Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms

Zhen-Qing Chen, **Takashi Kumagai**[*] and **Jian Wang**[†]

Abstract

In this paper, we establish stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms on metric measure spaces under general volume doubling condition. We obtain their stable equivalent characterizations in terms of the jumping kernels, variants of cutoff Sobolev inequalities, and Poincaré inequalities. In particular, we establish the connection between parabolic Harnack inequalities and two-sided heat kernel estimates, as well as with the Hölder regularity of parabolic functions for symmetric non-local Dirichlet forms.

1 Introduction and Main Results

Harnack inequalities are inequalities that control the growth of non-negative harmonic functions and caloric functions (solutions of heat equations) on domains. The inequalities were first proved for harmonic functions for Laplacian in the plane by Carl Gustav Axel von Harnack, and later became fundamental in the theory of harmonic analysis, partial differential equations and probability. One of the most significant implications of the inequalities is that (at least for the cases of local operators/diffusions) they imply Hölder continuity of harmonic/caloric functions. We refer readers to [K1] for the history and the basic introduction of Harnack inequalities.

Because of their fundamental importance, there has been a long history of research on Harnack inequalities. Harnack inequalities and Hölder regularities for harmonic functions are important components of the celebrated De Giorgi-Nash-Moser theory in harmonic analysis and partial differential equations. In early 90's, equivalent characterizations for parabolic Harnack inequalities (that is, Harnack inequalities for caloric functions) were obtained by Grigor'yan [Gr] and Saloff-Coste [Sa1] for Brownian motions (or equivalently, Laplace-Beltrami operators) on complete Riemannian manifolds. They showed that

[*]Research partially supported by the Grant-in-Aid for Scientific Research (A) 25247007.

[†]Research partially supported by the National Natural Science Foundation of China (No. 11522106), Fok Ying Tung Education Foundation (No. 151002) the JSPS postdoctoral fellowship (26·04021), National Science Foundation of Fujian Province (No. 2015J01003), the Program for Nonlinear Analysis and Its Applications (No. IRTL1206), and Fujian Provincial Key Laboratory of Mathematical Analysis and its Applications (FJKLMAA).

parabolic Harnack inequalities are equivalent to doubling condition of the volume measures plus Poincaré inequalities, which are also equivalent to the two-sided Gaussian-type heat kernel estimates. An important consequence of this equivalence is that the parabolic Harnack inequalities are stable under transformations of the Riemannian manifolds by quasi-isomorphisms. This result was later extended to symmetric diffusions on metric measure spaces by Sturm [St] and to random walks on graphs by Delmotte [De]. It has been further extended to symmetric anomalous diffusions on metric measure spaces including fractals in [BBK1].

In this paper, we consider the stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms (or equivalent, symmetric jump processes) on metric measure spaces. Let (M, d, μ) be a metric measure space where d is a metric and μ is a Radon measure (see Section 1.1 for a precise setting). We consider a symmetric regular *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(M;\mu)$ of pure jump type; that is,

$$\mathcal{E}(f,g) = \int_{M\times M\setminus \mathrm{diag}} (f(x)-f(y))(g(x)-g(y))\, J(dx,dy), \quad f,g\in\mathcal{F}, \tag{1.1}$$

where diag denotes the diagonal set $\{(x,x) : x \in M\}$ and $J(\cdot,\cdot)$ is a symmetric jumping measure on $M \times M \setminus \mathrm{diag}$. Let X be Hunt process corresponding to $(\mathcal{E}, \mathcal{F})$. An important example of the jumping kernel J is $J(dx, dy) = \frac{c(x,y)}{d(x,y)^{d+\alpha}}\,\mu(dx)\,\mu(dy)$, where $c(x, y)$ is a symmetric function bounded between two positive constants and $\alpha > 0$. The corresponding process is called a symmetric α-stable-like process. When $M = \mathbb{R}^d$, or more general, an Ahlfors d-regular space, μ is the Hausdorff measure on M and $\alpha \in (0, 2)$, various properties of the symmetric α-stable-like processes including two-sided heat kernel estimates and parabolic Harnack inequalities have been studied in [CK1]. In particular, when $M = \mathbb{R}^d$, μ is the Lebesgue measure on $\mathbb{R}^d$ and $c(x, y)$ is a constant function, this corresponds simply to a rotationally symmetric α-stable Lévy process. However, on some metric measure spaces M such as the Sierpinski gasket and the Sierpinski carpet, the index α can be larger than 2.

Let ϕ be a strictly increasing continuous function on $[0,\infty)$ with $\phi(0) = 0$.

Definition 1.1. We say that the *parabolic Harnack inequality* $\mathrm{PHI}(\phi)$ holds for the process X, if there exist constants $0 < C_1 < C_2 < C_3 < C_4$, $0 < C_5 < 1$ and $C_6 > 0$ such that for every $x_0 \in M$, $t_0 \geq 0$, $R > 0$ and for every non-negative function $u = u(t, x)$ on $[0,\infty)\times M$ that is parabolic on cylinder $Q(t_0, x_0, C_4\phi(R), R) := (t_0, t_0 + C_4\phi(R)) \times B(x_0, R)$,

$$\operatorname{ess\,sup}_{Q_-} u \leq C_6 \operatorname{ess\,inf}_{Q_+} u, \tag{1.2}$$

where $Q_- := (t_0 + C_1\phi(R), t_0 + C_2\phi(R)) \times B(x_0, C_5R)$ and $Q_+ := (t_0 + C_3\phi(R), t_0 + C_4\phi(R)) \times B(x_0, C_5R)$.

We call the function ϕ the scale function for $\mathrm{PHI}(\phi)$. The $\mathrm{PHI}(\phi)$ results obtained in [Gr, Sa2, St, De] are for $\phi(r) = r^2$. It is proved in [CK1] that symmetric α-stable-like processes with $\alpha \in (0, 2)$ enjoy $\mathrm{PHI}(\phi)$ for $\phi(r) = r^\alpha$. In [CK2], $\mathrm{PHI}(\phi)$ is obtained for mixed stable processes on metric measure spaces with variable scale ϕ.

Here is the question we consider in this paper.

(Q) Suppose $(\mathcal{E}, \mathcal{F})$ and $(\widehat{\mathcal{E}}, \mathcal{F})$ are regular Dirichlet forms on $L^2(M;\mu)$ of the form (1.1), whose corresponding jumping measures and processes are J, $\widehat{J}$ and X, $\widehat{X}$, respectively. Suppose further there exist constants $c_1, c_2 > 0$ such that $c_1 J(A,B) \le \widehat{J}(A,B) \le c_2 J(A,B)$ for all $A, B \subset M$ with $A \cap B = \emptyset$. If $\mathrm{PHI}(\phi)$ holds for X, does $\mathrm{PHI}(\phi)$ also hold for the process $\widehat{X}$?

We will answer the question affirmatively in Theorem 1.17, the main result of this paper, by giving an equivalent characterization of $\mathrm{PHI}(\phi)$ that is stable under such perturbations:

$$\mathrm{PHI}(\phi) \Longleftrightarrow \mathrm{PI}(\phi) + \mathrm{J}_{\phi,\le} + \mathrm{CSJ}(\phi) + \mathrm{UJS};$$

see (1.18), (1.9), (1.10) and (1.17) for related notations and definitions. Moreover, Theorem 1.17 also gives the precise relations among the parabolic Harnack inequality $\mathrm{PHI}(\phi)$, the Hölder regularity $\mathrm{PHR}(\phi)$ of caloric functions, and the elliptic Hölder regularity (EHR) of harmonic functions:

$$\mathrm{PHI}(\phi) \Longleftrightarrow \mathrm{PHR}(\phi) + \mathrm{E}_{\phi,\le} + \mathrm{UJS} \Longleftrightarrow \mathrm{EHR} + \mathrm{E}_{\phi} + \mathrm{UJS};$$

see (1.12), (1.14) and (1.15) for definitions.

