Stochastic Processes on Fractals

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Stochastic Analysis and Related Topics

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Plan

(L1) Brownian motion on fractals

Construction of BM on the Sierpinski gasket using Dirichlet forms,

Some basic properties

(L2) Properties of Brownian motion on fractals

Some spectral properties, Characterization of the domain of Dirichlet forms

(L3) Jump type processes on d-sets (Alfors d-regular sets)

Relations of some jump-type processes on d-sets, Heat kernel estimates

1 Brownian motion on fractals

1.1 A quick view of the theory of Dirichlet forms

General Theory (see Fukushima-Oshima-Takeda '94 etc.)

$$\{X_t\}_t : \text{ Sym. Hunt proc. on } (K,\mu) \oplus \text{ cont. path (diffusion)}$$

$$\Leftrightarrow \; -\Delta : \text{ non-neg. def. self-adj. op. on } \mathbb{L}^2 \text{ s.t. } P_t := \exp(t\Delta) \text{ Markovian } \oplus \text{ local}$$

$$P_t f(x) = E^x[f(X_t)], \; \lim_{t\to 0}(P_t - I)/t = \Delta$$

$$\Leftrightarrow \; (\mathcal{E},\mathcal{F}) : \text{ regular Dirichlet form (i.e. sym. closed Markovian form) on } \mathbb{L}^2$$

$$\mathcal{E}(u,v) = \int_K \sqrt{-\Delta}u\sqrt{-\Delta}v d\mu, \; \mathcal{F} = \mathcal{D}(\sqrt{-\Delta}) \oplus \text{ local}$$

•$(\mathcal{E},\mathcal{F})$: regular $\overset{\text{Def}}{\Leftrightarrow}$ $\exists C \subset \mathcal{F} \cap C_0(K)$ linear space which is dense

i) in $\mathcal{F}$ w.r.t. $\mathcal{E}_1$-norm and ii) in $C_0(K)$ w.r.t. $\|\cdot\|_\infty$-norm.

•$(\mathcal{E},\mathcal{F})$: local $\overset{\text{Def}}{\Leftrightarrow}$ $(u,v \in \mathcal{F}, \operatorname{Supp} u \cap \operatorname{Supp} v = \emptyset \Rightarrow \mathcal{E}(u,v) = 0)$.

Example

BM on $\mathbb{R}^n \Leftrightarrow$ Laplace op. on $\mathbb{R}^n \Leftrightarrow \mathcal{E}(f,f) = \frac{1}{2}\int |\nabla f|^2 dx,\ \mathcal{F} = H^1(\mathbb{R}^n)$

1.2 Sierpinski gaskets

$\{p_0, p_1, \cdots, p_n\}$: vertices of the n-dimensional simplex, p_0: the origin.

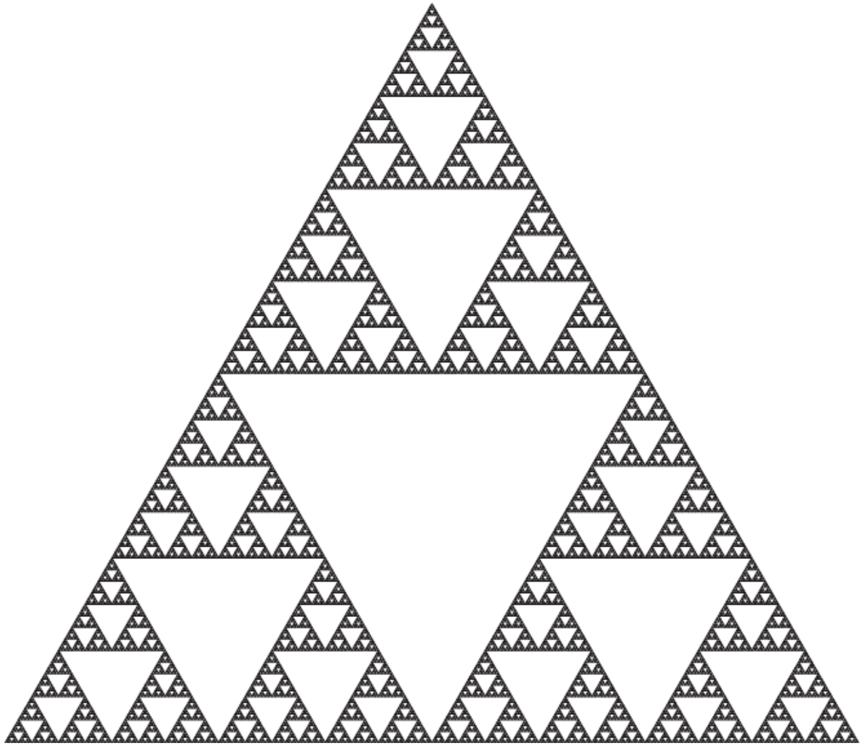
$F_i(z) = (z - p_{i-1})/2 + p_{i-1},\ z \in \mathbb{R}^n,\quad i = 1, 2, \cdots, n+1$

$\exists 1$ non-void compact set K s.t. $K = \cup_{i=1}^{n+1} F_i(K)$.

