Jump-type Markov Processes and Related Topics

Organizer: Takashi Kumagai (Kyoto University)

DATES: June 26, 2007 (Tue) 10:00–17:00

PLACE: Room 127 (Bldg. 3), Dept. of Math., Kyoto University
Sakyo-ku, Kyoto 606-8502

Program

10:00–10:50	Kim Panki (Seoul National University) Boundary behavior of harmonic functions for truncated stable processes
11:00–11:45	Yoshihiro Tawara (Tohoku University) L^p-independence of spectral bounds of non-local Feynman-Kac semigroups
13:20–14:05	Takahiro Tsuchiya (Tokyo Institute of Technology) Stochastic flows of SDEs with non-Lipschtzian coefficients driven by symmetric α stable processes
14:15–15:00	Yuichi Shiozawa (Ritsumeikan University) Asymptotic properties of branching symmetric Markov processes
15:15–16:00	Kazuhiro Kuwae (Kumamoto University) On the strong Feller property of Feynman-Kac semigroup for CAF of zero energy under heat kernel estimates
16:10–17:00	Zhen-Qing Chen (University of Washington) Schramm-Löwner's equation driven by stable processes

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Boundary behavior of harmonic functions for truncated stable processes

Kim, Panki, Seoul National University

June 26, 2007

Abstract: Recently there has been a lot of interest in studying discontinuous stable processes due to their importance in theory as well as applications. Many deep results have been established. However in a lot of applications one needs to use discontinuous Markov processes which are not stable processes. For example, in mathematical finance, it has been observed that even though discontinuous stable processes provide better representations of financial data than Gaussian processes financial data tend to become more Gaussian over a longer time-scale. The so called tempered stable processes have this required property: they behave like discontinuous stable processes in small scale and behave like Brownian motion in large scale. These processes are obtained by "tempering" stable processes, that is, by multiplying the Levy densities of stable processes with strictly positive and completely monotone decreasing factors. However, no matter how much we temper the stable process, it still can make any size of jumps with positive probability.

In this talk, we considered an extreme case of "tempering": we truncated the Levy densities of stable processes and obtained a class of Levy processes called truncated stable processes. For any $\alpha \in (0,2)$, a truncated symmetric α-stable process is a symmetric Lévy process with no diffusion part and with a Lévy density $l(x)$ given by $c|x|^{-d-\alpha} 1_{\{|x|<1\}}$ for some constant c. Levy density $l(x)$ coincides with the Levy density of a symmetric stable process for $|x| < 1$ and is equal to zero for $|x| \geq 1$. Truncated stable processes are very natural and important in applications where only jumps up to a certain size are allowed. Jointly with Renming Song, in [4] we have studied the potential theory of truncated symmetric stable processes. Among other things, we proved that the boundary Harnack principle is valid for the positive harmonic functions of this process in any bounded convex domain and showed that the Martin boundary of any bounded convex domain with respect

to this process is the same as the Euclidean boundary. However, for truncated symmetric stable processes, the boundary Harnack principle is not valid in non-convex domains.

In this talk, we show that, for a large class of not necessarily convex bounded open sets called bounded roughly connected κ-fat open sets (including bounded non-convex κ-fat domains), the Martin boundary with respect to any truncated symmetric stable process is still the same as the Euclidean boundary. The main tool for establishing this is the fact that, for any bounded roughly connected κ-fat open set, the Green function of a truncated symmetric α-stable process is comparable to that of a symmetric α-stable process.

Recently, a relative Fatou type theorem has been established for symmetric stable processes. It is known that if u and h are positive harmonic function for a symmetric α-stable process in a bounded κ-fat open set D with h vanishing on D^c, then the non-tangential limit of u/h exists almost everywhere with respect to the Martin measure of h [3] (see also [2, 6]. The assumption that h vanishes on D^c is necessary (see [1]). In this talk, we also show that, for truncated symmetric stable processes a relative Fatou type theorem is true in bounded roughly connected κ-fat open sets.

