LAWS OF THE ITERATED LOGARITHM FOR SYMMETRIC JUMP PROCESSES

PANKI KIM   TAKASHI KUMAGAI   JIAN WANG

Abstract. Based on two-sided heat kernel estimates for a class of symmetric jump processes on metric measure spaces, the laws of the iterated logarithm (LILs) for sample paths, local times and ranges are established. In particular, the LILs are obtained for β-stable-like processes on α-sets with $\beta > 0$.
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1. Introduction and Setting

The law of the iterated logarithm (LIL) describes the magnitude of the fluctuations of stochastic processes. The original statement of LIL for a random walk is due to Khinchin in [27]. In this paper we discuss various types of the LILs for a large class of symmetric jump processes.

We first recall some known results on LILs of stable processes, which are related to the topics of our paper. Let $X := (X_t)_{t\geqslant 0}$ be a strictly β-stable process on $\mathbb{R}$ in the sense of Sato [36, Definition 13.1] with $0 < \beta < 2$ and $\nu((0,\infty)) > 0$ for the Lévy measure ν of X. Then the following facts are well-known (see [36, Propositions 47.16 and 47.21]).

Proposition 1.1. (1) *Let h be a positive continuous and increasing function on $(0,\delta]$ for some $\delta > 0$. Then*

$$\limsup_{t\to 0} \frac{|X_t|}{h(t)} = 0 \quad a.s. \text{ or } \quad = \infty \quad a.s.$$

according to $\int_0^\delta h(t)^{-\beta}dt < \infty$ or $= \infty$, respectively.
(2) *Assume that X is not a subordinator. Then there exists a constant $c \in (0,\infty)$ such that*

$$\liminf_{t\to 0} \frac{\sup_{0<s\leqslant t} |X_s|}{(t/\log|\log t|)^{1/\beta}} = c \quad a.s..$$

Proposition 1.1(1) was obtained by Khinchin in [28]. A multidimensional version of Proposition 1.1(2) was first proved by Taylor in [39], and then a refined version of Proposition 1.1(2) for (non-symmetric) Lévy processes was established by Wee in [40]. We refer the reader to [1, 10, 11, 37] and the references therein. Recently the

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results in Proposition 1.1 have been extended to some class of Feller processes (see [29] and the references therein).

When $\beta > 1$, a local time of X exists, and various LILs for the local time are known. In the next result we concentrate on a symmetric β-stable process X on $\mathbb{R}$.

Proposition 1.2. *Assume $\beta \in (1,2)$. Then, there exist a local time $\{l(x,t) : x \in \mathbb{R}, t > 0\}$ and constants $c_1, c_2 \in (0,\infty)$ such that*

$$\limsup_{t\to\infty} \frac{\sup_y l(y,t)}{t^{1-1/\beta}(\log\log t)^{1/\beta}} = c_1 \quad a.s. \tag{1.1}$$

and

$$\liminf_{t\to\infty} \frac{\sup_y l(y,t)}{t^{1-1/\beta}(\log\log t)^{-1+1/\beta}} = c_2 \quad a.s.. \tag{1.2}$$

In [23] Griffin showed that (1.2) holds, and in [41] Wee has extended (1.2) to a large class of Lévy processes. As applications of the large deviation method, (1.1) was proved by Donsker and Varadhan in [17]. For the case of diffusions, LILs for the local time have further considered on metric measure spaces including fractals based on the large deviation technique (see [20, 8]); however, the corresponding work for (non-Lévy) jump processes is still not available. It would be very interesting to see to what extent the above results for Lévy processes are still true for general jump processes, e.g. see [42, p. 306]. Thus, we are concerned with the following;

Question 1.1. *If the generator of the process X is perturbed so that the corresponding process with new generator is no longer a Lévy process, do the results in Propositions* 1.1 *and* 1.2 *still hold?*

In this paper, we consider this problem for a large class of symmetric Markov jump processes on metric measure spaces via heat kernel estimates.

In order to explain our results explicitly, let us first give the framework. Let (M,d) be a locally compact, separable and connected metric space, and let μ be a Radon measure on M with full support. We assume that $B(x,r)$ is relatively compact for all $x \in M$ and $r > 0$. Let $(\mathscr{E},\mathscr{F})$ be a symmetric regular Dirichlet form on $L^2(M,\mu)$. By the Beurling-Deny formula, such form can be decomposed into three terms — the strongly local term, the pure-jump term and the killing term (see [19, Theorem 4.5.2]). Throughout this paper, we consider the form that consists of the pure-jump term only; namely there exists a symmetric Radon measure $n(\cdot,\cdot)$ on $M\times M\setminus \mathrm{diag}$, where diag denotes the diagonal set $\{(x,x) : x \in M\}$, such that

$$\mathscr{E}(u,v) = \int_{M\times M\setminus \mathrm{diag}} (u(x)-u(y))(v(x)-v(y))\, n(dx,dy) \tag{1.3}$$

for all $u,v \in \mathscr{F}\cap C_c(M)$. We denote the associated Hunt process by $X = (X_t, t \geqslant 0; \mathbf{P}^x, x \in M; \mathscr{F}_t, t \geqslant 0)$. Then there is a properly exceptional set $\mathscr{N} \subset M$ such that the associated Hunt process is uniquely determined up to any starting point outside $\mathscr{N}$. Let $(P_t)_{t\geqslant 0}$ be the semigroup corresponding to $(\mathscr{E},\mathscr{F})$, and set $\mathbb{R}_+ = (0,\infty)$. A heat kernel (a transition density) of X is a non-negative symmetric measurable function $p(t,x,y)$ defined on $\mathbb{R}_+\times M\times M$ such that

$$P_t f(x) = \int_M p(t,x,z)f(z)\,\mu(dz), \quad p(t+s,x,y) = \int_M p(t,x,z)p(s,z,y)\,\mu(dz),$$

for any Borel function f on M, for all $s,t>0$, all $x \in M \setminus \mathscr{N}$ and μ-almost all $y \in M$.

We will use ":=" to denote a definition, which is read as "is defined to be". For $a,b \in \mathbb{R}$, $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. The following is our main theorem for the case of β-stable like processes on α-sets.

Theorem 1.3. [**β-stable-like processes on α-sets**] *Let (M,d,μ) be as above. Consider a symmetric regular Dirichlet form $(\mathscr{E},\mathscr{F})$ on $L^2(M,\mu)$ that has the transition density function $p(t,x,y)$. We assume μ and $p(t,x,y)$ satisfy that*

(i) *there is a constant $\alpha>0$ such that*

$$c_1 r^\alpha \leqslant \mu(B(x,r)) \leqslant c_2 r^\alpha, \quad x \in M, r>0, \tag{1.4}$$

(ii) *there also exists a constant $\beta>0$ such that for all $x,y \in M$ and $t>0$,*

$$c_3\left(t^{-\alpha/\beta} \wedge \frac{t}{d(x,y)^{\alpha+\beta}}\right) \leqslant p(t,x,y) \leqslant c_4\left(t^{-\alpha/\beta} \wedge \frac{t}{d(x,y)^{\alpha+\beta}}\right). \tag{1.5}$$

Then, we have the following statements.

(1) *If φ is a strictly increasing function on $(0,1)$ satisfying*

$$\int_0^1 \frac{1}{\varphi(s)^\beta}\, ds < \infty \quad (\text{resp.} = \infty), \tag{1.6}$$

then

$$\limsup_{t \to 0} \frac{\sup_{0<s\leqslant t} d(X_s,x)}{\varphi(t)} = 0 \quad (\text{resp.} = \infty), \qquad \mathbf{P}^x\text{-a.e. } \omega,\ \forall x \in M. \tag{1.7}$$

Similarly, if φ is defined on $(1,\infty)$ and the integral in (1.6) is over $[1,\infty)$, then (1.7) holds for $t \to \infty$ instead of $t \to 0$.

(2) *There exist constants $c_5, c_6 \in (0,\infty)$ such that for all $x \in M$ and $\mathbf{P}^x$-a.e.,*

$$\liminf_{t \to 0} \frac{\sup_{0<s\leqslant t} d(X_s,x)}{(t/\log|\log t|)^{1/\beta}} = c_5, \quad \liminf_{t \to \infty} \frac{\sup_{0<s\leqslant t} d(X_s,x)}{(t/\log\log t)^{1/\beta}} = c_6.$$

(3) *Assume $\alpha<\beta$. Then, there exist a local time $\{l(x,t) : x \in M, t>0\}$ and constants $c_7,c_8,c_9,c_{10} \in (0,\infty)$ such that for all $x \in M$ and $\mathbf{P}^x$-a.e.,*

$$\limsup_{t \to \infty} \frac{\sup_y l(y,t)}{t^{1-\alpha/\beta}(\log\log t)^{\alpha/\beta}} = c_7, \quad \liminf_{t \to \infty} \frac{\sup_y l(y,t)}{t^{1-\alpha/\beta}(\log\log t)^{-1+\alpha/\beta}} = c_8,$$

$$\limsup_{t \to \infty} \frac{R(t)}{t^{\alpha/\beta}(\log\log t)^{1-\alpha/\beta}} = c_9, \quad \liminf_{t \to \infty} \frac{R(t)}{t^{\alpha/\beta}(\log\log t)^{-\alpha/\beta}} = c_{10},$$

where $R(t) := \mu(X([0,t]))$ is the range of the process X.

Note that in [13], (1.5) is proved for stable-like processes, that is

$$\mathscr{E}(u,v) = \int_{M\times M\setminus\{x=y\}} (\widetilde{u}(x)-\widetilde{u}(y))(\widetilde{v}(x)-\widetilde{v}(y))\, n(dx,dy), \quad \forall u,v \in \mathscr{F}, \tag{1.8}$$

where $\widetilde{u}$ is a quasi-continuous version of $u \in \mathscr{F}$, and the Lévy measure $n(\cdot,\cdot)$ satisfies

$$c_1' \frac{\mu(dx)\mu(dy)}{d(x,y)^{\alpha+\beta}} \leqslant n(dx,dy) \leqslant c_2' \frac{\mu(dx)\mu(dy)}{d(x,y)^{\alpha+\beta}},$$

for $\beta \in (0,2)$. β-stable-like processes are perturbations of β-stable processes, and clearly they are no longer Lévy processes in general. Stable-like processes are analogues of uniformly elliptic divergence forms in the framework of jump processes. – We emphasize here that, in Theorem 1.3 above, we do not assume $\beta < 2$ in general (see Example 5.3). Indeed, in this paper we will consider more general jump processes that include jump processes of mixed types on metric measure spaces, which are given in Section 5.

For the case of diffusions that enjoy the so-called sub-Gaussian heat kernel estimates, LILs corresponding to Theorem 1.3 have been established in [8, 20]. However, since the proof uses Donsker-Varadhan's large deviation theory for Markov processes, some self-similarity of the process is assumed in these papers (see [8, (4.4)] and [20, (1.7)]). In the present paper, we will not assume such a self-similarity on the process X. Instead we consider a family of scaling processes and take a (somewhat classical) "bare-hands" approach.

The remainder of the paper is organized as follows. In Section 2, we give the assumptions on estimates of heat kernels we will use, and present their consequences. In Section 3, we establish LILs for sample paths. Section 4 is devoted to the LILs of maximums of local times and ranges of processes. The LILs for jump processes of mixed types on metric measure spaces are given in Section 5 to illustrate the power of our results. Some of the proofs and technical lemmas are left in Appendix A.

Throughout this paper, we will use c, with or without subscripts and superscripts, to denote strictly positive finite constants whose values are insignificant and may change from line to line. We write $f \asymp g$ if there exist constants $c_1, c_2 > 0$ such that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$ for all x.

2. Heat Kernel Estimates and Their Consequences

Let (M, d) be a locally compact, separable and connected metric space, and let μ be a Radon measure on M with full support such that for any $x \in M$ and $r > 0$,

$$C_*^{-1} V(r) \leqslant \mu(B(x,r)) \leqslant C_* V(r), \tag{2.1}$$

where $C_* \geqslant 1$ and $V : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function satisfying that there exists a constants $c > 1$ so that

$$V(0) = 0, \quad V(\infty) = \infty \quad \text{and} \quad V(2r) \leqslant cV(r) \quad \text{for every } r > 0. \tag{2.2}$$

Note that (2.2) is equivalent to the following: there exist constants $c, d > 0$ such that

$$V(0) = 0, \quad V(\infty) = \infty \quad \text{and} \quad \frac{V(R)}{V(r)} \leqslant c\left(\frac{R}{r}\right)^d \quad \text{for all } 0 < r < R. \tag{2.3}$$

Let $(\mathscr{E}, \mathscr{F})$ be a symmetric regular Dirichlet form on $L^2(M, \mu)$. In this paper we will consider the following type of estimates for heat kernels: there exists a properly exceptional set $\mathscr{N}$ and, for given $T \in (0, \infty]$, there exist positive constants C_1 and C_2 such that for all $x \in M \setminus \mathscr{N}$, μ-almost all $y \in M$ and $t \in (0, T)$,

$$p(t,x,y) \leqslant C_1 \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(d(x,y))\phi(d(x,y))} \right), \tag{2.4}$$

$$C_2 \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(d(x,y))\phi(d(x,y))} \right) \leqslant p(t,x,y), \tag{2.5}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function.

We now state the first set of our assumptions on heat kernels.

Assumption 2.1. There exists a transition density $p(t,x,y) : \mathbb{R}_+ \times M \times M \to [0,\infty]$ of the semigroup of $(\mathscr{E},\mathscr{F})$ satisfying (2.4) and (2.5) with $T = \infty$, and (2.2).

Assumption 2.2. $\phi(0) = 0$, and there exist constants $c_0 \in (0,1)$ and $\theta > 1$ such that for every $r > 0$

$$\phi(r) \leqslant c_0 \phi(\theta r). \tag{2.6}$$

It is easy to see that under (2.6), $\lim_{r\to\infty} \phi(r) = \infty$, and there exist constants $c_0, d_0 > 0$ such that

$$c_0 \Big(\frac{R}{r}\Big)^{d_0} \leqslant \frac{\phi(R)}{\phi(r)} \quad \text{for all } 0 < r < R,$$

e.g. the proof of [24, Proposition 5.1].

In this section, we assume the above heat kernel estimates and discuss the consequences. Sometimes we only consider two-sided estimates on the heat kernel for short time. We say that Assumption 2.1 holds with $T < \infty$, if there exists a transition density $p(t,x,y) : \mathbb{R}_+ \times M \times M \to [0,\infty]$ of the semigroup of $(\mathscr{E},\mathscr{F})$ satisfying (2.4) and (2.5) with $T < \infty$, and (2.2). We emphasize that the constants appearing in the statements of this section only depend on heat kernel estimates (2.4) and (2.5).

Before we go on, let us note that (2.4) and (2.5) can be proved in a rather wide framework.

Theorem 2.3. ([14, Theorem 1.2]) *Let (M,d,μ) be a metric measure space given above with $\mu(M) = \infty$. We assume that $\mu(B(x,r)) \asymp V(r)$ for all $x \in M$ and $r > 0$ where V satisfies* (2.8) *below. We also assume that there exist $x_0 \in M$, $\kappa \in (0,1]$ and an increasing sequence $r_n \to \infty$ as $n \to \infty$ so that for every $n \geqslant 1$, $0 < r < 1$ and $x \in \overline{B(x_0,r_n)}$, there is some ball $B(y,\kappa r) \subset B(x,r) \cap \overline{B(x_0,r_n)}$. Let $(\mathscr{E},\mathscr{F})$ be a symmetric regular Dirichlet form on $L^2(M,\mu)$ such that $\mathscr{E}$ is given by* (1.8) *and the Lévy measure $n(\cdot,\cdot)$ satisfies*

$$c_1 \frac{\mu(dx)\mu(dy)}{V(d(x,y))\phi(d(x,y))} \leqslant n(dx,dy) \leqslant c_2 \frac{\mu(dx)\mu(dy)}{V(d(x,y))\phi(d(x,y))}. \tag{2.7}$$

Assume further that ϕ satisfies (2.10) *below and that $\int_0^r (s/\phi(s))ds \leqslant c_3 r^2/\phi(r)$ for all $r > 0$. Then there exists a jointly continuous heat kernel $p(t,x,y)$ that enjoys the estimates* (2.4) *and* (2.5) *with $T = \infty$.*

Remark 2.4. In [14, Theorem 1.2], an additional assumption was made on the space (M,d) such that it enjoys some scaling property (see [14, p. 282]). However, such assumption can be removed by introducing a family of scaled distances as in (4.17) below instead of assuming the existence of a family of scaled spaces, and by discussing similarly to the proof of Proposition 4.8 below.

2.1. **General case.** In this subsection, we state consequences of Assumptions 2.1 and 2.2. The proofs of next two propositions are given in Appendix A.1. We note that Proposition 2.5 and its proof are due to [15].

Proposition 2.5. *If $p(t,x,y)$ satisfies* (2.5) *with $T=\infty$ (in particular, if Assumption* 2.1 *is satisfied), then the process X is conservative, i.e. for any $x\in M\setminus\mathscr{N}$ and $t>0$,*

$$\int p(t,x,y)\,\mu(dy)=1.$$

Proposition 2.6. *Let $p(t,x,y)$ satisfy Assumptions* 2.1 *and* 2.2 *above. Then, we have Diam $(M)=\infty$ and $\mu(M)=\infty$. Moreover, there exist constants $c_1,c_2>0$, $d_2\geqslant d_1>0$ such that*

$$c_1\Big(\frac{R}{r}\Big)^{d_1}\leqslant\frac{V(R)}{V(r)}\leqslant c_2\Big(\frac{R}{r}\Big)^{d_2}\quad\text{for every}\ \ 0<r<R<\infty. \tag{2.8}$$

Proposition 2.7. *Assume that the regular Dirichlet form $(\mathscr{E},\mathscr{F})$ given by* (1.3) *enjoys the heat kernel $p(t,x,y)$ such that Assumption* 2.1 *is satisfied. Then, the jump measure $n(dx,dy)$ satisfies* (2.7)*.*

For the assertion of $n(dx,dy)$, using the heat kernel estimates, we can follow the proof of [6, Theorem 1.2, (a)⇒ (c)].

2.2. **The case that ϕ satisfies the doubling property.** Throughout this subsection, we assume that ϕ satisfies the doubling property.

Assumption 2.8. There is a constant $c>1$ so that

$$\phi(2r)\leqslant c\phi(r)\qquad\text{for every } r>0. \tag{2.9}$$

Note that, (2.9) implies that for any $\theta>1$ there exists $c_0=c_0(\theta)>1$ such that for every $r>0$, $\phi(\theta r)\leqslant c_0\phi(r)$. If Assumptions 2.2 and 2.8 are satisfied, then it is easy to see (also see the proof of [24, Proposition 5.1]) that ϕ satisfies the following inequality

$$c_3\Big(\frac{R}{r}\Big)^{d_3}\leqslant\frac{\phi(R)}{\phi(r)}\leqslant c_4\Big(\frac{R}{r}\Big)^{d_4} \tag{2.10}$$

for all $0<r\leqslant R$ and some positive constants $c_i,d_i (i=3,4)$.

In this subsection, we state consequences of Assumptions 2.1, 2.2 and 2.8. The proofs of Propositions 2.9, 2.11 and 2.12 in this subsection are also given in Appendix A.1.

We first prove the Hölder estimates for $p(t,x,y)$. As a result, under Assumptions 2.1, 2.2 and 2.8, even in the case that Assumption 2.1 holds with $T<\infty$ and that the process X is conservative, the property exceptional set $\mathscr{N}$ can be taken to be the empty set, and so (2.4) and (2.5) hold for all $x,y\in M$ and $t>0$. We will frequently use this fact without explicitly mentioning it.