K:(n-dimensional) *Sierpinski gasket.*

When $n = 1$, $K = [p_0, p_1]$.

For simplicity, we will consider the 2-dimensional gasket.



$V_0 = \{p_0, p_1, p_2\}$, $V_n = \cup_{i_1,\cdots,i_n \in I} F_{i_1\cdots i_n}(V_0)$

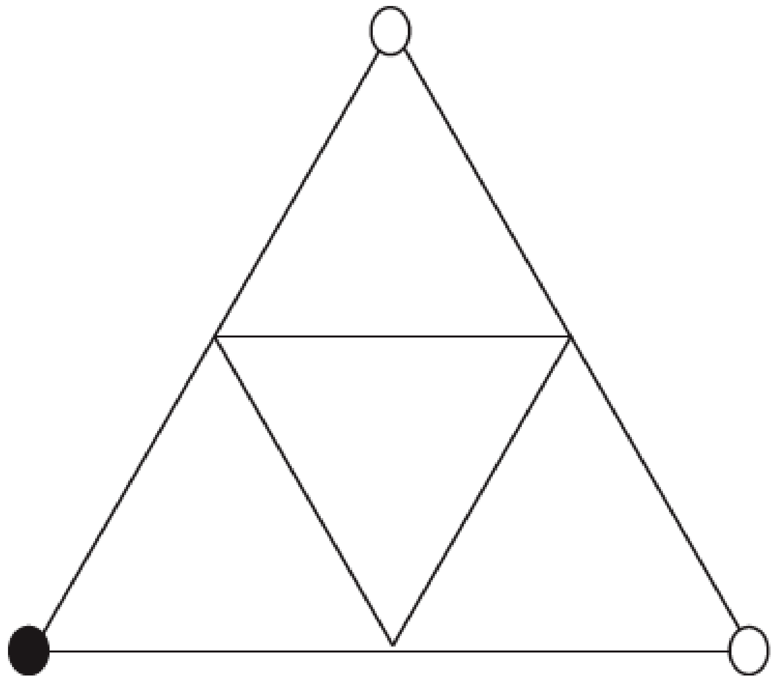
where $I := \{1, 2, 3\}$ and $F_{i_1\cdots i_n} := F_{i_1} \circ \cdots \circ F_{i_n}$.

Let $V_* = \cup_{n\in\bar{\mathbb{N}}} V_n$, where $\bar{\mathbb{N}} := \mathbb{N} \cup \{0\}$. Then $K = Cl(V_*)$.

$d_f := \log 3/\log 2$: Hausdorff dimension of K (w.r.t. the Euclidean metric)

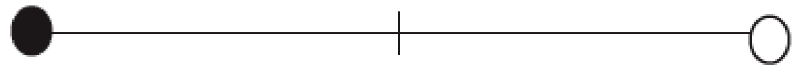
μ: (normalized) Hausdorff measure on K, i.e. a Borel measure on K s.t.

$$\mu(F_{i_1\cdots i_n}(K)) = 3^{-n} \qquad \forall i_1, \cdots, i_n \in I.$$



$E^{\bullet}[\sigma_{\circ}] = 5$

Cf.



$E^{\bullet}[\sigma_{\circ}] = 4$

1.3 Construction of Brownian motion on the gasket (Ideas)

(Goldstein '87, Kusuoka '87) X_n: simple random walk on V_n

$$X_n([5^n t]) \xrightarrow{n \to \infty} B_t: \text{Brownian motion on } K$$

1.4 Construction of Dirichlet forms on the gaskets

For $f, g \in \mathbb{R}^{V_n} := \{h : h \text{ is a real-valued function on } V_n\}$, define

$$\mathcal{E}_n(f,g) := \frac{b_n}{2} \sum_{i_1\cdots i_n \in I} \sum_{x,y\in V_0} (f\circ F_{i_1\cdots i_n}(x) - f\circ F_{i_1\cdots i_n}(y))(g\circ F_{i_1\cdots i_n}(x) - g\circ F_{i_1\cdots i_n}(y)),$$

where $\{b_n\}$ is a sequence of positive numbers with $b_0 = 1$ (conductance on each bond).

Choose $\{b_n\}$ s.t. $\exists$ nice relations between the $\mathcal{E}_n$'s

Elementary computations yield

$$\inf\{\mathcal{E}_1(f,f) : f \in \mathbb{R}^{V_1},\ f|_{V_0} = u\} = \frac{3}{5}\cdot b_1\mathcal{E}_0(u,u) \qquad \forall u \in \mathbb{R}^{V_0}. \tag{1.1}$$

So, taking $b_n = (5/3)^n$, we have

$$\mathcal{E}_n(f|_{V_n}, f|_{V_n}) \le \mathcal{E}_{n+1}(f,f) \quad \forall f \in \mathbb{R}^{V_{n+1}}$$

("=" $\Leftrightarrow$ f is 'harmonic' on $V_{n+1}\setminus V_n$).

