 0} \sup_{x \in B(0,R)} \epsilon \, |\chi(x/\epsilon, \omega)| = 0, \quad \mu\text{-a.s.}$$

   Then one shows $\epsilon\chi(X_{\cdot/\epsilon^2}, \omega) \to 0$ in $\mathbb{P}_0^\omega$-probability.

Conclusion: $\epsilon X_{\cdot/\epsilon^2} = \epsilon M_{\cdot/\epsilon^2} + \epsilon\chi(X_{\cdot/\epsilon^2}, \omega) \xrightarrow{d} \mathbf{D}^{1/2} W_\cdot \quad \mu\text{-a.s.}$ $\square$

Lecture 2. Harmonic coordinates and the environment process.

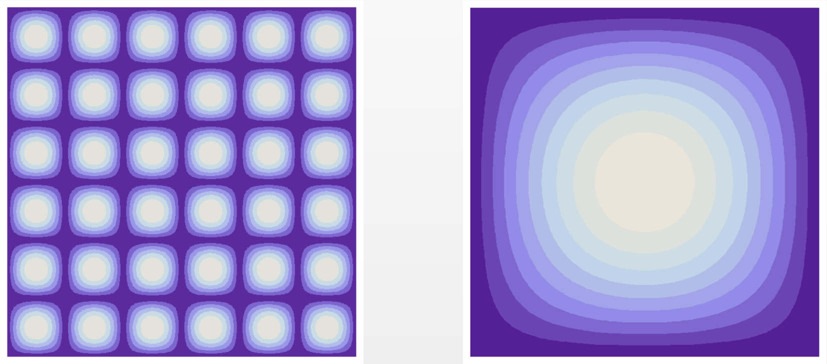


Figure: a periodic field.

The harmonic coordinates and the correctors

We prove that there exist "correctors" $\chi(x,\omega)$ such that $\nabla\chi(x,\omega)$ is a stationary field and such that $y(x,\omega) := x - \chi(x,\omega)$ (harmonic coordinates)

$$L^\omega y(x,\omega) = 0.$$

Thus X_t can be decomposed as $X_t = y(X_t,\omega) + \chi(X_t,\omega)$;

$M_t^\epsilon := \epsilon y(X_{t/\epsilon^2},\omega)$ is a $\mathbb{P}_0^\omega$-martingale (Itô formula) with quadratic variation given by

$$\langle M_h^\epsilon, M_k^\epsilon\rangle_t = 2\epsilon^2 \int_0^{t/\epsilon^2} \sum_{i,j}^{d} a_{ij}(X_s,\omega)\,\partial_i y^h(X_s,\omega)\partial_j y^k(X_s,\omega)ds$$

The Environment Process

The environment process corresponding to the diffusion $\{X_t, t \geq 0\}$ under $\mathbb{P}_0^\omega$ is an Ω valued process defined by

$$t \to \eta_t^\omega := \tau_{X_t}\omega \in \Omega, \quad t \geq 0.$$

Proposition

The measure μ is ergodic and invariant for the process η_t^ω.

Remark: for $f \in L^1(\Omega, \mu)$, if $f(x, \omega) = f(\tau_x \omega)$,

$$\frac{1}{t}\int_0^t f(X_s, \omega)ds = \frac{1}{t}\int_0^t f(\tau_{X_t}\omega)ds = \frac{1}{t}\int_0^t f(\eta_t^\omega)ds \to \mathbb{E}_\mu[\, f\,],$$

$\mathbb{P}_0^\omega$-a.s. for μ-a.a. ω as $t \to \infty$.