Proposition 2.9. *Suppose Assumptions* 2.1, 2.2 *and* 2.8 *hold. Then there exist constants $\theta\in(0,1]$ and $c>0$ such that for all $t\geqslant s>0$ and $x_i,y_i\in M$ with $i=1,2$*

$$\begin{aligned}&|p(t,x_1,y_1)-p(s,x_2,y_2)|\\ &\leqslant\frac{c}{V(\phi^{-1}(s))\phi^{-1}(s)^\theta}\left(\phi^{-1}(t-s)+d(x_1,x_2)+d(y_1,y_2)\right)^\theta.\end{aligned} \tag{2.11}$$

In particular, for all $t>0$ and $x_i,y_i\in M$ with $i=1,2$

$$|p(t,x_1,y_1)-p(t,x_2,y_2)|\leqslant\frac{c}{V(\phi^{-1}(t))}\left(\frac{d(x_1,x_2)+d(y_1,y_2)}{\phi^{-1}(t)}\right)^\theta. \tag{2.12}$$

Furthermore, (2.11) *and* (2.12) *still hold true for any* $0 < s < t \leqslant T$*, if Assumptions* 2.2 *and* 2.8 *are satisfied, Assumption* 2.1 *only holds with* $T < \infty$ *and the process* X *is conservative.*

Using Proposition 2.9, we can get

Theorem 2.10 (Zero-One Law for Tail Events). *Let* $p(t,x,y)$ *satisfy Assumptions* 2.1*,* 2.2 *and* 2.8 *above, and let* A *be a tail event. Then, either* $\mathbf{P}^x(A)$ *is* 0 *for all* x *or else it is* 1 *for all* $x \in M$*.*

For an open set D, we define

$$p^D(t,x,y) := p(t,x,y) - \mathbf{E}^x\big(\, p(t-\tau_D, X_{\tau_D}, y) : \tau_D < t\big), \quad t > 0, x, y \in D \tag{2.13}$$

where $\tau_D := \inf\{s > 0 : X_s \notin D\}$. Using the strong Markov property of X, it is easy to verify that $p^D(t,x,y)$ is the transition density for X^D, the subprocess of X killed upon leaving an open set D. $p^D(t,x,y)$ is also called the Dirichlet heat kernel of the process X killed on exiting D. The following two statements present a lower bound for the near diagonal estimate of Dirichlet heart kernels and detailed controls of the distribution of the maximal process.

Proposition 2.11. *If Assumptions* 2.1*,* 2.2 *and* 2.8 *hold, then there exist constants* $\delta_0, c_0 > 0$ *such that for any* $x \in M$ *and* $r > 0$*,*

$$p^{B(x,r)}(\delta_0\phi(r), x', y') \geqslant c_0 V(r)^{-1}, \quad x', y' \in B(x, r/2). \tag{2.14}$$

Furthermore, if Assumptions 2.2 *and* 2.8 *are satisfied, Assumption* 2.1 *only holds for* $T < \infty$ *and the process* X *is conservative, then* (2.14) *holds for all* $x \in M$ *and* $r \geqslant 0$ *with* $\delta_0\phi(r) \in (0, T)$*.*

Proposition 2.12. *If Assumptions* 2.1*,* 2.2 *and* 2.8 *hold, then there exist some constants* $c_0 > 0$ *and* $a_1^*, a_2^* \in (0,1)$ *such that for all* $x \in M$*,* $r > 0$ *and* $n \geqslant 1$*,*

$$a_1^{*n} \leqslant \mathbf{P}^x\big(\sup_{0 \leqslant s \leqslant c_0 n\phi(r)} d(X_s, x) \leqslant r\big) \leqslant a_2^{*n}. \tag{2.15}$$

Furthermore, if Assumptions 2.2 *and* 2.8 *are satisfied, Assumption* 2.1 *only holds for* $T < \infty$ *and the process* X *is conservative, then* (2.15) *holds for all* $x \in M$*,* $n \geqslant 1$ *and* $r > 0$ *with* $c_0 n\phi(r) \leqslant T$*.*

Let us introduce a space-time process $Z_s = (V_s, X_s)$, where $V_s = V_0 + s$. The law of the space-time process $s \mapsto Z_s$ starting from (t,x) will be denoted by $\mathbf{P}^{(t,x)}$. For any $r, t, \delta > 0$ and $x \in M$, we define

$$Q_\delta(t,x,r) = [t, t + \delta\phi(r)] \times B(x,r).$$

We say that a non-negative Borel measurable function $h(t,x)$ on $[0,\infty) \times M$ is parabolic in a relatively open subset D of $[0,\infty) \times M$, if for every relatively compact open subset $D_1 \subset D$, $h(t,x) = \mathbf{E}^{(t,x)} h(Z_{\hat\tau_{D_1}})$ for every $(t,x) \in D_1$, where $\hat\tau_{D_1} = \inf\{s > 0 : Z_s \notin D_1\}$.

We now state the following parabolic Harnack inequality.

Proposition 2.13. *Assume that Assumptions* 2.1*,* 2.2 *and* 2.8 *hold. For every* $0 < \delta < 1$*, there exists* $c_1 > 0$ *such that for every* $z \in M$*,* $R > 0$ *and every non-negative function* h *on* $[0,\infty) \times M$*, that is parabolic on* $[0, 3\delta\phi(R)] \times B(z, 2R)$*,*

$$\sup_{(t,y) \in Q_\delta(\delta\phi(R), z, R)} h(t,y) \leqslant c_1 \inf_{y \in B(z,R)} h(0,y).$$

By Assumptions 2.1, 2.2 and 2.8 and Proposition 2.7, the density $J(x,y)$ of the jump measure $n(dx,dy)$ satisfies the following upper jump smoothness (**UJS**): there exists a constant $c_1>0$ such that for μ-a.e. $x,y\in M$,

$$J(x,y)\leqslant \frac{c_1}{V(r)}\int_{B(x,r)} J(z,y)\,\mu(dz)\qquad \text{whenever } r\leqslant \tfrac{1}{2}d(x,y).$$

Noting that $J(x,y)=\lim_{r\to 0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} J(z,y)\,\mu(dz)$ for μ-a.e. $x,y\in M$, (**UJS**) is a kind of smooth assumption on the upper bound of jump kernel $J(x,y)$. Let c be the constant in Assumption 2.8, and $c_0\in(0,1)$ be the constant such that for almost all $x\in M$ and $r>0$,

$$\mathbf{P}^x(\tau_{B(x,r/2)}\leqslant c_0\phi(r))\leqslant 1/2, \tag{2.16}$$

see e.g. (3.4) below. Since the density $J(x,y)$ of the jump measure $n(dx,dy)$ satisfies (**UJS**), Proposition 2.13 can be proved by following the arguments of [14, Theorem 4.12] and [12, Theorem 5.2]. See [14, Appendix B] and [12, Section 5] for more details. In fact, as explained in the first paragraph of [12, Theorem 5.2] one can first consider the case that h is non-negative and bounded on $[0,\infty)\times F$ and establish the result for $\delta\leqslant c_0/c$. Once this is done, one can extend it to all $\delta<1$ and any non-negative parabolic function (not necessarily bounded) by a simple chaining argument and the argument in the step 3 of the proof of [12, Theorem 5.2], respectively.

3. Laws of the Iterated Logarithm for Sample Paths

In this section, we discuss LILs for sample paths of the process X. Instead of assuming full heat kernel estimates as in Assumption 2.1, we give the estimates that are needed in each statement. Throughout this paper (except Proposition A.4 below), we will always assume that the reference measure μ satisfies the uniform volume doubling property in (2.1) and that V is a strictly increasing function that satisfies (2.2).

3.1. **Upper bound for limsup behavior.** In this subsection we assume that the heat kernel $p(t,x,y)$ on (M,d,μ) satisfies the following upper bound estimate for all $x\in M\setminus\mathscr{N}$, μ-almost all $y\in M$ and all $t\in(a,b)$ with $a<b$,

$$p(t,x,y)\leqslant \frac{C\,t}{V(d(x,y))\phi(d(x,y))}, \tag{3.1}$$

where $C>0$, and $\phi:\mathbb{R}_+\to\mathbb{R}_+$ is a strictly increasing functions satisfying (2.10).

Theorem 3.1. *Assume that the process X is conservative. Then the following statements hold.*

(1) *If $a=0$ and φ is an increasing function on $(0,1)$ such that*

$$\int_0^1 \frac{1}{\phi\big(\varphi(t)\big)}\,dt<\infty, \tag{3.2}$$

then

$$\limsup_{t\to 0}\frac{\sup_{0\leqslant s\leqslant t} d(X_s,x)}{\varphi(t)}=0,\qquad \mathbf{P}^x\text{-}a.e.\ \omega,\ \forall x\in M\setminus\mathscr{N}.$$

(2) *If $b=\infty$ and φ is an increasing function on on $(1,\infty)$ such that*

$$\int_1^\infty \frac{1}{\phi\big(\varphi(t)\big)}\,dt<\infty,$$

then

$$\limsup_{t\to\infty}\frac{\sup_{0\leqslant s\leqslant t} d(X_s,x)}{\varphi(t)}=0,\qquad \mathbf{P}^x\text{-a.e. } \omega,\ \forall x\in M\setminus\mathscr{N}.$$

Proof. We only prove (1), since (2) can be verified similarly. Let us first check that there is a constant $c_1>0$ such that for all $x\in M\setminus\mathscr{N}$, $r>0$ and $t\in(0,b)$,

$$\int_{B(x,r)^c} p(t,x,z)\,\mu(dz)\leqslant \frac{c_1 t}{\phi(r)}. \tag{3.3}$$

If $t\geqslant\phi(r)$, then the right hand side of (3.3) is greater than 1 by taking $c_1>1$, so we may assume that $t\leqslant\phi(r)$. Without loss of generality, we also assume that $b=1$. It follows from (3.1) and the increasing property of V that, for all $x\in M\setminus\mathscr{N}$, μ-almost all $z\in M$ with $d(x,z)\geqslant s$ and each $t\in(0,1)$,

$$p(t,x,z)\leqslant\frac{Ct}{V(s)\phi(s)}.$$

This upper bound, along with the uniform volume doubling property of μ (e.g. (2.1) and (2.3)) and (2.10), yields that

$$\begin{aligned}\int_{B(x,r)^c} p(t,x,z)\,\mu(dz)&\leqslant\sum_{k=0}^\infty\int_{B(x,\theta^{k+1}r)\setminus B(x,\theta^k r)} p(t,x,z)\,\mu(dz)\\&\leqslant\sum_{k=0}^\infty\frac{C}{V(\theta^k r)}\frac{t}{\phi(\theta^k r)}\mu\Big(B(x,\theta^{k+1}r)\setminus B(x,\theta^k r)\Big)\\&\leqslant\sum_{k=0}^\infty\frac{c_2V(\theta^{k+1}r)}{V(\theta^k r)}\frac{t}{\phi(\theta^k r)}\leqslant c_3\sum_{k=0}^\infty c_0^k\frac{t}{\phi(r)}\leqslant\frac{c_4 t}{\phi(r)}.\end{aligned}$$

Recall that $\tau_{B(x,r)}=\inf\{t>0: X_t\notin B(x,r)\}$. By (3.3) and the strong Markov property and the conservativeness of X, for all $x\in M\setminus\mathscr{N}$, $t\in(0,1)$ and $r>0$,

$$\begin{aligned}&\mathbf{P}^x(\tau_{B(x,r)}\leqslant t)\\&=\mathbf{P}^x(\tau_{B(x,r)}\leqslant t, X_{2t}\in B(x,r/2)^c)+\mathbf{P}^x(\tau_{B(x,r)}\leqslant t, X_{2t}\in B(x,r/2))\\&\leqslant\mathbf{P}^x(\tau_{B(x,r)}\leqslant t, d(X_{2t},x)\leqslant r/2)+\mathbf{P}^x(d(X_{2t},x)\geqslant r/2)\\&\leqslant\mathbf{P}^x(\tau_{B(x,r)}\leqslant t, d(X_{2t},X_{\tau_{B(x,r)}})\geqslant r/2)+\frac{2c_1t}{\phi(r/2)}\\&\leqslant\sup_{s\leqslant t,d(z,x)\geqslant r}\mathbf{P}^z(d(X_{2t-s},z)\geqslant r/2)+\frac{2c_1t}{\phi(r/2)}\leqslant\frac{c_5t}{\phi(r/2)}.\end{aligned}\tag{3.4}$$

(Note that the conservativeness is used in the equality above. Indeed, without the assumption of the conservativeness, there must be an extra term

$$\mathbf{P}^x(\tau_{B(x,r)}\leqslant t,\zeta\leqslant 2t)$$

in the right hand side of the equality above, where ζ is the lifetime of the process X.)

Set $s_k = 2^{-k-1}$ for all $k \geqslant 1$. By (3.4), we have that, for all $x \in M \setminus \mathscr{N}$

$$\mathbf{P}^x(\sup_{0<s\leqslant s_k} d(X_s, x) \geqslant 2\varphi(s_k)) = \mathbf{P}^x(\tau_{B(x,2\varphi(s_k))} \leqslant s_k) \leqslant \frac{c_5 s_k}{\phi(\varphi(s_{k+1}))}.$$

By the assumption (3.2) and the Borel-Cantelli lemma,

$$\mathbf{P}^x(\sup_{0<s\leqslant s_k} d(X_s, x) \leqslant 2\varphi(s_k)) \text{ except finite } k \geqslant 1) = 1,$$

which implies that

$$\limsup_{t\to 0} \frac{\sup_{0\leqslant s\leqslant t} d(X_s, x)}{\varphi(t)} \leqslant 2, \qquad \mathbf{P}^x\text{-a.e. } \omega, \ \forall x \in M \setminus \mathscr{N}.$$

Therefore, the required assertion follows by considering $\varepsilon\varphi(r)$ for small $\varepsilon > 0$ instead of $\varphi(r)$ and using (2.10). □

Remark 3.2. From (3.3), one can easily get similar statements for the limsup behavior of $d(X_t, x)$ for both $t \to 0$ and $t \to \infty$.

3.2. **Lower bound for limsup behavior.** We begin with the assumption that the heat kernel $p(t, x, y)$ on (M, d, μ) satisfies the following off-diagonal lower bound estimate: there are constants $a, C > 0$ such that for every $x \in M \setminus \mathscr{N}$, μ-almost all $y \in M$ and all $t \in (a, \infty)$,

$$p(t, x, y) \geqslant \frac{C\, t}{V(d(x, y))\phi(d(x, y))}, \quad d(x, y) \geqslant \phi^{-1}(t), \tag{3.5}$$

where V and ϕ are strictly increasing functions satisfying (2.8) and (2.9), respectively. The statement below presents lower bound for the limsup behavior of maximal process for $t \to \infty$.

Theorem 3.3. *Let $p(t, x, y)$ satisfy the lower bound estimate (3.5) above. If φ is an increasing function on $(1, \infty)$ satisfying*

$$\int_1^\infty \frac{1}{\phi(\varphi(t))}\, dt = \infty, \tag{3.6}$$

then for all $x \in M \setminus \mathscr{N}$

$$\limsup_{t\to\infty} \frac{\sup_{0<s\leqslant t} d(X_s, x)}{\varphi(t)} = \limsup_{t\to\infty} \frac{d(X_t, x)}{\varphi(t)} = \infty, \qquad \mathbf{P}^x\text{-a.e. } \omega. \tag{3.7}$$

Proof. Without loss of generality, we can assume that $a = 1$ and $\phi(1) = 1$. First, choose $r_0 \geqslant 2$ such that $r_0^{-d_1} < c_1$, where d_1 and c_1 are constants given in (2.8). By (2.8) and (2.9), we have that for all $s \geqslant 1$

$$\begin{aligned}
\int_{r\geqslant s} \frac{1}{V(r)\phi(r)}\, dV(r) &= \sum_{k=0}^\infty \int_{r\in[r_0^k s, r_0^{k+1} s)} \frac{1}{V(r)\phi(r)}\, dV(r) \\
&\geqslant \sum_{k=0}^\infty \frac{V(r_0^{k+1}s) - V(r_0^k s)}{V(r_0^{k+1}s)\phi(r_0^{k+1}s)} \\
&\geqslant \left(1 - \frac{1}{c_1 r_0^d}\right) \sum_{k=0}^\infty \frac{1}{\phi(r_0^{k+1}s)} \\
&\geqslant \frac{1}{c_0}\left(1 - \frac{1}{c_1 r_0^d}\right) \sum_{k=0}^\infty c^{-(1+\log_2 r_0)(k+1)} \frac{1}{\phi(s)}
\end{aligned}$$

$$=: c_2 \frac{1}{\phi(s)}.$$

In particular,

$$\inf_{t\geqslant 1} \int_{r\geqslant \phi^{-1}(t)} \frac{t}{V(r)\phi(r)}\, dV(r) > 0, \tag{3.8}$$

and by (3.6),

$$\int_1^\infty dt \int_{r\geqslant \varphi(t)} \frac{1}{V(r)\phi(r)}\, dV(r) = \infty. \tag{3.9}$$

For any $k \geqslant 1$, set $B_k = \{d(X_{2^{k+1}}, X_{2^k}) \geqslant \varphi(2^{k+1}) \vee \phi^{-1}(2^{k+1})\}$. Then for every $x \in M \setminus \mathscr{N}$ and $k \geqslant 1$, by the Markov property,

$$\begin{aligned}\mathbf{P}^x(B_k|\mathscr{F}_{2^k}) \geqslant& \inf_z \mathbf{P}^z(d(X_{2^k}, z) \geqslant \varphi(2^{k+1}) \vee \phi^{-1}(2^{k+1}))\\ \geqslant& C\int_{r\geqslant \varphi(2^{k+1})\vee\phi^{-1}(2^{k+1})} \frac{2^k}{V(r)\phi(r)}\, dV(r).\end{aligned}$$

If there exist infinitely many $k \geqslant 1$ such that $\varphi(2^{k+1}) \leqslant \phi^{-1}(2^{k+1})$, then, by (3.8), for infinitely many $k \geqslant 1$,

$$\begin{aligned}\mathbf{P}^x(B_k|\mathscr{F}_{2^k}) \geqslant& C\int_{r\geqslant \phi^{-1}(2^{k+1})} \frac{2^k}{V(r)\phi(r)}\, dV(r)\\ \geqslant& \frac{C}{2}\inf_{t\geqslant 1}\int_{r\geqslant\phi^{-1}(t)} \frac{t}{V(r)\phi(r)}\, dV(r) =: c_3 > 0\end{aligned}$$

and so

$$\sum_{k=1}^\infty \mathbf{P}^x(B_k|\mathscr{F}_{2^k}) = \infty. \tag{3.10}$$

If there is $k_0 \geqslant 1$ such that for all $k \geqslant k_0$, $\varphi(2^{k+1}) > \phi^{-1}(2^{k+1})$, then

$$\mathbf{P}^x(B_k|\mathscr{F}_{2^k}) \geqslant C\int_{r\geqslant\varphi(2^{k+1})} \frac{2^k}{V(r)\phi(r)}\, dV(r) = \frac{C}{2}\int_{r\geqslant\varphi(2^{k+1})} \frac{2^{k+1}}{V(r)\phi(r)}\, dV(r).$$

Combining this with (3.9), we also get (3.10). Therefore, by the second Borel-Cantelli lemma, $\mathbf{P}^x(\limsup B_n) = 1$. Whence, for infinitely many $k \geqslant 1$,

$$d(X_{2^{k+1}}, x) \geqslant \frac{1}{2}(\varphi(2^{k+1}) \vee \phi^{-1}(2^{k+1}))$$

or

$$d(X_{2^k}, x) \geqslant \frac{1}{2}(\varphi(2^{k+1}) \vee \phi^{-1}(2^{k+1})) \geqslant \frac{1}{2}(\varphi(2^k) \vee \phi^{-1}(2^k)).$$

In particular,

$$\limsup_{t\to\infty} \frac{d(X_t, x)}{\varphi(t) \vee \phi^{-1}(t)} \geqslant \limsup_{k\to\infty} \frac{d(X_{2^k}, x)}{\varphi(2^k) \vee \phi^{-1}(2^k)} \geqslant \frac{1}{2}.$$

By the inequality above, we immediately get that for all $x \in M \setminus \mathscr{N}$

$$\limsup_{t\to\infty} \frac{\sup_{0<s\leqslant t} d(X_s, x)}{\varphi(t)} \geqslant \limsup_{t\to\infty} \frac{d(X_t, x)}{\varphi(t)} \geqslant \frac{1}{2}, \quad \mathbf{P}^x\text{-a.e. } \omega.$$

Therefore, (3.7) follows by considering $k\varphi(r)$ for large enough $k > 1$ instead of $\varphi(r)$ and using (2.9). □

To consider the lower bound for limsup behavior of maximal process for $t \to 0$, we need the following two-sided off-diagonal estimate for the heat kernel $p(t,x,y)$ on (M,d,μ), i.e. for every $x \in M \setminus \mathscr{N}$, μ-almost all $y \in M$ and each $t \in (0,b)$ with some constant $b>0$,

$$\frac{C_1 t}{V(d(x,y))\phi(d(x,y))} \leqslant p(t,x,y) \leqslant \frac{C_2 t}{V(d(x,y))\phi(d(x,y))}, \quad d(x,y) \geqslant \phi^{-1}(t), \tag{3.11}$$

where V and ϕ are strictly increasing functions satisfying (2.8) and (2.9), respectively.