Define

$$\mathcal{F}_* := \{f \in \mathbb{R}^{V_*} : \lim_{n\to\infty} \mathcal{E}_n(f,f) < \infty\}, \qquad \mathcal{E}(f,g) := \lim_{n\to\infty} \mathcal{E}_n(f,g) \qquad \forall f,g \in \mathcal{F}_*.$$

$(\mathcal{E}, \mathcal{F}_*)$: quadratic form on $\mathbb{R}^{V_*}$.

Further, $\forall f \in \mathbb{R}^{V_m}$, $\exists 1 P_m f \in \mathcal{F}_*$ s.t. $\mathcal{E}(P_m f, P_m f) = \mathcal{E}_m(f,f)$.

Want: to extend this form to a form on $\mathbb{L}^2(K,\mu)$.

Define $R(p,q)^{-1} := \inf\{\mathcal{E}(f,f) : f \in V_*, f(p) = 1, f(q) = 0\} \quad \forall p,q \in V_*,\ p \neq q.$

$R(p,q)$: effective resistance between p and q. Set $R(p,p) = 0$ for $p \in V_*$.

Proposition 1.1 *1) $R(\cdot,\cdot)$ is a metric on V_*. It can be extended to a metric on K, which gives the same topology on K as the one from the Euclidean metric.*

2) For $p \neq q \in V_*$, $R(p,q) = \sup\{|f(p) - f(q)|^2/\mathcal{E}(f,f) : f \in \mathcal{F}_*, f(p) \neq f(q)\}$.

$$\textit{So, } |f(p) - f(q)|^2 \le R(p,q)\mathcal{E}(f,f), \qquad \forall f \in \mathcal{F}_*, p,q \in V_*. \tag{1.2}$$

Remark. $R(p,q) \asymp \|p-q\|^{d_w - d_f}$, where $d_w = \log 5/\log 2$ (Walk dimension).

(Here $f(x) \asymp g(x) \Leftrightarrow c_1 f(x) \le g(x) \le c_2 f(x), \quad \forall x.$)

By (1.2), $f \in \mathcal{F}_*$ can be extended conti. to K.

$\mathcal{F}$: the set of functions in $\mathcal{F}_*$ extended to K $\quad \Rightarrow \mathcal{F} \subset C(K) \subset \mathbb{L}^2(K,\mu)$.

Theorem 1.2 (Kigami) *$(\mathcal{E},\mathcal{F})$ is a local regular D-form on $\mathbb{L}^2(K,\mu)$.*

$$|f(p)-f(q)|^2 \le R(p,q)\mathcal{E}(f,f) \qquad \forall f \in \mathcal{F},\ \forall p,q \in K \tag{1.3}$$

$$\mathcal{E}(f,g) = \frac{5}{3}\sum_{i\in I}\mathcal{E}(f\circ F_i, g\circ F_i) \quad \forall f,g \in \mathcal{F} \tag{1.4}$$

$\{B_t\}$: corresponding diffusion process (Brownian motion)

Δ: corresponding self-adjoint operator on $\mathbb{L}^2(K,\mu)$.

Uniqueness (Barlow-Perkins '88) Any self-similar diffusion process on K whose law is invariant under local translations and reflections of each small triangle is a constant time change of $\{B_t\}$. — Metz, Peirone, Sabot, ...

Unbounded Sierpinski gaskets $\hat{K} := \cup_{n \geq 1} 2^n K$: the unbdd Sierpinski gasket

We can construct Brownian motion similarly to Thm 1.2.

2 Properties of Brownian motion on fractals

(A) Spectral properties (Fukushima-Shima '92) $-\Delta$ on K has a compact resolvent.

Set $\rho(x) = \sharp\{\lambda \leq x : \lambda \text{ is an eigenvalue of } -\Delta\}$. Then

$$0 < \liminf_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \limsup_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \infty. \tag{2.1}$$

(Barlow-Kigami '97) $<$ above is because

$\exists$ 'many' localized eigenfunctions that produce eigenvalues with high multiplicities

u: a localized eigenfunction $\overset{\text{Def}}{\Leftrightarrow}$ u: is an eigenfunction of $-\Delta$ s.t.

$$\text{Supp } u \subset K \setminus V_0.$$

$d_s = 2\log 3/\log 5 = 2d_f/d_w$: spectral dimension

— Kigami-Lapidus, Lindstrøm, Mosco, Strichartz, Teplyaev, . . .

(B) Heat kernel estimates (Barlow-Perkins '88)

$\exists p_t(x,y)$: jointly continuous sym. transition density of $\{X_t\}$ w.r.t. μ

$(P_t f(x) = \int_K p_t(x,y) f(y)\mu(dy)\ \forall x,\quad \frac{\partial}{\partial t} p_t(x_0,x) = \Delta_x p_t(x_0,x)\ )$ s.t.

$$c_1 t^{-\frac{d_s}{2}} \exp\Big(-c_2\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\Big) \le p_t(x,y) \le c_3 t^{-\frac{d_s}{2}} \exp\Big(-c_4\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\Big). \quad (HK(d_w))$$

— Barlow-Bass, Hambly-K, Grigor'yan-Telcs, ...