This is a joint work with Renming Song [5].

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L^p-independence of Spectral Bounds of Non-local Feynman-Kac Semigroups

Yoshihiro TAWARA*†

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Let $\mathbb{M} = (\Omega, \mathbb{P}_x, X_t)$ be the symmetric α-stable process on $\mathbb{R}^d$, the pure jump process generated by $\mathcal{L} := -(1/2)(-\Delta)^{\alpha/2}$ $(0 < \alpha < 2)$. Let F be a symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal $\triangle$ and define an operator $\mathbf{F}$ by

$$\mathbf{F}f(x) = K(d,\alpha)\int_{\mathbb{R}^d}\left(e^{F(x,y)} - 1\right) f(y)|x-y|^{-(d+\alpha)}dy,$$

where

$$K(d,\alpha) = \frac{\alpha\Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}.$$

We consider a Schrödinger type operator, $\mathcal{H}^F := \mathcal{L}+\mathbf{F}$ on $L^p(\mathbb{R}^d)$, and study the growth of the operator norm of its semigroup $p_t^F := \exp(-t\mathcal{H}^F)$. The semigroup p_t^F is expressed as a non-local Feynman-Kac semigroup:

$$p_t^F f(x) = \mathbb{E}_x\left(\exp(A_t(F))f(X_t)\right), \quad A_t(F) = \sum_{0<s\le t} F(X_{s-}, X_s).$$

We then define the spectral bound of p_t^F by

$$\lambda_p(F) = -\lim_{t\to\infty}\frac{1}{t}\log\|p_t^F\|_{p,p}, \quad 1 \le p \le \infty,$$

where $\|p_t^F\|_{p,p}$ is the operator norm of p_t^F from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.

Definition. (1) A positive Radon measure μ on $\mathbb{R}^d$ is said to be in the *Kato class* (in notation $\mu \in \mathcal{K}$), if

$$\lim_{\epsilon\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le\epsilon}\frac{d\mu y}{|x-y|^{d-\alpha}} = 0.$$

(2) A measure $\mu \in \mathcal{K}$ is said to be *Green tight* (in notation $\mu \in \mathcal{K}_\infty$), if

$$\lim_{R\to\infty}\sup_{x\in\mathbb{R}^d}\int_{|y|\ge R}\frac{d\mu(y)}{|x-y|^{d-\alpha}} = 0.$$

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(3) A bounded symmetric Borel function F on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal $\triangle$ is said to be in $\mathcal{J}_\infty$, if

$$\int_{\mathbb{R}^d} \frac{|F(x,y)|}{|x-y|^{d+\alpha}} dy \in \mathcal{K}_\infty.$$

The main theorem in this talk is as follows:

Theorem. *If the function $F \in \mathcal{J}_\infty$, then*

$$\lambda_p(F) = \lambda_2(F), \quad 1 \le p \le \infty.$$

Corollary. *If the function $F \in \mathcal{J}_\infty$, then*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x(\exp(A_t(F))) = -\lambda_2(F).$$

We use arguments in Donsker-Varadhan's large deviation theory. To prove this theorem, we extend $\mathbb{M}$ to the Markov process $\bar{\mathbb{M}}$ on the one-point compactification $\mathbb{R}^d_\infty$ by making the adjoined point ∞ a trap. Then $\bar{\mathbb{M}}$ has the Feller property. In the proof of the large deviation upper bound for Markov processes with compact state space, we need only the Feller property. We thus have the following upper bound; let $\mathcal{P}(\mathbb{R}^d_\infty)$ be the set of probability measures on $\mathbb{R}^d_\infty$ and define a function I_F on $\mathcal{P}(\mathbb{R}^d_\infty)$ by

$$I_F(\nu) = -\inf_{\phi \in \mathcal{D}_{++}(\mathcal{H}^F)} \int_{\mathbb{R}^d} \frac{\mathcal{H}^F \phi}{\phi}(x) d\nu(x),$$

where

$$\mathcal{D}_{++}(\mathcal{H}^F) := \{\phi = R^F_\alpha g : \alpha > \kappa(F), g \in C_u(\mathbb{R}^d) \text{ with } g \ge \exists \epsilon > 0\}.$$