Theorem 3.4. *Let $p(t,x,y)$ satisfy two-sided off-diagonal estimate* (3.11) *above. If φ is an increasing function on $(0,1)$ satisfying*

$$\int_0^1 \frac{1}{\phi(\varphi(t))}\,dt = \infty, \tag{3.12}$$

then for all $x \in M \setminus \mathscr{N}$,

$$\limsup_{t\to 0} \frac{\sup_{0<s\leqslant t} d(X_s,x)}{\varphi(t)} = \limsup_{t\to 0} \frac{d(X_t,x)}{\varphi(t)} = \infty, \qquad \mathbf{P}^x\text{-a.e. } \omega. \tag{3.13}$$

To prove Theorem 3.4, we will adopt the following generalized Borel-Cantelli lemma.

Lemma 3.5. ([35, Theorem 2.1] *or* [43, Theorem 1]) *Let A_1, A_2, ... be a sequence of events satisfying conditions $\sum_{n=1}^\infty \mathbf{P}(A_n) = \infty$ and $\mathbf{P}(A_k \cap A_j) \leqslant C\mathbf{P}(A_k)\mathbf{P}(A_j)$ for all $k,j > L$ such that $k \neq j$ and for some constants $C \geqslant 1$ and L. Then, $\mathbf{P}(\limsup A_n) \geqslant 1/C$.*

Proof of Theorem 3.4. For simplicity, we may and will assume that $b=1$, $\phi(1)=1$ and $2^{-d_1} < c_1$, where d_1 and c_1 are constants given in (2.8). Then, similar to the proof of Theorem 3.3, under assumptions of the theorem, we have

$$\inf_{t\in(0,1]} \int_{r\geqslant\phi^{-1}(t)} \frac{t}{V(r)\phi(r)}\,dV(r) > 0, \tag{3.14}$$

and, by (3.12),

$$\int_0^1 dt \int_{r\geqslant\varphi(t)} \frac{1}{V(r)\phi(r)}\,dV(r) = \infty. \tag{3.15}$$

For some $t \in (0,1)$ and any $k \geqslant 1$, set $s_k = 2^{-k}t$ and

$$A_k = \Big\{ d(X_{s_k}, X_{s_{k+1}}) \geqslant \varphi(s_k) \vee \phi^{-1}(s_k) \Big\}.$$

By the Markov property and the lower bound in (3.11), for all $x \in M \setminus \mathscr{N}$,

$$\begin{aligned}\mathbf{P}^x(A_k) \geqslant& \inf_z \mathbf{P}^z(d(X_{s_{k+1}},z) \geqslant \varphi(s_k) \vee \phi^{-1}(s_k))\\ \geqslant& C_1 \inf_z \int_{d(y,z)\geqslant\varphi(s_k)\vee\phi^{-1}(s_k)} \frac{s_{k+1}}{V(d(z,y))\phi(d(z,y))}\mu(dy)\\ \geqslant& c_2 \int_{r\geqslant\varphi(s_k)\vee\phi^{-1}(s_k)} \frac{s_k}{V(r)\phi(r)}\,dV(r) =: c_2 c_{1,s_k}.\end{aligned}$$

In particular, if $\varphi(\theta) \geqslant \phi^{-1}(\theta)$, then

$$c_{1,\theta} = \int_{r \geqslant \varphi(\theta)} \frac{\theta}{V(r)\phi(r)}\, dV(r);$$

if $\varphi(\theta) \leqslant \phi^{-1}(\theta)$, then

$$c_{1,\theta} = \int_{r \geqslant \phi^{-1}(\theta)} \frac{\theta}{V(r)\phi(r)}\, dV(r). \tag{3.16}$$

Combining these two estimates above with (3.14) and (3.15) yields that

$$\sum_{k=1}^{\infty} \mathbf{P}^x(A_k) = \infty.$$

On the other hand, for any $k < j$, by the Markov property and the upper bound for the heat kernel (3.11),

$$\begin{aligned}\mathbf{P}^x(A_k \cap A_j) \leqslant &\mathbf{E}^x\Big(\mathbb{1}_{A_j} \mathbf{P}^{X_{s_k}}\big(d(X_{s_{k+1}}, X_0) \geqslant \varphi(s_k) \vee \phi^{-1}(s_k)\big)\Big)\\ \leqslant &\mathbf{P}^x(A_j) \sup_z \mathbf{P}^z\big(d(X_{s_{k+1}}, z) \geqslant \varphi(s_k) \vee \phi^{-1}(s_k)\big)\\ \leqslant &c_3 \mathbf{P}^x(A_j) c_{1,s_k} \leqslant c_3^2 c_{1,s_j} c_{1,s_k}.\end{aligned}$$

From this and (3.16), we can easily see that there is a constant $C_0 \geqslant 1$ such that

$$\mathbf{P}^x(A_k \cap A_j) \leqslant C_0 \mathbf{P}^x(A_k)\mathbf{P}^x(A_j).$$

Therefore, according to Lemma 3.5, $\mathbf{P}^x(\limsup A_n) \geqslant 1/C_0$, which along with the Blumenthal 0-1 law implies that $\mathbf{P}^x(\limsup A_n) = 1$. Whence, for infinitely many $k \geqslant 1$,

$$d(X_{s_k}, x) \geqslant \frac{1}{2}(\varphi(s_k) \vee \phi^{-1}(s_k))$$

or

$$d(X_{s_{k+1}}, x) \geqslant \frac{1}{2}(\varphi(s_k) \vee \phi^{-1}(s_k)) \geqslant \frac{1}{2}\left(\varphi\left(s_{k+1}\right) \vee \phi^{-1}\left(s_{k+1}\right)\right).$$

In particular,

$$\limsup_{t \to 0} \frac{d(X_t, x)}{\varphi(t) \vee \phi^{-1}(t)} \geqslant \limsup_{k \to \infty} \frac{d(X_{s_k}, x)}{\varphi(s_k) \vee \phi^{-1}(s_k)} \geqslant \frac{1}{2}.$$

Hence, (3.13) follows by considering $k\varphi(r)$ for large $k > 1$ instead of $\varphi(r)$ and using (2.9). □

Remark 3.6. The proof of Theorem 3.3 is based only on off-diagonal lower bound of the heat kernel estimate for long time, while in the proof of Theorem 3.4 explicit two-sided off-diagonal estimate of the heat kernel for small time is used. Unlike the case of Theorem 3.3, we do not know how to prove Theorem 3.4 by using only the off-diagonal lower bound of the heat kernel estimate.

3.3. **Liminf laws of the iterated logarithm.** In this part, we discuss Chung-type liminf laws of the iterated logarithm. To this end, we assume that the heat kernel $p(t,x,y)$ on (M,d,μ) satisfies the following two-sided estimates with $T\in(0,\infty]$: for every $x\in M\setminus\mathscr{N}$, μ-almost all $y\in M$ and each $0<t<T$,

$$\begin{aligned} &C_1\left(\frac{1}{V(\phi^{-1}(t))}\wedge\frac{t}{V(d(x,y))\phi(d(x,y))}\right)\leqslant p(t,x,y),\\ &p(t,x,y)\leqslant C_2\left(\frac{1}{V(\phi^{-1}(t))}\wedge\frac{t}{V(d(x,y))\phi(d(x,y))}\right),\end{aligned}\tag{3.17}$$

where V and ϕ are strictly increasing functions satisfying (2.8) and (2.10) respectively.

Theorem 3.7. *Assume that the process X is conservative. Let $p(t,x,y)$ satisfy two-sided estimate* (3.17) *above with $0<T<\infty$. Then there exists a constant $c\in(0,\infty)$ such that*

$$\liminf_{t\to0}\frac{\sup_{0<s\leqslant t}d(X_s,x)}{\phi^{-1}(t/\log|\log t|)}=c,\qquad \mathbf{P}^x\text{-}a.e.\ \omega,\ \forall x\in M.$$

Proof. The following proof is based on the idea of proofs in [18, Chapter 3] (see also the proof of [29, Theorem 2]). Without loss of generality, we can assume that $T=1$, and $\mathscr{N}=\emptyset$ due to Proposition 2.9.

Let $(a_k)_{k\geqslant1}$ be the sequence defined by $a_k=\phi^{-1}(e^{-k^2})$ so that $\phi(a_k)=e^{-k^2}$. For any $k\geqslant1$, set $\lambda_k=\frac{2}{3|\log a_1^*|}\log(1+k)$, $u_k=c_0\lambda_ke^{-k^2}$ and $\sigma_k=\sum_{i=k+1}^\infty u_i$, where $c_0>0$ and $a_1^*\in(0,1)$ are the constants in Proposition 2.12. We will prove that there are $\xi,c_1\in(0,\infty)$ such that for all $x\in M$

$$\mathbf{P}^x\left(\sup_{2a_{2m}\leqslant r\leqslant 2a_m}\frac{\tau_{B(x,r)}}{\phi(r)\log|\log\phi(r)|}\leqslant\xi\right)\leqslant c_1\exp(-m^{1/4}),\quad m\geqslant1.$$

For $k\geqslant1$, let $G_k=\left\{\sup_{\sigma_k\leqslant s\leqslant\sigma_{k-1}}d(X_s,X_{\sigma_k})>a_k\right\}$. By the Markov property, the conservativeness of the process X and Proposition 2.12, for all $x\in M$,

$$\begin{aligned}\mathbf{P}^x(G_k)&\leqslant\sup_z\mathbf{P}^z\big(\sup_{0\leqslant s\leqslant u_k}d(X_s,z)>a_k\big)\\&=1-\inf_z\mathbf{P}^z\big(\sup_{0\leqslant s\leqslant u_k}d(X_s,z)\leqslant a_k\big)\\&=1-{a_1^*}^{\lambda_k}=1-(1+k)^{-2/3}\leqslant\exp(-c_2k^{-2/3}).\end{aligned}$$

For $k\geqslant1$, let $H_k=\left\{\sup_{0<s\leqslant\sigma_k}d(X_s,x)>a_k\right\}$. Then, for all $x\in M$ and for all $k\geqslant1$,

$$\mathbf{P}^x(H_k)\leqslant\frac{c_3\sigma_k}{\phi(a_k)}\leqslant\frac{c_4\sum_{i=1}^\infty e^{-(k+i)^2}\log(1+k+i)}{e^{-k^2}}\leqslant c_5e^{-k},$$

where the first inequality follows from (3.4) and the doubling property of ϕ.

For $m\geqslant1$, define $A_m=\bigcap_{k=m}^{2m}D_k$, where $D_k=\left\{\sup_{0<s\leqslant\sigma_{k-1}}d(X_s,x)>2a_k\right\}$. Since $D_k\subset G_k\cup H_k$, $A_m\subset(\cap_{k=m}^{2m}G_k)\cup(\cup_{k=m}^{2m}H_k)$. By using the Markov property again, we find that for all $x\in M$,

$$\mathbf{P}^x(A_m)\leqslant\mathbf{P}^x(\cap_{k=m}^{2m}G_k)+\mathbf{P}^x(\cup_{k=m}^{2m}H_k)$$

$$\leqslant \prod_{k=m}^{2m} \exp(-c_2 k^{-2/3}) + c_5 \sum_{k=m}^{2m} e^{-k} \leqslant c_6 \exp(-m^{1/4}).$$

Therefore,

$$\begin{aligned}
c_6 \exp(-m^{1/4}) \geqslant & \mathbf{P}^x\Big(\bigcap_{k=m}^{2m} \Big\{ \frac{\sup_{0<s\leqslant\sigma_{k-1}} d(X_s,x)}{2a_k} > 1 \Big\} \Big) \\
= & \mathbf{P}^x\Big(\inf_{m\leqslant k\leqslant 2m} \frac{\sup_{0\leqslant s\leqslant\sigma_{k-1}} d(X_s,x)}{2a_k} > 1 \Big) \\
= & \mathbf{P}^x\Big(\sup_{m\leqslant k\leqslant 2m} \frac{\tau_{B(x,2a_k)}}{\sigma_{k-1}} < 1 \Big) \geqslant \mathbf{P}^x\big(\sup_{m\leqslant k\leqslant 2m} \frac{\tau_{B(x,2a_k)}}{u_k} < 1 \big) \\
\geqslant & \mathbf{P}^x\Big(\sup_{2a_{2m}\leqslant r\leqslant 2a_m} \frac{\tau_{B(x,r)}}{\phi(r)\log|\log\phi(r)|} \leqslant \xi \Big)
\end{aligned}$$

for some $\xi \in (0,\infty)$. Using this equality, by the Borel-Cantelli lemma, we conclude that

$$\limsup_{r\to 0} \frac{\tau_{B(x,r)}}{\phi(r)\log|\log\phi(r)|} \geqslant \xi.$$

On the other hand, with $l_k := \phi^{-1}(e^{-k})$ for $k \geqslant 1$, we have

$$B_k := \Big\{ \sup_{l_{k+1}\leqslant r\leqslant l_k} \frac{\tau_{B(x,r)}}{\phi(r)\log|\log\phi(r)|} \geqslant b \Big\} \subset \Big\{ \tau_{B(x,l_k)} \geqslant b e^{-1} \phi(l_k) \log|\log\phi(l_k)| \Big\}.$$

Taking $b = -4/\log a_2^*$ where $a_2^* \in (0,1)$ is the constant in Proposition 2.12, we know from Proposition 2.12 that $\mathbf{P}^x(B_k) \leqslant k^{-4/e}$. Thus, by the Borel-Cantelli lemma again,

$$\limsup_{r\to 0} \frac{\tau_{B(x,r)}}{\phi(r)\log|\log\phi(r)|} \in [\xi, b],$$

which implies that

$$\limsup_{r\to 0} \frac{\tau_{B(x,r)}}{\phi(r)\log|\log\phi(r)|} = C, \qquad \mathbf{P}^x\text{-a.e. } \omega,\ \forall x \in M,$$

for some constant $C > 0$, also thanks to the Blumenthal 0-1 law. The desired assertion follows from the equality above. □

For the behavior of liminf for maximal process with $t \to \infty$, we have the following conclusion similar to Theorem 3.7.

Theorem 3.8. *Let $p(t,x,y)$ satisfy two-sided estimate (3.17) for all $t > 0$, i.e. $T = \infty$. Then there exists a constant $c \in (0,\infty)$ such that*

$$\liminf_{t\to\infty} \frac{\sup_{0<s\leqslant t} d(X_s,x)}{\phi^{-1}(t/\log\log t)} = c, \qquad \mathbf{P}^x\text{-a.e. } \omega,\ \forall x \in M.$$

Proof. Since the proof is the same as that of Theorem 3.7 with some modifications, we just highlight a few differences. Note that, by Proposition 2.5, the process X is conservative. With the notions in the argument above, we define the sequences a_k, σ_k and sets G_k, D_k as $\phi(a_k) = e^{k^2}$, $\sigma_k = \sum_{i=1}^{k-1} u_i$ and

$$G_k = \big\{ \sup_{\sigma_k\leqslant s\leqslant\sigma_{k+1}} d(X_s, X_{\sigma_k}) > a_k \big\}, \quad D_k = \big\{ \sup_{0<s\leqslant\sigma_{k+1}} d(X_s,x) > 2a_k \big\},$$

respectively. To conclude the proof, we use Theorem 2.10 instead of Blumenthal 0-1 law. □

Remark 3.9. It can be easily observed that the behavior of lim sup does not change if we consider $\sup_{0<s\leqslant t} d(X_s,x)$ instead of $d(X_t,x)$. However, the lim inf behavior for $d(X_t,x)$ can be different from that of $\sup_{0<s\leqslant t} d(X_s,x)$. For instance, if the process X is recurrent, i.e. $\int_1^\infty \frac{1}{V(\phi^{-1}(t))}\,dt=\infty$, then for all $x\in M\backslash\mathcal{N}$, $\liminf_{t\to\infty} d(X_t,x)=0$.

4. Laws of the Iterated Logarithm for Local Times

In this section, we discuss the LILs for local time. We assume Assumptions 2.1, 2.2 and 2.8 throughout the section. Recall that, under Assumptions 2.1, 2.2 and 2.8, (2.8) holds for V by Proposition 2.6, and (2.10) is satisfied for ϕ by the remark below Assumption 2.8. Note that (2.8) and (2.10) are equivalent to the existence of constants $c_5,\cdots,c_8>1$ and $L_0>1$ such that for every $r>0$,

$$c_5\phi(r)\leqslant\phi(L_0r)\leqslant c_6\,\phi(r)\quad\text{and}\quad c_7V(r)\leqslant V(L_0r)\leqslant c_8\,V(r).$$

In particular,

$$\int_r^\infty \frac{dV(s)}{V(s)\phi(s)}\asymp\frac{1}{\phi(r)},\quad r>0. \tag{4.1}$$

4.1. Estimates for resolvent densities.

For $\lambda>0$, we define the λ-resolvent density (i.e. the density function of the λ-resolvent operator) by

$$u^\lambda(x,y)=\int_0^\infty e^{-\lambda t}p(t,x,y)\,dt.$$

For each $A\subset M$, set

$$\tau_A:=\inf\{t>0:X_t\notin A\},\quad \sigma_A:=\inf\{t>0:X_t\in A\}$$

and

$$\sigma_A^0:=\inf\{t\geqslant 0:X_t\in A\}.$$

For simplicity, we write $\sigma_x^0:=\sigma_{\{x\}}^0$.

For an open subset $A\subset M$ with $A\neq M$, define

$$u_A(x,y)=\int_0^\infty p^A(t,x,y)\,dt,\ \ x,y\in A,$$

where $p^A(t,\cdot,\cdot)$ is the Dirichlet heat kernel of the process X killed on exiting A, see (2.13).

Proposition 4.1. *Suppose that*

$$\int_0^\infty e^{-\lambda t}\frac{1}{V(\phi^{-1}(t))}\,dt\asymp\frac{\lambda^{-1}}{V(\phi^{-1}(\lambda^{-1}))},\quad \lambda>0. \tag{4.2}$$

Then the following three statements hold.

(i) *There exist $c_1,c_2>0$ such that*

$$c_1\frac{\phi(r)}{V(r)}\leqslant u_{B(x,r)}(x,x)\leqslant c_2\frac{\phi(r)}{V(r)}\quad \textit{for all}\ \ x\in M,\ r>0.$$

(ii) *There exists $c_3>0$ such that for any $x_0\in M$, $R>0$ and any $x,y\in B(x_0,R/4)$,*

$$\mathbf{P}^x(\sigma_y^0>\tau_{B(x_0,R)})\leqslant c_3\frac{\phi(d(x,y))}{V(d(x,y))}\frac{1}{u_{B(x_0,R)}(y,y)}.$$

(iii) *It holds that*

$$1-\mathbf{E}^y[e^{-\sigma^0_x}]\leqslant c_4\frac{\phi(d(x,y))}{V(d(x,y))}$$

for all $x,y\in M$.

Remark 4.2. The exponent on the right hand side of (iii) (which is $\beta-\alpha$ when $d_1=d_2=\alpha$ and $d_3=d_4=\beta$ in (2.8) and (2.10)) is sharp in general, and we do need this exponent later. We may be able to obtain the Hölder continuity by using the Harnack inequality in Proposition 2.13, but we cannot get the sharp exponent with that approach (cf. Proposition 2.9). Another possible approach is to use the properties of the so-called resistance form (see for example, [26]), but they require various preparations, so we take this "bare-hands" approach.