By the ergodic theorem for the environment process, for $\epsilon \to 0$

$$\frac{\langle M_i^\epsilon, M_j^\epsilon \rangle_t}{t} = \frac{2\epsilon^2}{t} \int_0^{t/\epsilon^2} \sum_{i,j}^{d} a_{ij}(0,\eta_t^\omega)\, \partial_i y^h(0,\eta_t^\omega)\, \partial_j y^k(0,\eta_t^\omega)\, ds$$

$$\to 2\,\mathbb{E}_\mu \Big[\sum_{i,j}^{d} a_{ij}(x,\omega) (\delta_{h,i} - \partial_i \chi^h(x,\omega)) (\delta_{k,j} - \partial_j \chi^k(x,\omega)) \Big] \doteq \mathbf{d}_{i,j}$$

CLT for Martingales (Helland 1982) $\Rightarrow$ the finite dimensional distributions of M^ϵ under $\mathbb{P}_0^\omega$ converges for μ a.a. ω to those of $\mathbf{D}^{1/2} W$ where W is a Brownian motion and $\mathbf{D} = [\mathbf{d}_{ij}]$.

We are left to prove the sublinearity of the corrector χ on balls, that is,

$$\lim_{\epsilon \to 0} \sup_{x \in B(0,R)} \epsilon \, |\chi(x/\epsilon, \omega)| = 0, \quad \mu\text{-a.s.}$$

This will imply $\epsilon \chi(X_{\cdot/\epsilon^2}, \omega) \to 0$ in $\mathbb{P}_0^\omega$-probability.

Lecture 3. The sublinearity of the corrector via Moser iteration.

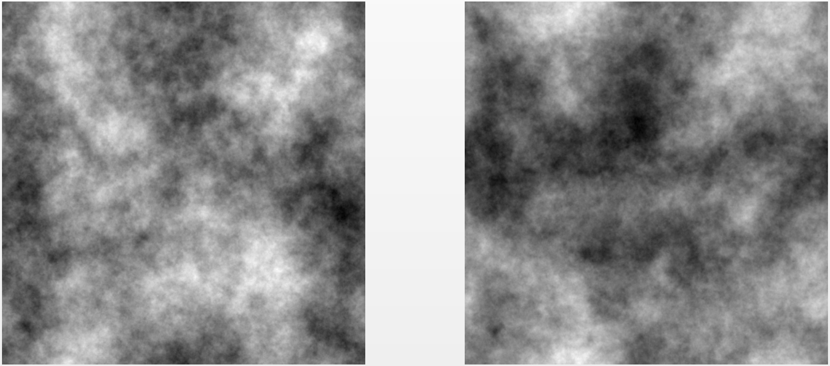


Figure: two samples of a gaussian random field.

Sublinearity of the corrector

Aim:

$$\lim_{\epsilon\to 0} \sup_{x\in B(0,R)} \epsilon\,|\chi(x/\epsilon,\omega)| = 0, \quad \mu\text{-a.s.}$$

Solution: a priori estimates for solutions to PDE.
The corrector $\chi = (\chi^1,\dots,\chi^d)$ satisfies a Poisson equation

$$\mathrm{div}(a^\omega(x)\nabla\chi^k(x,\omega)) = \mathrm{div}(a^\omega(x)e_k), \quad \mu\text{-a.s.}$$

Using "Moser iteration" we derive a *maximal inequality* for the correctors

$$\sup_{z\in B(0,R/2)} |\chi(z,\omega)| \le \left(1 \vee C_M^{B(0,R)}\right)^\kappa \|\chi(\cdot,\omega)\|_{2p^*,B(0,R)}^\gamma$$

$$C_M^{B(0,R)} := \|1/\lambda^\omega\|_{B(0,R),q}\|\Lambda^\omega\|_{B(0,R),p}.$$

is a random constant, bounded for large R by the ergodic theorem.

$$\lim_{R\to\infty} \|1/\lambda^\omega\|_{q,B(0,R)} = \mathbb{E}_\mu[\lambda^{-q}]^{1/q}, \quad \lim_{R\to\infty} \|\Lambda^\omega\|_{p,B(0,R)} = \mathbb{E}_\mu[\Lambda^p]^{1/p}.$$