Here $\kappa(F)$ is a certain constant defined by F and $C_u(\mathbb{R}^d)$ is the space of bounded and uniformly continuous functions on $\mathbb{R}^d$. Then

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}^d} \mathbb{E}_x\left(\exp(A_t(F))\right) \le -\inf_{\nu\in\mathcal{P}(\mathbb{R}^d_\infty)} I_F(\nu). \tag{1}$$

For the proof of (1), we can not directly use the Donsker-Varadhan large deviation theory because the functional $A_t(F)$ is not a function of the normalized occupation time $L_t \in \mathcal{P}(\mathbb{R}^d_\infty)$ defined for each Borel set $A \in \mathbb{R}^d$ by

$$L_t(A) = \frac{1}{t} \int_0^t 1_A(X_s) ds.$$

However, D. Kim extended it to Markov processes with multiplicative functional $\exp(A_t(F))$. We here apply the upper bound. We also emphasis that the function I_F is defined on the space of probability measures on $\mathbb{R}^d_\infty$ not on $\mathbb{R}^d$. In this sense, the adjoined point ∞ makes a contribution to the rate function I_F.

We will prove that if $\lambda_2(F) \le 0$, then

$$\inf_{\nu\in\mathcal{P}(\mathbb{R}^d_\infty)} I_F(\nu) = \lambda_2(F),$$

which implies that $\lambda_\infty(F) \ge \lambda_2(F)$ because the left hand side of (1) is equal to $-\lambda_\infty(F)$. On the other hand, the inequality, $\lambda_\infty(F) \le \lambda_2(F)$, always holds by the symmetry and the positivity of p^F_t. As a result, we see that if $\lambda_2(F) \le 0$, then $\lambda_p(F)$ is independent of p. We will prove that $\lambda_2(F) \le 0$ for $F \in \mathcal{J}_\infty$. In this way, we obtain the main theorem stated above.

STOCHASTIC FLOWS OF SDES WITH NON-LIPSCHTZIAN COEFFICIENTS DRIVEN BY SYMMETRIC α STABLE PROCESSES

TAKAHIRO TSUCHIYA

The construction of flows in the case of Brownian motion was investigated in the beginning of the 80's by lots of people. They considered the SDE,

$$X_t = \sigma(X_t)dB_t, \quad X_0 = x,$$

and assumed that the coefficients σ are sufficiently smooth. Then we obtain that the mapping, from the initial value to the solution at time t, $x \mapsto X_t(x)$, has modification of homeomorphism. In the case of more general Lévy driven SDE, the problem of construction of a stochastic flow was studied in depth by Fujiwara-Kunita and Applebaum-Kunita etc. They considered especially the stochastic flows of diffeomorphism when the coefficients σ are sufficiently smooth.

In this presentation, we consider the stochastic flow problem in the case of jump-type SDE,

$$Y_t = \sigma(Y_{t-})dZ_t, \quad Y_0 = y$$

where Z is a symmetric α stable process under non-Lipschitz conditions of the coefficients. The case of Brownian motion is studied by Yamada and Ogura and developed by Fang and Zhang. The reason why we select symmetric α processes to replace Brownian motions is due to its interesting property depending on index α. When α is equal to two it is just Brownian motion SDE, and when α is smaller than two, it turns to be a jump type SDE. The goal is to describe how stochastic flows are affected by the index of α.

