Recent developments of analysis on fractals

Takashi Kumagai

1. Introduction

"Fractal" is a word invented by French mathematician B.B. Mandelbrot around 1970s. He claimed that many patterns of Nature, such as clouds, mountains and coastlines are not lines nor circles which are smooth, but are so irregular, fragmented and exhibit an altogether different level of complexity. From this viewpoint, he called the family of those shapes as fractals. Especially, he made special attention to the self-similar property, i.e. similarity between global parts and local parts, of fractals. Around the same time, mathematical physicists began to analyse properties of disordered media such as the structure of polymers and networks, growth of molds and crystals, using some self-similar properties of the media (see, for instance, [**30, 31**]).

Motivated by these works, mathematicians got interested in the analytical properties of fractals such as heat transfer and wave transfer. Fractals are typical ideal examples of the disordered media. Since there is no smoothness on fractals, one cannot define the notion of differentials directly. So the biggest problem was how to treat such physical phenomena in a rigorous way. In the middle eighties, probabilists solved the problem by constructing a diffusion process on the Sierpinski gasket, which is a typical fractal. The first works are by Goldstein ([**21**]) and Kusuoka ([**50**]); Barlow-Perkins ([**11**]) further obtained detailed heat kernel estimates of the diffusion which we will discuss in more details later. After that, Kigami ([**41**]) constructed a Laplace operator on the gasket as a limit of difference operators. This analytical approach motivated a work by Fukushima-Shima ([**18**]), which made it clear that the theory of Dirichlet forms was well-applicable to this area.

Since then, for the last several decades, diffusion processes (and the corresponding self-adjoint operators) have been constructed on various classes of fractals and their properties have been deeply studied. It is now getting clear that the diffusions on fractals have completely different properties from diffusions on Euclidean spaces. For instance, it is understood that such processes typically have sub-diffusive behaviour and heat kernels for Brownian motion on 'nice' fractals enjoy sub-Gaussian estimates (see (2.2)). Through substantial amount of work, stochastic processes on fractals have been related to various other fields. Recently, there are new developments of this area, namely to aim for analysis on "fractal-like spaces" instead of

1991 *Mathematics Subject Classification.* Primary 60J60, 60J35; Secondary 31B05, 46E35.

Key words and phrases. Brownian motion, fractals, Sierpinski gaskets, Dirichlet forms, heat kernel estimates, parabolic Harnack inequalities, stability, Besov spaces, jump processes.

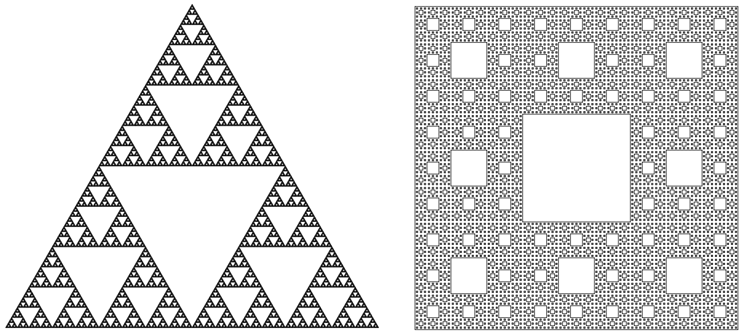

FIGURE 1. 2-dimensional Sierpinski gasket (left) and Sierpinski carpet (right)

working only on "ideal fractals". More concretely, the following two directions of research are getting active:

- Stability of parabolic Harnack inequalities and heat kernel estimates
- Function spaces and stochastic processes on fractals

In this article, we will restrict ourselves to the Sierpinski gasket and overview what is known for Brownian motion (the Laplace operator) on the gasket. We will then introduce recent developments of analysis on fractals, based on the two directions (especially the first one) mentioned above. For analysis on general finitely ramified fractals[1] which include the Sierpinski gasket as a typical example, there are nice lecture notes and a book [**3, 38, 48**], see also surveys [**40, 45, 53, 64**]. For analysis on infinitely ramified fractals[2] which include the Sierpinski carpet (the right of Figure 1) as a typical example, readers may refer [**5, 6, 12, 61**]. For analysis on random recursive fractals, [**26**] is a good survey.

2. Brownian motion on the Sierpinski gasket

Let a, b, c be the vertices of an equilateral triangle in $\mathbb{R}^2$, and set $F_1(x) = (x-a)/2 + a$, $F_2(x) = (x-b)/2 + b$ and $F_3(x) = (x-c)/2 + c$. Then, there exists unique non-void compact set which satisfies $K = F_1(K) \cup F_2(K) \cup F_3(K)$; we call K the 2-dimensional Sierpinski gasket (we will abbreviate it as S.G.). On $\mathbb{R}^N$, we can define K similarly from the family of $(N+1)$-th contraction maps with contraction rate $1/2$. (For $N = 1$, K is a line segment $[a, b]$.) The Hausdorff dimension of the S.G. is $d = \log(N+1)/\log 2$. By a parallel transfer, we can let $a = (0, 0)$. Define the non-compact set $\hat{K}$ which corresponds to K as $\hat{K} = \cup_{n=0}^{\infty} 2^n K$.

We first give an overview of the construction of Brownian motion on the S.G.. Let

$$V_0 = \{a, b, c\},\ V_m = \bigcup_{i_1, \cdots, i_m = 1}^{3} F_{i_1} \circ \cdots \circ F_{i_m}(V_0)$$

The closure of $\cup_{m \geq 0} V_m$ is the S.G.. Let $\{X_m(i)\}_i$ be the simple random walk on V_m. That is, it is a random walk such that $X_m(i+1)$ is in one of the adjacent neighbours of $X_m(i)$ (i.e. points in the same small triangles as those $X_m(i)$ belongs

[1] self-similar sets that can be disconnected by removing a specific finite number of points
[2] fractals that are not finitely ramified

to) with equal probabilities. X_m moves distance 2^{-m} per second, so if we take $m \to \infty$, then $X_m(i) \to 0$. In other word, the limit process does not move from the origin. Therefore, we must speed up the random walks. In order to obtain a non-trivial limit, the time scale for each step should be the average time for the random walk on V_{m+1} starting from a point in V_m to arrive at one of the points in V_m except the starting point. By the self-similarity and symmetry of K, this average time is independent of m and it is equal to the average time for the random walk on V_1 starting from a to arrive at either b or c. By a simple calculation, this value turns out to be 5. (For the case of $[0,1]$, it is 4. For the N-dimensional S.G., it is $N+3$.) Now, let $X_m([5^m t])$. Then, we can prove that the process converges to a non-trivial diffusion on K as $m \to \infty$, which is called a Brownian motion on K. It is known that any self-similar diffusion process on K whose law is invariant under local translations and reflections of each small triangle is a constant time change of this diffusion ([**11**]). This guarantees that the diffusion is the most natural one on the S.G., so it is worth to be called as Brownian motion. One can construct Brownian motion on $\hat{K}$ in a similar way.