By integrating (HK(d_w)), we have $E^0[d(0,X_t)] \asymp t^{1/d_w}$.

$d_w = \log 5/\log 2 > 2, d_s = 2\log 3/\log 5 = 2d_f/d_w < 2$

As $d_w > 2$, we say the process is sub-diffusive.

n-dim. Sierpinski gasket $(n \ge 2)$

$d_f = \log(n+1)/\log 2, d_w = \log(n+3)/\log 2 > 2, d_s = 2\log(n+1)/\log(n+3) < 2$

Proof of on-diagonal upper bound on K.

Theorem 2.1 *Let* $d_s = 2\log 3/\log 5$. *Then*

$$\|u\|_2^{2+4/d_s} \leq c(\mathcal{E}(u,u) + \|u\|_2^2)\|f\|_1^{4/d_s}, \qquad \forall u \in \mathcal{F}. \tag{2.2}$$

Note. This is equivalent to $p_t(x,y) \leq c' t^{-d_s/2}\exp(t)$, $\forall t > 0, x, y \in K$. (CKS '87)

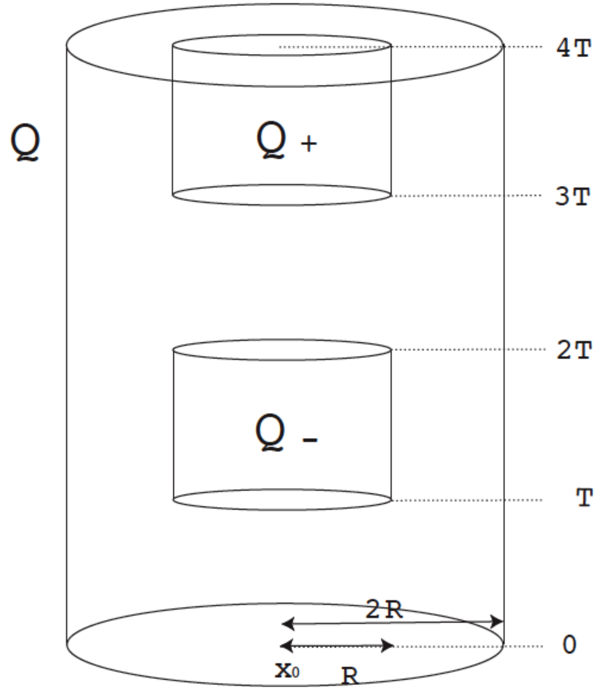
Proof. By integrating (1.3), $\|u\|_2^2 \leq c(\mathcal{E}(u,u) + \|u\|_1^2) \ \cdots \ (*)$. So, for $\forall m \in \bar{\mathbb{N}}$,

$$\|u\|_2^2 \overset{\text{S.S.}}{=} \sum_{w\in I^m} (\frac{1}{3})^m \int_K (u_w)^2 d\mu \overset{(*)}{\leq} c\sum_w (\frac{1}{3})^m \{\mathcal{E}(u_w) + \|u_w\|_1^2\}$$

$$= c\sum_w (\frac{1}{5})^m (\frac{5}{3})^m \mathcal{E}(u_w) + c\sum_w 3^m \{(\frac{1}{3})^m \int_K |u_w| d\mu\}^2 \leq C\{(\frac{1}{5})^m \mathcal{E}(u,u) + 3^m\|u\|_1^2\}$$

where $u_w := u \circ F_w$. We thus obtain $\|u\|_2^2 \leq C\{\lambda^{2/d_s}\mathcal{E}(u,u) + \lambda^{-1}\|u\|_1^2\}$, $\forall \lambda \in (0,1)$.

- If $\mathcal{E}(u,u) > \|u\|_1^2$, then taking $\lambda^{2/d_s+1} = \|u\|_1^2/\mathcal{E}(u,u)$, we obtain (2.2).
- If $\mathcal{E}(u,u) \leq \|u\|_1^2$, then by (*), $\|u\|_2^2 \leq 2c\|u\|_1^2$. Thus (2.2) holds. □



- Let $Q = Q(x_0, T, R) = (0, 4T) \times B(x_0, 2R)$,

$$Q_-(T, 2T) \times B(x_0, R) \ \text{ and } \ Q_+ = (3T, 4T) \times B(x_0, R).$$

Parabolic Harnack inequality $(PHI(d_w))$: $\exists c_1 > 0$ s.t. the following holds.

Let $R > 0$, $T = R^{d_w}$, and $u = u(t, x) : Q \to \mathbb{R}_+$ satisfies $\frac{\partial u}{\partial t} = \Delta u$ in Q. Then,

$$\sup_{Q_-} u \le c_1 \inf_{Q_+} u. \qquad (PHI(d_w))$$

$(HK(d_w)) \Leftrightarrow (PHI(d_w)) \Rightarrow$ Various properties of the process.