Moser iteration scheme

1. Bound from above the L^r-norm of powers of a solution by the Dirichlet form. (Sobolev inequality)

$$\|u\|_{\alpha r} \lesssim \mathcal{E}(u^\alpha, u^\alpha), \quad \alpha \geq 1.$$

2. Bound the Dirichlet norm of u^α by a L^s-norm of u^α with $r > s$. (energy-like estimate for solutions to PDEs.)

$$\mathcal{E}(u^\alpha, u^\alpha) \lesssim \|u\|_{\alpha s}.$$

Introduce $\alpha_k := (r/s)^k \geq 1$ then $\alpha_k r = \alpha_{k+1} s$ and

$$\|u\|_{\alpha_k r} \leq C^{1/\alpha_k} \|u\|_{\alpha_k s} \quad \Rightarrow \quad \|u\|_{\alpha_{k+1} s} \leq C^{1/\alpha_k} \|u\|_{\alpha_k s}$$

Now iterate to get

$$\|u\|_\infty = \limsup_{k\to\infty} \|u\|_{\alpha_{k+1} s} \leq C^{\sum_{j=0}^\infty 1/\alpha_j} \|u\|_s.$$

Our setting

Sobolev inequality: let $\eta \in C_0^\infty(B)$ and $\alpha \geq 1$, then

$$\|u^\alpha \eta\|_{\rho,B}^2 \lesssim \|1/\lambda^\omega\|_{q,B} |B|^{2/d} \left[\frac{\mathcal{E}_\eta^\omega(u^\alpha, u^\alpha)}{|B|} + \text{ boundary terms} \right],$$

where $\rho := \frac{2d}{d-2+d/q}$ and $\mathcal{E}_\eta^\omega(u,u) = \int a^\omega \nabla u \cdot \nabla u \, \eta^2 dx$.

A priori estimate: in the PDE take as test function $\phi = \eta u^{2\alpha-1}$, then

$$\frac{\mathcal{E}_\eta^\omega(u^\alpha, u^\alpha)}{|B|} \lesssim \|\Lambda^\omega\|_{p,B} \|\nabla \eta\|_\infty^2 \|u^\alpha\|_{2p/(p-1),B}^2$$

For Moser iteration to work we need

$$\rho > 2p/(p-1) \iff \frac{1}{p} + \frac{1}{q} < \frac{2}{d}.$$

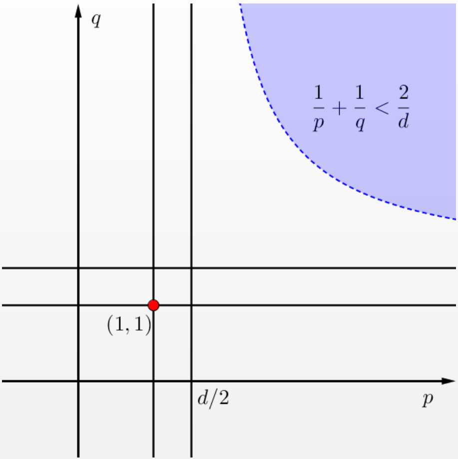
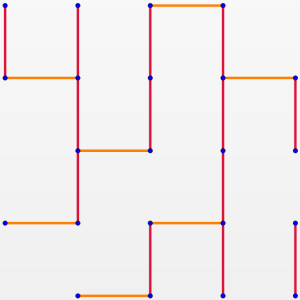


Figure: p, q condition.

Lecture 4. The local limit theorem and the parabolic Harnack inequality.



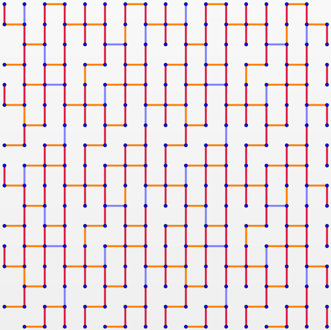


Figure: the random conductance model.