The corresponding Laplace operator Δ is defined as follows. When f is in a suitable function space,

$$\Delta f(x) = \lim_{m\to\infty} 5^m \Big(\sum_{x \overset{m}{\sim} x_i} f(x_i) - 4f(x) \Big), \; x \in \cup_{m\geq 0} V_m \setminus V_0. \tag{2.1}$$

Here $x \overset{m}{\sim} y$ means that x and y are adjacent in V_m. For the case of $[0,1]$, the approximation which corresponds to (2.1) is $\Delta f(x) = \lim_{m\to\infty} 2^{2m}(f(x+2^{-m}) + f(x-2^{-m}) - 2f(x))$ for $f \in C^2([0,1])$. Here the square appears because the Laplace operator on $[0,1]$ is the second order differential operator. Set $d_w = \log 5/\log 2$ for the S.G. case, then $5 = 2^{d_w}$. So, naively the Laplace operator on the S.G. is a "differential operator of order d_w".

It is well-known that there is a one to one correspondence between symmetric diffusion processes and the self adjoint operators which are generators of them. It is also well-known that they have a one to one correspondence to some quadratic forms called Dirichlet forms (for the Laplace operator on $\mathbb{R}^n$, it is $\int_{\mathbb{R}^n} |\nabla f|^2 dx$), see [**17**]. So, investigating the properties of the self-adjoint operators means investigating the properties of the corresponding diffusions and Dirichlet forms. Here we have given an overview of the construction of Brownian motion since it is intuitively easier, but practically constructing Dirichlet forms is the most systematical way to construct diffusions (self-adjoint operators) on fractals.

In the following, we will introduce typical properties of Brownian motion on the S.G..

<u>Domains of the Dirichlet forms</u> For $1 \leq p < \infty,\ 1 \leq q \leq \infty,\ \beta \geq 0$ and $m \in \mathbb{N} \cup \{0\}$, set

$$a_m(\beta, f) := L^{m\beta}(L^{md} \int\int_{|x-y|<c_0 L^{-m}} |f(x) - f(y)|^p d\mu(x) d\mu(y))^{1/p}, \; f \in \mathbb{L}^p(K, \mu),$$

where $1 < L,\ 0 < c_0$ and d is the Hausdorff dimension of K. Define a *Lipschitz space* $\mathrm{Lip}(\beta, p, q)(K)$ as a set of $f \in \mathbb{L}^p(K, \mu)$ such that $\bar{a}(\beta, f) := \{a_m(\beta, f)\}_{m=0}^{\infty} \in l^q$. $\mathrm{Lip}(\beta, p, q)(K)$ is a Banach space with the norm $\|f\|_{\mathrm{Lip}} := \|f\|_{\mathbb{L}^p} + \|\bar{a}(\beta, f)\|_{l^q}$.

β is an exponent which gives the "smoothness order" of the functions. We are interested in $p = 2$. When $K = \mathbb{R}^n$, $\mathrm{Lip}(\beta, 2, q)(\mathbb{R}^n) = B_{\beta}^{2,q}(\mathbb{R}^n)$ if $0 < \beta < 1$, $= \{0\}$

if $\beta > 1$. Here $B^{2,q}_\beta(\mathbb{R}^n)$ is the Besov space on $\mathbb{R}^n$. When $\beta = 1$, $\mathrm{Lip}(1,2,\infty)(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$, i.e. the Sobolev space, and $\mathrm{Lip}(1,2,2)(\mathbb{R}^n) = \{0\}$.

THEOREM 2.1. ([**34, 44**]) *Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form on the S.G.. Then,*

$$\mathcal{F} = Lip(\frac{d_w}{2}, 2, \infty)(K).$$

Here $d_w = \log 5/\log 2$ ($\log(N+3)/\log 2$ for the N-dimensional S.G.). Since $\mathrm{Lip}(1,2,\infty)(\mathbb{R}^n)$ is the domain of the Dirichlet form on $\mathbb{R}^n$ determined by the second order differential operator, this theorem supports that the Laplace operators on fractals are "differential operators of order d_w".

Sub-Gaussian heat kernel estimates and the parabolic Harnack inequality

For Brownian motion on the S.G., there exists a continuous heat kernel $p_t(\cdot,\cdot)$ w.r.t. the Hausdorff measure. It is a fundamental solution of the heat equation, i.e. the following holds,

$$\frac{\partial p_t(x,\cdot)}{\partial t} = \Delta p_t(x,\cdot), \ \lim_{t\to 0} p_t(x,\cdot) = \delta_x(\cdot).$$

Here δ_x is the Dirac delta function with mass at x. For the case of Brownian motion on $\mathbb{R}^n$, $p_t(x,y)$ is the Gauss-kernel $\frac{1}{(2\pi t)^{n/2}}\exp(-|x-y|^2/(2t))$. For the Brownian motion on S.G., it is known that $p_t(x,y)$ enjoys the following estimates for $0 < t \le 1, x, y \in K$ (resp. $t > 0, x, y \in \hat{K}$) – see [**11**].

$$\begin{aligned} &c_1 t^{-d_s/2}\exp\Big(-c_2\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\Big) \le p_t(x,y) \\ &\le c_3 t^{-d_s/2}\exp\Big(-c_4\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\Big). \end{aligned} \tag{2.2}$$

Here $d(x,y)$ is the shortest distance between x and y in $\hat{K}$ (in this case it is equivalent to the Euclidean distance). d_w expresses the order of the diffusion speed of particles. Indeed, by integrating (2.2), we have $c_5 t^{1/d_w} \le E^x[d(x,X_t)] \le c_6 t^{1/d_w}$. As $d_w > 2$, we say the process is sub-diffusive. This diffusion does not have the finite quadratic variation, so it is not a semi-martingale. $d_s = 2\log 3/\log 5$ is called a spectral dimension (we will explain more about this exponent later).

(2.2) is a very useful estimate. Various properties of Brownian motion such as laws of the iterated logarithm can be deduced from this estimate. Concerning the properties of harmonic functions, the following (generalized) parabolic Harnack inequality can be obtained from (2.2).

THEOREM 2.2. *For $s, r \in (0,1], x_0 \in K$ (resp. $s, r \in (0,\infty), x_0 \in \hat{K}$), set*

$$Q_- = (s + r^{d_w}, s + 2r^{d_w}) \times B(x_0, r), \ Q_+ = (s + 3r^{d_w}, s + 4r^{d_w}) \times B(x_0, r).$$

There exists $c_1 > 0$ such that, for any $s, r \in (0,1], x_0 \in K$ (resp. $s, r \in (0,\infty), x_0 \in \hat{K}$), if u is a non-negative function on $(s, s+4r^{d_w}) \times B(x_0, 2r)$ with $\frac{\partial u}{\partial t} = \Delta u$, then

$$\sup_{(t,x)\in Q_-} u(t,x) \le c_1 \inf_{(t,x)\in Q_+} u(t,x). \tag{$PHI(d_w)$}$$

Once we have (2.2), $(PHI(d_w))$ can be deduced through well-known arguments (see, for instance, [**16**]). In fact, they are equivalent as we will see in Section 3.