Proof of Proposition 4.1. The following arguments are based on [4, Section 4] and [7, Section 5], but with highly non-trivial modifications due to the generality and the effects of jumps.

(i) The lower bound is easy. Set $A=B(x,r)$. By (3.4) and (2.10), there exists a constant $c_1>0$ such that for all $x\in M$ and $r>0$,

$$\mathbf{P}^x\left(\tau_A\leqslant c_1\phi(r)\right)\leqslant\frac{1}{2}$$

and so, by conservativeness of the process (Proposition 2.5), we have

$$\mathbf{E}^x(\tau_A)\geqslant c_1\phi(r)\mathbf{P}^x\left(\tau_A\geqslant c_1\phi(r)\right)\geqslant\frac{c_1}{2}\phi(r).$$

We then have

$$\frac{c_1}{2}\phi(r)\leqslant\mathbf{E}^x(\tau_A)=\int_A u_A(x,y)\,\mu(dy)\leqslant u_A(x,x)\mu(A)\leqslant c_2V(r)u_A(x,x),$$

where we used the fact $u_A(x,y)=u_A(y,x)=\mathbf{P}^y(\sigma^0_x<\sigma^0_{A^c})u_A(x,x)\leqslant u_A(x,x)$. Thus, the lower bound is established.

Next, we prove the upper bound. Let Exp_λ be an independent exponential distributed random variable with mean λ^{-1}. In the following, with some abuse of notation, we also use $\mathbf{P}^x$ for the product probability of $\mathbf{P}^x$ and the law of Exp_λ. We claim that there exists a constant $c_3>0$ such that

$$\mathbf{P}^x(\mathrm{Exp}_\lambda\leqslant\tau_A)\leqslant(c_3\lambda\phi(r))\wedge 1,\quad r,\lambda>0,x\in M. \tag{4.3}$$

To prove this, we first note that

$$\mathbf{P}^x(\tau_A\geqslant t)\leqslant\exp(-t/(c_3\phi(r))),\quad r,t>0,x\in M. \tag{4.4}$$

Indeed, since for any $x\in M$ and $t,r>0$,

$$\mathbf{P}^x(\tau_{B(x,2r)}\geqslant t)\leqslant\int_{B(x,2r)}p(t,x,y)\,\mu(dy)\leqslant\frac{c_4V(2r)}{V(\phi^{-1}(t))},$$

by (2.8) and (2.10), there is a constant $c_5>0$ such that

$$\mathbf{P}^x(\tau_{B(x,2r)}\geqslant c_5\phi(r))\leqslant 1/2$$

for all $x\in M$ and $r>0$. So, by induction and the Markov property, we have for each $k\in\mathbb{N}$,

$$\mathbf{P}^x(\tau_A\geqslant c_5(k+1)\phi(r))\leqslant\mathbf{E}^x\left[1_{\{\tau_A\geqslant c_5k\phi(r)\}}\mathbf{P}^{X_{c_5k\phi(r)}}(\tau_{B(X_0,2r)}\geqslant c_5\phi(r))\right]\leqslant(1/2)^{k+1},$$

which immediately yields (4.4). Using (4.4), we have

$$\begin{aligned}\mathbf{P}^x(\mathrm{Exp}_\lambda \leqslant \tau_A) = \int_0^\infty \lambda e^{-\lambda t}\mathbf{P}^x(\tau_A \geqslant t)\,dt \leqslant \int_0^\infty \lambda e^{-\lambda t}\exp(-t/(c_3\phi(r)))\,dt \\ =\lambda(\lambda+1/(c_3\phi(r)))^{-1} \leqslant c_3\lambda\phi(r),\end{aligned}$$

so (4.3) is established.

Now using (4.3) with the choice of $\lambda = (2c_3\phi(r))^{-1}$, the fact that $u_A(y,x) \leqslant u_A(x,x)$ and the strong Markov property, we have

$$u_A(x,x) \leqslant u^\lambda(x,x) + \mathbf{P}^x(\mathrm{Exp}_\lambda \leqslant \tau_A)u_A(x,x) \leqslant u^\lambda(x,x) + (1/2)u_A(x,x).$$

This, along with (2.4), (4.2) and (2.10), gives us

$$u_A(x,x) \leqslant 2u^\lambda(x,x) \leqslant 2\int_0^\infty e^{-\lambda t}\frac{1}{V(\phi^{-1}(t))}\,dt \leqslant c_6\frac{\phi(r)}{V(r)}.$$

(ii) Write $A = B(x_0,R)$ and $B = B(y, c_* d(x,y))$, where $0 < c_* < 1$ is chosen later. Using the strong Markov property and Proposition 2.5,

$$u_A(y,y) = u_B(y,y) + \mathbf{E}^y\left(1 - f_y(X_{\tau_B})\right)u_A(y,y),$$

where $f_y(x) := \mathbf{P}^x(\sigma_y^0 > \tau_A)$. Thus,

$$u_B(y,y) = u_A(y,y)\mathbf{E}^y[f_y(X_{\tau_B})]. \tag{4.5}$$

Since $f_y(\cdot)$ is harmonic on $A\setminus\{y\}$, by Proposition 2.13 (we only use the elliptic Harnack inequality here), there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \leqslant f_y(z)/f_y(z') \leqslant c_2, \qquad \forall z, z' \in B(y, c_* k d(x,y)) \setminus B, \tag{4.6}$$

where we choose $k > 0$ to satisfy $1 < c_* k < 3/2$. Note that $1 < c_* k$ is required in order to guarantee that $x \in B(y, c_* k d(x,y)) \setminus B$. Using the jump kernel of the process X (see Proposition 2.7) and the Lévy system formula (see for example [14, Appendix A]), we have

$$\begin{aligned}\mathbf{P}^y(X_{\tau_B\wedge t} \notin B(y, c_* k d(x,y))) =&\mathbf{E}^y\Big[\int_0^{\tau_B\wedge t}\int_{B(y,c_*kd(x,y))^c} J(X_s,u)\,\mu(du)\,ds\Big] \\ \leqslant&\mathbf{E}^y\Big[\int_0^{\tau_B\wedge t}\int_{B(y,c_*kd(x,y))^c} \frac{c_3\,\mu(du)\,ds}{V(d(X_s,u))\phi(d(X_s,u))}\Big] \\ \leqslant&\frac{c_4\mathbf{E}^y[\tau_B\wedge t]}{\phi(c_*(k-1)d(x,y))} \leqslant c_5(k-1)^{-d_3},\end{aligned}$$

where in the last line we have used (2.10), (4.1) and the fact that for any $x, y \in M$, $\mathbf{E}^y(\tau_B) \leqslant c_0\phi(c_* d(x,y))$ due to (4.4) (e.g. see (A.2)). Note that the constant $c_5 > 0$ is independent of c_* and k. We choose k large enough and c_* small enough such that $c_5(k-1)^{-d_3} < 1/2$ and $1 < c_* k < 3/2$. Taking $t \to \infty$ in the inequality above, we have

$$\mathbf{P}^y(X_{\tau_B} \notin B(y, c_* k d(x,y))) \leqslant 1/2.$$

Using this, (4.5) and (4.6), we find that

$$\begin{aligned}\mathbf{P}^x(\sigma_y^0 > \tau_A)/2 = f_y(x)/2 \leqslant& c_2\mathbf{E}^y[1_{\{X_{\tau_B}\in B(y,c_*kd(x,y))\}}f_y(X_{\tau_B})] \leqslant c_2\mathbf{E}^y[f_y(X_{\tau_B})] \\ =&c_2\frac{u_B(y,y)}{u_A(y,y)} \leqslant c_6\frac{1}{u_A(y,y)}\frac{\phi(d(x,y))}{V(d(x,y))},\end{aligned}$$

where we use (i) in the last inequality. We thus obtain (ii).

(iii) From (4.2), we know that

$$c^{-1}\frac{\lambda^{-1}}{V(\phi^{-1}(\lambda^{-1}))}\leqslant\int_0^\infty e^{-\lambda t}\frac{1}{V(\phi^{-1}(t))}\,dt\leqslant c\frac{\lambda^{-1}}{V(\phi^{-1}(\lambda^{-1}))}$$

for some constant $c\geqslant 1$ and $\lambda>0$. Then, for all $r>0$,

$$c^{-1}\frac{\phi(r)}{V(r)}\leqslant\int_0^\infty e^{-t/\phi(r)}\frac{1}{V(\phi^{-1}(t))}\,dt\leqslant c\frac{\phi(r)}{V(r)},$$

which implies that for any $s,t>0$,

$$\frac{\phi(s)}{V(s)}\leqslant c^2\frac{\phi(s+t)}{V(s+t)}. \tag{4.7}$$

Using (4.7), the desired inequality is trivial when $d(x,y)\geqslant e^{-1}$ by taking $c_4=\frac{c^2V(e^{-1})}{\phi(e^{-1})}$. Let $n\in\mathbb{N}$ be such that $e^{-n-1}\leqslant d(x,y)<e^{-n}$ and set $\tau_m=\tau_{B(y,e^{-m})}$ for each $m\in\mathbb{N}$. Then,

$$\begin{aligned}
&1-\mathbf{E}^y[e^{-\sigma_x^0}]=\mathbf{P}^y(\sigma_x^0\geqslant \mathrm{Exp}_1)\\
&\leqslant \mathbf{P}^y(\sigma_x^0\geqslant \mathrm{Exp}_1,\mathrm{Exp}_1<\tau_n)+\sum_{m=1}^n\mathbf{P}^y(\sigma_x^0\geqslant \mathrm{Exp}_1,\tau_m\leqslant\mathrm{Exp}_1<\tau_{m-1})\\
&\quad+\mathbf{P}^y(\sigma_x^0\geqslant \mathrm{Exp}_1,\mathrm{Exp}_1\geqslant\tau_0)\\
&\leqslant \mathbf{P}^y(\mathrm{Exp}_1<\tau_n)+\sum_{m=1}^n\mathbf{P}^y(\sigma_x^0\geqslant \mathrm{Exp}_1,\tau_m\leqslant\mathrm{Exp}_1<\tau_{m-1})+\mathbf{P}^y(\sigma_x^0\geqslant\tau_0)\\
&\leqslant \mathbf{P}^y(\mathrm{Exp}_1<\tau_n)+\sum_{m=1}^n\mathbf{P}^y(1_{\{\sigma_x^0\geqslant\tau_m,\mathrm{Exp}_1\geqslant\tau_m,X_{\tau_m}\in B(y,e^{-m+1})\}}\mathbf{P}^{X_{\tau_m}}(\mathrm{Exp}_1<\tau_{m-1}))\\
&\quad+\mathbf{P}^y(\sigma_x^0\geqslant\tau_0)\\
&\leqslant \mathbf{P}^y(\mathrm{Exp}_1<\tau_n)+\sum_{m=1}^n\mathbf{P}^y(\sigma_x^0\geqslant\tau_m)\sup_{z\in B(y,e^{-m+1})}\mathbf{P}^z(\mathrm{Exp}_1<\tau_{B(y,e^{-m+1})})\\
&\quad+\mathbf{P}^y(\sigma_x^0\geqslant\tau_0)\\
&\leqslant c_1\phi(e^{-n})+c_2\sum_{m=1}^n\phi(e^{-n})V(e^{-m})/V(e^{-n})+c_3\phi(e^{-n})/V(e^{-n})\\
&\leqslant c_4\phi(e^{-n})/V(e^{-n})\leqslant c_5\phi(d(x,y))/V(d(x,y)),
\end{aligned}$$

where we used (i), (ii), (4.3), (2.8) and (2.10) in the fifth inequality, and (2.8) and (2.10) in the last line. □

4.2. **Existence and estimates for local times.** Let $(A_t)_{t\geqslant 0}$ be a continuous additive functional of the process X, i.e.

- $t\mapsto A_t$ is almost surely continuous and nondecreasing with $A_0=0$;
- $A_t\in\mathscr{F}_t$;
- $A_{t+s}(\omega)=A_t(\omega)+A_s(\theta_t\omega)$ for all $s,t\geqslant 0$.

Set $T_A=\inf\{t>0:A_t>0\}$. A_t is called a local time of the process X at x, if $\mathbf{P}^x(T_A=0)=1$ and $\mathbf{P}^y(T_A=0)=0$ for all $y\notin x$. The reason that A_t is called a local time at x for the process X is that the function $t\mapsto A_t$ is the distribution function of a measure supported on the set $\{t|X_t=x\}$, see e.g. [9, V. 3]. The next proposition gives us a necessary and sufficient condition for the existence of a local time.

Proposition 4.3. *The process X has a local time for all $x \in M$, if and only if*

$$\int_0^1 \frac{1}{V(\phi^{-1}(t))}\, dt < \infty. \tag{4.8}$$

Moreover, we can choose a version of the local time at x, which will denote by $l(x,t)$, by requiring the following property.

1. *The function $(\omega, t, x) \mapsto l(x,t)(\omega)$ is jointly measurable such that the following density of occupation formula holds for all non-negative Borel measurable function f,*

$$\int_0^t f(X_s)\, ds = \int_M f(x) l(x,t)\, \mu(dx). \tag{4.9}$$

2. *For any $x, y \in M$ and $\lambda > 0$,*

$$\mathbf{E}^x \left(\int_0^\infty e^{-\lambda t}\, dl(y,t) \right) = u^\lambda(x,y). \tag{4.10}$$

Proof. According to [33, Theorem 3.2], the process X has a local time for all $x \in M$ if and only if

$$u^\lambda(x,x) < \infty \quad \text{for all } x \in M \text{ and some } \lambda > 0.$$

Using Assumption 2.1 and the doubling properties of V and ϕ,

$$u^\lambda(x,x) = \int_0^\infty e^{-\lambda t} p(t,x,x)\, dt < \infty \quad \text{for all } x \in M$$

if and only if

$$\int_0^1 e^{-\lambda t} \frac{1}{V(\phi^{-1}(t))}\, dt < \infty,$$

which in turn is equivalent to (4.8).

Local times are defined up to a multiplicative constant, see [9, V. 3.13]. By [22, Theorem 1] and [9, VI. 4.18], we can choose a version of local times satisfying the desired properties (i) and (ii), also see the remark below [33, Theorem 3.2]. □

Below we suppose that the local time $l(x,t)$ is always chosen to satisfy (1) and (2) in Proposition 4.3, if (4.8) is satisfied. Note that, (4.2) implies (4.8). By the strong Markov property and (4.10),

$$\begin{aligned} u^\lambda(x,y) =& \mathbf{E}^x \int_0^\infty e^{-\lambda t}\, dl(y,t) = \mathbf{E}^x \int_{\sigma_y^0}^\infty e^{-\lambda t}\, dl(y,t) \\ =& \mathbf{E}^x e^{-\lambda \sigma_y^0} \mathbf{E}^y \int_0^\infty e^{-\lambda t}\, dl(y,t) = \mathbf{E}^x e^{-\lambda \sigma_y^0} u^\lambda(y,y). \end{aligned}$$

So,

$$\mathbf{E}^x[e^{-\sigma_y^0}] = u^1(x,y)/u^1(y,y), \tag{4.11}$$

which is continuous because of the continuity of $p(t,x,y)$, see Proposition 2.9. □

Let d_2 and d_3 be the constants in (2.8) and (2.10) respectively. Throughout the remainder of this section, we always assume the following

Assumption 4.4. $d_3 > d_2$.

The functions V and ϕ respectively characterize the underlying space and the process in question. Assumption 4.4 means that the walk dimension of the process is greater than the dimension of the space, which implies that the process could stay at every point for efficiently long time; that is, the local time of the process exists.

The following lemma is easy.

Lemma 4.5. *Under Assumption* 4.4, (4.2) *holds. In particular,*

$$\int_0^t \frac{1}{V(\phi^{-1}(s))}\,ds \asymp \frac{t}{V(\phi^{-1}(t))}, \quad t>0, \tag{4.12}$$

and so (4.8) *is satisfied.*

Proof. Let $f(t) := \frac{1}{V(\phi^{-1}(t))}$ and

$$w(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,dt = \lambda^{-1}\int_0^\infty e^{-s} f(s/\lambda)\,ds.$$

Since f is decreasing, we see that

$$w(\lambda) \geqslant \lambda^{-1}\int_{1/2}^1 e^{-s} f(s/\lambda)\,ds \geqslant \lambda^{-1} f(1/\lambda)\int_{1/2}^1 e^{-s}\,ds = c_0\lambda^{-1} f(1/\lambda).$$

On the other hand, it follows from (2.8) and (2.10) that

$$c_1\Big(\frac{R}{r}\Big)^{d_1/d_4} \leqslant \frac{V(\phi^{-1}(R))}{V(\phi^{-1}(r))} \leqslant c_2\Big(\frac{R}{r}\Big)^{d_2/d_3} \tag{4.13}$$

holds for all $0<r\leqslant R$ and some constants $c_1, c_2>0$. This along with the assumption $d_3>d_2$ yields that

$$\begin{aligned}\frac{\lambda w(\lambda)}{f(1/\lambda)} &= \int_0^1 e^{-s}\frac{f(s/\lambda)}{f(1/\lambda)}\,ds + \int_1^\infty e^{-s}\frac{f(s/\lambda)}{f(1/\lambda)}\,ds\\ &\leqslant c_2\int_0^1 e^{-s}s^{-d_2/d_3}\,ds + \int_1^\infty e^{-s}\,ds < \infty.\end{aligned}$$

We have proved (4.2).

We now verify (4.12). By the increasing properties of V and ϕ, for any $t>0$,

$$\int_0^t \frac{1}{V(\phi^{-1}(s))}\,ds \geqslant \frac{t}{V(\phi^{-1}(t))}.$$

The upper bound of (4.12) can be obtained from (4.2) as follows:

$$\int_0^t \frac{1}{V(\phi^{-1}(s))}\,ds \leqslant e\int_0^t e^{-s/t}\frac{1}{V(\phi^{-1}(s))}\,ds \leqslant e\int_0^\infty e^{-s/t}\frac{1}{V(\phi^{-1}(s))}\,ds \leqslant \frac{c_3 t}{V(\phi^{-1}(t))}.$$

The proof is complete. □

From now on, we will always consider versions of the local time at x, denote by $l(x,t)$, satisfying the results in Proposition 4.3. The following statement is Kac's moment formula of the local time. Since (4.2) implies (4.10), this directly follows from [34, Theorem 3.10.1].

Proposition 4.6. *For any $x, y_i \in M$ with $1\leqslant i\leqslant n$ and $t>0$,*

$$\mathbf{E}^x\Pi_{i=1}^n l(y_i,t) = \sum_\pi \mathbf{E}^x l(y_{\pi_1},t)\cdots\mathbf{E}^{y_{\pi_{n-1}}} l(y_{\pi_n},t),$$

where the sum runs over all permutations π of $\{1,\dots,n\}$. In particular, for any $x,y\in M$ and $n\geqslant 1$,

$$\mathbf{E}^x(l(y,t))^n = n!\mathbf{E}^x l(y,t)\big(\mathbf{E}^y l(y,t)\big)^{n-1}.$$

Proposition 4.1 combining with some general theory yields the following. (See [16, Theorem 1.1] for the discrete version.)