Quenched Local CLT

We look again at the diffusion $(X_t, \mathbb{P}_x^\omega)$, $x \in \mathbb{R}^d$, "formally" associated to

$$L^\omega u(x) = \mathrm{div}(a^\omega(x)\nabla u(x)).$$

(Again we make sense of $(X_t, \mathbb{P}_x^\omega)$, $x \in \mathbb{R}^d$ with Dirichlet forms theory).

We introduce

- the transition densities $p_t^\omega(\cdot,\cdot)$ of $(X_t, \mathbb{P}_x^\omega)$ with respect to the Lebesgue measure dx
- and the gaussian kernel with positive definite covariance matrix Σ

$$k_t^\Sigma(x) := \frac{1}{\sqrt{(2\pi t)^d \det \Sigma}} \exp\left(-\frac{x \cdot \Sigma x}{2t}\right).$$

Recall the assumptions:

A.1 there exist $\lambda, \Lambda : \Omega \to [0, +\infty]$ such that

$$\lambda(\omega)|\xi|^2 \leq a(\omega)\xi \cdot \xi \leq \Lambda(\omega)|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \ \mu\text{-a.s.}$$

A.2 there exist $p, q \in [1, +\infty]$ such that $1/p + 1/q < 2/d$ and

$$\mathbb{E}_\mu[\lambda^{-q}] < +\infty, \quad \mathbb{E}_\mu[\Lambda^p] < +\infty.$$

And consider further the following:

A.4 for μ-a.a. ω, for a.a $z \in \mathbb{R}^d$, all balls $B \subset \mathbb{R}^d$ and all compacts $I \subset (0, +\infty)$

$$\lim_{\epsilon \to 0} \mathbb{P}_z^\omega \left[\epsilon X_{t/\epsilon^2} \in B \right] = \int_B k_t^\Sigma(x)\, dx$$

uniformly in $t \in I$.

Theorem (Chiarini, Deuschel 2015)

Assume A.1, A.2 and A.4. Let $R > 0$ and $I \subset (0, \infty)$ be compact. Then for μ-almost all $\omega \in \Omega$ we have that for almost all $z \in \mathbb{R}^d$

$$\lim_{\epsilon \to 0} \sup_{x \in B(z,R)} \sup_{t \in I} \left| \epsilon^{-d} p^{\omega}_{t/\epsilon^2}(z, x/\epsilon) - k^{\Sigma}_t(x) \right| = 0. \tag{1}$$

If we further assume that $\lambda^{\omega}(\cdot)^{-1}, \Lambda^{\omega}(\cdot) \in L^{\infty}_{loc}(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$, then (1) is satisfied for all $z \in \mathbb{R}^d$.

Related works:

- Andres, Deuschel, Slowik (2015), quenched local CLT on $\mathbb{Z}^d$ with degenerate conductances. They show in particular that the condition $1/p + 1/q < 2/d$ is sharp.
- Barlow, Hambly (2009), quenched local CLT on $\mathbb{Z}^d$ based on Harnack inequality.
- Croydon, Hambly (2008), quenched local CLT on graphs based on resistance estimates.

The key estimate

The key tool is a *parabolic Harnack inequality*

$$\sup_{(s,z)\in Q^-} u(s,z) \le C_{PH}^{B(x,R)} \inf_{(s,z)\in Q^+} u(s,z)$$

where u is positive and caloric (i.e. $\partial_t u = L^\omega u$) in

$$Q := (t,\ t+R^2) \times B(x,R).$$

The Harnack constant $C_{PH}^{B(x,R)}$ depends increasingly on

$$\|1/\lambda^\omega\|_{q,B(x,R)} \text{ and } \|\Lambda^\omega\|_{p,B(x,R)}.$$

The ergodic theorem grants good control of $C_{PH}^{B(x/\epsilon,R/\epsilon)}$ as $\epsilon \to 0$.