Short time asymptotics of the heat kernels and large deviations Let $t_0 = 2^{1-d_w} = 2/5$. For each $z \in [t_0, 1)$, let $\epsilon_{n,z} = t_0^n z$. Then the following holds ([**46**]):

$$-\lim_{n\to\infty} \epsilon_{n,z}^{\frac{1}{d_w-1}} \log p_{\epsilon_{n,z}}(x,y) = d(x,y)^{\frac{d_w}{d_w-1}} F\Big(\frac{z}{d(x,y)}\Big), \qquad \forall x, y \in K.$$

Here F is a periodic non-constant positive continuous function with period t_0^{-1}.

Let P_ϵ^x be the law of B_ϵ. starting at x. For each $z \in [t_0, 1)$, $A \subset \Omega_x := \{f \in C([0,T] \to K : f(0) = x\}$,

$$
\begin{aligned}
-\inf_{\phi \in \mathrm{Int}A} I_x^z(\phi) &\leq \liminf_{n\to\infty} \epsilon_{n,z}^{1/(d_w-1)} \log P_{\epsilon_{n,z}}^x(A) \\
&\leq \limsup_{n\to\infty} \epsilon_{n,z}^{1/(d_w-1)} \log P_{\epsilon_{n,z}}^x(A) \leq -\inf_{\phi \in \mathrm{Cl}A} I_x^z(\phi),
\end{aligned} \tag{2.3}
$$

([**13**]). Here $\{I_x^z\}_{z\in[t_0,1)}$ is a sequence of rate functions defined as follows for each $\phi \in \Omega_x$,

$$
I_x^z(\phi) = \begin{cases} \displaystyle\int_0^T \Big(\dot{\phi}(t)\Big)^{d_w/(d_w-1)} F\Big(\frac{z}{\dot{\phi}(t)}\Big) dt & \text{if } \phi \text{ is absolutely continuous,} \\ \infty & \text{otherwise,} \end{cases}
$$

where F is the same periodic function as above and $\dot{\phi}(t) := \lim_{s\to t} \frac{d(\phi(s),\phi(t))}{|s-t|}$ for $t \in [0,T]$. For the case of Brownian motion on $\mathbb{R}^n$, F is a constant function and (2.3) holds independently of z with $d_w = 2$ (especially, I_x^z is independent of z and $I_x(\phi) := \frac{1}{2}\int_0^T \dot{\phi}(t)^2 dt$). This fact is called the Schilder-type large deviation principle. (2.3) tells us that the classical Schilder-type large deviation does not hold when $\epsilon \to 0$. Instead, for each fixed z, it holds via the sequence $\epsilon_{n,z}$ as $n \to \infty$.

When $A = \{f \in \Omega_x : f(T) = y\}$, $\inf\{I_x^z(\phi) : \phi \in A\}$ is attained independently of z by the path(s) which moves on the geodesic(s) between x and y homogeneously. Thus "the most probable path" should be this path, but the energy (action functional) of the path depends on time sequences determined by z.

<u>Spectral properties</u> Let Δ be the Laplace operator on K. It is known that $-\Delta$ has a compact resolvent. Set $\rho(x) = \sharp\{\lambda \leq x : \lambda \text{ is an eigenvalue of } -\Delta\}$. Then the following holds ([**18**]).

$$
0 < \liminf_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \limsup_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \infty. \tag{2.4}
$$

d_s is the spectral dimension mentioned above. Remark that on (2.4), the limit supremum and the limit infimum do not coincide. This is because there exist 'many' localized eigenfunctions that produce eigenvalues with high multiplicities ([**9**]). Here, a localized eigenfunction is an eigenfunction u of $-\Delta$ such that $\mathrm{Supp}\, u \subset O$ on some open set $O \subset K \setminus V_0$. For the case of the Laplace operator on $[0,1]^n$, it is well-known that $d_s = n$ and the limit supremum and the limit infimum in (2.4) coincides. There is no localized eigenfunctions in this case.

For the N-dimensional S.G., the Hausdorff dimension is $\log(N+1)/\log 2$ which can be any large number as N increases, but the spectral dimension is $2\log(N+1)/\log(N+3)$ which is less than 2 (so, Brownian motion is always point recurrent). In other word, the N-dimensional S.G. is analytically less than 2-dimensional due to the finite ramified property.

Let $\hat{\Delta}$ be the Laplace operator on $\hat{K}$. Eigenfunctions with compact supports are complete in $\mathbb{L}^2(\hat{K}, \hat{\mu})$; especially $\hat{\Delta}$ on $\hat{K}$ consists only on point spectra ([**66**]).

<u>Energy measure</u> By the general theory of Dirichlet forms (cf. [**17**]), for each $f \in \mathcal{F}$, there is a uniquely determined Borel measure $\mu_{<f>}$ on K (called an energy measure) satisfying

$$
\int_K g d\mu_{<f>} = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \qquad \forall g \in \mathcal{F}.
$$

For the case of the Dirichlet form on $\mathbb{R}^n$ which corresponds to Brownian motion, the energy measure is $d\mu_{<f>} = |\nabla f|^2 dm$ (m is the Lebesgue measure), so it is absolutely continuous w.r.t. the Lebesgue measure. On the other hand, the signed measure $\mu_{<f>}$ on the S.G. is mutually singular to the Hausdorff measure ([**32, 49**]). By this fact, we can observe that the well-known perturbation method for heat kernel estimates (sometimes called the Davies method) does not work on S.G.. (2.2) is established using probabilistic methods.

Homogenization Let $\Gamma_{\text{S.G.}} = \cup_{n\geq 0} 2^n V_n$ (cf. the left of Figure 2) be the S.G. graph and put conductances randomly on each edge of triangles in $\Gamma_{\text{S.G.}}$. We assume that the conductances are i.i.d. on each triangle and they are bounded from above and below by some positive constants (this corresponds to the uniform elliptic condition). Let ν be a probability measure which governs the randomness of conductances. Define $Y^\omega(n)$ as the corresponding Markov chain. We have

$$2^{-n} Y^\omega([5^n t]) \stackrel{n\to\infty}{\longrightarrow} B_{ct} \qquad \text{in law}$$

where $\{B_t\}$ is the Brownian motion on the S.G. ([**47**]). The convergence is in probability w.r.t. ν. In a word, the long time behaviour of the random walk on the S.G. graph with random environments converges to that of the Brownian motion on S.G. by homogenization. There is a naturally defined renormalization map, and the key part for this homogenization problem is to study the renormalization map in details ([**62, 57**]).