Proposition 4.7. *There exists a positive constant $c_1>0$ such that for all $x,y,z\in M$ and $u,\delta>0$,*

$$\mathbf{P}^z\big(\sup_{0\leqslant t\leqslant u}|l(x,t)-l(y,t)|>\delta\big)\leqslant 2e^u e^{-c_1\delta\sqrt{V(d(x,y))/\phi(d(x,y))}}. \tag{4.14}$$

Proof. Let

$$q(x,y):=(1-\mathbf{E}^x[e^{-\sigma_y^0}]\mathbf{E}^y[e^{-\sigma_x^0}])^{1/2}. \tag{4.15}$$

Note that, since $y\mapsto \mathbf{E}^y[e^{-\sigma_x}]$ is continuous (see (4.11)), by [9, V. 3.25 and 3.28]

$$\mathbf{P}^z\big(\sup_{0\leqslant t\leqslant u}|l(x,t)-l(y,t)|>\delta\big)\leqslant 2e^u e^{-\delta/(2q(x,y))}.$$

Since Proposition 4.1(iii) implies that

$$q(x,y)\leqslant (1-\mathbf{E}^x[e^{-\sigma_y^0}])+(1-\mathbf{E}^y[e^{-\sigma_x^0}])\leqslant c_1\phi(d(x,y))/V(d(x,y)), \tag{4.16}$$

the proof is complete. □

The next proposition is an analogue of [20, Lemma 5.5]. Since we do not have self-similarity of the process, serious modifications of the proof are needed. We will also use a version of Garsia's lemma (Lemma A.1), which is proved in Appendix A.2.

Proposition 4.8. *There exist a version of the local time $l(x,t)(\omega)$ such that almost surely $(x,t)\to l(x,t)(\omega)$ is continuous; moreover, there exist constants $c_1,c_2>0$ such that for all $z\in M$, $L,u,A>0$,*

$$\begin{aligned}&\mathbf{P}^z\big(\sup_{d(x,y)\leqslant L}\sup_{0\leqslant t\leqslant u}|l(x,t)-l(y,t)|\geqslant A\big)\\&\leqslant \frac{c_1V(\phi^{-1}(u)\vee L)^2}{V(L)^2}\exp\left(-c_2A\frac{V(\phi^{-1}(u)\vee L)}{\phi(\phi^{-1}(u)\vee L)}\sqrt{\frac{V((L/\phi^{-1}(u))\wedge 1)}{\phi((L/\phi^{-1}(u))\wedge 1)}}\right)\end{aligned}$$

Proof. First note that Assumption 4.4 and (4.16) (where q is defined by (4.15)) imply that the local time $l(x,t)(\omega)$ exists and it is jointly continuous almost surely.

In fact, since $\sup_{z\in M}u^1(z,z)<\infty$, by (4.11) we see that for any $x,y\in M$,

$$d_1(x,y)^2:=u^1(x,x)+u^1(y,y)-2u^1(x,y)\leqslant 2(\sup_{z\in M}u^1(z,z))q(x,y)^2\leqslant (c_0')^2q(x,y)^2.$$

Moreover, by (2.8), (2.10), Assumption 4.4 and (4.16), for any $x_0\in M$ and any $x,y\in B(x_0,1)$ it holds that

$$q(x,y)\leqslant c_1'\frac{\phi(d(x,y))}{V(d(x,y))}\leqslant c_2'd(x,y)^{d_3-d_2}.$$

Thus, for all $x\in B(x_0,1)$ and $\varepsilon\in(0,c_0'c_2')$ small enough so that $(\varepsilon/(c_0'c_2'))^{1/(d_3-d_2)}+d(x_0,x)<1$, we have

$$\mu(\{y\in B(x_0,1):d_1(x,y)<\varepsilon\})\geqslant \mu\big(B(x,(\varepsilon/(c_0'c_2'))^{1/(d_3-d_2)})\big)$$

$$\geqslant c_3' V\big((\varepsilon/(c_0'c_2'))^{1/(d_3-d_2)}\big) \geqslant c_4' \varepsilon^{d_2/(d_3-d_2)}.$$

Therefore, by [34, Theorem 6.3.3] (with $T = B(x_0,1)$, $d_X = d_1$ and μ being $\frac{1}{\mu(B(x_0,1))}\mu$), we have the almost sure continuity of the mean zero Gaussian process $\{G_1(x) : x \in B(x_0,1)\}$ with covariance $u^1(\cdot,\cdot)$, and so by [34, Theorem 9.4.1], $\{l(x,t) : x \in B(x_0,1), t \geqslant 0\}$ is jointly continuous almost surely. Since this is satisfied for any $x_0 \in M$, $\{l(x,t) : x \in M, t \geqslant 0\}$ is jointly continuous almost surely.

Since we will use a scaling argument in the remainder of the proof, we prepare a scaled distance and a scaled measure. Below, without loss of generality, we assume $\phi(1) = 1$. For each $\delta > 0$, define a metric $d_{(\delta)}$ and a measure $\mu_{(\delta)}$ on M by

$$\begin{aligned} d_{(\delta)}(x,y) &:= \delta^{-1} d(x,y), \quad \forall x,y \in M, \\ \mu_{(\delta)}(J) &:= V(\delta)^{-1}\mu(J), \quad \forall J \subset \mathcal{B}(M). \end{aligned} \tag{4.17}$$

For $\delta > 0$, let $(M, d_{(\delta)}, \mu_{(\delta)})$ be the scaled metric measure space defined by (4.17), and $X^{(\delta)} := \{X_{\phi(\delta)t} : t \geqslant 0\}$ be the scaled process in $(M, d_{(\delta)}, \mu_{(\delta)})$. We also let

$$V_{(\delta)}(r) = V(\delta r)/V(\delta), \quad \phi_{(\delta)}(r) = \phi(\delta r)/\phi(\delta)$$

and

$$B_{d_{(\delta)}}(x,r) = \{x \in M : d_{(\delta)}(x,y) < r\}.$$

Then, $\mu_{(\delta)}(B_{d_{(\delta)}}(x,r)) \asymp V_{(\delta)}(r)$ uniformly on $\delta, r > 0$ and $x \in M$,

$$c_1\Big(\frac{R}{r}\Big)^{d_1} \leqslant \frac{V_{(\delta)}(R)}{V_{(\delta)}(r)} \leqslant c_2\Big(\frac{R}{r}\Big)^{d_2} \qquad \text{for every} \quad \delta > 0, 0 < r < R < \infty, \tag{4.18}$$

and

$$c_3\Big(\frac{R}{r}\Big)^{d_3} \leqslant \frac{\phi_{(\delta)}(R)}{\phi_{(\delta)}(r)} \leqslant c_4\Big(\frac{R}{r}\Big)^{d_4} \qquad \text{for every} \quad \delta > 0, 0 < r < R < \infty. \tag{4.19}$$

In particular, if (M, d, μ) is an α-set, i.e. satisfies (1.4), then it is easy to see that $(M, d_{(\delta)}, \mu_{(\delta)})$ with $V(r) = r^\alpha$ is also an α-set, and $\mu_{(\delta)}$ satisfies (1.4) with the same constants $c_1, c_2 > 0$.

Note that the transition density function $p^{(\delta)}(t,x,y)$ of $X^{(\delta)}$ with respect to the measure $\mu_{(\delta)}$ is related to that of X by the formula

$$p^{(\delta)}(t,x,y) = V(\delta)p(\phi(\delta)t,x,y)$$

for all $t > 0$ and $x, y \in M$. Thus, from Assumptions 2.1 we have that all $x, y \in M$ and $t, \delta \in (0,\infty)$,

$$\begin{aligned} &p^{(\delta)}(t,x,y) \leqslant C_1\bigg(\frac{1}{V_{(\delta)}(\phi_{(\delta)}^{-1}(t))} \wedge \frac{t}{V_{(\delta)}(d_{(\delta)}(x,y))\phi_{(\delta)}(d_{(\delta)}(x,y))}\bigg), \\ &C_2\bigg(\frac{1}{V_{(\delta)}(\phi_{(\delta)}^{-1}(t))} \wedge \frac{t}{V_{(\delta)}(d_{(\delta)}(x,y))\phi_{(\delta)}(d_{(\delta)}(x,y))}\bigg) \leqslant p^{(\delta)}(t,x,y). \end{aligned}$$

Let $l^{(\delta)}(x,t)$ be its local time with respect to the measure $\mu_{(\delta)}$, which exists by Proposition 4.3, (4.18), (4.19) and the assumption $d_2 < d_3$. Let $\mathbf{P}^{\cdot}_{(\delta)}$ be its probability space.

In the following, set $\delta' = \delta^{-1}$. Then, from (4.9) we see that $(V(\delta')/\phi(\delta'))l(y, \phi(\delta')t)$ under $\mathbf{P}^x$ corresponds to $l^{(\delta')}(y,t)$ under $\mathbf{P}^x_{(\delta')}$. Thus, choosing $\delta = (1/\phi^{-1}(u)) \wedge L^{-1}$, we have

$$\begin{aligned}
&\mathbf{P}^z\Big(\sup_{d(x,y)\leqslant L}\sup_{0\leqslant t\leqslant u}|l(x,t)-l(y,t)|\geqslant A\Big)\\
&=\mathbf{P}^z\Big(\sup_{d(x,y)\leqslant L}\sup_{0\leqslant t\leqslant u/\phi(\delta')}V(\delta')/\phi(\delta')\\
&\qquad |l(x,\phi(\delta')t)-l(y,\phi(\delta')t)|\geqslant AV(\delta')/\phi(\delta')\Big) \qquad (4.20)\\
&\leqslant \mathbf{P}^z_{(\delta')}\Big(\sup_{d_{(\delta')}(x,y)\leqslant \delta L}\sup_{0\leqslant t\leqslant u/\phi(\delta')}|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|\geqslant AV(\delta')/\phi(\delta')\Big)\\
&\leqslant \mathbf{P}^z_{(\delta')}\Big(\sup_{d_{(\delta')}(x,y)\leqslant \delta L}\sup_{0\leqslant t\leqslant 1}|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|\geqslant AV(\delta')/\phi(\delta')\Big).
\end{aligned}$$

Set $U(r)=\sqrt{\phi(r)/V(r)}$ and $H=B_{d_{(\delta')}}(x_0,1/2)$ for some $x_0\in M$, and define

$$\Gamma_{\delta'}(H):=\iint_{H\times H}\Big(\exp\Big(c_*\frac{\sup_{0\leqslant t\leqslant 1}|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|}{U(d_{(\delta')}(x,y))}\Big)-1\Big)\,\mu_{(\delta')}(dx)\,\mu_{(\delta')}(dy),$$

$$F_{\delta'}:=\iint_{d_{(\delta')}(x,y)\leqslant 1}\Big(\exp\Big(c_*\frac{\sup_{0\leqslant t\leqslant 1}|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|}{U(d_{(\delta')}(x,y))}\Big)-1\Big)\,\mu_{(\delta')}(dx)\,\mu_{(\delta')}(dy),$$

for small constant $c_*>0$. Clearly $\Gamma_{\delta'}(H)\leqslant F_{\delta'}$ and, by (2.8) and (2.10),

$$c_L\Big(\frac{R}{r}\Big)^{(d_3-d_2)/2}\leqslant\frac{U(R)}{U(r)}\leqslant c_U\Big(\frac{R}{r}\Big)^{(d_4-d_1)/2} \qquad (4.21)$$

holds for all $0<r\leqslant R$ and some positive constants c_L,c_U. We will prove in the end of this proof that $\mathbf{E}^z_{(\delta')}[F_{\delta'}]$ is uniformly bounded (with respect to δ) so that $\Gamma_{\delta'}(H)\leqslant F_{\delta'}<\infty$. Assuming this fact for the moment, we can apply Lemma A.1 with $\Psi(x)=e^{c_*x}-1$ and $q(u)=U(u)$, and deduce

$$|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|\leqslant c_0\int_0^{d_{(\delta')}(x,y)}\log(c_1\Gamma_{\delta'}(H)V_{(\delta')}(u)^{-2}+1)\,\frac{U(u)du}{u}$$

for $\mu_{(\delta)}$-almost all $x,y\in B_{d_{(\delta')}}(x_0,1/16)$ and $t\leqslant 1$, and c_0,c_1 are independent of x_0. Due to (4.18) and (4.21), as stated in Lemma A.1 the above estimate holds for $l^{(\delta')}(y,t)$ under $\mathbf{P}^z_{(\delta')}$ uniformly (i.e. with the same constants $c_0,c_1>0$ for all $\delta>0$). By (4.21) again, there exist constants $c_2,c_3>0$ independent of δ such that for $\mu_{(\delta)}$-almost all $x,y\in M$ with $d_{(\delta')}(x,y)\leqslant\delta L$ and $t\leqslant 1$,

$$\begin{aligned}
|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|&\leqslant c_0\int_0^{\delta L}\log(c_1F_{\delta'}V_{(\delta')}(u)^{-2}+1)\,\frac{U(u)du}{u}\\
&\leqslant c_2U(\delta L)\big(\log(1+c_3F_{\delta'}V_{(\delta')}(\delta L)^{-2})\big).
\end{aligned} \qquad (4.22)$$

Indeed, by (4.18) and (4.21),

$$\begin{aligned}
&\int_0^{\delta L}\log(c_1F_{\delta'}V_{(\delta')}(u)^{-2}+1)\,\frac{U(u)du}{u}\\
&\leqslant\sum_{k=0}^\infty\big(\log(1+c_1F_{\delta'}V_{(\delta')}(\delta L/2^{k+1})^{-2})\big)U(\delta L/2^k)\\
&\leqslant c_2'\big(\log(1+c_3F_{\delta'}V_{(\delta')}(\delta L)^{-2})\big)U(\delta L)\sum_{k=0}^\infty 2^{-k(d_3-d_2)/2}
\end{aligned}$$

$$\leqslant c_2 U(\delta L)\big(\log(1+c_3 F_{\delta'} V_{(\delta')}(\delta L)^{-2})\big).$$

Plugging this into (4.20), we have

$$\begin{aligned}
&\mathbf{P}^z\Big(\sup_{d(x,y)\leqslant L}\sup_{0\leqslant t\leqslant u}|l(x,t)-l(y,t)|\geqslant A\Big)\\
&\leqslant \mathbf{P}^z_{(\delta')}\Big(c_2U(\delta L)\log(1+c_3F_{\delta'}V_{(\delta')}(\delta L)^{-2})>AV(\delta')/\phi(\delta')\Big)\\
&=\mathbf{P}^z_{(\delta')}\Big(\log(1+c_3F_{\delta'}V_{(\delta')}(\delta L)^{-2})\geqslant c_2^{-1}AV(\delta')/(U(\delta L)\phi(\delta'))\Big)\\
&\leqslant e^{-c_2^{-1}AV(\delta')/(U(\delta L)\phi(\delta'))}\Big(1+c_3\mathbf{E}^z_{(\delta')}[F_{\delta'}]/V_{(\delta')}(\delta L)^2\Big)\\
&\leqslant \frac{c_4}{V_{(\delta')}(\delta L)^2}e^{-c_2^{-1}AV(\delta')/(U(\delta L)\phi(\delta'))}\Big(1+\mathbf{E}^z_{(\delta')}[F_{\delta'}]\Big)\\
&=\frac{c_4V(\phi^{-1}(u)\vee L)^2}{V(L)^2}e^{-c_2^{-1}AV(\phi^{-1}(u)\vee L)/(U((L/\phi^{-1}(u))\wedge 1)\phi(\phi^{-1}(u)\vee L))}\Big(1+\mathbf{E}^z_{(\delta')}[F_{\delta'}]\Big),
\end{aligned}$$

where we used Chebyshev's inequality in the second inequality, the fact that $\delta L\leqslant 1$ (so that $V_{(\delta')}(\delta L)\leqslant 1$) in the third inequality and put $\delta=(1/\phi^{-1}(u))\wedge L^{-1}$ in the last equality.

Finally, we will check the integrability of $F_{\delta'}$. Using (4.14) for $l^{(\delta')}(y,t)$ under $\mathbf{P}^z_{(\delta')}$ (note that (4.14) holds uniformly, i.e. with the same constant $c_5>0$ for all $\delta'>0$), we have

$$\mathbf{P}^z_{(\delta')}\Big(\sup_{0\leqslant t\leqslant 1}|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|\geqslant kU(d_{(\delta')}(x,y))\Big)\leqslant 2e^{1-c_5k}.$$

Let $c_*=c_5/2$, and

$$I_{(\delta')}(x,y,s)=\exp\Big(c_*\frac{\sup_{0\leqslant t\leqslant s}|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|}{U(d_{(\delta')}(x,y))}\Big).$$

Thus, we have

$$\begin{aligned}
&\mathbf{E}^z_{(\delta')}[I_{(\delta')}(x,y,1)]\\
&\quad\leqslant\sum_{k=0}^\infty e^{c_*(k+1)}\mathbf{P}^z_{(\delta')}\left(k\leqslant\frac{\sup_{0\leqslant t\leqslant 1}|l^{(\delta')}(x,t)-l^{(\delta')}(y,t)|}{U(d_{(\delta')}(x,y))}\leqslant k+1\right)\\
&\quad\leqslant 2e^{1+c_*}\sum_{k=0}^\infty e^{-c_5k/2}=:K<\infty.
\end{aligned}$$

Note that this value is uniformly bounded for all $\delta'>0$. Take an open covering

$$\big\{d_{(\delta')}(x,y)\leqslant 1\big\}\subset\cup_i(B_{(\delta')}(x_i,2)\times B_{(\delta')}(x_i,2))$$

such that each point in $\{d_{(\delta')}(x,y)\leqslant 1\}$ is covered by at most a (uniformly) finite number of $\{B_{(\delta')}(x_i,2)\times B_{(\delta')}(x_i,2)\}_i$, say C_0. Using the doubling property of the volume and the assumption that balls are relatively compact, such a covering is possible. For each x,y with $d_{(\delta')}(x,y)\leqslant 1$,

$$\begin{aligned}
\mathbf{E}^z_{(\delta')}[I_{(\delta')}(x,y,1)-1]&=\mathbf{E}^z_{(\delta')}\Big[1_{\{\sigma_{B_{(\delta')}(x_i,2)}\leqslant 1\}}\mathbf{E}^{X^{(\delta')}_{\sigma_{B_{(\delta')}(x_i,2)}}}\Big[I_{(\delta')}(x,y,1-\sigma_{B_{(\delta')}(x_i,2)})-1\Big]\Big]\\
&\leqslant (K-1)\mathbf{P}^z_{(\delta')}(\sigma_{B_{(\delta')}(x_i,2)}\leqslant 1).
\end{aligned}$$

So

$$\begin{aligned}\mathbf{E}^z_{(\delta')}[F_{\delta'}] &= \iint_{d_{(\delta')}(x,y)\leqslant 1} \mathbf{E}^z_{(\delta')}[I_{(\delta')}(x,y,1)-1]\, d\mu_{(\delta')}(x)\, d\mu_{(\delta')}(y)\\ &\leqslant c_6(K-1)\sum_i \mathbf{P}^z_{(\delta')}(\sigma_{B_{(\delta')}(x_i,2)}\leqslant 1).\end{aligned}$$

Here we note that $\mu_{(\delta')}(B_{(\delta')}(x_i,2)) \leqslant c_6' V(2)$, i.e. $\mu_{(\delta')}(B_{(\delta')}(x_i,2))$ is uniformly bounded. Noting that

$$\begin{aligned}&\mathbf{E}^z_{(\delta')}\left[\int_{B_{(\delta')}(x_i,4)} l^{(\delta')}(y,4)\,\mu_{(\delta')}(dy)\right] = \mathbf{E}^z_{(\delta')}\left[\int_0^4 1_{B_{(\delta')}(x_i,4)}(X_s^{(\delta')})\,ds\right]\\ &\geqslant \mathbf{E}^z_{(\delta')}\left[\int_{\sigma_{B_{(\delta')}(x_i,2)}}^{3+\sigma_{B_{(\delta')}(x_i,2)}} 1_{B_{(\delta')}(x_i,4)}(X_s^{(\delta')})\,ds : \sigma_{B_{(\delta')}(x_i,2)}\leqslant 1\right]\\ &\geqslant \mathbf{E}^z_{(\delta')}\left[\mathbf{E}^{X^{(\delta')}_{\sigma_{B_{(\delta')}(x_i,2)}}}_{(\delta')}\Big[\int_0^3 1_{B_{(\delta')}(x_i,4)}(X_s^{(\delta')})\,ds\Big] 1_{\{\sigma_{B_{(\delta')}(x_i,2)}\leqslant 1\}}\right]\\ &\geqslant c_7\mathbf{P}^z_{(\delta')}(\sigma_{B_{(\delta')}(x_i,2)}\leqslant 1),\end{aligned}$$

where the last inequality is due to the fact that

$$\mathbf{E}^{X^{(\delta')}_{\sigma_{B_{(\delta')}(x_i,2)}}}_{(\delta')}\left[\int_0^3 1_{B_{(\delta')}(x_i,4)}(X_s^{(\delta')})\,ds\right]$$

is uniformly bounded from below. Indeed, since $\phi_{(\delta')}(1)=1$ for all $\delta'>0$, using Proposition 2.11 for the scaled process and the semigroup property for the Dirichlet heart kernel, we have

$$\begin{aligned}&\inf_{w\in B_{(\delta')}(x_i,2)} \mathbf{E}^w_{(\delta')}\left[\int_0^3 1_{B_{(\delta')}(x_i,4)}(X_s^{(\delta')})\,ds\right] \geqslant 3\inf_{w\in B_{(\delta')}(x_i,2)} \mathbf{P}^w_{(\delta')}(\tau_{B_{(\delta')}(x_i,4)}\geqslant 3)\\ &= 3\inf_{w\in B_{(\delta')}(x_i,4)}\int_{B_{(\delta')}(x_i,2)} p^{(\delta),B_{(\delta')}(x_i,4)}(3,w,y)\,\mu_{(\delta')}(dy)\geqslant c_8.\end{aligned}$$

We thus obtain

$$\begin{aligned}\sum_i \mathbf{P}^z_{(\delta')}(\sigma_{B_{(\delta')}(x_i,2)}\leqslant 1) &\leqslant c_9\sum_i \mathbf{E}^z_{(\delta')}\left[\int_{B_{(\delta')}(x_i,4)} l^{(\delta')}(y,4)\,\mu_{(\delta')}(dy)\right]\\ &\leqslant c_{10}\mathbf{E}^z_{(\delta')}\left[\int_{M^{(\delta')}} l^{(\delta')}(y,4)\,\mu_{(\delta')}(dy)\right] = 4c_{10},\end{aligned}$$

so we conclude $\mathbf{E}^z_{(\delta')}[F_{\delta'}]$ is uniformly bounded. □

Remark 4.9. In lines 8 and 12 of [20, p. 526], $(N/(1-c))^{n(t)}$ should be changed to $N^{n(t)\rho}/(1-c)^{n(t)\rho/2}$. Because of the typos, in the statement of [20, Lemma 5.5], $\exp\big(-c_{55}ta^\rho\delta^{-\rho\theta/2}\big)$ should be changed to $\exp\big(-c_{55}t^{(1+d_s/2)\rho/2}a^\rho\delta^{-\rho\theta/2}\big)$.