To our knowledge, there has been no literature on the equivalence of parabolic Harnack inequalities for non-local Dirichlet forms on general metric measure spaces despite of the importance of parabolic Harnack inequalities. We note that when the underlying space is a graph satisfying the Ahlfors regular condition, some equivalence conditions for $\mathrm{PHI}(\phi)$ with $\phi(r) = r^\alpha$ for $\alpha \in (0,2)$ are obtained in Barlow, Bass and Kumagai [BBK2]. In some general metric measure spaces including certain fractals mentioned above, it is known that $\mathrm{PHI}(\phi)$ may hold for $\phi(r) = r^\alpha$ with $\alpha \ge 2$ (see, for instance, [CKW, Section 6.1]). In this paper, we establish the stability of $\mathrm{PHI}(\phi)$ for a large class of scale functions ϕ including those $\phi(r) = r^\alpha$ with $\alpha \ge 2$. We also emphasize that our metric measure spaces are only assumed to satisfy general volume doubling and reverse volume doubling properties; see Definition 1.2 for definitions. These make the study of stability of $\mathrm{PHI}(\phi)$ extremely challenging.

Parabolic Harnack inequalities are closely related to heat kernel estimates. In the very recent paper [CKW], we obtained stability of two-sided heat kernel estimates and upper bound heat kernel estimates for symmetric jump processes of mixed type on general metric measure spaces (see Section 1.2 for a brief survey of the results of [CKW]). In contrast to the cases of local operators/diffusions, parabolic Harnack inequalities are no longer equivalent to (in fact weaker than) the two-sided heat kernel estimates. In fact Corollary 1.18 of this paper asserts

$$\mathrm{HK}(\phi) \Longleftrightarrow \mathrm{PHI}(\phi) + \mathrm{J}_{\phi,\ge};$$

see (1.9) and (1.13) for definitions. This discrepancy is caused by the heavy tail of the jumping kernel. This heavy tail phenomenon is also one of main sources of difficulties in analyzing non-local operators/jump processes.

Due to the above difficulties and differences, obtaining the stability of $\mathrm{PHI}(\phi)$ for non-local operators/jump processes requires new ideas. Our approach contains the following two key ingredients, and both of them are highly non-trivial:

(i) We make full use of the probabilistic properties of jump process X (in particular the Lévy system of X that describes how the process X jumps) to connect PHI(ϕ) with the properties of the associated heat kernel and jumping kernel. See the equivalence condition (3) in main result Theorem 1.17.

(ii) We adopt some PDE's techniques from the recent study of fractional p-Laplacian operators in [CKP1] to derive some useful properties of the process X. We emphasis that, to get the stability of PHI(ϕ) in our general framework we should use cutoff Sobolev inequalities CSJ(ϕ) for non-local Dirichlet forms, instead of the fractional Poincaré inequalities or Sobolev inequalities in the existing literature (e.g. see [CKP1, DK, K2]), since the latter two functional inequalities require some regularity of state space and non-local operators. See the equivalence condition (7) in Theorem 1.17.

Finally, we should mention that, even though non-local operators appear naturally in the study of stochastic processes with jumps, there are huge amount of interests among analysts to study Harnack inequalities and related properties for non-local operators; see [CS, CKP1, CKP2, DK, K1, K2, Sil] and the references therein. Combining probabilistic methods with analytic methods in the study of heat kernel estimates and parabolic Harnack inequalities for non-local operators proves to be quite powerful and fruitful, as is the case for this paper and for [CKW].

In the following, we give the framework of this paper in details and present the main results of this paper. We also recall some theorems from [CKW] that will be used in this paper.

1.1 Setting

Let (M, d) be a locally compact separable metric space, and μ a positive Radon measure on M with full support. A triple (M, d, μ) is called a *metric measure space*, and we denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(M; \mu)$. For simplicity, we assume that $\mu(M) = \infty$ throughout the paper. Let us emphasize that we do not assume M to be connected nor (M, d) to be geodesic.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(M; \mu)$ given in (1.1). We assume throughout this paper that, for each $x \in M$, there is a kernel $J(x, dy)$ so that

$$J(dx, dy) = J(x, dy)\, \mu(dx).$$

In this paper, we will abuse notation and always take the quasi-continuous version for an element of $\mathcal{F}$ (note that since $(\mathcal{E}, \mathcal{F})$ is regular, each function in $\mathcal{F}$ admits a quasi-continuous version). Denote by $\mathcal{L}$ the (negative definite) L^2-*generator* of $(\mathcal{E}, \mathcal{F})$. Let $\{P_t\}$ be the associated *semigroup* on $L^2(M; \mu)$. Associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M; \mu)$ is an μ-symmetric *Hunt process* $X = \{X_t, t \geq 0, \mathbb{P}^x, x \in M \backslash \mathcal{N}\}$, where $\mathcal{N}$ is a properly exceptional set for $(\mathcal{E}, \mathcal{F})$ in that $\mu(\mathcal{N}) = 0$ and $\mathbb{P}^x(X_t \in \mathcal{N}$ for some $t > 0) = 0$ for all $x \in M \setminus \mathcal{N}$. This Hunt process is unique up to a properly exceptional set (see [FOT, Theorem 4.2.8]). A more precise version of $\{P_t\}$ with better regularity properties can be obtained as follows: for any bounded Borel measurable function f on M,

$$P_t f(x) = \mathbb{E}^x f(X_t), \quad x \in M_0 := M \setminus \mathcal{N}.$$

The *heat kernel* associated with $\{P_t\}$ (if it exists) is a measurable function $p(t, x, y) : M_0 \times M_0 \to (0, \infty)$ for every $t > 0$, such that

$$\begin{aligned}
\mathbb{E}^x f(X_t) = P_t f(x) &= \int p(t, x, y) f(y)\, \mu(dy), \quad x \in M_0, f \in L^\infty(M; \mu), \\
p(t, x, y) &= p(t, y, x) \quad \text{for all } t > 0,\ x, y \in M_0, \\
p(s + t, x, z) &= \int p(s, x, y) p(t, y, z)\, \mu(dy) \quad \text{for all } s, t > 0 \text{ and } x, z \in M_0.
\end{aligned}$$

We call $p(t, x, y)$ the *heat kernel* on $(M, d, \mu, \mathcal{E})$. Note that we can extend $p(t, x, y)$ to all x, $y \in M$ by setting $p(t, x, y) = 0$ if x or y is outside M_0.

The goal of this paper is to present stable characterizations of parabolic Harnack inequality for the symmetric jump process X. To state our results precisely and show the relations between heat kernel estimates and parabolic Harnack inequalities, we need a number of definitions and also recall the stable characterizations of two-sided estimates and upper bound estimates for heat kernels from [CKW].

Definition 1.2. Denote by $B(x, r)$ the ball in (M, d) centered at x with radius r, and set

$$V(x, r) = \mu(B(x, r)).$$

(i) We say that (M, d, μ) satisfies the *volume doubling property* (VD) if there exists a constant $C_\mu \geq 1$ such that for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq C_\mu V(x, r). \tag{1.3}$$

(ii) We say that (M, d, μ) satisfies the *reverse volume doubling property* (RVD) if there exist positive constants d_1 and c_μ such that for all $x \in M$ and $0 < r \leq R$,

$$\frac{V(x, R)}{V(x, r)} \geq c_\mu \Big(\frac{R}{r}\Big)^{d_1}. \tag{1.4}$$

VD condition (1.3) is equivalent to the following: there exist $d_2, \widetilde{C}_\mu > 0$ so that

$$\frac{V(x, R)}{V(x, r)} \leq \widetilde{C}_\mu \Big(\frac{R}{r}\Big)^{d_2} \quad \text{for all } x \in M \text{ and } 0 < r \leq R. \tag{1.5}$$

RVD condition (1.4) is equivalent to the existence of positive constants l_μ and $\widetilde{c}_\mu > 1$ so that

$$V(x, l_\mu r) \geq \widetilde{c}_\mu V(x, r) \quad \text{for all } x \in M \text{ and } r > 0. \tag{1.6}$$

It is known that VD implies RVD if M is connected and unbounded (see, for example [GH, Proposition 5.1 and Corollary 5.3]).