3. Stability of parabolic Harnack inequalities and heat kernel estimates

In this section, we will focus on (2.2), which is equivalent to $(PHI(d_w))$ as we will see below. Estimates like (2.2) and $(PHI(d_w))$ are extremely powerful and important properties in the harmonic analysis. Indeed, one can deduce from them the elliptic Harnack inequality, and in particular (when the diameter of the space is infinite), one can deduce the Liouville property of harmonic functions that says positive harmonic functions on the space are constants. Here is a naive question: Is (2.2) stable under perturbations of the operator? For the $d_w > 2$ case, the answer was given quite recently.

Let us first briefly overview the history for the case of $d_w = 2$. For any divergence operator $\mathcal{L} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on $\mathbb{R}^n$ satisfying a uniform elliptic condition, Aronson ([**1**]) proved (2.2) with $d_s = n$ and $d_w = 2$ (so (2.2) is sometimes called the Aronson-type estimate). Later in the 20th century, there were various outstanding results in the field of global analysis on manifolds. Let Δ be the Laplace-Beltrami operator on a complete Riemannian manifold M with the Riemannian metric d and with the Riemannian measure μ. Li-Yau ([**54**]) proved the remarkable fact that if M has non-negative Ricci curvature, then the heat kernel $p_t(x, y)$ satisfies

$$c_1\Phi(x, c_2 d(x, y), t) \leq p_t(x, y) \leq c_3\Phi(x, c_4 d(x, y), t), \tag{3.1}$$

where $\Phi(x, r, t) = \mu(B(x, t^{1/2}))^{-1}\exp(-r^2/t)$. A few years later, Grigor'yan ([**23**]) and Saloff-Coste ([**63**]) elegantly refined the result and proved, in conjunction with the results by Fabes-Stroock ([**16**]) and Kusuoka-Stroock ([**51**]), that (3.1) is equivalent to a volume doubling condition (VD) plus Poincaré inequalities (PI(2)) –see Definition 3.1 for definitions in the graph setting. The results were then extended to the framework of Dirichlet forms and graphs (see, for instance, [**15, 65**]). Detailed heat kernel estimates are strongly related to the control of harmonic functions. The origin of ideas and techniques used in this field goes back to Nash ([**60**]), Moser ([**58, 59**]) and there are many other significant works in this area. Summarizing, the following equivalence holds.

$$(VD) + (PI(2)) \Leftrightarrow (PHI(2)) \Leftrightarrow (3.1). \tag{3.2}$$

An important corollary of this fact is, since (VD) and (PI(2)) are stable under certain perturbations of the operator, that (3.1) and (PHI(2)) are also stable under these perturbations.

Now let us discuss about the extension of (3.2) to the $d_w > 2$ case. For simplicity, we will discuss on graphs. Let G be an infinite connected locally finite graph. Assume that the graph G is endowed with a weight (conductance) μ_{xy}, which is a symmetric nonnegative function on $G \times G$ such that $\mu_{xy} > 0$ if and only if x and y are connected by a bond (in which case we write $x \sim y$). We call the pair (G, μ) a weighted graph. We can regard it as an electrical network. We define a quadratic form on (G, μ) as follows. Set

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{\substack{x,y \in G \\ x \sim y}} (f(x) - f(y))(g(x) - g(y))\mu_{xy}, \qquad \forall f, g \in \mathbb{R}^G.$$

We say (G, μ') is a bounded perturbation of (G, μ) if $c_1 \mu_{xy} \le \mu'_{xy} \le c_2 \mu_{xy}$ for all $x \sim y$.

For each $x \in G$, let $\mu_x = \sum_{y \in G} \mu_{xy}$ and for each $A \subset G$, set $\mu(A) = \sum_{x \in A} \mu_x$; μ is then a measure on G. Let $\{X_n\}_{n \ge 0}$ be a discrete time Markov chain which moves at unit time intervals to any vertex y in the neighbourhood of x with probabilities given by $p_{xy} := \mu_{xy}/\mu_x$. The heat kernel of $\{X_n\}_n$ can be written as $p_n(x,y) := \mathbb{P}^x(X_n = y)/\mu_y$. Clearly, $p_n(x,y) = p_n(y,x)$. The discrete Laplace operator corresponding to the Markov chain is

$$\mathcal{L}f(x) = \sum_{\substack{y \in G \\ y \sim x}} P(x,y)f(y) - f(x) = \frac{1}{\mu_x} \sum_{\substack{y \in G \\ y \sim x}} \Big(f(y) - f(x)\Big)\mu_{xy}.$$

The natural metric on the graph obtained by counting the number of steps in the shortest path between points is written $d(x,y)$ for $x, y \in G$. For $x \in G$ and $r \ge 0$, denote $B(x,r) = \{y \in G : d(x,y) \le r\}$, $V(x,r) = \mu(B(x,r))$.

Definition 3.1. Let (G, μ) be a weighted graph and let $\beta > 0$.
(1) We say (G, μ) satisfies a p_0-condition if there exists $p_0 > 0$ such that

$$p_{xy} = \mu_{xy}/\mu_x \ge p_0 \qquad \forall x \sim y \in G.$$

(2) We say (G, μ) satisfies a volume doubling condition (VD) if there exists $c_1 > 1$ such that

$$V(x, 2R) \le c_1 V(x,R) \qquad \forall x \in G, R \ge 1. \tag{3.3}$$

(3) We say (G, μ) satisfies sub-Gaussian heat kernel estimates $(HK(\beta))$ if for $x, y \in G, n \ge d(x,y)$, $p_n(\cdot,\cdot)$ satisfies

$$p_n(x,y) \le \frac{c_1}{V(x, n^{1/\beta})} \exp\Big(- \Big(\frac{d(x,y)^\beta}{c_1 n}\Big)^{1/(\beta-1)}\Big),$$

$$p_n(x,y) + p_{n+1}(x,y) \ge \frac{c_2}{V(x, n^{1/\beta})} \exp\Big(- \Big(\frac{d(x,y)^\beta}{c_2 n}\Big)^{1/(\beta-1)}\Big).$$

(4) Fix $x_0 \in G$ and let $B_R := B(x_0, R)$. We say (G, μ) satisfies $(PI(\beta))$, a scaled Poincaré inequality with parameter $\beta \geq 2$, if there exists a constant $c_1 > 0$ such that for any ball $B_R \subset G$ with $R \geq 1$ and $f : B_R \to \mathbb{R}$,

$$\sum_{x \in B_R} (f(x) - \bar{f}_{B_R})^2 \mu_x \leq c_1 R^\beta \sum_{x \in B_R} \Gamma(f, f)(x).$$

Here $\bar{f}_{B_R} := \mu(B_R)^{-1} \sum_{y \in B_R} f(y)\mu_y$, and $\Gamma(f, f)(x) := \sum_{y \sim x} (f(x) - f(y))^2 \mu_{xy}$.
(5) Let $\beta \geq 2$. We say (G, μ) satisfies $(CS(\beta))$, a cut-off Sobolev inequality with exponent β, if there exist constants $c_1, c_2 > 0$ and $\theta \in (0, 1]$ such that for every $x_0 \in G, R \geq 1$, there exists a cut-off function $\varphi (= \varphi_{x_0,R})$ satisfying the following properties.