(i) $c_1 t^{1/d_w} \le E^x[d(x,X_t)] \le c_2 t^{1/d_w}$ ($d_w > 2$: subdiffusive)

(ii) Law of the iterated logarithm (i.e. $\limsup_{t\to\infty} \frac{d(X_t,X_0)}{t^{1/d_w}(\log\log t)^{1-1/d_w}} = C$, P^x-a.s.)

(iii) Hölder continuity of the sol. of the heat equation

(iv) Elliptic Harnack inequality: (EHI)

(v) Liouville property (i.e. positive harm. function on X is const.)

Indeed, if $m_u := \inf_X u$, then by (EHI),

$\sup_B (u - m_u) \le c \inf_B (u - m_u) \to 0$ as $B \to \infty$. So $u \equiv m_u$, μ-a.e.

(vi) Estimates of the Green kernel etc.

*Note that (ii), (v) are consequences of $(HK(d_w))$ for all $t > 0$ (i.e. on $\hat{K}$).

(C) Domains of the Dirichlet forms

For $1 \le p < \infty,\ 1 \le q \le \infty,\ \beta \ge 0$ and $m \in \bar{\mathbb{N}}$, set

$$a_m(\beta, f) := L^{m\beta}(L^{md_f} \int\int_{|x-y|<c_0 L^{-m}} |f(x) - f(y)|^p d\mu(x) d\mu(y))^{1/p}, \quad f \in \mathbb{L}^p(K, \mu),$$

where $1 < L < \infty,\ 0 < c_0 < \infty$.

$\Lambda^{\beta}_{p,q}(K)$: a set of $f \in \mathbb{L}^p(K, \mu)$ s.t. $\bar{a}(\beta, f) := \{a_m(\beta, f)\}_{m=0}^{\infty} \in l^q$.

$\Lambda^{\beta}_{p,q}(K)$ is a *Besov-Lipschitz space*. It is a Banach space.

$\underline{p = 2}$ $\Lambda^{\beta}_{2,q}(\mathbb{R}^n) = B^{\beta}_{2,q}(\mathbb{R}^n)$ if $0 < \beta < 1$, $= \{0\}$ if $\beta > 1$.

$\underline{p = 2, \beta = 1}$ $\Lambda^{1}_{2,\infty}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, $\Lambda^{1}_{2,2}(\mathbb{R}^n) = \{0\}$.

Theorem 2.2 (Jonsson '96, K, Paluba, Grigor'yan-Hu-Lau, K-Sturm)

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on the gasket. Then,

$$\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K).$$

Proof. Proof of $\mathcal{F} \subset \Lambda^{d_w/2}_{2,\infty}$. Let $\mathcal{E}_t(f,f) := (f - P_t f, f)_{\mathbb{L}^2}/t, \quad f \in \mathbb{L}^2(K,\mu)$. Then,

$$\begin{aligned}
\mathcal{E}_t(f,f) &= \frac{1}{2t}\int\int_{K\times K}(f(x)-f(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\geq \frac{1}{2t}\int\int_{|x-y|\leq c_0 t^{1/d_w}}(f(x)-f(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\geq \frac{c_1}{2t}\int\int_{|x-y|\leq c_0 t^{1/d_w}} t^{-d_s/2}(f(x)-f(y))^2\mu(dx)\mu(dy), \qquad (2.3)
\end{aligned}$$

where (HK(d_w)) was used in the last inequality.

Take $t = L^{-md_w}$ and use $d_s/2 = d_f/d_w \quad \Rightarrow \quad$ (2.3)$= c_1 a_m(d_w/2, f)^2$.

$\mathcal{E}_t(f,f) \nearrow \mathcal{E}(f,f)$ as $t \downarrow 0$. So we obtain $\sup_m a_m(d_w/2,f) \leq c_2\sqrt{\mathcal{E}(f,f)}$.

Proof of $\mathcal{F} \supset \Lambda_{2,\infty}^{d_w/2}$. Set $\gamma = 1/(d_w - 1)$, diam $(K) = 1$. Then, $\forall g \in \Lambda_{2,\infty}^{d_w/2}$

$$\begin{aligned}
\mathcal{E}_t(g,g) &= \frac{1}{2t}\int\int_{\substack{x,y\in K\\|x-y|\le 1}} (g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy)\\
&\le \frac{1}{2t}\sum_{m=1}^{\infty} c_3 t^{-d_s/2} e^{-c_4(tL^{md_w})^{-\gamma}} \int\int_{L^{-m}<|x-y|\le L^{-m+1}} (g(x)-g(y))^2 \mu(dx)\mu(dy)\\
&\le c_3 t^{-(1+d_s/2)} \sum_{m=1}^{\infty} e^{-c_4(tL^{md_w})^{-\gamma}} L^{-m(d_w+d_f)} a_{m-1}(d_w/2, g)^2, \qquad (2.4)
\end{aligned}$$

where (HK(d_w)) was used in the first inequality. Let $\Phi_t(x) = e^{-c_4(tL^{xd_w})^{-\gamma}} L^{-x(d_w+d_f)}$.