4.3. **Laws of the iterated logarithm for the maximum of local times and ranges of processes.** In the subsection, we always assume that Assumption 4.4 is satisfied. In particular, according to Proposition 4.8, the joint continuous version of the local time of the process X, which is denoted by $l(x,t)$ as before, exists for all $x \in M$. Denote by

$$L^*(t) = \sup_{x\in M} l(x,t), \quad t > 0.$$

We will establish two LILs for $L^*(t)$.

Remark 4.10. Even for one-dimensional Lévy process, some mild assumptions like Assumption 4.4 on characteristic exponent (also called symbol) are required to establish LILs of associated local times, see [41].

First, we have the following LIL for $L^*(t)$.

Theorem 4.11. *Under Assumption* 4.4*, there exists a constant* $c_0 \in (0,\infty)$ *such that*

$$\limsup_{t\to\infty} \frac{L^*(t)}{t/V(\phi^{-1}(t/\log\log t))} = c_0, \qquad \mathbf{P}^x\text{-}a.e.\ \omega,\ \forall x \in M.$$

We need the following tail probability estimate for the local time $l(x,t)$.

Lemma 4.12. *Under Assumption* 4.4*, there exists a constant* $c_1 > 0$ *such that for all* $x, y \in M$ *and* $t, b > 0$*,*

$$\mathbf{P}^y\left(l(x,t) \geqslant \frac{bt}{V(\phi^{-1}(t))}\right) \leqslant 2e^{-c_1 b}.$$

Proof. For any $\varepsilon > 0$, by Assumption 2.1,

$$\mathbf{P}^y(d(X_s,x) \leqslant \varepsilon) = \int_{B(x,\varepsilon)} p(s,y,z)\,\mu(dz) \leqslant \frac{C_2}{V(\phi^{-1}(s))}\mu(B(x,\varepsilon)),$$

and so

$$\int_0^t \mathbf{P}^y(d(X_s,x) \leqslant \varepsilon)\,ds \leqslant C_2\mu(B(x,\varepsilon)) \int_0^t \frac{1}{V(\phi^{-1}(s))}\,ds.$$

Combining this with the fact

$$l(x,t) = \lim_{\varepsilon\to 0} \frac{1}{\mu(B(x,\varepsilon))} \int_0^t \mathbb{1}_{B(x,\varepsilon)}(X_s)\,ds,$$

we have

$$\mathbf{E}^y(l(x,t)) \leqslant C_2 \int_0^t \frac{1}{V(\phi^{-1}(s))}\,ds. \tag{4.23}$$

Furthermore, according to the estimate above and Proposition 4.6, we find that

$$\mathbf{E}^y\Big(l(x,t)^n\Big) \leqslant n!\left(C_2\int_0^t \frac{1}{V(\phi^{-1}(s))}\,ds\right)^n, \quad n \geqslant 0,$$

which implies that

$$\mathbf{E}^y\left(\exp\left(\frac{l(x,t)}{2C_2\int_0^t \frac{1}{V(\phi^{-1}(s))}\,ds}\right)\right) \leqslant 2. \tag{4.24}$$

The desired assertion is a direct consequence of the inequality above, the Chebyshev inequality and (4.12). □

Remark 4.13. Alternatively one can obtain the exponential integrability (4.24) directly from (4.23), by applying Khas'misnkii's lemma, e.g. see [38, Lemma B.1.2].

Proposition 4.14. *There are constants $c_1, c_2 > 0$ such that for $b \geqslant 1$,*

$$\sup_{t>0, x\in M} \mathbf{P}^x \left(L^*(t) \geqslant \frac{bt}{V(\phi^{-1}(t))} \right) \leqslant c_1 b^{-c_2}.$$

Proof. Let f be an increasing function such that $f(1) = 1$ and $\lim_{r\to\infty} f(r) = \infty$. By (3.4), the doubling property of ϕ and (2.10), we find that for any $x \in M$ and $t > 0$ and $b \geqslant 1$,

$$\begin{aligned}
&\mathbf{P}^x \left(L^*(t) \geqslant \frac{2bt}{V(\phi^{-1}(t))} \right) \\
&\leqslant \mathbf{P}^x \left(\sup_{d(z,x)\leqslant f(b)\phi^{-1}(t)} l(z,t) \geqslant \frac{2bt}{V(\phi^{-1}(t))} \right) + \mathbf{P}^x v \left(\sup_{0<s\leqslant t} d(X_s, x) \geqslant f(b)\phi^{-1}(t) \right) \\
&\leqslant \mathbf{P}^x \left(\sup_{d(z,x)\leqslant f(b)\phi^{-1}(t)} l(z,t) \geqslant \frac{2bt}{V(\phi^{-1}(t))} \right) + \frac{c_0 t}{\phi(f(b)\phi^{-1}(t))} \\
&\leqslant \mathbf{P}^x \left(\sup_{d(z,x)\leqslant f(b)\phi^{-1}(t)} l(z,t) \geqslant \frac{2bt}{V(\phi^{-1}(t))} \right) + c_1 f(b)^{-d_3}
\end{aligned}$$

for some constant $c_1 > 0$.

On the one hand, by Lemma 4.12, there is a constant $c_2 > 0$ such that for all $x \in M$, $t > 0$ and $b \geqslant 1$

$$\begin{aligned}
&\mathbf{P}^x \left(\sup_{d(z,x)\leqslant f(b)\phi^{-1}(t)} l(z,t) \geqslant \frac{2bt}{V(\phi^{-1}(t))} \right) \\
&\leqslant \mathbf{P}^x \left(\sup_{d(z,x)\leqslant f(b)\phi^{-1}(t)} |l(z,t) - l(x,t)| \geqslant \frac{bt}{V(\phi^{-1}(t))} \right) + \mathbf{P}^x \left(l(x,t) \geqslant \frac{bt}{V(\phi^{-1}(t))} \right) \\
&\leqslant \mathbf{P}^x \left(\sup_{d(z,x)\leqslant f(b)\phi^{-1}(t)} |l(z,t) - l(x,t)| \geqslant \frac{bt}{V(\phi^{-1}(t))} \right) + 2e^{-c_2 b}.
\end{aligned}$$

On the other hand, according to Proposition 4.8, there are constants $c_3, c_4 > 0$ such that for all $t > 0$ and $b \geqslant 1$,

$$\begin{aligned}
&\mathbf{P}^x \left(\sup_{d(z,x)\leqslant f(b)\phi^{-1}(t)} |l(z,t) - l(x,t)| \geqslant \frac{bt}{V(\phi^{-1}(t))} \right) \\
&\leqslant c_3 \exp \left(-c_4 b \frac{t}{V(\phi^{-1}(t))} \frac{V(f(b)\phi^{-1}(t))}{\phi(f(b)\phi^{-1}(t))} \right) \\
&= c_3 \exp \left(-c_4 b \frac{V(f(b)\phi^{-1}(t))}{V(\phi^{-1}(t))} \frac{\phi(\phi^{-1}(t))}{\phi(f(b)\phi^{-1}(t))} \right) \\
&\leqslant c_5 \exp \left(-c_6 b f(b)^{d_1} f(b)^{-d_4} \right) = c_5 \exp \Big(- \frac{c_6 b}{f(b)^\theta} \Big),
\end{aligned}$$

where $\theta := d_4 - d_1 > 0$.

Combining with all the estimates above, we find that

$$\sup_{t>0} \mathbf{P}^x \left(L^*(t) \geqslant \frac{bt}{V(\phi^{-1}(t))} \right) \leqslant c_7 \left[f(b)^{-d_3} + e^{-c_2 b} + \exp \left[-\Big(\frac{c_6 b}{f(b)^\theta} \Big) \right] \right].$$

The proof is finished by taking $f(r) = r^{1/(2\theta)}$ in the inequality above . □

Now, we are ready to prove Theorem 4.11.

Proof of Theorem 4.11. (i)**(Upper bound)**: According to Proposition 4.14, we find that

$$\sup_{t>0,x\in M} \mathbf{P}^x\left(L^*(t) \geqslant \frac{bt}{V(\phi^{-1}(t))}\right) \to 0, \quad b\to\infty.$$

Then, according to Proposition A.2 and the (stronger) doubling properties of V and ϕ, we know that

$$\limsup_{t\to\infty} \frac{L^*(t)}{t/V(\phi^{-1}(t/\log\log t))} = \limsup_{t\to\infty} \frac{L^*(t)}{\frac{(t/\log\log t)}{V(\phi^{-1}(t/\log\log t))}(\log\log t)} \leqslant c_0.$$

(ii)**(Lower bound)**: Let $R(t) = \mu(X([0,t]))$ be the range of the process. By Theorem 3.8, there is a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$, and

$$\sup_{0\leqslant s\leqslant t_n} d(X_s, x) \leqslant c_1\phi^{-1}\left(\frac{t_n}{\log\log t_n}\right).$$

Since $R(t) \leqslant c_2 V(\sup_{0\leqslant s\leqslant t} d(X_s,x))$,

$$R(t_n) \leqslant c_3 V\left(\phi^{-1}\left(\frac{t_n}{\log\log t_n}\right)\right).$$

In particular,

$$\liminf_{t\to\infty} \frac{R(t)}{V\left(\phi^{-1}\left(t/\log\log t\right)\right)} \leqslant c_3. \tag{4.25}$$

By the fact that

$$t = \int_{X([0,t])} l(x,t)\,\mu(dx) \leqslant L^*(t)R(t), \tag{4.26}$$

we get

$$\limsup_{t\to\infty} \frac{L^*(t)}{t/V\left(\phi^{-1}\left(t/\log\log t\right)\right)} \geqslant \limsup_{t\to\infty} \frac{t}{R(t)t/V\left(\phi^{-1}\left(t/\log\log t\right)\right)} \geqslant \frac{1}{c_3}.$$

From those two inequalities above, we have proved the desired assertion by zero-one law for tail events (see Theorem 2.10). ☐

Next, we turn to another LIL.

Theorem 4.15. *Under Assumption* 4.4, *there exists a constant* $c_0 \in (0,\infty)$ *such that*

$$\liminf_{t\to\infty} \frac{L^*(t)}{(t/\log\log t)/V(\phi^{-1}(t/\log\log t))} = c_0, \qquad \mathbf{P}^x\text{-}a.e.\ \omega,\ \forall x\in M.$$

Proof. (i)**(Lower bound)**: Let $R(t)$ be the range of the process. Then, by (3.4),

$$\mathbf{P}^x(R(t)\geqslant r) \leqslant \mathbf{P}^x(\sup_{0\leqslant s\leqslant t} d(X_s,x) \geqslant V^{-1}(c_1 r)) \leqslant \frac{c_2 t}{\phi(V^{-1}(c_1 r))}.$$

According to the doubling properties of V and ϕ,

$$\sup_{x\in M,t>0} \mathbf{P}^x(R(t)\geqslant bV(\phi^{-1}(t))) \to 0, \quad b\to\infty.$$

This, along with Proposition A.2 and the doubling properties of V and ϕ again, yields that

$$\limsup_{t\to\infty} \frac{R(t)}{V(\phi^{-1}(t/\log\log t))\log\log t} \leqslant c_3. \tag{4.27}$$

Also due to (4.26), we get that

$$\begin{aligned}&\liminf_{t\to\infty} \frac{L^*(t)}{(t/\log\log t)/V(\phi^{-1}(t/\log\log t))}\\ &\geqslant \liminf_{t\to\infty} \frac{t}{R(t)(t/\log\log t)/V(\phi^{-1}(t/\log\log t))} \geqslant \frac{1}{c_3}.\end{aligned}$$

(ii)**(Upper bound)**: Below, we turn to prove that

$$\liminf_{t\to\infty} \frac{L^*(t)}{(t/\log\log t)/V(\phi^{-1}(t/\log\log t))} \leqslant c_4,$$

which along with the inequality above and zero-one law for tail events (see Theorem 2.10) yields the required assertion.

Let $t_k = e^{k^2}$. Then,

$$\begin{aligned}&\liminf_{t\to\infty}\frac{L^*(t)}{(t/\log\log t)/V(\phi^{-1}(t/\log\log t))}\\ &\leqslant \limsup_{k\to\infty} \frac{L^*(t_k)}{(t_{k+1}/\log\log t_{k+1})/V(\phi^{-1}(t_{k+1}/\log\log t_{k+1}))}\\ &\quad + \liminf_{k\to\infty}\sup_{x\in M} \frac{l(x,t_{k+1})-l(x,t_k)}{(t_{k+1}/\log\log t_{k+1})/V(\phi^{-1}(t_{k+1}/\log\log t_{k+1}))}.\end{aligned}$$

From Theorem 4.11, (4.13) and the assumption $d_3 > d_2$, we know that

$$\limsup_{k\to\infty} \frac{L^*(t_k)}{(t_{k+1}/\log\log t_{k+1})/V(\phi^{-1}(t_{k+1}/\log\log t_{k+1}))} = 0.$$

So, by the Markov property and the second Borel-Cantelli lemma, it suffices to prove that there is a constant $C > 0$ such that for any $x \in M$,

$$\sum_{k=1}^{\infty} \mathbf{P}^x\Big(\sup_{x\in M}(l(x,t_{k+1})-l(x,t_k)) < C\frac{t_{k+1}/\log\log t_{k+1}}{V(\phi^{-1}(t_{k+1}/\log\log t_{k+1}))}|\mathscr{F}_{t_k}\Big) = \infty.$$

For this, we follow the proofs of [8, Proposition 4.8] and [41, Theorem 3.2] but with some significant modifications. Note that, using Assumption 2.1, we have that there is a constant $c_0 = c_0(d_3) \in (0,1)$ such that for every $t > 0$ and balls B_1 and B_2 of radius $2\phi^{-1}(t)$ with $B_1 \cap B_2 \neq \emptyset$,

$$\begin{aligned}&\inf_{t>0,z\in B_1}\int_{B_2} p(t,z,y)\,\mu(dy)\\ &\geqslant c\inf_{t>0,z\in B_1}\int_{B_2}\left(\frac{1}{V(\phi^{-1}(t))}\wedge\frac{t}{V(d(z,y))\phi(d(z,y))}\right)\mu(dy)\\ &\geqslant c\inf_{t>0}\left(\frac{1}{V(\phi^{-1}(t))}\wedge\frac{t}{V(8\phi^{-1}(t))\phi(8\phi^{-1}(t))}\right)\mu(B_2) \geqslant c_0,\end{aligned}\tag{4.28}$$

where in the last inequality we used the doubling properties of V and ϕ.

Let $\gamma = -4\log(c_0/2)$ and constants $\rho > 2$ and $c_* > 0$ will be chosen later. Set $s = \gamma t/\log\log t$ for $t > e^2$. According to Lemma A.4, there exists a sequence $\{A_i\}_{i=0}^{\infty}$

depending on x and s such that each A_i is a ball of radius $2\phi^{-1}(s)$, $\lim_{i\to\infty} d(x, A_i) = \infty$, and the following hold:

$$x \in A_0, \quad A_i \cap A_{i+1} \neq \emptyset \ \text{ for all } \ i \in \mathbb{N}, \ A_i \cap A_j = \emptyset \ \text{ for all } \ |i-j| \geqslant 2.$$

For $k \geqslant 1$, set

$$E_k = \Big\{ \sup_{x\in M} (l(x, ks) - l(x,(k-1)s)) \leqslant c_*(t/\log\log t)/V(\phi^{-1}(t/\log\log t)),$$
$$\sup_{0\leqslant u<s} d(X_{(k-1)s+u}, X_{(k-1)s}) \leqslant \rho\phi^{-1}(s), \ X_{ks} \in A_{2k} \Big\}.$$

Let

$$B_1 := \big\{ L^*(s) \leqslant c_*(t/\log\log t)/V(\phi^{-1}(t/\log\log t) \big\},$$
$$B_2 := \big\{ \sup_{0<u<s} d(X_u, X_0) \leqslant \rho\phi^{-1}(s) \big\} \ \text{ and } \ B_{3,k} := \big\{ X_s \in A_{2k} \big\}.$$

By the strong Markov property, for all $x \in M$,

$$\begin{aligned}\mathbf{P}^x \left(\bigcap_{k=1}^{n_0} E_k | \mathscr{F}_{(n_0-1)s} \right) &= \Big(\prod_{k=1}^{n_0-1} \mathbb{1}_{E_k} \Big) \mathbf{P}^{X_{(n_0-1)s}}(E_{n_0}) \\ &= \Big(\prod_{k=1}^{n_0-1} \mathbb{1}_{E_k} \Big) \mathbf{P}^{X_{(n_0-1)s}}(B_1 \cap B_2 \cap B_{3,n_0}).\end{aligned} \tag{4.29}$$

First, let c_1, d_1 and d_4 be the constants in (4.13). For $s > 0$ and $c_* > 0$ with $c_* c_1 \gamma^{-1+(d_1/d_4)} \geqslant 1$, using Proposition 4.14, we have

$$\begin{aligned}\sup_{z\in M} \mathbf{P}^z(B_1^c) &\leqslant \sup_{z\in M} \mathbf{P}^z \left(L^*(s) \geqslant c_* c_1 \gamma^{-1+(d_1/d_4)} s/V(\phi^{-1}(s)) \right) \\ &\leqslant c_2 \big(c_* c_1 \gamma^{-1+(d_1/d_4)} \big)^{-c_3},\end{aligned}$$

where in the first inequality we have used (4.13), and c_2, c_3 are positive constants independent of s and c_*. Second, according to Propositions 2.5 and 2.12, there is a constant $c_4 \in (0,1)$ such that for all $s > 0$ and $\rho \geqslant 1$,

$$\sup_{z\in M} \mathbf{P}^z(B_2^c) \leqslant \sup_{z\in M} \mathbf{P}^z \left(\sup_{0<u<s} d(X_u, z) \geqslant \rho\phi^{-1}(s) \right) \leqslant c_4^\rho.$$