(a) $\varphi(x) \geq 1$ if $x \in B_{R/2}$, $\varphi(x) = 0$ if $x \in B_R^c$. Further, $|\varphi(x) - \varphi(y)| \leq c_1 (d(x, y)/R)^\theta$, for each $x, y \in G$.

(b) For any ball B_s with $1 \leq s \leq R$ and $f : B_{2s} \to \mathbb{R}$,

$$\sum_{x \in B_s} f(x)^2 \Gamma(\varphi, \varphi)(x) \leq c_2 (\frac{s}{R})^{2\theta} \Big(\sum_{x \in B_{2s}} \Gamma(f, f)(x) + s^{-\beta} \sum_{x \in B_{2s}} f(x)^2 \mu_x \Big).$$

Theorem 3.2. ([**4**]) *Let (G, μ) be a weighted graph satisfying the p_0-condition. Then,*

$$(VD) + (PI(\beta)) + (CS(\beta)) \Leftrightarrow (PHI(\beta)) \Leftrightarrow (HK(\beta)).$$

Here $(PHI(\beta))$ means the discrete version of Theorem 2.2 with $d_w = \beta$.

Remark 3.3. 1) There are preceding works by Grigor'yan-Telcs [**24, 25**]. The proof of Theorem 3.2 relies on their results.
2) When one of (thus all) the above conditions holds, then $\beta \geq 2$.
3) $(CS(2))$ always holds. (Indeed, take $\varphi(x) = 2d(x, B(x_0, R)^c)/R, \theta = 1$.) Thus Theorem 3.2 is an extension of (3.2) to the cases of $\beta > 2$ for graphs.

For the $\beta = 2$ case, there is a well-known method called Moser's iteration to deduce the Harnack inequality in (3.2). In order the method to work, it is necessary that the best order estimates can be deduced using Lipschitz continuous cut-off functions, and there the fact that $\beta = 2$ is very important. (Indeed, if we adopt similar arguments using the Lipschitz continuous cut-off functions for the $\beta > 2$ case, then the estimates obtained are not sharp enough to establish the Harnack inequality.) Roughly saying, $(CS(\beta))$ guarantees the existence of 'nice' cut-off functions $\varphi = \varphi_{x_0,R}$ that satisfies $\mathcal{E}(\varphi, \varphi) \leq c_1 R^{-\beta} V(x_0, R)$ (i.e. the order of its energy is less than $R^{-\beta} V(x_0, R)$) for each $x_0 \in G, R \geq 1$. The order of the energy for the Lipschitz continuous cut-off function is $R^{-2} V(x_0, R)$, so the energy of the cut-off function given by $(CS(\beta))$ is strictly smaller. The idea of the proof of the Harnack inequality when $\beta > 2$ is, to apply similar arguments as Moser's iteration for weighted measures $\nu_x := \mu_x + R^\beta \Gamma(\varphi, \varphi)(x)$ using $(CS(\beta))$.

Clearly, (VD), $(PI(\beta))$ and $(CS(\beta))$ are stable under bounded perturbations of Dirichlet forms. Moreover, they are stable under rough isometries, as we will discuss later.

On the other hand, in general it is not easy to check $(CS(\beta))$. Very recently, simpler equivalent conditions are given under a stronger volume growth condition.

Definition 3.4. (1) We say (G, μ) satisfies a volume growth condition $(VG(\beta))$ if there exist $K > 1, c_1 > 0$ with $\log c_1 / \log K < \beta$ such that

$$V(x, KR) \leq c_1 V(x, R) \qquad \forall x \in G, R \geq 1.$$

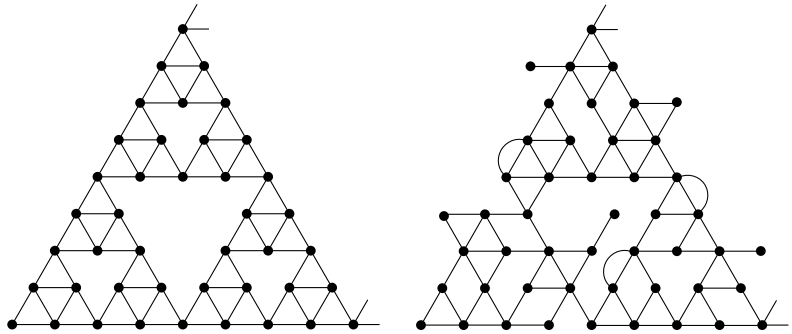

FIGURE 2. S.G. graph and its image by a rough isometry

(2) For each $x \neq y \in G$, define the effective resistance between them by

$$R(x,y)^{-1} = \inf\Big\{\mathcal{E}(f,f) : f(x) = 1, f(y) = 0, f \in \mathbb{R}^G\Big\}.$$

(We define $R(x,x) = 0$.) We say (G,μ) satisfies $(RE(\beta))$, effective resistance bounds of order β, if there exist $c_1, c_2 > 0$ such that

$$\frac{c_1 d(x,y)^\beta}{V(x,d(x,y))} \leq R(x,y) \leq \frac{c_2 d(x,y)^\beta}{V(x,d(x,y))}, \ \forall x,y \in G.$$

THEOREM 3.5. ([**8**]) *Let (G,μ) be a weighted graph satisfying the p_0-condition and assume $(VG(\beta))$. Then,*

$$(RE(\beta)) \Leftrightarrow (PHI(\beta)) \Leftrightarrow (HK(\beta))$$

When the above conditions hold, then the Markov chain is recurrent.

A weighted graph (G,μ) is called a *tree* if the graph has no loop. If (G,μ) is a tree such that $\mu_{xy} = 1$ for all $x \sim y$, then $R(x,y) = d(x,y)$ for all $x,y \in G$. Thus, we have the following corollary to Theorem 3.5.

COROLLARY 3.6. Let (G,μ) be a tree satisfying the p_0-condition and $\mu_{xy} = 1$ for all $x \sim y$. Then,

$$(VG(\beta)) + (HK(\beta)) \Leftrightarrow [c_1 d(x,y)^{\beta-1} \leq V(x,d(x,y)) \leq c_2 d(x,y)^{\beta-1}, \ \forall x,y \in G].$$

Finally, we will discuss stability of parabolic Harnack inequalities under rough isometries.