- $\Phi_t(0) > 0$, $\lim_{x\to\infty} \Phi_t(x) = 0$ and $\int_0^\infty \Phi_t(x)dx = c_5 t^{1+d_s/2}$.
- $\exists x_t > 0$ s.t. $\Phi_t(x) \uparrow$ $(0 \le \forall x < x_t)$, $\Phi_t(x) \downarrow$ $(x_t < \forall x < \infty)$, and $\Phi_t(x_t) = c_6 t^{1+d_s/2}$.

Thus, $\sum_{m=1}^{\infty} \Phi_t(m) \le \int_0^\infty \Phi_t(x)dx + 2\Phi_t(x_t) \le c_7 t^{1+d_s/2}$.

Since $(2.4) \le c_3 t^{-(1+d_s/2)} (\sup_m a_m(d_w/2, f))^2 \sum_{m=1}^{\infty} \Phi_t(m)$,

we conclude that $\sup_{t>0} \mathcal{E}_t(g,g) = \lim_{t\to 0} \mathcal{E}_t(g,g) \le c_8 (\sup_m a_m(d_w/2, f))^2$. □

More general fractals

- Nested fractals (Lindstrøm '90): Similar constructions, similar results.
- P.c.f. self-similar sets (Kigami '93): Under the existence of the 'reg. harm. structure', similar constructions, generalized versions for (A), (B) and (C).
- Sierpinski carpets: Construction of D-forms, much harder, but possible (Barlow-Bass etc). Similar results for (B) and (C).

3 Jump type processes on Alfors d-regular set

K: compact Alfors d-regular set in $\mathbb{R}^n$ $(n \geq 2, 0 < d \leq n)$. I.e., $K \subset \mathbb{R}^n$, $\exists c_1, c_2 > 0$ s.t.

$$c_1 r^d \leq \mu(B(x,r)) \leq c_2 r^d \qquad \text{for all } \ x \in K, \ 0 < r \leq 1, \tag{3.1}$$

$B(x,r)$: ball center x, radius r w.r.t. Euclidean metric.

d: Hausdorff dimension of K, μ: Hausdorff measure on K.

$\hat{K}$: unbounded Alfors d-regular set in $\mathbb{R}^n$, i.e. (3.1) holds for all $r > 0$.

For $0 < \alpha < 2$, let

$$\mathcal{E}_{Y(\alpha)}(u,u) = \int\int_{K \times K} \frac{c(x,y)|u(x) - u(y)|^2}{|x-y|^{d+\alpha}} \mu(dx)\mu(dy),$$

where $c(x,y)$ is jointly measurable, $c(x,y) = c(y,x)$ and $c(x,y) \asymp 1$.

–We denote $\mathcal{E}_{\hat{Y}(\alpha)}$ if we integrate over $\hat{K}$.

A Besov space $\Lambda^{\alpha/2}_{2,2}(K)$ is defined as follows,

$$\begin{aligned}\|u|\Lambda^{\alpha/2}_{2,2}(K)\| &= \|u\|_{\mathbb{L}^2(K,\mu)} + \left(\int\int_{K\times K} \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\mu(dx)\mu(dy)\right)^{1/2} \\ \Lambda^{\alpha/2}_{2,2}(K) &= \{u : u \text{ is measurable}, \|u|\Lambda^{\alpha/2}_{2,2}(K)\| < \infty\}.\end{aligned}$$

Theorem 3.1 $(\mathcal{E}_{Y(\alpha)}, \Lambda^{\alpha/2}_{2,2}(K))$ *is a regular Dirichlet space on* $\mathbb{L}^2(K,\mu)$.

Denote $\{Y^{(\alpha)}_t\}_{t\geq 0}$ the corresponding Hunt process on K.

Examples $\underline{c(x,y) \equiv 1}$ (Fukushima-Uemura '03, Stós '00)

$^*K = \mathbb{R}^n \Rightarrow \{Y^{(\alpha)}_t\}$ is a α-stable process on $\mathbb{R}^n$.

$^*K$: an open n-set $\Rightarrow \{Y^{(\alpha)}_t\}$ is a reflected α-stable process on K.

Proof of $\Lambda^{\alpha/2}_{2,2}(K) \cap C_0(K)$ dense in $C_0(K)$. First, note that (using $\alpha < 2$)

$$\sup_z \int_{B(z,r)^c} \frac{\mu(dy)}{|z-y|^{d+\alpha}} \le cr^{-\alpha}, \quad \sup_z \int_{B(z,r)} \frac{|z-y|^2 \mu(dy)}{|z-y|^{d+\alpha}} \le c \int_0^r s^{1-\alpha} ds \le c' r^{2-\alpha}. \quad (*)$$

For $x \neq y \in K$, let $r := |x-y|$ and $\psi(\xi) = 1 - \frac{|\xi - x| \wedge r}{r}$.