Let $\mathbb{R}_+ := [0, \infty)$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing continuous function with $\phi(0) = 0$, $\phi(1) = 1$ that satisfies the following: there exist $c_1, c_2 > 0$ and $\beta_2 \geq \beta_1 > 0$ such that

$$c_1\Big(\frac{R}{r}\Big)^{\beta_1} \leq \frac{\phi(R)}{\phi(r)} \leq c_2\Big(\frac{R}{r}\Big)^{\beta_2} \quad \text{for all } 0 < r \leq R. \tag{1.7}$$

Definition 1.3. We say J_ϕ holds if for any $x, y \in M$ there exists a non-negative symmetric function $J(x, y)$ so that for $\mu \times \mu$-almost all $x, y \in M$,

$$J(dx, dy) = J(x, y)\, \mu(dx)\, \mu(dy), \tag{1.8}$$

and

$$\frac{c_1}{V(x, d(x, y))\phi(d(x, y))} \leq J(x, y) \leq \frac{c_2}{V(x, d(x, y))\phi(d(x, y))} \tag{1.9}$$

for some constants $c_2 \geq c_1 > 0$. We say that $\mathrm{J}_{\phi,\leq}$ (resp. $\mathrm{J}_{\phi,\geq}$) if (1.8) holds and the upper bound (resp. lower bound) in (1.9) holds.

For the non-local Dirichlet form $(\mathcal{E}, \mathcal{F})$, we define the carré du-Champ operator $\Gamma(f, g)$ for $f, g \in \mathcal{F}$ by

$$\Gamma(f, g)(dx) = \int_{y\in M} (f(x) - f(y))(g(x) - g(y))\, J(dx, dy).$$

1.2 Heat kernel estimates

The following CSJ(ϕ) and SCSJ(ϕ) conditions that control the energy of cutoff functions are first introduced in [CKW]. See [CKW, Remark 1.6] for background on these conditions. Recall that ϕ is a strictly increasing continuous function on $\mathbb{R}_+$ satisfying $\phi(0) = 0$, $\phi(1) = 1$ and (1.7).

Definition 1.4. (i) Let $U \subset V$ be open sets in M with $U \subset \overline{U} \subset V$. We say a non-negative bounded measurable function φ is a *cutoff function for* $U \subset V$, if $\varphi = 1$ on U, $\varphi = 0$ on V^c and $0 \leq \varphi \leq 1$ on M.

(ii) We say that CSJ(ϕ) holds if there exist constants $C_0 \in (0, 1]$ and $C_1, C_2 > 0$ such that for every $0 < r \leq R$, almost all $x \in M$ and any $f \in \mathcal{F}$, there exists a cutoff function $\varphi \in \mathcal{F}_b := \mathcal{F} \cap L^\infty(M, \mu)$ for $B(x, R) \subset B(x, R + r)$ so that

$$\begin{aligned}\int_{B(x,R+(1+C_0)r)} f^2\, d\Gamma(\varphi, \varphi) \leq& C_1 \int_{U\times U^*} (f(x) - f(y))^2\, J(dx, dy)\\ &+ \frac{C_2}{\phi(r)} \int_{B(x,R+(1+C_0)r)} f^2\, d\mu,\end{aligned} \tag{1.10}$$

where $U = B(x, R + r) \setminus B(x, R)$ and $U^* = B(x, R + (1 + C_0)r) \setminus B(x, R - C_0r)$.

(iii) We say that SCSJ(ϕ) holds if there exist constants $C_0 \in (0, 1]$ and $C_1, C_2 > 0$ such that for every $0 < r \leq R$ and almost all $x \in M$, there exists a cutoff function $\varphi \in \mathcal{F}_b$ for $B(x, R) \subset B(x, R + r)$ so that (1.10) holds for any $f \in \mathcal{F}$.

Clearly SCSJ(ϕ) $\Longrightarrow$ CSJ(ϕ).

Remark 1.5. As is pointed out in [CKW, Rermark 1.7], under VD, (1.7) and $J_{\phi,\le}$, SCSJ(ϕ) always holds if $\beta_2 < 2$, where β_2 is the exponent in (1.7). In particular, SCSJ(ϕ) holds for $\phi(r) = r^\alpha$ always when $0 < \alpha < 2$.

We next introduce the Faber-Krahn inequality. For any open set $D \subset M$, $\mathcal{F}_D$ is defined to be the $\|\cdot\|_{\mathcal{E}_1}$-closure in $\mathcal{F}$ of $\mathcal{F} \cap C_c(D)$, where $\|\cdot\|^2_{\mathcal{E}_1} = \|\cdot\|^2_{\mathcal{E}} + \|\cdot\|^2_2$. Here $C_c(D)$ is the space of continuous functions on M with compact support in D. Define

$$\lambda_1(D) = \inf\left\{\mathcal{E}(f,f) : \, f \in \mathcal{F}_D \text{ with } \|f\|_2 = 1\right\},$$

the bottom of the Dirichlet spectrum of $-\mathcal{L}$ on D.

Definition 1.6. $(M, d, \mu, \mathcal{E})$ satisfies the *Faber-Krahn inequality* FK(ϕ), if there exist positive constants C and ν such that for any ball $B(x,r)$ and any open set $D \subset B(x,r)$,

$$\lambda_1(D) \ge \frac{C}{\phi(r)}(V(x,r)/\mu(D))^\nu. \tag{1.11}$$

For a set $A \subset M$, define the exit time $\tau_A = \inf\{t > 0 : X_t \in A^c\}$.

Definition 1.7. We say that E_ϕ holds if there is a constant $c_1 > 1$ such that for all $r > 0$ and all $x \in M_0$,

$$c_1^{-1}\phi(r) \le \mathbb{E}^x[\tau_{B(x,r)}] \le c_1\phi(r). \tag{1.12}$$

We say that $E_{\phi,\le}$ (resp. $E_{\phi,\ge}$) holds if the upper bound (resp. lower bound) in the inequality above holds.

Definition 1.8. (i) We say that HK(ϕ) holds if there exists a heat kernel $p(t,x,y)$ of the semigroup $\{P_t\}$ for $(\mathcal{E},\mathcal{F})$, which has the following estimates for all $t > 0$ and all $x, y \in M_0$,

$$\begin{aligned} c_1\Big(\frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))}\Big) \\ \le p(t,x,y) \\ \le c_2\Big(\frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))}\Big), \end{aligned} \tag{1.13}$$

where $c_1, c_2 > 0$ are constants independent of $x, y \in M_0$ and $t > 0$. Here $\phi^{-1}(t)$ is the inverse function of the strictly increasing function $t \mapsto \phi(t)$.

(ii) We say UHK(ϕ) (resp. LHK(ϕ)) holds if the upper bound (resp. the lower bound) in (1.13) holds.

(iii) We say UHKD(ϕ) holds if there is a constant $c > 0$ such that

$$p(t,x,x) \le \frac{c}{V(x,\phi^{-1}(t))} \quad \text{for all } t > 0 \text{ and } x \in M_0.$$

It is pointed out in [CKW, Remark 1.12] that

$$\frac{1}{V(y,\phi^{-1}(t))}\wedge\frac{t}{V(y,d(x,y))\phi(d(x,y))}\asymp\frac{1}{V(x,\phi^{-1}(t))}\wedge\frac{t}{V(x,d(x,y))\phi(d(x,y))}.$$

We may thus replace $V(x,\phi^{-1}(t))$ and $V(x,d(x,y))$ by $V(y,\phi^{-1}(t))$ and $V(y,d(x,y))$ in (1.13) by modifying the values of c_1 and c_2. On the other hand, it follows from [CKW, Theorem 1.13 and Lemma 5.6] that if HK(ϕ) holds, then the heat kernel $p(t,x,y)$ is Hölder continuous on (x,y) for every $t>0$, and so (1.13) holds for all $x,y\in M$.