DEFINITION 3.7. Let $(G^{(1)},\mu^{(1)})$, $(G^{(2)},\mu^{(2)})$ be weighted graphs satisfying the p_0-condition. A map $T : G^{(1)} \to G^{(2)}$ is called a rough isometry if there exist positive constants $a, c > 1, b > 0$ and $M > 0$ such that the following holds for all $x,y \in G^{(1)}$ and $y' \in G^{(2)}$.

$$a^{-1}d^{(1)}(x,y) - b \leq d^{(2)}(T(x),T(y)) \leq a d^{(1)}(x,y) + b,$$
$$d^{(2)}(T(G^{(1)}),y') \leq M, \ c^{-1}\mu_x^{(1)} \leq \mu_{T(x)}^{(2)} \leq c\mu_x^{(1)}.$$

Rough isometry is a notion introduced by M. Kanai ([**36**]).

The next theorem shows that $(PHI(\beta))$ is stable under rough isometries.

THEOREM 3.8. ([**27**]) *Let $(G^{(1)},\mu^{(1)})$, $(G^{(2)},\mu^{(2)})$ be weighted graphs satisfying the p_0-condition. If $(G^{(1)},\mu^{(1)})$ satisfies $(PHI(\beta))$ w.r.t. the graph distance and*

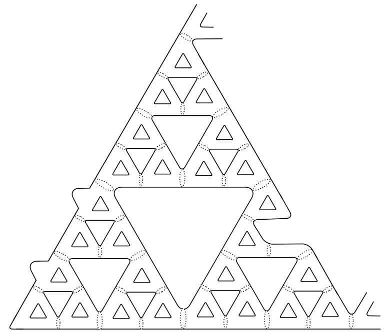

FIGURE 3. Fractal-like manifold

there exists a rough isometry between $(G^{(1)}, \mu^{(1)})$ *and* $(G^{(2)}, \mu^{(2)})$*, then* $(G^{(2)}, \mu^{(2)})$ *also satisfies* $(PHI(\beta))$ *w.r.t. the graph distance.*

Likewise the case of Brownian motion on the S.G., the simple random walk on the S.G. graph (the left of Figure 2) satisfies $(HK(\log 5/\log 2))$. The graph on the right of Figure 2 is an image of the S.G. graph by a rough isometry. So the simple random walk on the graph also satisfies $(HK(\log 5/\log 2))$, and thus satisfies $(PHI(\log 5/\log 2))$.

Note that Theorem 3.2 and Theorem 3.8 are extended to the framework of local regular Dirichlet forms on metric measure spaces (especially Riemannian manifolds) in [**2, 7**]. (In general, the time scale order which corresponds to β may change for short times and long times. In order Theorem 3.8 to hold in this framework, one needs to assume some conditions for the local structures of the spaces and the Dirichlet forms.) Figure 3 is a 2-dimensional Riemannian manifold whose global structure is like that of the S.G.. This can be constructed from the left of Figure 2 by changing each bond to the cylinder and putting projections and dents locally. The diffusion corresponding to the Dirichlet form moves on the surface of the cylinders. Using the generalization of Theorem 3.8, one can show that any divergence operator $\mathcal{L} = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on the manifold which satisfies the uniform elliptic condition enjoys $(HK(2))$ for $t \le 1 \vee d(x,y)$ and $(HK(\log 5/\log 2))$ for $t \ge 1 \vee d(x,y)$. Theorem 3.5 is also generalized to the framework called resistance forms on metric measure spaces ([**43**]).

Using the methods developed in the theory discussed above, one can obtain detailed heat kernel estimates for stochastic processes on some random media. For instance, let $\{Y_n\}$ be a simple random walk on a critical branching processes conditioned on non-extinction. It is proved that the heat kernel of $\{Y_n\}$ has on-diagonal estimates of order $n^{-2/3}$ with probability 1, and $\{Y_n\}$ has sub-diffusive behaviour with $d_w = 3$ ([**10**]). For this random walk, Kesten ([**37**]) established the scaling order $d_w = 3$ for the annealed case (i.e. to take average over the randomness of the clusters) by completely different methods. On the other hand, in [**10**], detailed heat kernel estimates are obtained both for the quenched case (i.e. to discuss for each random cluster) and the annealed case.

4. Function spaces and stochastic processes on fractals

The trace theory of Sobolev and Besov spaces on $\mathbb{R}^n$ have been studied in various directions as generalizations of the Sobolev imbedding theorem. Since 80's, these problems have been also studied for Besov-type spaces on more complicated spaces, namely on the Alfors d-regular sets defined below (see, for instance, [**35, 67**]). Recently, there are several fruitful attempts to deduce detailed properties of the function spaces by analysing properties of stochastic processes and self-adjoint operators naturally defined on these spaces. In this section, we will discuss briefly on the recent developments in this direction. Readers interested in more details may refer to surveys [**22, 42**].

Let K be a closed set on $\mathbb{R}^n$. K is called the Alfors d-regular set if there exists a Radon measure μ on K such that for each $x \in K$, the following holds,

$$c_1 r^d \le \mu\Big(B(x,r)\cap K\Big) \le c_2 r^d.$$

Let m be the Lebesgue measure, and for each $f \in \mathbb{L}^1_{loc}(\mathbb{R}^n, m)$, define its trace into K as

$$Tr_K f(x) = \lim_{r\to 0} \frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)dy, \quad \forall x \in K.$$

Note that the limit exists m-a.e.. In [**67**], Triebel defined a Besov space $B^{2,2}_{\alpha/2}(K)$ on K for each $\alpha > 0$ as $Tr_K\ H^{(\alpha+n-d)/2}(\mathbb{R}^n)$, i.e. as a trace of the Sobolev space on $\mathbb{R}^n$ into K. When K is compact, he studied the spectral properties of the self-adjoint operators naturally defined by the norm $\inf_{Tr_K g = \cdot} \|g|H^{(\alpha+n-d)/2}(\mathbb{R}^n)\|$ of the Besov space. When $0 < \alpha < 2$, this function space coincides to the following space,

$$\begin{aligned}(4.1)\quad \|u|B^{2,2}_{\alpha/2}(K)\| &= \|u\|_{\mathbb{L}^2(K,\mu)} + \Big(\int\int_{K\times K} \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\mu(dx)\mu(dy)\Big)^{1/2}\\ B^{2,2}_{\alpha/2}(K) &= \{u : u \text{ is measurable, } \|u|B^{2,2}_{\alpha/2}(K)\| < \infty\}\end{aligned}$$

and the two norms are equivalent. (When $0 < \alpha < 2$, this function space also coincides to the Besov space defined by Jonsson-Wallin ([**35**]).)