Then, $\psi \in C_0(K)$, $\mathrm{supp}[\psi] \subset B(x,r) =: B$ and $|\psi(\xi) - \psi(\eta)| \le |\eta - \xi|/r$. So, using (*),

$$\begin{aligned}
\mathcal{E}(\psi, \psi) &= \int_B \int_B \frac{(\psi(\xi) - \psi(\eta))^2}{|\xi - \eta|^{d+\alpha}} \mu(d\xi)\mu(d\eta) + 2 \int_{B^c} \mu(d\xi) \int_B \frac{\psi(\eta)^2}{|\xi - \eta|^{d+\alpha}} \mu(d\eta) \\
&\le \frac{1}{r^2} \int_B \int_B \frac{|\xi - \eta|^2}{|\xi - \eta|^{d+\alpha}} \mu(d\xi)\mu(d\eta) + cr^{-\alpha} \int_B \psi(\eta)^2 \mu(d\eta) \\
&\le c' r^{-\alpha} \mu(B) < \infty.
\end{aligned}$$

Thus $\psi \in \Lambda^{\alpha/2}_{2,2}(K) \cap C_0(K)$. Since this holds for all $x \neq y \in K$, using Stone-Weierstrass' theorem we see that $\Lambda^{\alpha/2}_{2,2}(K) \cap C_0(K)$ is dense in $C_0(K)$. □

Cf. Jump process as a subordination of a diffusion on th gasket (Restrictive)

K: the Sierpinski gasket, $\{B_t^K\}_{t\geq 0}$: Brownian motion on K. Recall

$$c_1 t^{-\frac{d}{d_w}} \exp(-c_2(\frac{|x-y|^{d_w}}{t}))^{\frac{1}{d_w-1}} \leq p_t(x,y) \leq c_3 t^{-\frac{d}{d_w}} \exp(-c_4(\frac{|x-y|^{d_w}}{t}))^{\frac{1}{d_w-1}}. \qquad (HK(d_w))$$

(Other examples: nested fractals, Sierpinski carpets)

$\{\xi_t\}_{t>0}$: strictly $(\alpha/2)$-stable subordinator $(0 < \alpha < 2)$.

I.e., 1-dim. non-neg. Lévy process, indep. of $\{B_t^K\}_{t\geq 0}$, $E[\exp(-u\xi_t)] = \exp(-tu^{\alpha/2})$.

$\{\eta_t(u) : t > 0, u \geq 0\}$: distribution density of $\{\xi_t\}_{t>0}$. Define

$$q_t(x,y) := \int_0^\infty p_u(x,y)\eta_t(u)du \qquad \text{for all } t > 0,\ x, y \in K. \qquad (3.2)$$

$\{X_t^{(\alpha)}\}_{t\geq 0}$: the subordinate process (with the transition density $q_t(x,y)$).

$$P_t^{X^{(\alpha)}} f := \mathbb{E}^{(\xi)}[P_{\xi_t}^{B^K} f] = \int_0^\infty P_s^{B^K} f \cdot \eta_t(s) ds.$$

Then, $\{X_t^{(\alpha)}\}_{t\geq 0}$ is a μ-symmetric Hunt process. (Stós '00, Bogdan-Stós-Sztonyk '02)

$(\mathcal{E}_{X^{(\alpha)}}, \mathcal{F}_{X^{(\alpha)}})$: the corresponding Dirichlet form on $\mathbb{L}^2(K,\mu)$.

Remark.

1) If we start from the BM on $\mathbb{R}^n$, the resulting process is the α- stable process on $\mathbb{R}^n$.

2) For d-sets on $\mathbb{R}^n$, we can also construct jump-type processes by a time change of the α-stable process on $\mathbb{R}^n$. (Triebel, K, Zähle etc.)

Comparison of the forms K: Sierpinski gasket, $\bar{\alpha} =: \alpha d_w/2$

Proposition 3.2 *For* $0 < \alpha < 2$,

$$\mathcal{E}_{X(\alpha)}(f,f) \asymp \mathcal{E}_{Y(\bar{\alpha})}(f,f) \qquad \textit{for all } f \in \mathbb{L}^2(K,\mu).$$

In particular, $\mathcal{F}_{X(\alpha)} = \Lambda^{\bar{\alpha}/2}_{2,2}(K)$.

Further, the densities of the Levy measures are also compatible.

Note. On the gasket, the two Dirichlet forms introduced are different and the corresponding processes cannot be obtained by time changes of others by PCAFs.

Heat kernel estimates   The HK estimates for $X^{(\alpha)}$ is easy to obtain using (3.2).

So, for the gasket case, we have the sharp HK estimates for $Y^{(\alpha)}$ as well.

How about the general d-set case?