We say $(\mathcal{E},\mathcal{F})$ *is conservative* if its associated Hunt process X has infinite lifetime. This is equivalent to $P_t1=1$ a.e. on M_0 for every $t>0$.

The following are the main results of [CKW], which will be used later in this paper.

Theorem 1.9. ([CKW, Theorem 1.13]) *Assume that the metric measure space (M,d,μ) satisfies* VD *and* RVD*, and ϕ satisfies* (1.7)*. Then the following are equivalent:*
(1) HK(ϕ).
(2) J_ϕ *and* E_ϕ.
(3) J_ϕ *and* SCSJ(ϕ).
(4) J_ϕ *and* CSJ(ϕ).

Theorem 1.10. ([CKW, Theorem 1.15]) *Assume that the metric measure space (M,d,μ) satisfies* VD *and* RVD*, and ϕ satisfies* (1.7)*. Then the following are equivalent:*
(1) UHK(ϕ) *and $(\mathcal{E},\mathcal{F})$ is conservative.*
(2) UHKD(ϕ), $J_{\phi,\le}$ *and* E_ϕ.
(3) FK(ϕ), $J_{\phi,\le}$ *and* SCSJ(ϕ).
(4) FK(ϕ), $J_{\phi,\le}$ *and* CSJ(ϕ).

As a consequence of [CKW, Proposition 3.1(ii)] (recalled in Proposition 2.4 of this paper), LHK(ϕ) implies that X has infinite lifetime. As is remarked in [CKW], UHK(ϕ) alone does not imply the conservativeness of the associated Dirichlet form $(\mathcal{E},\mathcal{F})$.

1.3 Parabolic Harnack inequalities

We first give probabilistic definitions of harmonic and parabolic functions in the general context of metric measure spaces.

Let $Z:=\{V_s,X_s\}_{s\ge0}$ be the space-time process corresponding to X where $V_s=V_0-s$. The filtration generated by Z satisfying the usual conditions will be denoted by $\{\widetilde{\mathcal{F}}_s;s\ge 0\}$. The law of the space-time process $s\mapsto Z_s$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. For every open subset D of $[0,\infty)\times M$, define $\tau_D=\inf\{s>0:Z_s\notin D\}$.

Recall that a set $A\subset[0,\infty)\times M$ is said to be nearly Borel measurable if for any probability measure μ_0 on $[0,\infty)\times M$, there are Borel measurable subsets A_1, A_2 of $[0,\infty)\times M$ so that $A_1\subset A\subset A_2$ and that $\mathbb{P}^{\mu_0}(Z_t\in A_2\setminus A_1 \text{ for some } t\ge0)=0$. The collection of all nearly Borel measurable subsets of $[0,\infty)\times M$ forms a σ-field, which is called nearly Borel measurable σ-field.

Definition 1.11. (i) We say that a nearly Borel measurable function $u(t, x)$ on $[0, \infty) \times M$ is *parabolic* (or *caloric*) on $D = (a, b) \times B(x_0, r)$ for the Markov process X if there is a properly exceptional set $\mathcal{N}_u$ of the Markov process X so that for every relatively compact open subset U of D, $u(t, x) = \mathbb{E}^{(t,x)} u(Z_{\tau_U})$ for every $(t, x) \in U \cap ([0, \infty) \times (M \backslash \mathcal{N}_u))$.

(ii) A nearly Borel measurable function u on M is said to be *subharmonic* (resp. *harmonic, superharmonic*) in D (with respect to the process X) if for any relatively compact subset $U \subset D$, $t \mapsto u(X_{t \wedge \tau_U})$ is a uniformly integrable submartingale (resp. martingale, supermartingale) under $\mathbb{P}^x$ for q.e. $x \in U$.

Definition 1.12. (i) We say that the *parabolic Harnack inequality* $\mathrm{PHI}^+(\phi)$ holds for Markov the process X if Definition 1.1 holds for some constants $C_1 > 0$, $C_k = kC_1$ for $k = 2, 3, 4$, $0 < C_5 < 1$ and $C_6 > 0$.

(ii) We say that the *elliptic Harnack inequality* (EHI) holds for the Markov process X if there exist constants $c > 0$ and $\delta \in (0, 1)$ such that for every $x_0 \in M$, $r > 0$ and for every non-negative function u on M that is harmonic in $B(x_0, r)$,

$$\text{ess sup}_{B(x_0,\delta r)} h \leq c \, \text{ess inf}_{B(x_0,\delta r)} h.$$

(iii) We say that the *parabolic Hölder regularity* $\mathrm{PHR}(\phi)$ holds for the Markov process X if there exist constants $c > 0$, $\theta \in (0, 1]$ and $\varepsilon \in (0, 1)$ such that for every $x_0 \in M$, $t_0 \geq 0$, $r > 0$ and for every bounded measurable function $u = u(t, x)$ that is caloric in $Q(t_0, x_0, \phi(r), r)$, there is a properly exceptional set $\mathcal{N}_u \supset \mathcal{N}$ so that

$$|u(s, x) - u(t, y)| \leq c \left(\frac{\phi^{-1}(|s - t|) + d(x, y)}{r} \right)^{\theta} \text{ess sup}_{[t_0, t_0 + \phi(r)] \times M} |u| \tag{1.14}$$

for every $s, t \in (t_0, t_0 + \phi(\varepsilon r))$ and $x, y \in B(x_0, \varepsilon r) \setminus \mathcal{N}_u$.

(vi) We say that the *elliptic Hölder regularity* (EHR) holds for the process X, if there exist constants $c > 0$, $\theta \in (0, 1]$ and $\varepsilon \in (0, 1)$ such that for every $x_0 \in M$, $r > 0$ and for every bounded measurable function u on M that is harmonic in $B(x_0, r)$, there is a properly exceptional set $\mathcal{N}_u \supset \mathcal{N}$ so that

$$|u(x) - u(y)| \leq c \left(\frac{d(x, y)}{r} \right)^{\theta} \text{ess sup}_M |u| \tag{1.15}$$

for any x, $y \in B(x_0, \varepsilon r) \setminus \mathcal{N}_u$.

Clearly $\mathrm{PHI}^+(\phi) \Longrightarrow \mathrm{PHI}(\phi) \Longrightarrow$ EHI and $\mathrm{PHR}(\phi) \Longrightarrow$ EHR. We point out that $\mathrm{PHR}(\phi)$ implies $\mathrm{E}_{\phi,\geq}$; see Proposition 3.9.

Remark 1.13. (i) $\mathrm{PHI}(\phi)$ in Definition 1.1 is called a weak parabolic Harnack inequality in [BGK], in the sense that (1.2) holds for some $C_1, \cdots, C_5$. It is called a parabolic Harnack inequality in [BGK] if (1.2) holds for any choice of positive constants $C_4 > C_3 > C_2 > C_1 > 0$, $0 < C_5 < 1$ with $C_6 = C_6(C_1, \ldots, C_5) < \infty$. Since our underlying metric measure space may not be geodesic, one can not expect to deduce parabolic Harnack inequality from weak parabolic Harnack inequality. See [BGK] for related discussion on diffusions.

(ii) We will show in Proposition 4.4 that under VD, RVD and (1.7), PHI$^+(\phi)$ and PHI(ϕ) are equivalent.

(iii) Clearly, PHI(ϕ) holds if and only if the desired property holds for every bounded parabolic function on cylinder $Q(t_0, x_0, C_4\phi(R), R)$. Same for PHI$^+(\phi)$ and EHI.