This talk is organized as follows. In the first section, we discuss non-contact problems of solutions where the Riesz potential operator plays an essential role. The Riesz potential I_α is defined by

$$I_\alpha(f)(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy = \left(\frac{1}{|\cdot|^{d-\alpha}} * f \right)(x),$$

for $f \in C_0(\mathbf{R}^d)$. In the second section, we summarize the results of the pathwise uniqueness property. Pathwise uniqness guarantees the well-definedness of the mapping from initial data to the solution, $y \mapsto Y_t(y)$. In the third section, we show the continuity of the map with respect to initial data using following the lemma, Here, hypergeometric functions and Bessel functions are the key to the proof of the lemma. The fourth section is devoted to the behavior of the mapping at infinity, $\limsup_{|y|\to\infty} |\boldsymbol{Y}_t(y)| = \infty$ a.s.. Finally, in the last section, combining these properties and applying Jordan's curve theorem, we construct stochastic flows.

Theorem 1. *If $\rho(u) = u(\log \frac{1}{u})^{\frac{1}{\alpha}}$, then, for every σ such that*

$$|\sigma(x) - \sigma(y)| \leq \rho(|x-y|) \ for \ x, y \in \mathbf{R}^1,$$

the map $y \mapsto Y_t(y)$ has a homeomorphic modification for every $t \geq 0$.

This is an answer to determine the possibility of constructing stochastic flows $y \mapsto Y_t(y)$ under non-Lipschitz coefficients and characterize the possibility by index α.

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Asymptotic properties of branching symmetric Markov processes

Yuichi Shiozawa*

In this talk, we study the asymptotic properties of a branching symmetric Markov process such as the extinction problem, the growth of the numbers of particles and the asymptotic distribution of particles. In particular, we characterize them in terms of the principal eigenvalue and the ground state of the Schrödinger operator associated with underlying Markov process, branching rate measure and branching mechanism function. We finally apply our results to branching Brownian motions and branching symmetric α-stable processes.

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. We denote by $\mathbf{M} = (X_t, P_x, \zeta)$ an m-symmetric Hunt process on X. Throughout this talk, we assume that the associated semigroup $p_t f(x) = E_x [f(X_t)]$ satisfies the following:

Assumption 0.1. (i) (Irreducibility) If a Borel set A is p_t-invariant, that is, $p_t(\chi_A f) = \chi_A p_t f(x)$ for any $f \in L^2(X; m) \cap \mathcal{B}_b(X)$ and $t > 0$, then $m(A) = 0$ or $m(X \setminus A) = 0$. Here $\mathcal{B}_b(X)$ stands for the set of bounded Borel measurable functions on X.
(ii) (Strong Feller property) For any $f \in \mathcal{B}_b(X)$, $p_t f$ is a bounded and continuous function on X.
(iii) (Ultracontractivity) For any $t > 0$, it holds that $\|p_t\|_{1,\infty} < \infty$, where $\| \cdot \|_{p,q}$ denotes the operator norm from $L^p(X; m)$ to $L^q(X; m)$.

Note that Assumption 0.1 (ii) implies the absolutely continuity of the transition probability of $\mathbf{M}$ with respect to m by the m-symmetry of p_t. Let $G_\alpha(x, y)$, $\alpha > 0$, be the α-resolvent of $\mathbf{M}$. If $\mathbf{M}$ is transient, then we set $G(x, y) := G_0(x, y)$.

Definition 0.2. *A positive smooth Radon measure on X is said to be in $\mathcal{K}_\infty(G_\alpha)$, if for any $\varepsilon > 0$, there exist a compact set $K \subset X$ and a positive constant $\delta > 0$ such that $\sup_{x \in X} \int_{X \setminus K} G_\alpha(x, y)\, \mu(dy) < \varepsilon$, and for all measurable sets $B \subset K$ with $\mu(B) < \delta$, $\sup_{x \in X} \int_B G_\alpha(x, y)\, \mu(dy) < \varepsilon$. Further, the class $\mathcal{K}_\infty$ is defined by*

$$\mathcal{K}_\infty = \begin{cases} \mathcal{K}_\infty(G), & \mathbf{M} \textit{ is transient} \\ \bigcap_{\alpha > 0} \mathcal{K}_\infty(G_\alpha), & \mathbf{M} \textit{ is recurrent.} \end{cases}$$