In this case, the quadratic form $(\mathcal{E}, B^{2,2}_{\alpha/2}(K))$ equipped with the norm (4.1) is a regular Dirichlet form and the corresponding stochastic process is a jump-type process ([**20**]). When $n = 2, K = \mathbb{R}^1$ and $\alpha = 1$, the corresponding process is the well known Cauchy process. By combining the probabilistic and analytic methods, it is possible to obtain detailed estimates of the spectra and heat kernels for $(\mathcal{E}, B^{2,2}_{\alpha/2}(K))$ –see [**14, 19, 20**]. For example, in [**14**] it is proved that the corresponding jump-type process enjoys the continuous heat kernel $p_t(x,y)$ which satisfies the following estimate for each $x, y \in K$,

$$c_1\Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big) \le p_t(x,y) \le c_2\Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big).$$

The parabolic Harnack inequality is also established for the process. In this way, one can obtain detailed information of the function spaces on fractals by analysing the associated jump-type processes and pseudo-differential operators.

Finally, as an application of the trace theory on fractals, we will introduce diffusion processes penetrating fractals. We will consider the following question: Let K be a medium in $\mathbb{R}^n$. Is it possible to construct a diffusion process which moves over the whole of $\mathbb{R}^n$, whose behaviour is like that of the Brownian motion on K when the particle is inside K, and like that of Brownian motion on $\mathbb{R}^n$ outside K? If so, how does such a diffusion behave? Roughly speaking, the problem is how the heat transfers in media that contain impurities. Our interest is when K is a disordered medium like a fractal. In that case, when the domain of the Dirichlet form on K determined by the Brownian motion is the Besov-Lipschitz type space, the answer of the question is 'yes' ([**44**]). Especially, when K is the S.G., one can construct the diffusion process penetrating K ([**33, 44, 55**]). Further, there is a recent work which establishes detailed short time heat kernel bounds and large deviation estimates of the process ([**28**]). Let $\mathcal{E}_K$ be the Dirichlet form on K. Then the quadratic form which may correspond to the penetrating diffusion we want is the following,

$$\tilde{\mathcal{E}}(f,f) = \mathcal{E}_K(f|_K, f|_K) + \int_{\mathbb{R}^n\setminus K} \left|\nabla(f|_{\mathbb{R}^n\setminus K})\right|^2 dx.$$

In order to confirm that there is a corresponding diffusion to this quadratic form, one has to prove the regularity of the form. In other word, the key point is to prove that there are "sufficiently enough" f which satisfies $\tilde{\mathcal{E}}(f,f) < \infty$. It is exactly this point that the trace theory of Besov spaces can be used effectively.

Acknowledgements. The author thanks anonymous referees for fruitful comments.

References

[1] D.G. Aronson, *Bounds on the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc., **73** (1967), 890–896.

[2] M.T. Barlow, *Anomalous diffusion and stability of Harnack inequalities*, Surveys in Differential Geometry IX, 1–25, Int. Press, Somerville, MA, 2004.

[3] M.T. Barlow, *Diffusions on fractals*, L.N.M. **1690**, Springer, 1998.

[4] M.T. Barlow and R.F. Bass, *Stability of parabolic Harnack inequalities*, Trans. Amer. Math. Soc., **356** (2003), 1501–1533.

[5] M.T. Barlow and R.F. Bass, *Brownian motion and harmonic analysis on Sierpiński carpets*, Canad. J. Math., **51** (1999), 673–744.

[6] M.T. Barlow and R.F. Bass, *Construction of Brownian motion on the Sierpiński carpet,* Ann. Inst. Henri Poincaré, **25** (1989), 225–257.

[7] M.T. Barlow and R.F. Bass and T. Kumagai, *Stability of parabolic Harnack inequalities on measure metric spaces*, J. Math. Soc. Japan, **58** (2006), 485–519.

[8] M. T. Barlow, T. Coulhon and T. Kumagai, *Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs*, Comm. Pure Appl. Math. **58** (2005), 1642–1677.

[9] M.T. Barlow and J. Kigami, *Localized eigenfunctions of the Laplacian on p.c.f. self-similar sets*, J. London math. Soc., **56** (1997), 320-332.

[10] M.T. Barlow and T. Kumagai, *Random walk on the incipient infinite cluster on trees*, Illinois J. Math. **50** (2006), 33–65.

[11] M.T. Barlow and E.A. Perkins, *Brownian Motion on the Sierpiński gasket*, Probab. Theory Relat. Fields, **79** (1988), 543–623.

[12] R.F. Bass, *Diffusions on the Sierpinski carpet*, Trends in probability and related analysis (Taipei, 1996), 1–34, World Sci. Publ., River Edge, NJ, 1997.

[13] G. Ben Arous and T. Kumagai, *Large deviations for Brownian motion on the Sierpinski gasket*, Stoch. Proc. Their Appl., **85** (2000), 225–235.

[14] Z.-Q. Chen and T.Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stoch. Proc. Their Appl., **108** (2003), 27–62.

[15] T. Delmotte, *Parabolic Harnack inequality and estimates of Markov chains on graphs*, Rev. Math. Iberoamericana, **15** (1999), 181–232.

[16] E.B. Fabes and D.W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rational Mech. Anal., **96** (1986), 327–338.

[17] M. Fukushima, Y. Oshima and M.Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter, Berlin, 1994.

[18] M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket*, Potential Anal., **1** (1992), 1–35.

[19] M. Fukushima and T. Uemura, *Capacity bounds of measures and ultracontractivity of time changed processes*, J. Math. Pures Appl., **82** (2003), 553–572.

[20] M. Fukushima and T. Uemura, *On Sobolev and capacitary inequalities for contractive Besov spaces over d-sets*, Potential Anal., **18** (2003), 59–77.

[21] S. Goldstein, *Random walks and diffusions on fractals*, Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), 121–129, IMA Vol. Math. Appl. **8**, Springer, New York, 1987.

[22] A. Grigor'yan, *Heat kernels and function theory on metric measure spaces,* Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 143–172, Contemp. Math. **338**, A.M.S., Providence, RI, 2003.

[23] A. Grigor'yan, *The heat equation on non-compact Riemannian manifolds*, (in Russian) Matem. Sbornik., **182** (1991), 55–87. (English transl.) Math. USSR Sb., **72** (1992), 47–77.

[24] A. Grigor'yan and A. Telcs, *Harnack inequalities and sub-Gaussian estimates for random walks*, Math. Annalen, **324** (2002), 521–556.

[25] A. Grigor'yan and A. Telcs, *Sub-Gaussian estimates of heat kernels on infinite graphs*, Duke Math. J., **109** (2001), 451–510.

[26] B.M. Hambly, *Heat kernels and spectral asymptotics for some random Sierpinski gaskets*, Fractal geometry and stochastics II (Greifswald/Koserow, 1998), 239–267, Progr. Probab. **46**, Birkhäuser, Basel, 2000.