(iv) Note that in the definition of PHR(ϕ) (resp. EHR) if the inequality (1.14) (resp. (1.15)) holds for some $\varepsilon \in (0,1)$, then it holds for all $\varepsilon \in (0,1)$ (with possibly different constant c). We take EHR for example. For every $x_0 \in M$ and $r > 0$, let u be a bounded function on M such that it is harmonic in $B(x_0, r)$. Then, for any $\varepsilon' \in (0,1)$ and $x \in B(x_0, \varepsilon' r) \setminus \mathcal{N}_u$, u is harmonic on $B(x, (1-\varepsilon')r)$. Applying (1.15) for u on $B(x, (1-\varepsilon' r))$, we find that for any $y \in B(x_0, \varepsilon' r) \setminus \mathcal{N}_u$ with $d(x,y) \leq (1-\varepsilon')\varepsilon r$,

$$|u(x) - u(y)| \leq c \left(\frac{d(x,y)}{r} \right)^\theta \text{ess sup}_{z \in M} |u(z)|.$$

This implies that for any $x, y \in B(x_0, \varepsilon' r) \setminus \mathcal{N}_u$, (1.15) holds with $c' = c \vee \frac{2}{[(1-\varepsilon')\varepsilon]^\theta}$.

Below we discuss stability of parabolic Harnack inequalities. This requires further definitions.

Definition 1.14. We say that *a lower bound near diagonal estimate for Dirichlet heat kernel* NDL(ϕ) holds, i.e. there exist $\varepsilon \in (0,1)$ and $c_1 > 0$ such that for any $x_0 \in M$, $r > 0$, $0 < t \leq \phi(\varepsilon r)$ and $B = B(x_0, r)$,

$$p^B(t, x, y) \geq \frac{c_1}{V(x_0, \phi^{-1}(t))}, \quad x, y \in B(x_0, \varepsilon\phi^{-1}(t)) \cap M_0. \tag{1.16}$$

Under VD, we may replace $V(x_0, \phi^{-1}(t))$ in the definition by either $V(x, \phi^{-1}(t))$ or $V(y, \phi^{-1}(t))$. Under (1.7), we also may replace $\phi(\varepsilon r)$ and $\varepsilon\phi^{-1}(t)$ in the definition above by $\varepsilon\phi(r)$ and $\phi^{-1}(\varepsilon t)$, respectively.

The following inequality was introduced in [BBK2] in the setting of graphs. See [CKK1] for the general setting of metric measure spaces.

Definition 1.15. We say that UJS holds if there is a symmetric function $J(x,y)$ so that $J(x, dy) = J(x,y)\,\mu(dy)$, and there is a constant $c > 0$ such that for μ-a.e. $x, y \in M$ with $x \neq y$,

$$J(x,y) \leq \frac{c}{V(x,r)} \int_{B(x,r)} J(z,y)\,\mu(dz) \quad \text{for every } 0 < r \leq d(x,y)/2. \tag{1.17}$$

Note that UJS is implied by the following pointwise comparability condition of the jump kernel $J(x,y)$: there is a constant $c>0$ such that $J(x,y)\le cJ(z,y)$ for μ-a.e. $x,y,z\in M$ with $x\neq y$ and $0<d(x,z)\le d(x,y)/2$.

Definition 1.16. We say that the (*weak*) *Poincaré inequality* $\mathrm{PI}(\phi)$ holds if there exist constants $C>0$ and $\kappa\ge 1$ such that for any ball $B_r=B(x,r)$ with $x\in M$ and for any $f\in\mathcal{F}_b$,

$$\int_{B_r}(f-\overline{f}_{B_r})^2\,d\mu\le C\phi(r)\int_{B_{\kappa r}\times B_{\kappa r}}(f(y)-f(x))^2\,J(dx,dy), \tag{1.18}$$

where $\overline{f}_{B_r}=\frac{1}{\mu(B_r)}\int_{B_r}f\,d\mu$ is the average value of f on B_r.

If the integral on the right hand side of (1.18) is over $B_r\times B_r$ (i.e. $\kappa=1$), then it is called strong Poincaré inequality. If the metric is geodesic, it is known that (weak) Poincaré inequality implies strong Poincaré inequality (see for instance [Sa2, Section 5.3]), but in general they are not the same. In this paper, we only use weak Poincaré inequality. Note also that the left hand side of (1.18) is equal to $\inf_{a\in\mathbb{R}}\int_{B_r}(f-a)^2\,d\mu$.

The following is the main result of this paper.

Theorem 1.17. *Suppose that the metric measure space (M,d,μ) satisfies* VD *and* RVD*, and ϕ satisfies* (1.7)*. Then the following are equivalent:*
(1) $\mathrm{PHI}(\phi)$.
(2) $\mathrm{PHI}^+(\phi)$.
(3) $\mathrm{UHK}(\phi)$, $\mathrm{NDL}(\phi)$ *and* UJS.
(4) $\mathrm{NDL}(\phi)$ *and* UJS.
(5) $\mathrm{PHR}(\phi)$, $\mathrm{E}_{\phi,\le}$ *and* UJS.
(6) EHR, E_ϕ *and* UJS.
(7) $\mathrm{PI}(\phi)$, $\mathrm{J}_{\phi,\le}$, $\mathrm{CSJ}(\phi)$ *and* UJS.

We note that any of the conditions above implies the conservativeness of the process $\{X_t\}$; see Proposition 2.4 and [CKW, Lemma 4.22], Proposition 3.2 and Proposition 4.9.

As a corollary of Theorem 1.9 and Theorem 1.17 (noting that J_ϕ implies UJS), we have the following.

Corollary 1.18. *Suppose that the metric measure space (M,d,μ) satisfies* VD *and* RVD*, and ϕ satisfies* (1.7)*. Then*

$$\mathrm{HK}(\phi)\Longleftrightarrow\mathrm{PHI}(\phi)+\mathrm{J}_{\phi,\ge}.$$

In addition to the papers mentioned above, for other related work on Harnack inequalities and Hölder regularities for harmonic functions of non-local operators, we mention [BL, ChZ, LS, Kom, MK, SU, SV] and the references therein. We emphasize this is only a partial list of the vast literature on the subject.

The rest of the paper is organized as follows. The proof of Theorem 1.17 is given in Section 4. In Section 2, we present some preliminary results. Various consequences of parabolic Harnack inequalities are given in Section 3. The proof of (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) is given in Subsection 4.1, the proof of (1) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) is given in Subsection 4.2,

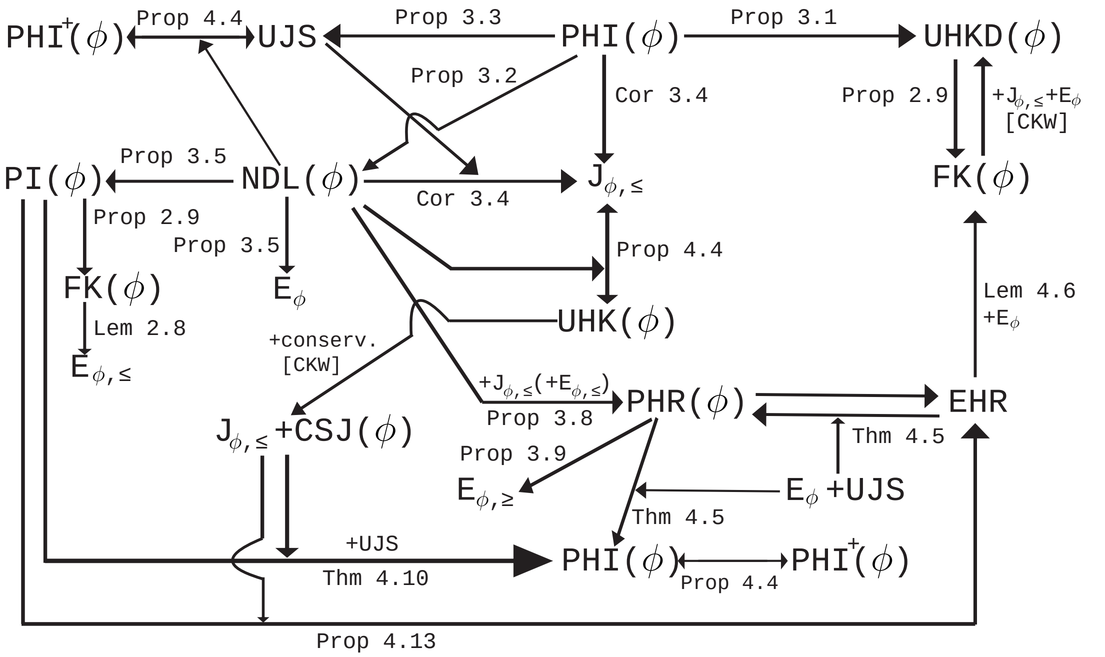

Figure 1: diagram

while (1) $\Longleftrightarrow$ (7) is shown in Subsection 4.3. Figure 1 illustrates implications of various conditions and flow of our proofs.