Let $\overline{\mathbf{M}} = (\mathbf{\Omega}, \mathbf{X}_t, \mathbf{P}_x)$ be the branching symmetric Markov process on X with *motion component* $\mathbf{M}$, *branching rate measure* $\mu \in \mathcal{K}_\infty$ and *branching mechanism function* $\{p_n(x)\}_{n=0}^\infty$ so that $\sum_{n=0}^\infty p_n(x) = 1$. Namely, if we denote by T the splitting time of a particle, then the law of T is determined by

$$\mathbf{P}_x(T > t \,|\, \sigma(X)) = \exp\left(-A_t^\mu\right),$$

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where A_t^μ is the positive continuous additive functional corresponding to the measure $\mu \in \mathcal{K}_\infty$. A particle of $\overline{\mathbf{M}}$ starts at $x \in X$ according to the law P_x and then splits into n particles with probability $p_n(X_{T-})$ at time T. After that, each particle moves independently according to the law of $P_{X_{T-}}$.

Let Z_t denote the total number of particles at time t and $e_0 := \inf\{t > 0 : Z_t = 0\}$. Here we say that $\overline{\mathbf{M}}$ *goes extinct*, if $\mathbf{P}_x(e_0 < \infty) = 1$ for any $x \in X$. Denote by $Q(x) := \sum_{n=0}^\infty n p_n(x)$ the expected number of particles at branching site $x \in X$. We assume that $\sup_{x\in X} Q(x) < \infty$. Since $Q\mu$ and μ denote the intensity of creations and the intensity of killings respectively, we say that the operator $\mathcal{L}^{(Q-1)\mu} := \mathcal{L} + (Q-1)\mu$ expresses the balance between these intensities, where $\mathcal{L}$ denotes the $L^2(X)$-generator of $\mathbf{M}$. In fact, if we denote by λ_1 the principal eigenvalue of $-\mathcal{L}^{(Q-1)\mu}$, then we obtain the following:

Theorem 0.3. *Assume that $P_x(\zeta < \infty) = 1$ for all $x \in X$. Then the branching process $\overline{\mathbf{M}}$ goes extinct if and only if $\lambda_1 \geq 1$. Moreover, if $\lambda_1 < 1$, then it holds that $\mathbf{P}_x\left(\lim_{t\to\infty} Z_t = \infty \mid e_0 = \infty\right) = 1$ for all $x \in X$.*

We are now concerned with the growth rate of Z_t and the asymptotic distribution of particles. In the sequel, we assume that $\lambda_1 < 0$. If we denote by h the corresponding ground state, then $M_t := e^{\lambda_1 t}\int_X h(x)\, Z_t(dx)$ is a $\mathbf{P}_x$-martingale for any $x \in X$, and thus $M_\infty := \lim_{t\to\infty} M_t$ exists $\mathbf{P}_x$-a.s. Let $R(x) := \sum_{n=0}^\infty n(n-1)p_n(x)$ and assume that $\sup_{x\in X} R(x) < \infty$. We then have

Theorem 0.4. (with Z.-Q. Chen) *There exists a subspace $\mathbf{\Omega}_0 \subset \mathbf{\Omega}$ of full $\mathbf{P}_x$-probability for any $x \in X$ such that, for any $\omega \in \mathbf{\Omega}_0$ and for every bounded Borel measurable function f on X with compact support whose set of discontinuous points has zero m-measure, it holds that*

$$e^{\lambda_1 t} Z_t(f)(\omega) = M_\infty(\omega) \int_X f h\, dm.$$

Theorem 0.4 says that the number of particles on a bounded set grows exponentially with rate $-\lambda_1$ and the ground state determines the asymptotic distribution of particles.