We can control oscillations

$p^\omega_t(z,x)$ is caloric in $(0,+\infty)\times\mathbb{R}^d$ for a.a. $z\in\mathbb{R}^d$.

$$\boxed{\text{Parabolic Harnack Inequality}}$$

$$\Downarrow$$

We find $\epsilon_0^\omega(x,R)>0$ such that for $\sqrt{t}/2>R$ and all $\epsilon<\epsilon_0^\omega(x,R)$

$$\sup_{y\in B(x,R)} \epsilon^{-d}\left|p^\omega_{t/\epsilon^2}(z,x/\epsilon)-p^\omega_{t/\epsilon^2}(z,y/\epsilon)\right|\le c\left(\frac{R}{\sqrt{t}}\right)^\theta t^{-d/2}$$

where c,θ depend only on $\limsup_{\epsilon\to 0} C_{PH}^{B(x/\epsilon,R/\epsilon)}<\infty$.

How to get Harnack inequality

$$\sup_{(s,z)\in Q^-} u(s,z) \leq C_{PH}^{B(x,R)} \inf_{(s,z)\in Q^+} u(s,z)$$

- Control the L^∞-norm of a caloric function u on a cylinder by the L^α-norm for $\alpha > 0$ on a slightly larger cylinder (Moser iteration).

$$\sup u \lesssim \|u\|_\alpha$$

- Control the L^∞-norm of u^{-1} on a cylinder by the L^α-norm for $\alpha > 0$ on a slightly larger cylinder (Moser iteration).

$$\|u^{-1}\|_\alpha^{-1} \lesssim \inf u$$

- Link the L^α-norm of u and the L^α-norm of u^{-1} (Bombieri-Giusti).

Remark: in the **uniformly elliptic** case the link is provided by John-Niremberg inequality for BMO functions.

Bombieri-Giusti's lemma

Fix a measure space $(U, \mathcal{B}, m)$ and sets $U_{\sigma'} \subset U_\sigma$ with $0 < \sigma' < \sigma \leq 1$ and $U_1 := U$. Fix $\delta \in (0,1)$ and $0 < \alpha_0 \leq \infty$. Let $f : U \to \mathbb{R}$ measurable. Assume that:

- (Mean Value Inequality) $\exists C, \gamma > 0$ such that

  $$\|f\|_{\alpha_0, U_{\sigma'}} \leq \left[C(\sigma - \sigma')^{-\gamma} m(U)^{-1}\right]^{1/\alpha - 1/\alpha_0} \|f\|_{\alpha, U_\sigma}$$

  for all $\delta \leq \sigma' < \sigma \leq 1$ and all $0 < \alpha \leq \min\{1, \alpha_0/2\}$.
- (Mean Value Inequality for the Logarithm). For all $\lambda > 0$

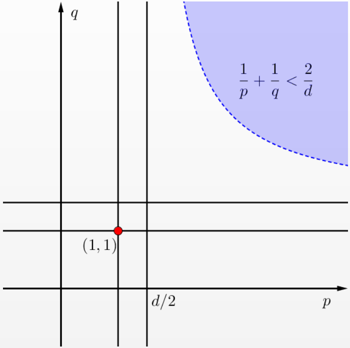
  $$m(\log f > \lambda) \leq Cm(U)\lambda^{-1}.$$

Then,

$$\|f\|_{\alpha_0, U_\delta} \leq A\, m(U)^{1/\alpha_0}$$

where A depends only on $\delta,\ \gamma,\ C$ and a lower bound on α_0.

Still an open problem!



Conjecture. It is believed that

$$\mathbb{E}_\mu[\lambda^{-1}] < \infty, \quad \mathbb{E}_\mu[\Lambda] < \infty$$

is a sufficient condition for the quenched invariance principle to hold.
(Periodic case: Ba, Mathieu 2015)

On the other hand the condition

$$\frac{1}{p}+\frac{1}{q}<\frac{2}{d}$$

is sharp for a quenched local CLT to hold
Andres, Deuschel Slowik (2015).

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