Throughout this paper, we will use c, with or without subscripts, to denote strictly positive finite constants whose values are insignificant and may change from line to line. For functions f and g defined on a set D, we write $f \asymp g$ if there exists a constant $c \geq 1$ such that $c^{-1}f(x) \leq g(x) \leq c\,f(x)$ for all $x \in D$. For $p \in [1, \infty]$, we will use $\|f\|_p$ to denote the L^p-norm in $L^p(M;\mu)$. For any $D \subset M$, denote by $C(D)$ (resp. $C_c(D)$) the set of continuous functions (resp. continuous functions with compact support) on D.

2 Preliminaries

In this section we present some preliminary results that will be used in the sequel.

We first recall the analytic characterization of harmonic and subharmonic functions. Let D be an open subset of M. Recall that a function f is said to be locally in $\mathcal{F}_D$, denoted as $f \in \mathcal{F}_D^{loc}$, if for every relatively compact subset U of D, there is a function $g \in \mathcal{F}_D$ such that $f = g$ m-a.e. on U. The following is established in [C].

Lemma 2.1. ([C, Lemma 2.6]) *Let D be an open subset of M. Suppose u is a function in $\mathcal{F}_D^{loc}$ that is locally bounded on D and satisfies that*

$$\int_{U\times V^c} |u(y)|\, J(dx, dy) < \infty \tag{2.1}$$

for any relatively compact open sets U and V of M with $\bar U \subset V \subset \bar V \subset D$. Then for every $v \in C_c(D) \cap \mathcal{F}$, the expression

$$\int (u(x) - u(y))(v(x) - v(y))\, J(dx, dy)$$

is well defined and finite; it will still be denoted as $\mathcal{E}(u, v)$.

As noted in [C, (2.3)], since $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M; \mu)$, for any relatively compact open sets U and V with $\bar U \subset V$, there is a function $\psi \in \mathcal{F} \cap C_c(M)$ such that $\psi = 1$ on U and $\psi = 0$ on V. Consequently,

$$\int_{U\times V^c} J(dx, dy) = \int_{U\times V^c} (\psi(x) - \psi(y))^2\, J(dx, dy) \le \mathcal{E}(\psi, \psi) < \infty,$$

so each bounded function u satisfies (2.1).

We say that a nearly Borel measurable function u on M is *$\mathcal{E}$-subharmonic* (resp. *$\mathcal{E}$-harmonic, $\mathcal{E}$-superharmonic*) in D if $u \in \mathcal{F}_D^{loc}$ that is locally bounded on D, satisfies (2.1) for any relatively compact open sets U and V of M with $\bar U \subset V \subset \bar V \subset D$, and that

$$\mathcal{E}(u, \varphi) \le 0 \quad (\text{resp. } = 0, \ge 0) \quad \text{for any } 0 \le \varphi \in \mathcal{F} \cap C_c(D).$$

The following is established in [C, Theorem 2.11 and Lemma 2.3] first for harmonic functions, and then extended in [ChK, Theorem 2.9] to subharmonic functions.

Theorem 2.2. *Let D be an open subset of M, and u be a bounded function. Then u is $\mathcal{E}$-harmonic* (*resp. $\mathcal{E}$-subharmonic*) *in D if and only if u is harmonic* (*resp. subharmonic*) *in D.*

We next recall four results from [CKW]. Lemma 2.3 is essentially given in [CK2, Lemma 2.1].

Lemma 2.3. ([CKW, Lemma 2.1]) *Assume that* VD, (1.7) *and* $\mathrm{J}_{\phi,\le}$ *hold. Then there exists a constant $c_1 > 0$ such that*

$$\int_{B(x,r)^c} J(x, y)\, \mu(dy) \le \frac{c_1}{\phi(r)} \quad \textit{for every } x \in M \textit{ and } r > 0.$$

Proposition 2.4. ([CKW, Proposition 3.1(ii)]) *Suppose that* VD *holds. Then either* LHK(ϕ) *or* NDL(ϕ) *implies $\zeta = \infty$ a.s., where ζ denotes the lifetime of the process X.*

For a Borel measurable function u on M, following [CKP1], we define its *nonlocal tail* $\mathrm{Tail}(u; x_0, r)$ in the ball $B(x_0, r)$ by

$$\mathrm{Tail}\,(u; x_0, r) := \phi(r) \int_{B(x_0,r)^c} \frac{|u(z)|}{V(x_0, d(x_0, z))\phi(d(x_0, z))}\, \mu(dz). \tag{2.2}$$

In the following, for any $x \in M$ and $r > 0$, set $B_r(x) = B(x, r)$.

Lemma 2.5. ([CKW, Lemma 4.8]) *Suppose* VD, (1.7), FK(ϕ), CSJ(ϕ) *and* $\mathrm{J}_{\phi,\le}$ *hold. Let* $x_0 \in M$, $R, r_1, r_2 > 0$ *with* $r_1 \in [R/2, R]$ *and* $r_1 + r_2 \le R$, *and* u *be an* $\mathcal{E}$*-subharmonic function in* $B_R(x_0)$. *For* $\theta > 0$, *set* $v := (u-\theta)_+$. *We have*

$$\int_{B_{r_1}(x_0)} v^2\, d\mu \le \frac{c_1}{\theta^{2\nu} V(x_0,R)^\nu} \left(\int_{B_{r_1+r_2}(x_0)} u^2\, d\mu \right)^{1+\nu} \times \left(1+\frac{r_1}{r_2}\right)^{\beta_2} \left[1 + \left(1+\frac{r_1}{r_2}\right)^{d_2+\beta_2-\beta_1} \frac{\mathrm{Tail}\,(u; x_0, R/2)}{\theta} \right],$$

where ν *is the constant in* FK(ϕ), d_2 *is the constant in* (1.5), β_1, β_2 *are the constants in* (1.7), *and* c_1 *is a constant independent of* θ, x_0, R, r_1 *and* r_2.

Proposition 2.6. ([CKW, Proposition 4.10]) **(L^2-mean value inequality)** *Assume* VD, (1.7), FK(ϕ), CSJ(ϕ) *and* $\mathrm{J}_{\phi,\le}$ *hold. For any* $x_0 \in M$ *and* $r > 0$, *let* u *be a bounded* $\mathcal{E}$*-subharmonic in* $B_r(x_0)$. *Then there is a constant* $C_0 > 0$ *independent of* x_0 *and* r *so that*

$$\mathrm{ess\,sup}_{\,B_{r/2}(x_0)} u \le C_0 \left(\left(\frac{1}{V(x_0,r)} \int_{B_r(x_0)} u^2\, d\mu \right)^{1/2} + \mathrm{Tail}\,(u; x_0, r/2) \right). \tag{2.3}$$

The following three results are proved in [CKW].

Proposition 2.7. ([CKW, Proposition 4.14]) *Assume* VD, (1.7), FK(ϕ), $\mathrm{J}_{\phi,\le}$ *and* CSJ(ϕ) *hold. Then,* E_ϕ *holds.*

Lemma 2.8. ([CKW, Lemma 4.15]) *Assume that* VD, (1.7) *and* FK(ϕ) *hold. Then,* $\mathrm{E}_{\phi,\le}$ *holds.*

Proposition 2.9. ([CKW, Proposition 7.6]) *Assume that* VD, RVD *and* (1.7) *are satisfied. Then either* PI(ϕ) *or* UHKD(ϕ) *implies* FK(ϕ).

We also record the following elementary iteration lemma, see, e.g., [G, Lemma 7.1] or [CKW, Lemma 4.9].