Example 0.5. Let $1 < \alpha \leq 2$ and denote by $\mathbf{M}$ the absorbing symmetric α-stable process on an interval $(-R, R)$. We consider the binary branching process with motion component $\mathbf{M}$. Let us take the Dirac measure at the origin as branching rate. We then see from Theorem 0.3 that $\overline{\mathbf{M}}$ goes extinct if and only if

$$0 < R \leq \left\{(\alpha-1)2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)^2\right\}^{1/(\alpha-1)}.$$

Furthermore, if $R > \left\{(\alpha-1)2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)^2\right\}^{1/(\alpha-1)}$, then Theorem 0.4 implies that for any $x \in (-R, R)$, $\mathbf{P}_x$-a.s.

$$\begin{aligned}&\lim_{t\to\infty} e^{\lambda_1 t} Z_t((-r,r)) \\ &= \begin{cases} \dfrac{C(a,R)}{\sqrt{-2\lambda_1}} \sinh\{2\sqrt{-2\lambda_1}R\}\left(\sinh^2\{\sqrt{-2\lambda_1}R\} - \sinh^2\{\sqrt{-2\lambda_1}(R-r)\}\right) M_\infty & \alpha = 2 \\ \left(\displaystyle\int_{-\infty}^{\infty} h(x)\,dx - O\left((R-r)^{(\alpha+2)/2}\right)\right) M_\infty & 1 < \alpha < 2. \end{cases}\end{aligned}$$

We can further show that M_∞ is strictly positive $\mathbf{P}_x$-a.s. on the event that $\{e_0 = \infty\}$.

ON THE STRONG FELLER PROPERTY OF FEYNMAN-KAC SEMIGROUP FOR CAF OF ZERO ENERGY UNDER HEAT KERNEL ESTIMATES

Kazuhiro Kuwae (Kumamoto University)

1. Framework

Throughout this talk, we fix $d, \beta \in]0,\infty[$ and $t_0 \in]0,\infty]$. Let (X,d) be a locally compact separable metric space, m a positive Radon measure on X with full support. We consider a one point compactification X_Δ with a cemetery point Δ. Let $\mathcal{B}_b(X)$ be a family of bounded Borel measurable functions. In what follows, f denotes a function in $\mathcal{B}_b(X)$. Let $(\mathcal{E},\mathcal{F})$ be a symmetric regular Dirichlet form on $L^2(X;m)$ and $\mathbf{M} = (\Omega, X_t, \zeta, \mathbb{P}_x)$ the associated m-symmetric Hunt process. Here $\zeta := \inf\{t \geq 0 \mid X_t = \Delta\}$ is the life time. We further assume that there exists a properly exceptional set N and $\mathbf{M}|_{X\setminus N}$ admits a heat kernel $p_t(x,y)$, $x,y \in X \setminus N, t > 0$: $\mathbb{E}_x[f(X_t)] = \int_X p_t(x,y)f(y)m(dy)$, $x \in X \setminus N$, $t > 0$, $f \in \mathcal{B}_b(X)$.

Condition 1.1 (**Ahlfors regularity**). $^\exists r_0 \in]0,\infty]$, $^\exists C > 0$ s.t. $C^{-1}r^d \leq m(B_r(x)) \leq Cr^d$ for $^\forall x \in X$, $^\forall r \in]0, r_0[$.

Condition 1.2 (**Heat Kernel Estimates**). Let Φ, Ψ be positive decreasing functions on $[0,\infty[$ with $\Phi(0) = \Psi(0) = 1$. Further assume the condition $H(\Phi)$: $\sup_{s\in[0,\infty[} s^d\Phi(s) < \infty$ and the condition $(\Phi E_{d,\beta})$: $^\exists M > 0$, $^\forall x, y \in X\setminus N$, $t \in]0, t_0[$

$$\frac{M^{-1}}{t^{d/\beta}}\Psi\left(\frac{d(x,y)}{t^{1/\beta}}\right) \leq p_t(x,y) \leq \frac{M}{t^{d/\beta}}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right).$$

Condition 1.3. For $V(r) := \sup_{x\in X} m(B_r(x)) < \infty$, $r > 0$, it holds that $^\exists\gamma > 0$ s.t. $\int_1^\infty V(s)s^{\gamma-1}\Phi(s)ds < \infty$, where Φ is the function specified in Condition 1.2.