Third, by (4.28), for any $k \geqslant 1$,

$$\inf_{z\in A_{2(k-1)}} \mathbf{P}^z(B_{3,k}) = \inf_{z\in A_{2(k-1)}} \int_{A_{2k}} p(s,z,y)\,\mu(dy) \geqslant c_0.$$

Combining with all the estimates above and the fact

$$\mathbf{P}(D_1 \cap D_2 \cap D_3) \geqslant \mathbf{P}(D_3) - \mathbf{P}(D_1^c) - \mathbf{P}(D_2^c),$$

we find that

$$\inf_{z\in A_{2(k-1)}} \mathbf{P}^z(E_k) = \inf_{z\in A_{2(k-1)}} \mathbf{P}^z(B_1 \cap B_2 \cap B_{3,k}) \geqslant c_0 - c_2 \big(c_* c_1 \gamma^{-1+(d_1/d_4)} \big)^{-c_3} - c_4^\rho.$$

Now we choose c_* and ρ depending on d_1, d_4 and c_i, $i = 1, \ldots 4$, large enough such that $\inf_{z\in A_{2(k-1)}} \mathbf{P}^x(E_k) \geqslant c_0/2$. By this and (4.29), we find that for all $x \in M$ and $t > e^2$,

$$\mathbf{P}^x \Big(\bigcap_{k=1}^{n_0} E_k \Big) \geqslant (c_0/2)^{n_0} \geqslant (c_0/2) \Big(\log t \Big)^{-1/4},$$

where $n_0 = [\frac{\log\log t}{\gamma}] + 1 = [\frac{\log\log t}{-4\log(c_0/2)}] + 1$. Since there is a constant $C = C(c_*, \rho) > 0$ such that

$$\bigcap_{k=1}^{n_0} E_k \subset \Big\{L^*(t) < C(t/\log\log t)/V(\phi^{-1}(t/\log\log t))\Big\},$$

we get for all $x \in M$ and $t > e^2$,

$$\mathbf{P}^x\Big\{L^*(t) < C(t/\log\log t)/V(\phi^{-1}(t/\log\log t))\Big\} \geqslant (c_0/2)\Big(\log t\Big)^{-1/4},$$

Therefore,

$$\begin{aligned}&\mathbf{P}^x\Big(\sup_{x\in M}(l(x,t_{k+1}) - l(x,t_k)) < C(t_{k+1}/\log\log t_{k+1})/V(\phi^{-1}(t_{k+1}/\log\log t_{k+1}))|\mathscr{F}_{t_k}\Big)\\ &\geqslant \inf_{z\in M}\mathbf{P}^z\Big(L^*(t_{k+1}) < C(t_{k+1}/\log\log t_{k+1})/V(\phi^{-1}(t_{k+1}/\log\log t_{k+1}))\Big)\\ &\geqslant (c_0/2)(k+1)^{-1/2},\end{aligned}$$

whose summation on k diverges. This completes the proof. □

As in the proofs of Theorems 4.11 and 4.15, let $R(t) = \mu(X([0,t]))$ be the range of the process X. As a direct application of previous theorems, we have the following statements for the ranges.

Theorem 4.16. *Under Assumption* 4.4*, there exist constants* $c_0, c_1 \in (0,\infty)$ *such that*

$$\limsup_{t\to\infty} \frac{R(t)}{V(\phi^{-1}(t/\log\log t))\log\log t} = c_0, \qquad \mathbf{P}^x\text{-a.e. } \omega,\ \forall x \in M, \tag{4.30}$$

$$\liminf_{t\to\infty} \frac{R(t)}{V\left(\phi^{-1}\left(t/\log\log t\right)\right)} = c_1, \qquad \mathbf{P}^x\text{-a.e. } \omega,\ \forall x \in M. \tag{4.31}$$

Proof. First, the upper bound of (4.30) is already obtained in (4.27). The lower bound of (4.30) is a consequence of (4.26) and Theorem 4.15. Next, the upper bound of (4.31) is already obtained in (4.25). The lower bound of (4.31) is a consequence of (4.26) and Theorem 4.11. Finally, the zero-one law for tail events (Theorem 2.10) yields the desired results. □

5. Examples: Jump Processes of Mixed Types on Metric Measure Spaces

We now give three examples. The first one is the β-stable-like processes on α-set. This is the case $d_1 = d_2 = \alpha$ and $d_3 = d_4 = \beta$ in (2.8) and (2.10), and our results can be written simply as Theorem 1.3 in Section 1.

The other two examples below are essentially taken from [14, Example 2.3(1) and (2)]. We recall the framework on the metric measure space from here. Let (M, d, μ) be a locally compact, separable and connected metric space such that there is a strictly increasing function V satisfying (2.1) and (2.8), i.e. for any $x \in M$ and $r > 0$, $\mu(B(x,r)) \asymp V(r)$, and there exist constants $c_1, c_2 > 0$, $d_2 \geqslant d_1 > 0$ such that

$$c_1\Big(\frac{R}{r}\Big)^{d_1} \leqslant \frac{V(R)}{V(r)} \leqslant c_2\Big(\frac{R}{r}\Big)^{d_2} \quad \text{for every} \quad 0 < r < R < \infty.$$

Example 5.1. Assume that there exist $0 < \beta_1 \leqslant \beta_2 < \infty$ and a probability measure ν on $[\beta_1, \beta_2]$ such that

$$\phi(r) = \int_{\beta_1}^{\beta_2} r^{\beta}\, \nu(d\beta), \quad r > 0.$$

Clearly, ϕ is a continuous strictly increasing function such that (2.10) holds with $d_3 = \beta_1$ and $d_4 = \beta_2$. Consider a regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(M, \mu)$ such that $\mathscr{E}$ is given by (1.8) and the Lévy measure $n(\cdot, \cdot)$ satisfies (2.7) with the function ϕ given above. Then the associated Hunt process has the transition density function $p(t, x, y)$ satisfying Assumption 2.1 with the functions V and ϕ given above. Furthermore, we have the following assertions.

(i) All the statements of theorems in Section 3 hold for sample paths of the process X.
(ii) If $d_2 < \beta_1$, then the local time of the process X exists, and all the theorems in Section 4 hold for local times and the range of the process X.

Example 5.2. Consider the following increasing function

$$\phi(r) = \left(\int_{\beta_1}^{\beta_2} r^{-\beta}\, \nu(d\beta) \right)^{-1}, \quad r > 0,$$

where ν is a probability measure on $[\beta_1, \beta_2] \subset (0, \infty)$. We can check easily that for this example (2.10) also holds with $d_3 = \beta_1$ and $d_4 = \beta_2$. Consider a regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(M, \mu)$ such that $\mathscr{E}$ is given by (1.8) and the Lévy measure $n(\cdot, \cdot)$ satisfies (2.7) with the function ϕ given above. Then the associated Hunt process has the transition density function $p(t, x, y)$ satisfying Assumption 2.1 with the functions V and ϕ given above. Furthermore, we have the same conclusions for the process X as these in Example 5.1.

Example 5.3. We give an example where β could be strictly larger than 2. Assume that (M, d, μ) enjoys the following:

(i) μ is a α-set, namely $d_1 = d_2 = \alpha$.
(ii) There exists a μ-symmetric conservative diffusion on M which has a symmetric jointly continuous transition density $\{q(t, x, y) : t > 0, x, y \in M\}$ with the following estimates for all $t > 0, x, y \in M$:

$$\begin{aligned} c_1 t^{-\alpha/\beta_*} \exp\Big(- c_2 \Big(\frac{d(x, y)^{\beta_*}}{t} \Big)^{\frac{1}{\beta_* - 1}} \Big) &\leqslant q(t, x, y) \\ &\leqslant c_3 t^{-\alpha/\beta_*} \exp\Big(- c_4 \Big(\frac{d(x, y)^{\beta_*}}{t} \Big)^{\frac{1}{\beta_* - 1}} \Big), \end{aligned}$$

where $\beta_* \geqslant 2$.

It is known that various fractals including the Sierpinski gaskets and Sierpinski carpets satisfy the conditions and for those cases, typically $\beta_* > 2$. For example, for Sierpinski gaskets, $\beta_* = \log 5/\log 2$ and $\alpha = \log 5/\log 2$. (see [2, 31] for details.)

Now, for $0 < \gamma < 1$, let $\{\xi_t\}_{t>0}$ be the strictly γ-stable subordinator; namely let $\{\xi_t\}_{t>0}$ be a one dimensional non-negative Lévy process with the generating function $\mathbf{E}[\exp(-u\xi_t)] = \exp(-tu^{\gamma})$. Assume further that $\{\xi_t\}_{t>0}$ is independent of the diffusion process above. Then the subordinate process of the diffusion by the

γ-stable subordinator has the following heat kernel

$$p(t,x,y)=\int_0^\infty q(u,x,y)\eta_t(u)\,du \qquad \text{for all } t>0,\ x,y\in M,$$

where $\{\eta_t(u): t>0, u\geqslant 0\}$ is the transition density of $\{\xi_t\}_{t>0}$. It is easy to check that $p(t,x,y)$ satisfies (1.5) with $\beta=\gamma\beta_*$, so the conclusions of Theorem 1.3 hold (see [30] for details).

Appendix A. Some Proofs and Technical Lemmas

In this appendix, we give some proofs of the results in Section 2, and also present some technical lemmas that are used in the paper.

A.1. Proofs of some results in Section 2.

Proof of Proposition 2.5. Let ζ be the lifetime of the process X and $M_0=M\setminus\mathscr{N}$. By (2.5), we have that for any $t>0$ and every $x\in M_0$,

$$\mathbf{P}^x(\zeta>t)\geqslant\int_{B(x,\phi^{-1}(t))}p(t,x,y)\,\mu(dy)\geqslant\int_{B(x,\phi^{-1}(t))}\frac{C_1}{V(\phi^{-1}(t))}\,\mu(dy)\geqslant C_1C_*^{-1}>0.$$

Let $u(x):=\mathbf{P}^x(\zeta=\infty)$. Then $u(x)=\lim_{t\to\infty}\mathbf{P}^x(\zeta>t)\geqslant C_1C_*^{-1}>0$ for every $x\in M_0$. Note that $u(X_t)=1_{\{\zeta>t\}}u(X_t)=\mathbf{E}^x(1_{\{\zeta=\infty\}}|\mathscr{F}_t)$ is a bounded martingale with $\lim_{t\to\infty}u(X_t)=1_{\{\zeta=\infty\}}$. Let $\{K_j; j\geqslant 1\}$ be an increasing sequence of compact sets so that $\cup_{j=1}^\infty K_j=M$ and define $\tau_j=\inf\{t\geqslant 0: X_t\notin K_j\}$. Since X admits no killings inside M, we have $\tau_j<\zeta$ a.s. Clearly, $\lim_{j\to\infty}\tau_j=\zeta$. By the optional stopping theorem, we have for $x\in M_0$,

$$\begin{aligned}u(x)&=\lim_{j\to\infty}\mathbf{E}^x u(X_{\tau_j})=\mathbf{E}^x\big(\lim_{j\to\infty}u(X_{\tau_j})\big)\\&=\mathbf{E}^x\big(\lim_{j\to\infty}u(X_{\tau_j})1_{\{\zeta<\infty\}}+\lim_{t\to\infty}u(X_t)1_{\{\zeta=\infty\}}\big)\\&\geqslant C_1C_*^{-1}\mathbf{P}^x(\zeta<\infty)+\mathbf{P}^x(\zeta=\infty).\end{aligned}$$

It follows that $\mathbf{P}^x(\zeta<\infty)=0$ for every $x\in M_0$. The proof is complete. □

Proof of Proposition 2.6. Fix a point $x_0\in M$ and let $u_t(x)=p(t,x_0,x)$. By Proposition 2.5, $\|u_t\|_1=1$; on the other hand, $\|u_t\|_\infty\leqslant\frac{C_2}{V(\phi^{-1}(t))}$. Hence, noting $V(\infty)=\infty$, we have

$$\mu(M)\geqslant\frac{\|u_t\|_1}{\|u_t\|_\infty}\to\infty,\quad t\to\infty,$$

that is, $\mu(M)=\infty$. Due to (1) the measure of any ball is finite, and so M is not contained in any ball, which proves diam $(M)=\infty$. The last assertion immediately follows from [24, Corollary 5.3] and the fact that M is connected. □

Proof of Proposition 2.9. For simplicity, we only deal with the case that both Assumptions 2.1 and 2.8 hold true. The proof is essentially the same as that of [13, Theorem 4.11], and we shall highlight a few different steps.

For each $A\subset[0,\infty)\times M$, define $\sigma_A=\inf\{t>0: Z_t\in A\}$ and $A_s=\{y\in M: (s,y)\in A\}$. Let $Q(t,x,r)=[t,t+c_0\phi(r)]\times B(x,r)$, where $c_0\in(0,1)$ is the constant in (2.16). Then, following the argument of [14, Lemma 6.2] and using Proposition 2.7 and the Lévy system for the process X (see [14, Appendix A]), we can obtain that there is a constant $c_1 > 0$ such that for all $x \in M \setminus \mathscr{N}$, $t, r > 0$ and any compact subset $A \subset Q(t, x, r)$

$$\mathbf{P}^{(t,x)}(\sigma_A < \tau_{Q(t,x,r)}) \geqslant c_1 \frac{m \otimes \mu(A)}{V(r)\phi(r)}, \tag{A.1}$$

where $m \otimes \mu$ is a product measure of the Lebesgue measure m on $\mathbb{R}_+$ and μ on M. Note that unlike [14, Lemma 6.2], here (A.1) is satisfied for all $r > 0$ not only $r \in (0, 1]$, which is due to the fact (2.16) holds for all $r > 0$.

Also by the Lévy system of the process X, we find that there is a constant $c_2 > 0$ such that for all $x \in M \setminus \mathscr{N}$, $t, r > 0$ and $s \geqslant 2r$,

$$\begin{aligned}\mathbf{P}^{(t,x)}(X_{\tau_{Q(t,x,r)}} \notin B(x, s)) &= \mathbf{E}^{(t,x)} \int_0^{\tau_{Q(t,x,r)}} \int_{B(x,s)^c} J(X_v, u)\, \mu(du)\, dv \\ &\leqslant c_2 \left(\int_{r>s/2} \frac{dV(r)}{V(r)\phi(r)} \right) \mathbf{E}^x \tau_{B(x,r)}.\end{aligned}$$

On one hand, by the doubling properties of V and ϕ, we have

$$\int_{r>s/2} \frac{dV(r)}{V(r)\phi(r)} = \sum_{k=0}^{\infty} \int_{r \in (2^{k-1}s, 2^k s]} \frac{dV(r)}{V(r)\phi(r)} \leqslant \sum_{k=0}^{\infty} \frac{V(2^k s) - V(2^{k-1}s)}{V(2^{k-1}s)\phi(2^{k-1}s)} \leqslant c_3 \frac{1}{\phi(s)}.$$

On the other hand, for all $x \in M \setminus \mathscr{N}$ and $r, t > 0$, by (4.4) (which is proved by the doubling property (2.9) of ϕ only),

$$\mathbf{P}^x(\tau_{B(x,r)} \geqslant t) \leqslant \exp(-c_4 t/\phi(r)),$$

which implies that

$$\mathbf{E}^x(\tau_{B(x,r)}) = \int_0^{\infty} \mathbf{P}^x(\tau_{B(x,r)} \geqslant t)\, dt \leqslant c_5 \phi(r). \tag{A.2}$$

Therefore, there is a constant $c_6 > 0$ such that for all $x \in M \setminus \mathscr{N}$, $t, r > 0$ and $s \geqslant 2r$,

$$\mathbf{P}^{(t,x)}(X_{\tau_{Q(t,x,r)}} \notin B(X, s)) \leqslant c_6 \frac{\phi(r)}{\phi(s)}. \tag{A.3}$$

Having (A.1) and (A.3) at hand, one can follow the argument of [13, Theorem 4.11] to get that the Hölder continuity of bounded parabolic functions (see the definition before Proposition 2.13), and so the desired assertion (2.11) for the heart kernel $p(t, x, y)$. Furthermore, (2.12) is an immediately consequence of (2.11). □

Proof of Theorem 2.10. The proof is similar to the one of [8, Proposition 2.3]. For completeness, we provide the full proof here. (See [3] for the original proof.) Let $\varepsilon > 0$ and A be a tail event. Fix $x_0 \in M$. By the martingale convergence theorem, $\mathbf{E}^{x_0}[1_A | \mathcal{F}_t] \to 1_A$ a.s. as $t \to \infty$. Choose t_0 large enough so that

$$\mathbf{E}^{x_0} |\mathbf{E}^{x_0}[1_A | \mathcal{F}_{t_0}] - 1_A| < \varepsilon. \tag{A.4}$$

Set $Y := \mathbf{E}^{x_0}[1_A | \mathcal{F}_{t_0}]$. Then

$$|\mathbf{P}^{x_0}(A) - \mathbf{E}^{x_0}(Y; A)| = |\mathbf{E}^{x_0}(1_A; A) - \mathbf{E}^{x_0}(Y; A)| < \varepsilon. \tag{A.5}$$

On the other hand, using (3.4) and the doubling property of ϕ, we can take $c_1 > 0$ large so that

$$\mathbf{P}^{x_0}(\sup_{s \leqslant t_0} d(X_s, x_0) > c_1 \phi^{-1}(t_0)) < \varepsilon. \tag{A.6}$$

By Proposition 2.9, we now choose t_1 large so that for all $f \in L^\infty(M)$ and $x \in M$ with $d(x, x_0) \leqslant c_1\phi^{-1}(t_0)$,

$$|P_{t_1} f(x) - P_{t_1} f(x_0)| < \varepsilon \|f\|_\infty. \tag{A.7}$$

Since A is a tail event, there exists an event C such that $A = C \circ \theta_{t_0+t_1}$. Let $f(z) = \mathbf{P}^z(C)$. Then by the Markov property at time t_1,

$$\mathbf{E}^w(1_C \circ \theta_{t_1}) = \mathbf{E}^w \mathbf{E}^{X_{t_1}} 1_C = \mathbf{E}^w f(X_{t_1}) = P_{t_1} f(w). \tag{A.8}$$

Thus the Markov property at time t_0 and $(A.8)$ further give us

$$\mathbf{E}^{x_0}(Y; A) = \mathbf{E}^{x_0}[Y \mathbf{E}^{X_{t_0}}(1_C \circ \theta_{t_1})] = \mathbf{E}^{x_0}[Y P_{t_1} f(X_{t_0})] \tag{A.9}$$

and

$$\mathbf{P}^{x_0}(A) = \mathbf{E}^{x_0} 1_A = \mathbf{E}^{x_0} \mathbf{E}^{X_{t_0}}(1_C \circ \theta_{t_1}) = \mathbf{E}^{x_0}[P_{t_1} f(X_{t_0})]. \tag{A.10}$$

Let $A_{t_0} = \{d(X_{t_0}, x_0) \leqslant c_1 \phi^{-1}(t_0)\}$. Using (A.6) and (A.7), we see that

$$\begin{aligned} &|\mathbf{E}^{x_0}[Y P_{t_1} f(X_{t_0})] - P_{t_1} f(x_0) \mathbf{E}^{x_0} Y| \\ &\leqslant 2\mathbf{P}_x(A_{t_0}^c) + |\mathbf{E}^{x_0}[Y P_{t_1} f(X_{t_0}); A_{t_0}] - P_{t_1} f(x_0) \mathbf{E}^{x_0}[Y; A_{t_0}]| \\ &< 2\varepsilon + |\mathbf{E}^{x_0}[Y |P_{t_1} f(X_{t_0}) - P_{t_1} f(x_0)|; A_{t_0}]| \leqslant 3\varepsilon. \end{aligned} \tag{A.11}$$

Similarly

$$|\mathbf{E}^{x_0} P_{t_1} f(X_{t_0}) - P_{t_1} f(x_0)| \leqslant 3\varepsilon. \tag{A.12}$$