Lemma 2.10. *Let* $\beta > 0$ *and let* $\{A_j\}$ *be a sequence of real positive numbers such that* $A_{j+1} \le c_0 b^j A_j^{1+\beta}$ *for every* $j \ge 0$ *with* $c_0 > 0$ *and* $b > 1$. *If* $A_0 \le c_0^{-1/\beta} b^{-1/\beta^2}$, *then we have* $A_j \le b^{-j/\beta} A_0$ *for* $j \ge 1$, *which in particular yields* $\lim_{j\to\infty} A_j = 0$.

The following formula, often called the Lévy system formula, will be used many times in this paper. See, for example [CK2, Appendix A] for a proof.

Lemma 2.11. *Let* f *be a non-negative measurable function on* $\mathbb{R}_+ \times M \times M$ *that vanishes along the diagonal. Then for every* $t \ge 0$, $x \in M_0$ *and stopping time* T *(with respect to the filtration of* $\{X_t\}$*),*

$$\mathbb{E}^x \left[\sum_{s \le T} f(s, X_{s-}, X_s) \right] = \mathbb{E}^x \left[\int_0^T \int_M f(s, X_s, y)\, J(X_s, dy)\, ds \right].$$

3 Consequences of Harnack inequalities

3.1 Consequences of PHI(ϕ)

In this subsection (together with some of the results from next subsection), we prove that PHI(ϕ) implies UHK(ϕ), NDL(ϕ) and UJS. Without further mention, throughout the proof we will assume that μ and ϕ satisfy VD and (1.7), respectively. Noting that $V(y, r) > 0$ for every $y \in M$ and $r > 0$ (since μ has full support), we have from (1.5) that for all $x, y \in M$ and $0 < r \le R$,

$$\frac{V(x, R)}{V(y, r)} \le \frac{V(y, d(x, y) + R)}{V(y, r)} \le \widetilde{C}_\mu\Big(\frac{d(x, y) + R}{r}\Big)^{d_2}. \tag{3.1}$$

Proposition 3.1. *Under* VD *and* (1.7)*,* PHI(ϕ) *implies* UHKD(ϕ)*.*

Proof. Let C_i $(i = 1, \dots, 6)$ be the constants taken from the definition of PHI(ϕ). For any $x_0 \in M$, $r > 0$, $t = C_4\phi(r)$ and any $0 \le f \in L^2(M; \mu) \cap L^1(M; \mu)$, applying PHI($\phi$) to the caloric function $v(s, x) := P_s f(x)$ in $Q(0, x_0, t, r)$, we have for $x, y \in B(x_0, C_5 r) \setminus \mathcal{N}_v$,

$$P_{(C_1+C_2)\phi(r)/2} f(x) \le C_6 P_{(C_3+C_4)\phi(r)/2} f(y),$$

where $\mathcal{N}_v$ is the properly exceptional set associated with v. Then,

$$V(x_0, C_5 r) P_{(C_1+C_2)\phi(r)/2} f(x) \le C_6 \int_{B(x_0, C_5 r)} P_{(C_3+C_4)\phi(r)/2} f(y)\, \mu(dy) \le C_6 \int f(y)\, \mu(dy).$$

Therefore, there is a constant $c_1 > 0$ such that for almost all $x \in M$ and $t > 0$,

$$P_t f(x) \le \frac{c_1}{V(x, \phi^{-1}(t))} \|f\|_1, \tag{3.2}$$

where we have used VD and (1.7) in the inequality above. In particular, the semigroup $\{P_t\}$ is locally ultracontractive. According to [CKW, Proposition 7.7] (see also [BBCK, Theorem 3.1] and [GT, Theorem 2.12]), there exists a properly exceptional set $\mathcal{N} \subset M$ such that, the semigroup $\{P_t\}$ possesses the heat kernel $p(t, x, y)$ with domain $(0, \infty) \times (M \setminus \mathcal{N}) \times (M \setminus \mathcal{N})$.

By (3.2) again, for almost all x, $y \in M$,

$$p(t, x, y) \le \frac{c_1}{V(x, \phi^{-1}(t))}.$$

In the following, for any $x \in M$ and $t > 0$, define

$$\varphi(x, t) = \inf_{0 < r \le \phi^{-1}(t)} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} \frac{1}{V(z, \phi^{-1}(t))}\, \mu(dz).$$

On the one hand, by (3.1) from VD, there is a constant $c_2 > 1$ such that for all $x \in M$ and $t > 0$,

$$\frac{1}{c_2 V(x, \phi^{-1}(t))} \le \varphi(x, t) \le \frac{c_2}{V(x, \phi^{-1}(t))}.$$

On the other hand, for any $t > 0$, $x \mapsto \varphi(x,t)$ is an upper semi-continuous function on M. Indeed, for any $x \in M$,

$$\begin{aligned}\limsup_{y\to x} \varphi(y,t) =& \lim_{s\to 0} \sup_{0<d(y,x)\le s} \inf_{0<r\le \phi^{-1}(t)} \frac{1}{\mu(B(y,r))} \int_{B(y,r)} \frac{1}{V(z,\phi^{-1}(t))}\,\mu(dz) \\ \le& \inf_{0<r\le\phi^{-1}(t)} \lim_{s\to 0} \sup_{0<d(y,x)\le s} \frac{1}{\mu(B(y,r))} \int_{B(y,r)} \frac{1}{V(z,\phi^{-1}(t))}\,\mu(dz) \\ =& \inf_{0<r\le\phi^{-1}(t)} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \frac{1}{V(z,\phi^{-1}(t))}\,\mu(dz) \\ =&\varphi(x,t).\end{aligned}$$

Combining all the conclusions above with [CKW, Proposition 7.7] again, we have

$$p(t,x,y) \le \frac{c_3}{V(x,\phi^{-1}(t))} \quad \text{for all } (x,y) \in (M \setminus \mathcal{N}) \times (M \setminus \mathcal{N}).$$

This proves UHKD(ϕ). □

A key consequence of PHI(ϕ) is a near-diagonal lower bound estimate for $p^D(t,x,y)$. For the cases of diffusions, similar fact was proved in [BGK, Section 4.3.4], but there is a gap in the middle of Page 1129. (Indeed, the proof uses $B(x_0, R+\rho) = \cup_{x\in B(x_0,R)} B(x,\rho)$, which is not true in general unless the metric is geodesic.) Our proof below fixes the issue (see step (ii) in the proof) and proves NDL(ϕ) in the framework of general metric spaces.

Proposition 3.2. *Assume* VD, (1.7) *and* PHI(ϕ) *hold. Then* NDL(ϕ) *holds. Consequently, $X = \{X_t\}$ is conservative.*

Proof. Note that by VD and Proposition 2.4, NDL(ϕ) implies the conservativeness of the process X. We only need to verify that NDL(ϕ) holds. Below we will prove NDL(ϕ) with $\phi(\varepsilon r)$ and $\varepsilon\phi^{-1}(t)$ replaced by $\varepsilon\phi(r)$ and $\phi^{-1}(\varepsilon t)$ in the definition.

(i) For any open ball $B := B(x_0, r)$ with $x_0 \in M_0$ and $r > 0$, it follows from (3.2) and VD that for any $t > 0$

$$\|P_t^B f\|_\infty \le \frac{c_1}{V(x_0,\phi^{-1}(t))} \|f\|_1.$$

Then, by [BBCK, Theorem 3.1], the Dirichlet semigroup $\{P_t^B\}$ has the heat kernel $p^B(t,x,y)$ defined on $(0,\infty) \times (B \setminus \mathcal{N}_1) \times (B \setminus \mathcal{N}_1)$ such that

$$p^B(t,x,y) \le \frac{c_1}{V(x_0,\phi^{-1}(t))}, \quad x,y \in B \setminus \mathcal{N}_1,$$

where $\mathcal{N}_1 \subset B$ is a properly exceptional set of the killing process $\{X_t^B\}$ such that $\mathcal{N}_1 \supset \mathcal{N} \cap B$; moreover, there is an $\mathcal{E}^B$-nest $\{F_k\}$ consisting of an increasing sequence of compact sets of B so that $\mathcal{N}_1 = B \setminus \cup_{k=1}^\infty F_k$ and that for every $t > 0$, $y \in B \setminus \mathcal{N}$ and $k \ge 1$, $x \mapsto p^B(t,x,y)$ is continuous on each F_k (i.e. for every $t > 0$ and $y \in B \setminus \mathcal{N}_1$, the function $x \mapsto p^B(t,x,y)$ is quasi-continuous on B).