Theorem 1.1 (**Doubly Feller Property**). *Under* Conditions 1.1 and 1.2*, the heat kernel $p_t(x,y)$ has a continuous extension on $]0,\infty[\times X \times X$ and the semigroup $(P_t)_{t>0}$ defined by $P_tf(x) := \int_X p_t(x,y)f(y)m(dy)$ $x \in X$ using this extension has a doubly Feller property (i.e. it has strong Feller and Feller properties).*

2. Girsanov transformation

Definition 2.1 (**Kato class measure S_K^0**). A positive Borel measure μ on X is said to be in the *Kato class* (write $\mu \in S_K^0$) if $\displaystyle\lim_{t\to 0}\sup_{x\in X}\int_0^t\int_X p_s(x,y)\mu(dy)ds = 0$. In Lemmas 4.4 and 4.5 of [6], we show $S_K^0 \subset S_1$ under $(\Phi E_{d,\beta})$, where S_1 is the family of smooth measures in the strict sense ([3]).

Take $u \in \mathcal{F}_b$ and let $\rho := e^u$. Then $\rho - 1 \in \mathcal{F}$. Let M^ρ be the MAF of finite energy appears as the martingale part in the Fukushima decomposition for $\rho - 1$. Set $M_t := \int_0^t \frac{dM_s^\rho}{\rho(X_{s-})}, t < \zeta$ and let L_t^ρ be the solution of Doleans-Dade equation:

$L_t = 1 + \int_0^t L_{s-} dM_s$, $t < \zeta$. Define $\widetilde{P}_t f(x) := \mathbb{E}_x[L_t^\rho f(X_t)]$, $x \in X \setminus N_\rho$ with some properly exceptional set N_ρ appears in the above Fukushima decomposition. Chen-Zhang proved that the semigroup $(\widetilde{P}_t)$ is $\rho^2 m$-symmetric and determined the domain of the associated Dirichlet form on $L^2(X; \rho^2 m)$.

Theorem 2.1. *Under* Conditions 1.1 and 1.2, *if* $\mu_{\langle u \rangle} \in S_K^0$, *then* $(\widetilde{P}_t)$ *admits a continuous heat kernel* $p_t^\rho(x, y)$, $t > 0, x, y \in X$ *and the semigroup* (P_t^ρ) *defined by* $P_t^\rho f(x) := \int_X p_t^\rho(x, y) f(y) m(dy)$, $x \in X$ *has a doubly Feller property.*

3. Feynman-Kac semigroup for CAF of zero energy

Suppose that $(\mathcal{E}, \mathcal{F})$ has no killing part. If $u \in \mathcal{F} \cap C_b(X)$ satisfies $\mu_{\langle u \rangle} \in S_K^0$, then u admits the Fukushima decomposition in the strict sense: $u(X_t) - u(X_0) = M_t^u + N_t^u, {}^\forall t < \zeta$ $\mathbb{P}_x$-a.s. for all $x \in X$, where N^u is a CAF of zero energy, and a local CAF in the strict sense ([2]). Set $Q_t^u f(x) := \mathbb{E}_x[e^{N_t^u} f(X_t)]$, $x \in X$.

Theorem 3.1. *Under* Conditions 1.1 and 1.2, *the semigroup* (Q_t^u) *admits a continuous integral kernel* $q_t^u(x, y)$, $t > 0, x, y \in X$. *If further the* Condition 1.3 *is satisfied, then* (Q_t^u) *has the strong Feller property (write SFP).*

Remark 3.1. The SFP of (Q_t^u) yields the same result as in [8]. The existence of the continuous integral kernel $q_t^u(x, y)$, $t > 0, x, y \in X$ is shown in [4] for Brownian motions and in [7] for symmetric stable-like processes. The SFP of (Q_t^u) is also proved in [8] (resp. in [9]) for Brownian motions (resp. for symmetric Lévy processes under some additional conditions).