Combining $(A.5), (A.9), (A.10), (A.11)$ and $(A.12)$,

$$\begin{aligned} |\mathbf{P}^{x_0}(A) - \mathbf{P}^{x_0}(A)\mathbf{E}^{x_0} Y| \leqslant & |\mathbf{P}^{x_0}(A) - \mathbf{E}^{x_0}(Y; A)| \\ & + |\mathbf{E}^{x_0}[Y P_{t_1} f(X_{t_0})] - P_{t_1} f(x_0) \mathbf{E}^{x_0} Y| \\ & + |P_{t_1} f(x_0) \mathbf{E}^{x_0} Y - \mathbf{E}^{x_0} P_{t_1} f(X_{t_0}) \mathbf{E}^{x_0} Y| \leqslant 7\varepsilon. \end{aligned}$$

Using this and $(A.4)$, $|\mathbf{P}^{x_0}(A) - \mathbf{P}^{x_0}(A)\mathbf{P}^{x_0}(A)| \leqslant 8\varepsilon$. Since ε is arbitrary, we deduce $\mathbf{P}^{x_0}(A) = [\mathbf{P}^{x_0}(A)]^2$, and so $\mathbf{P}^{x_0}(A)$ is 0 or 1. Since $\mathbf{P}^x(A) = \mathbf{E}^x P_{t_1} f(X_{t_0}) = P_{t_0}(P_{t_1} f)(x)$ is continuous in x (which is easily seen from Proposition 2.9) and M is connected, we further conclude that either $\mathbf{P}^x(A)$ is 0 for all $x \in M$ or else it is 1 for all $x \in M$. The proof is complete. □

Proof of Proposition 2.11. For any $x', y' \in B(x, r/2)$ and $t > 0$,

$$p(t, x', y') = p^{B(x,r)}(t, x', y') + \mathbf{E}^x\Big(p(t - \tau_{B(x,r)}, X_{\tau_{B(x,r)}}, y') : \tau_{B(x,r)} < t\Big).$$

On the one hand,

$$\mathbf{E}^x\Big(p(t - \tau_{B(x,r)}, X_{\tau_{B(x,r)}}, y') : \tau_{B(x,r)} < t\Big) \leqslant \sup_{s \leqslant t; d(y,z) \geqslant r/2} p(s, z, y) \leqslant \frac{C_2 t}{V(r/2)\phi(r/2)}.$$

For any $\delta \in (0, 1/2)$, any $x', y' \in B(x, \frac{1}{2}\delta r)$ and $t = \phi(\delta r)$,

$$p(t, x', y') \geqslant C_1 \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(d(x', y'))\phi(d(x', y'))} \right) \geqslant \frac{C_1}{V(\delta r)},$$

so

$$p^{B(x,r)}(t, x', y') \geqslant \frac{C_1}{V(\delta r)} - \frac{C_2}{V(r/2)}.$$

By the doubling property of V, we find that

$$p^{B(x,r)}(t, x', y') \geqslant \frac{C_3}{V(r)}$$

providing that $\delta \in (0,1/2)$ is small enough. Having this at hand, one can follow the argument of [5, Lemma 2.3] and use the doubling property of ϕ to get the first required assertion. The second assertion of the proposition directly follows from the argument above. □

Proof of Proposition 2.12. Here we only prove the case that Assumption 2.1 and 2.8 hold. According to (3.4) and the doubling property of ϕ, for any $r > 0$ and all $x \in M$,

$$\mathbf{P}^x\Big(\sup_{0\leqslant s\leqslant c_0\phi(r)} d(X_s, X_0) \leqslant 2r\Big) \leqslant a_2^*$$

holds with some constants $c_0 > 0$ and $a_2^* \in (0,1)$ independent of x and r. Then, for any $n \geqslant 1$ and $x \in M$, by the Markov property,

$$\begin{aligned}
&\mathbf{P}^x(\sup_{0\leqslant s\leqslant c_0 n\phi(r)} d(X_s, x) \leqslant r)\\
&\leqslant \mathbf{E}^x\Big(\mathbb{1}_{\{\sup_{0\leqslant s\leqslant c_0(n-1)\phi(r)} d(X_s,x)\leqslant r\}}; \mathbf{P}^{X_{c_0(n-1)\phi^{-1}(r)}}(\sup_{0\leqslant s\leqslant c_0\phi(r)} d(X_s, X_0) \leqslant 2r)\Big)\\
&\leqslant a_2^*\mathbf{P}^x(\sup_{0\leqslant s\leqslant c_0(n-1)\phi(r)} d(X_s, x) \leqslant r).
\end{aligned}$$

This proves the upper bound.

On the other hand, according to Proposition 2.11, there are constants $\delta_0, c_1 > 0$ such that for all $x \in M$ and any $r > 0$,

$$p^{B(x,r)}(\delta_0\phi(r), x', y') \geqslant c_1 V(r)^{-1}, \quad x', y' \in B(x, r/2),$$

where $p^{B(x,r)}(t, x', y')$ denotes the Dirichlet heat kernel of the process killed by exiting $B(x,r)$. Then, choosing $m = [c_0/\delta_0] + 1$,

$$\begin{aligned}
&\mathbf{P}^x(\sup_{0\leqslant s\leqslant \delta_0 mn\phi(r)} d(X_s, x) \leqslant r)\\
&= \int_{B(x,r)} p^{B(x,r)}(\delta_0 mn\phi(r), x, y)\,\mu(dy)\\
&\geqslant \int_{B(x,r/2)}\int_{B(x,r/2)}\cdots\int_{B(x,r/2)} p^{B(x,r)}\Big(\delta_0\phi(r), x, x_1\Big)\,\mu(dx_1)\\
&\qquad p^{B(x,r)}\Big(\delta_0\phi(r), x_1, x_2\Big)\mu(dx_2)\ldots\int_{B(x,r/2)} p^{B(x,r)}\Big(\delta_0\phi(r), x_{mn-1}, y\Big)\,\mu(dy)\\
&\geqslant \Big(c_1 V(r)^{-1}\mu(B(x,r/2))\Big)^{mn}.
\end{aligned}$$

Thanks to the doubling property of V, there exists a constant $a_1^* \in (0,1)$ such that for all $x \in M$, $r > 0$ and $n \geqslant 1$,

$$\mathbf{P}^x(\sup_{0\leqslant s\leqslant \delta_0 mn\phi(r)} d(X_s, x) \leqslant r) \geqslant a_1^{*n}.$$

By the fact that

$$\mathbf{P}^x(\sup_{0\leqslant s\leqslant c_0 n\phi(r)} d(X_s, x) \leqslant r) \geqslant \mathbf{P}^x(\sup_{0\leqslant s\leqslant \delta_0 mn\phi(r)} d(X_s, x) \leqslant r),$$

the proof is complete. □

A.2. **Some technical results.** The first result is a extended version of Garsia's lemma ([21, Lemma 1]), see [7, Lemma 6.1] for a version of Garsia's lemma for a fractal.

Lemma A.1. *Let* (M, d, μ) *satisfy* (2.1) *and* (2.8). *Suppose* $q : [0, \infty) \to [0, \infty)$ *is a measurable function with* $q(0) = 0$ *and that there exist constants* C_1, C_2 *and* γ_1, γ_2 *such that*

$$C_1\left(\frac{r}{R}\right)^{\gamma_1} \leqslant \frac{q(r)}{q(R)} \leqslant C_2\left(\frac{r}{R}\right)^{\gamma_2} \quad \text{for every } 0 < r \leqslant R < \infty. \tag{A.13}$$

Let $\Psi : [0, \infty) \to [0, \infty)$ *be a non-negative strictly increasing convex function such that* $\lim_{u\to\infty} \Psi(u) = \infty$. *For any* $x_0 \in M$ *and* $R_0 > 0$, *let* $H = B(x_0, R_0)$ *and* $f : H \to \mathbb{R}^d$ *be a measurable function. If*

$$\Gamma(H) := \iint_{H\times H} \Psi\Big(\frac{|f(x) - f(y)|}{q(d(x, y))}\Big)\,\mu(dx)\,\mu(dy) < \infty,$$

then there exist $c_1, c_2 > 0$ *that depends only on the constants in* (2.8) *and* (A.13) *such that*

$$|f(x) - f(y)| \leqslant c_1 \int_0^{d(x,y)} \Psi^{-1}\Big(\frac{c_2\Gamma(H)}{V(u)^2}\Big)\,\frac{q(u)du}{u}, \tag{A.14}$$

for $\mu \times \mu$*-a.e.* $(x, y) \in B(x_0, R_0/8) \times B(x_0, R_0/8)$. *If* f *is continuous, then* (A.14) *holds every* $(x, y) \in B(x_0, R_0/8) \times B(x_0, R_0/8)$.

Proof. For fixed $(x, y) \in B(x_0, R_0/8)\times B(x_0, R_0/8)$ and $k \geqslant 0$, let $a_k := 2^{-k+1}d(x, y)$ and B_k' be open balls with radii a_k such that $B_{k+1} \subset B_k$ and $x, y \in B_0 \subset H$. We denote $f_k := \frac{1}{\mu(B_k)}\int_{B_k} f d\mu$. For $(z, w) \in B_{k-1}$, we have $d(z, w) \leqslant 2a_{k-1}$, so by (A.13), $C_2q(2a_{k-1}) \geqslant q(d(z, w))$. Thus, since Ψ is increasing,

$$\Psi\left(\frac{|f(z) - f(w)|}{C_0q(2a_{k-1})}\right) \leqslant \Psi\left(\frac{|f(z) - f(w)|}{q(d(z, w))}\right), \quad (z, w) \in B_{k-1} \times B_k.$$

Using this, the increasing property and the convexity of Ψ and the Jensen inequality,

$$\begin{aligned}
&\Psi\left(\frac{|f_{k-1} - f_k|}{C_2q(2a_{k-1})}\right)\\
&\leqslant \Psi\left(\frac{1}{\mu(B_{k-1})\mu(B_k)}\int_{B_{k-1}\times B_k}\frac{|f(z) - f(w)|}{C_2q(2a_{k-1})}\mu(dw)\mu(dz)\right)\\
&\leqslant \frac{1}{\mu(B_{k-1})\mu(B_k)}\int_{B_{k-1}\times B_k}\Psi\left(\frac{|f(z) - f(w)|}{q(d(z, w))}\right)\mu(dw)\mu(dz)\\
&\leqslant \frac{\Gamma(H)}{\mu(B_{k-1})\mu(B_k)} \leqslant c_1\frac{\Gamma(H)}{V(a_k)^2},
\end{aligned} \tag{A.15}$$

where in the last inequality we used (2.1) and (2.8).

On the other hand, for $k \geqslant 1$

$$\begin{aligned}
&\int_{a_{k+1}}^{a_k} \Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(u)^2}\Big)\,\frac{q(u)du}{u}\\
&\geqslant q(2a_{k-1})\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(a_k)^2}\Big)\int_{a_{k+1}}^{a_k}\frac{q(u)}{q(2a_{k-1})}\frac{du}{u}\\
&\geqslant q(2a_{k-1})\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(a_k)^2}\Big)\int_{a_{k+1}}^{a_k} C_1\left(\frac{u}{2a_{k-1}}\right)^{\gamma_1}\frac{du}{u} \qquad \text{(A.16)}\\
&= C_1 q(2a_{k-1})\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(a_k)^2}\Big)\,(2a_{k-1})^{-\gamma_1}\int_{a_{k+1}}^{a_k} u^{\gamma_1-1}du\\
&= c_2 q(2a_{k-1})\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(a_k)^2}\Big).
\end{aligned}$$

Thus, by (A.15) and (A.16), for $k \geqslant 1$,

$$|f_{k-1}-f_k| \leqslant C_0 q(2a_{k-1})\Psi^{-1}\left(\frac{c_1\Gamma(H)}{V(a_k)^2}\right) \leqslant c_3\int_{a_{k+1}}^{a_k}\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(u)^2}\Big)\,\frac{q(u)du}{u}$$

which implies

$$\limsup_{k\to\infty}|f_k-f_0| \leqslant \sum_{k=1}^{\infty}|f_{k-1}-f_k| \leqslant c_2\int_0^{d(x,y)}\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(u)^2}\Big)\,\frac{q(u)du}{u}. \qquad \text{(A.17)}$$

Suppose that f is continuous at x. Then, let $B_0 = B(x,a_0)$, so that $x,y \in B_0 = B(x,2d(x,y)) \subset B(x_0,R_0)$. By considering $B_k = B(x,a_k)$ for $k \geqslant 1$, we get from (A.17) that

$$|f(x)-f_0| \leqslant c_2\int_0^{d(x,y)}\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(u)^2}\Big)\,\frac{q(u)du}{u}.$$

Similarly, we get from (A.17) that, if f is continuous at y then

$$|f(y)-f_0| \leqslant c_2\int_0^{d(x,y)}\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(u)^2}\Big)\,\frac{q(u)du}{u}.$$

Thus, if f is continuous at both x and y,

$$|f(x)-f(y)| \leqslant |f(x)-f_0| + |f(y)-f_0| \leqslant 2c_2\int_0^{d(x,y)}\Psi^{-1}\Big(\frac{c_1\Gamma(H)}{V(u)^2}\Big)\,\frac{q(u)du}{u}.$$

The general case follows from Lebesgue differentiation theorem (e.g. see [25, Theorem 1.8]). □

The following proposition gives an upper bound for LILs. Since it can be proved by a simple modification of the proof of [8, Theorem 3.1], we skip the proof.

Proposition A.2. *Let X be a strong Markov process on (M,d,μ). Suppose $(F_t)_{t\geqslant 0}$ is a continuous adapted non-decreasing functional of X satisfying the following conditions.*

(1) *There exists an increasing function φ on $\mathbb{R}_+$ satisfying the doubling property and such that*

$$\sup_{x\in M, t>0}\mathbf{P}^x(F_t \geqslant b\varphi(t)) \to 0 \qquad \text{as } b\to\infty.$$

(2) $F_t - F_s \leqslant F_{t-s}\circ\theta_s, \quad 0 < s \leqslant t.$

Then, there exists a constant $C \in (0, \infty)$ *such that*

$$\limsup_{t\to\infty} \frac{F_t}{\varphi\,(t/\log\log t)\log\log t} \leqslant C, \qquad \mathbf{P}^x\text{-}a.e.\ \omega,\ \forall x \in M.$$

Remark A.3. Similar to the remark after the proof of [8, Theorem 3.1], Proposition A.2 can be used to derive upper bounds for LIL of $L^*(t) = \sup_{x\in M} l(x,t)$ and the range $R(t) = \mu(X([0,t]))$ of jump processes. Note that, in our setting the continuity of $L^*(t)$ is a consequence of Proposition 4.14, the strong Markov property and the Borel-Cantelli lemma; while one can use Theorem 3.7 and the fact $R(t) \leqslant c_1 V\big(\sup_{0\leqslant s\leqslant t} d(X_s, x)\big)$ for all $t > 0$ and some constant $c_1 > 0$ to obtain the continuity of $R(t)$.

Proposition A.4. *Let* (M, d, μ) *be a connected metric measure space such that diam* $M = \infty$ *and the volume doubling condition holds, i.e. there exists* $c_1 > 0$ *such that*

$$\mu(B(x, 2r)) \leqslant c_1 \mu(B(x, r)), \quad x \in M, r > 0.$$

Then, for each $x_0 \in M$ *and* $R > 0$*, there exists a sequence* $\{A_i\}_{i=0}^\infty$ *such that each* A_i *is a ball of radius* R*,* $\lim_{i\to\infty} d(x_0, A_i) = \infty$*, and the following hold:*

$$x_0 \in A_0, \quad A_i \cap A_{i+1} \neq \emptyset \ \text{ for all } \ i \in \mathbb{N}, \ A_i \cap A_j = \emptyset \ \text{ for all } \ |i-j| \geqslant 2.$$

Proof. First, by [32, Lemma 3.1 (i)], there exists a constant $N_0 \in \mathbb{N}$ such that for each $R > 0$, there exists an open covering $\{B(z_i, R)\}_{i=0}^\infty$ of M with the property that no point in M is more than N_0 of the balls. We say a subset Λ of $\{z_i\}_i$ is linked if for each $z_i, z_j \in \Lambda$, there is a chain $z^0 = z_i, z^1, \cdots, z^l = z_j \in \Lambda$ such that $z^k \sim z^{k+1}$ (by which we mean $B(z^k, R) \cap B(z^{k+1}, R) \neq \emptyset$) for all $k = 0, 1, \cdots, l-1$. Take $x_0 \in M$. We may assume without loss of generality that $x_0 = z_0$. For each $k \in \mathbb{N}$, we may take a linked set $G_k \subset \{z_i\}_i \cap B(x_0, 4kR)^c$ such that $\sharp G_k = \infty$. (Indeed, if there is no such linked sets, then because diam $M = \infty$ and M is connected, there are infinite number of mutually disjoint and non-empty linked sets $\{L_j\}$ such that $\sharp L_j < \infty$ and $L_j \subset \{z_i\}_i \cap B(x_0, 4kR)^c$. We may assume that each L_j is maximal (i.e. no elements in $\{z_i\}_i \cap B(x_0, 4kR)^c \cap L_j^c$ is linked to L_j). Because M is connected, from each L_j, there exists $\hat{x}_j \in L_j$ such that $B(\hat{x}_j, R) \cap B(x_0, 4kR) \neq \emptyset$. By construction, $\{B(\hat{x}_j, R)\}_j$ are mutually disjoint, but this contradicts to the volume doubling assumption.) We fix one such a linked set G_k which is maximal; we may choose $G_k \supset G_{k+1} \supset \cdots$. Set $G_0 = \{z_i\}_i$.

We now construct a desired chain inductively that contains a sequence $\{z_{m_k}\}_{k=0}^\infty \subset \{z_i\}$. Take $z_{m_0} = x_0$. For each $k \geqslant 0$, given $z_{m_k} \in G_k \cap B(x_0, (4k+2)R)$, take a chain $y_0^k = z_{m_k}, y_1^k, \cdots, y_{s_k}^k$ such that $y_i^k \sim y_{i+1}^k$ for $i = 0, \cdots s_k - 1$, $y_j^k \in G_k \setminus G_{k+1}$, $j = 0, \cdots s_k - 1$ and $y_{s_k}^k =: z_{m_{k+1}} \in G_{k+1}$. Then it holds that $z_{m_{k+1}} \in B(x_0, (4(k+1)+2)R)$. Now let $\tilde{y}_0^k = y_0^k$ and define $\tilde{y}_i^k$, $i \geqslant 1$ inductively as the maximum j such that $y_j^k \sim \tilde{y}_{i-1}^k$. Then we have a sequence $\tilde{y}_0^k = z_{m_k}, \tilde{y}_1^k, \cdots, \tilde{y}_{s_k'}^k = z_{m_{k+1}}$ such that $\tilde{y}_i^k \sim \tilde{y}_{i+1}^k$ and $\tilde{y}_i^k \not\sim \tilde{y}_j^k$ if $|i-j| \geqslant 2$. By doing this procedure iteratively, and doing the same procedure (i.e. procedure to produce $\{\tilde{y}_i^k\}$ from $\{y_i^k\}$) again for each adjacent sequences (this is necessary because the sequences of balls made by the adjacent sequences $\{\tilde{y}_0^k = z_{m_k}, \tilde{y}_1^k, \cdots, \tilde{y}_{s_k'}^k = z_{m_{k+1}}\}$ and $\{\tilde{y}_0^{k+1} = z_{m_{k+1}}, \tilde{y}_1^{k+1}, \cdots, \tilde{y}_{s_{k+1}'}^{k+1} = z_{m_{k+2}}\}$ could overlap many times), we have the desired chain. $\square$

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Panki Kim
Department of Mathematical Sciences and Research Institute of Mathematics,
Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu, Seoul 08826, Republic of Korea
E-mail: pkim@snu.ac.kr

Takashi Kumagai
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail: kumagai@kurims.kyoto-u.ac.jp

Jian Wang
School of Mathematics and Computer Science, Fujian Normal University, 350007 Fuzhou, P.R. China.
E-mail: jianwang@fjnu.edu.cn