Recall that for $0 < \alpha < 2$,

$$\mathcal{E}_{Y^{(\alpha)}}(u,u) := \int\int_{K\times K} \frac{c(x,y)|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\mu(dx)\mu(dy),$$

where $c(x,y)$ is jointly measurable, $c(x,y)=c(y,x)$ and $c(x,y)\asymp 1$.

Theorem 3.3 (Chen-K '03) *For all* $0<\alpha<2$,

$\exists p_t^{Y^{(\alpha)}}(x,y)$: *jointly continuous heat kernel s.t.*

$$c_1\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right) \le p_t^{Y^{(\alpha)}}(x,y) \le c_2\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

- Parabolic Harnack inequalities hold.
- Related works: Bass-Levin ('02)

Corollary 3.4 (Transience, recurrence) *For* $\hat{K}$,

$\hat{Y}^{(\alpha)}$ *is transient iff* $d > \alpha$, *point recurrent iff* $d < \alpha$.

For $d = \alpha$, $P^x(\sigma_y < \infty) = 0, P^x(\sigma_{B(y,r)} < \infty) = 1 \quad \forall x, y \in \hat{K}, r > 0.$

Application Hausdorff dim. for the range of the process

Proposition 3.5

$$dim_H\{\hat{Y}_t^{(\alpha)} : 0 < t < \infty\} = d \wedge \alpha \qquad \mu - a.e.$$

*More general version Y. Xiao ('04), R. Schilling-Y.Xiao ('05).

Theorem 3.6 *(Nash ineq.) Let $r_0 = diamK$. Then,*

$$\|u\|_2^{2\left(1+\frac{\alpha}{d}\right)} \leq C\left(r_0^{-\alpha}\|u\|_2^2 + \mathcal{E}(u,u)\right)\|u\|_1^{\frac{2\alpha}{d}}, \qquad \forall u \in \mathcal{F}. \tag{3.3}$$

Proof. Define the average function u_r of u by

$$u_r(x) := \frac{1}{\mu(B(x,r))}\int_{B(x,r)} u(z)\,d\mu(z),\ x \in K.$$

Then we have

$$\|u_r\|_\infty \leq c_0\, r^{-d}\|u\|_1,, \quad \|u_r\|_1 \leq C\,\|u\|_1, \qquad 0 < \forall r < r_0.$$

Thus

$$\|u_r\|_2^2 = \int_K |u_r(x)|^2\,d\mu(x) \leq \|u_r\|_\infty\|u_r\|_1 \leq C\,r^{-d}\|u\|_1^2, \quad 0 < r < r_0. \tag{3.4}$$

On the other hand,

$$\begin{aligned}
\|u-u_r\|_2^2 &= \int_K \Big|\frac{1}{\mu(B(x,r))}\int_{B(x,r)}(u(x)-u(y))\,d\mu(y)\Big|^2 d\mu(x)\\
&\overset{\text{Schwarz}}{\leq} c_0\, r^{-d}\int_K\int_{B(x,r)}|u(x)-u(y)|^2\,d\mu(y)d\mu(x)\\
&= c_0\, r^{-d}\int_K\int_{B(x,r)}\frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\cdot|x-y|^{d+\alpha}\,d\mu(y)d\mu(x)\\
&\leq c_0\, r^{\alpha}\mathcal{E}(u,u), \qquad 0<r<r_0. \qquad (3.5)
\end{aligned}$$

Therefore, it follows from (3.4) and (3.5) that

$$\|u\|_2^2 \leq 2(\|u_r\|_2^2+\|u-u_r\|_2^2) \leq C\,(r^{-d}\|u\|_1^2 + r^{\alpha}\mathcal{E}(u,u)), \quad 0<r<r_0. \qquad (3.6)$$

Noting that $\|u\|_2^2 \leq (\frac{r}{r_0})^{\alpha}\|u\|_2^2$ for $r\geq r_0$, we see from (3.6) that

$$\|u\|_2^2 \leq C\,(r^{-d}\|u\|_1^2 + r^{\alpha}\left(r_0^{-\alpha}\|u\|_2^2+\mathcal{E}(u,u)\right)) \qquad (3.7)$$

for all $r>0$. We obtain (3.3) by minimizing the right-hand side of (3.7). □

$$d_w := \sup\{\alpha : (\mathcal{E}_{Y(\alpha)}, \Lambda_{2,2}^{\alpha/2}(K)) \text{ is regular in } \mathbb{L}^2\}$$

Then, we can prove the former theorems for all $\alpha < d_w$

if $d_w > d$ (strongly recurrent case).

(Open Prob.) Does Them 3.3 hold $\forall \alpha < d_w$ when $d_w \leq d$?

Remark: $\bar{d}_w := \sup\{\alpha : \Lambda_{2,2}^{\alpha/2}(K) \text{ is dense in } \mathbb{L}^2\}$. (cf. Paluba '00, Stós '00)

Then $d_w \leq \bar{d}_w$. When there is a fractional diffusion on K, then $d_w = \bar{d}_w$.