We have many examples: Relativistic α-stable processes, stable like processes on d-sets, Brownian motion on smooth complete Riemannian manifolds with Ricci curvature lower bound and with positive injectivity radius, diffusions on nested fractals or Sierpinski Carpet and so on.

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Schramm-Löwner's Equation Driven by Stable Processes

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Stochastic Löwner evolution (SLE) driven by Brownian motion, based on the classical deterministic Löwner equation, was introduced by Oded Schramm in 2000. It played a fundamental role in recent breakthroughs by Greg Lawler, Oded Schramm, Wendelin Werner as well as others in settling the Brownian intersection exponent problem and in mathematically rigorous proof for the existence and conformal invariance of continuum limit of various two-dimensional lattice models in statistical physics. However the Brownian SLEs exclude physical models that exhibit branching phenomena such as branching polymers and diffusion-limited aggregation. Studying such models call for SLE driven by discontinuous Lévy processes, in particular, stable processes.

Stable processes is a family of discontinuous random processes indexed by a parameter $\alpha \in (0, 2)$, which itself is a focus of recent research activities in probability theory, due to its importance in theory and in applications. The importance of α-stable processes can be illustrated by the central limit theorem: stable distributions together with Gaussian distributions are the only limiting distribution of normalized sums of independent, identically distributed random variables. Stable distributions and stable processes have recently been used quite successfully to model certain physical systems that exhibit large deviation and high variability.

In this talk, we will present some recent results on Schramm-Löwner's equation driven by symmetric stable processes.

Let W_t be a right continuous real-valued function. For each initial point $z \in \mathbf{C} \setminus \{0\}$, the *Löwner differential equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}\ , \quad g_0(z) = z \tag{1}$$

has a unique solution up to a time $0 < T_z \leq \infty$ where $g_t(z) = W_t$. More precisely, let $T_z = \sup\{t : \inf_{s\in[0,t]} |g_s(z) - W_s| > 0\}$, then the initial value problem (1) has a unique solution on $[0, T_z)$ and if $T_z < \infty$ then $\liminf_{t\to T_z} |g_t(z) - W_t| = 0$. The subset

$$K_t = \{z \in \mathbf{H} : T_z \leq t\}$$

is a compact subset of the upper half plane $\mathbf{H}$ and is called the *hull* of the Löwner equation (1). It is well-known that the map $z \mapsto g_t(z)$ is a conformal map (i.e., analytic and one-to-one) from $\mathbf{H} \setminus K_t$ onto $\mathbf{H}$, with Laurent series $g_t(z) = z + \frac{2t}{z} + O(1/z^2)$ near ∞. By Riemann mapping theorem, the Löwner's equation is characterized by the compact hulls $\{K_t, t \geq 0\}$.

When $W_t = \kappa B_t$, where B_t is standard Brownian motion on $\mathbf{R}$ and $\kappa > 0$, equation (1) is called Schramm-Löwner's equation, whose hulls $\{K_t, t \geq 0\}$ are found to be related to various two-dimensional lattice models in statistical physics.

Now let W_t be the sample path of symmetric α-stable process on $\mathbf{R}$ with $0 < \alpha < 2$. To investigate the geometry of its random hulls $\{K_t, t \geq 0\}$, we look at the inverse map $f_t(z) := g_t^{-1}(z) : \mathbf{C} \to \mathbf{C} \setminus K_t$ of (1). We prove that for each fixed $t > 0$, almost surely, $z \mapsto f_t(z)$ is Hölder continuous in $z = x + iy$ with $0 < y < 1$ and so $z \mapsto f_t(z)$ can be extended continuously to $\partial\mathbf{H}$. We then use it to show that K_t has Hausdorff dimension 1 and that $\{K_t, t \geq 0\}$ has tree-like structure almost surely.

Joint work with Steffen Rohde.