[27] B.M. Hambly and T. Kumagai, *Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries*, In: Fractal geometry and applications: A Jubilee of B. Mandelbrot (M.L. Lapidus and M. van Frankenhuijsen (eds.)), Proc. of Sympos. Pure Math. **72**, Part 2, 233–260, Amer. Math. Soc. 2004.

[28] B.M. Hambly and T. Kumagai, *Diffusion processes on fractal fields and their large deviations*, Probab. Theory Relat. Fields, **127** (2003), 305–352.

[29] B.M. Hambly and T. Kumagai, *Transition density estimates for diffusion processes on post critically finite self-similar fractals*, Proc. London Math. Soc., **78** (1999), 431–458.

[30] K. Hattori, T. Hattori and H. Watanabe, *Gaussian field theories on general networks and the spectral dimensions*, Progr. Theoret. Phys. Suppl., **92** (1987), 108–143.

[31] S. Havlin and D. Ben-Avraham, *Diffusion in disordered media*, Adv. Phys., **36** (1987), 695–798.

[32] M. Hino, *On the singularity of energy measures on self-similar sets*, Probab. Theory Related Fields, **132** (2005), 265–290.

[33] A. Jonsson, *Dirichlet forms and Brownian motion penetrating fractals*, Potential Anal., **13** (2000), 69–80.

[34] A. Jonsson, *Brownian motion on fractals and function spaces*, Math. Z., **222** (1996), 496–504.

[35] A. Jonsson and H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$, Mathematical Reports, Vol. **2**, Part 1 (1984), Acad. Publ., Harwood.

[36] M. Kanai, *Rough isometries and combinatorial approximations of geometries of non-compact riemannian manifolds*, J. Math. Soc. Japan, **37** (1985), 391–413.

[37] H. Kesten, *Subdiffusive behavior of random walk on a random cluster*, Ann. Inst. Henri Poincaré, **22** (1986), 425–487.

[38] J. Kigami, *Analysis on Fractals*, Cambridge Univ. Press, 2001.

[39] J. Kigami, *Harmonic calculus on p.c.f. self-similar sets,* Trans. Amer. Math. Soc., **335** (1993), 721–755.

[40] J. Kigami, *Laplacians on self-similar sets –Analysis on fractals,* (in Japanese) Sugaku **44** (1992), 13–28. (English transl.) Amer. Math. Soc. Transl. Ser 2, **161** (1994), 75–93.

[41] J. Kigami, *A harmonic calculus on the Sierpiński space*, Japan J. Appl. Math., **6** (1989), 259–290.

[42] T. Kumagai, *Function spaces and stochastic processes on fractals,* Fractal geometry and stochastics III (Friedrichroda, 2003), Progr. Probab., **57**, Birkhäuser, Basel, 2004, 221–234.

[43] T. Kumagai, *Heat kernel estimates and parabolic Harnack inequalities on graphs and resistance forms*, Publ. RIMS, Kyoto Univ., **40** (2004), 793–818.

[44] T. Kumagai, *Brownian motion penetrating fractals -An application of the trace theorem of Besov spaces-*, J. Funct. Anal., **170** (2000), 69–92.

[45] T. Kumagai, *Stochastic processes on fractals and related topics,* (in Japanese) Sugaku **49** (1997), 158–172. (English transl.) Sugaku Expositions, Amer. Math. Soc., **13** (2000), 55–71.

[46] T. Kumagai, *Short time asymptotic behavior and large deviations for Brownian motion on some affine nested fractals*, Publ. RIMS. Kyoto Univ., **33** (1997), 223–240.

[47] T. Kumagai and S. Kusuoka, *Homogenization on nested fractals,* Probab. Theory Relat. Fields, **104** (1996), 375–398.

[48] S. Kusuoka, *Diffusion processes on nested fractals*, L.N.M. **1567**, Springer, 1993.

[49] S. Kusuoka, *Dirichlet forms on fractals and products of random matrices*, Publ. RIMS. Kyoto Univ., **25** (1989), 659–680.

[50] S. Kusuoka, *A diffusion process on a fractal*, Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 251–274, Academic Press, Boston, MA, 1987.

[51] S. Kusuoka and D.W. Stroock, *Applications of the Malliavin calculus. III*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **34** (1987), 391–442.

[52] S. Kusuoka and X.Y. Zhou, *Waves on fractal-like manifolds and effective energy propagation*, Probab. Theory Relat. Fields, **110** (1998), 473–495.

[53] M.L. Lapidus, *Analysis on fractals, Laplacians on self-similar sets, noncommutative geometry and spectral dimensions*, Topological Methods in Nonlinear Analysis, **4** (1994), 137–195.

[54] P. Li and S.T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math., **156** (1986), 153–201.

[55] T. Lindstrøm, *Brownian motion penetrating the Sierpiński gasket*, In: Asymptotic problems in probability theory: stochastic models and diffusions on fractals, Sanda/Kyoto, 1990, Pitman Res. Notes Math. Ser. **283**, Longman Sci. Tech., Harlow, 1993, 248–278.

[56] T. Lindstrøm, *Brownian motion on nested fractals*, Mem. Amer. Math. Soc., **83**, no. 420 (1990).

[57] V. Metz, *Renormalization contracts on nested fractals*, J. Reine Angew. Math., **480** (1996), 161–175.

[58] J. Moser, *On Harnack's inequality for parabolic differential equations*, Comm. Pure Appl. Math., **17** (1964), 101–134.

[59] J. Moser, *On Harnack's inequality for elliptic differential equations*, Comm. Pure Appl. Math., **14** (1961), 577–591.

[60] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. Math. J., **80** (1958), 931–954.

[61] H. Osada, *A family of diffusion processes on Sierpinski carpets*, Probab. Theory Relat. Fields, **119** (2001), 275–310.

[62] C. Sabot, *Existence and uniqueness of diffusions on finitely ramified self-similar fractals,* Ann. Sci. École Norm. Sup. (4), **30** (1997), 605–673.

[63] L. Saloff-Coste, *A note on Poincaré, Sobolev and Harnack inequality*, Internat. Math. Res. Notices, **2** (1992), 27–38.

[64] R.S. Strichartz, *Analysis on fractals,* Notices Amer. Math. Soc., **46** (1999), 1199–1208.

[65] K.T. Sturm, *Analysis on local Dirichlet spaces -III. the parabolic Harnack inequality*, J. Math. Pure Appl., **75** (1996), 273–297.

[66] A. Teplyaev, *Spectral analysis on infinite Sierpiński gaskets*, J. Funct. Anal., **159** (1998), 537–567.

[67] H. Triebel, *Fractals and spectra -related to Fourier analysis and function spaces-,* Monographs in Math., Vol. **91**, Birkhäuser, 1997.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: kumagai@math.kyoto-u.ac.jp