(ii) Choose an $\widehat{x}_0 \in B(x_0, C_5 r) \setminus \mathcal{N}_1$, where $C_5 \in (0,1)$ is the constant in PHI(ϕ). Define

$$\widehat{B} = \{y \in B \setminus \mathcal{N}_1 : p^B(t, \widehat{x}_0, y) > 0 \text{ for some } t > 0\}.$$

We will show that for every $x, y \in \widehat{B}$, there is some $t > 0$ so that $p^B(t,x,y) > 0$, and that

$$p^B(t,x,y) = 0 \quad \text{on } (0,\infty) \times \widehat{B} \times (B \setminus (\widehat{B} \cup \mathcal{N}_1)). \tag{3.3}$$

To prove these, first noting that since $\mathbb{P}^x(\lim_{t\downarrow 0} X_t^B = X_0^B = x) = 1$ implies $\mathbb{P}^x(\tau_B > 0) = 1$, we must have $p^B(t, \widehat{x}_0, \widehat{x}_0) = \int_B p^B(t/2, \widehat{x}_0, y)^2 \, \mu(dy) > 0$ for some $t > 0$. Thus $\widehat{x}_0 \in \widehat{B}$. By PHI($\phi$) applied to the caloric function $(s,y) \mapsto p^B(s, y, \widehat{x}_0) = p^B(s, \widehat{x}_0, y)$, we see that if $x \in \widehat{B}$, then there are constants $r_x > 0$ and $s_x > 0$ so that

$$p^B(s, \widehat{x}_0, z) > 0 \quad \text{ for every } z \in B(x, r_x) \setminus \mathcal{N}_1 \text{ and } s \geq s_x. \tag{3.4}$$

Hence, there is an open subset U of B containing $\widehat{x}_0$ so that $\widehat{B} = U \setminus \mathcal{N}_1$. Similarly, for every $x, y \in \widehat{B}$, by PHI(ϕ), there are constants $r_0 > 0$ and $s_0 > 0$ so that

$$p^B(s,x,z) > 0 \quad \text{and} \quad p^B(s,y,z) > 0 \quad \text{for every } z \in B(\widehat{x}_0, r_0) \setminus \mathcal{N}_1 \text{ and } s \geq s_0.$$

In particular, it follows that for every $s, t \geq s_0$,

$$p^B(t+s, x, y) \geq \int_{B(\widehat{x}_0, r_0)} p^B(s,x,z) p^B(t,z,y) \, \mu(dz) > 0. \tag{3.5}$$

For $x \in \widehat{B}$, define

$$\widehat{B}_x = \{y \in B \setminus \mathcal{N}_1 : p^B(t,x,y) > 0 \text{ for some } t > 0\}.$$

Then $\widehat{B} \subset \widehat{B}_x$. We claim $\widehat{B} = \widehat{B}_x$. Were $\widehat{B} \subsetneq \widehat{B}_x$, take $y \in \widehat{B}_x \setminus \widehat{B}$. By PHI($\phi$) applied to the caloric function $(s,z) \mapsto p^B(s,z,y) = p^B(s,y,z)$, there are constants $r_x > 0$ and $s_x > 0$ so that $p^B(s,y,z) > 0$ for every $z \in B(x, r_x) \setminus \mathcal{N}_1$ and $s \geq s_x$, and (3.4) holds. Hence, for every $t, s \geq s_x$, we have

$$p^B(t+s, \widehat{x}_0, y) \geq \int_{B(x, r_x)} p^B(t, \widehat{x}_0, z) p^B(s,z,y) \, \mu(dz) > 0,$$

which implies that $y \in \widehat{B}$. This contradiction shows that $\widehat{B}_x = \widehat{B}$ for every $x \in \widehat{B}$. We have thus established that for every $x, y \in \widehat{B}$, there is some $t > 0$ so that $p^B(t,x,y) > 0$, and that (3.3) holds. Consequently, for every $t > 0$ and $x, y \in \widehat{B} = U \setminus \mathcal{N}_1$,

$$p^U(t,x,y) = p^B(t,x,y) - \mathbb{E}_x \left[p^B(t - \tau_U, X_{\tau_U}^B, y); t < \tau_U \right] = p^B(t,x,y) \tag{3.6}$$

Observe that by the symmetry of $p^B(t,x,y)$, (3.3) implies that

$$\int_{B \setminus U} P_t^B \mathbf{1}_U(x) \, \mu(dx) = \int_{U \times (B \setminus U)} p^B(t,x,y) \, \mu(dx) \, \mu(dy) = 0;$$

in other words, for every $t > 0$,

$$P_t^B \mathbf{1}_U = 0 \quad \mu\text{-a.e. on } B \setminus U. \tag{3.7}$$

Let $\lambda_0 > 0$ be the bottom of the generator $\mathcal{L}^U$ associated with $\{P_t^U\}$ and $\psi \geq 0$ the corresponding eigenfunction with $\|\psi\|_{L^2(U;\mu)} = 1$. Note that $\psi = 0$ on $B \setminus U$. In view of (3.6) and (3.7), we have for every $t > 0$ and $x \in B \setminus \mathcal{N}_1$,

$$P_t^B \psi(x) = P_t^U \psi(x) = e^{-\lambda_0 t} \psi(x).$$

Since

$$e^{-\lambda_0 t} \|\psi\|_{L^\infty(B;\mu)} = \|P_t^B \psi\|_{L^\infty(B;\mu)} \leq \mu(B) \|\psi\|_{L^\infty(B;\mu)} \sup_{x,y \in B \setminus \mathcal{N}_1} p^B(t,x,y),$$

we have

$$\sup_{x,y \in B \setminus \mathcal{N}_1} p^B(t,x,y) \geq \frac{1}{\mu(B)} e^{-\lambda_0 t}. \tag{3.8}$$

We claim that $\psi > 0$ on $\widehat{B}$. Noticing that

$$v(t,x) := P_t^B \psi(x) = e^{-\lambda_0 t} \psi(x) \tag{3.9}$$

is a caloric function on $(0,\infty) \times B$ and $\psi > 0$ has unit $L^2(B;\mu)$-norm, by PHI(ϕ), there are some $y_0 \in \widehat{B}$ and $r_0 > 0$ so that $B(y_0, r_0) \setminus \mathcal{N}_1 \subset \widehat{B}$, and $\psi > 0$ on $B(y_0, r_0)$. On the other hand, for every $x \in \widehat{B}$, by (3.5) (and so $p^B(s,x,y_0) > 0$ for some $s > 0$) and PHI(ϕ) again, there are constants $s_0 > 0$ and $r_1 \in (0, r_0]$ so that $p^B(t,x,z) > 0$ for every $t \geq s_0$ and $z \in B(y_0, r_1) \setminus \mathcal{N}_1$. It follows then

$$\psi(x) = e^{\lambda_0 t} P_t^B \psi(x) \geq e^{\lambda_0 t} \int_{B(y_0,r_1)} p^B(t,x,z) \psi(z) \, \mu(dz) > 0.$$

The claim that $\psi > 0$ on $\widehat{B}$ is proved. In particular, $\psi(\widehat{x}_0) > 0$.

(iii) Let C_i ($i = 1, \ldots, 6$) be the constants in the definition of PHI(ϕ). Applying PHI(ϕ) to the function $v(t,x) = e^{-\lambda_0 t} \psi(x)$ in the cylinder $Q(0, x_0, C_4 \phi(r), r)$, we get that

$$v(t_-, \widehat{x}_0) \leq C_6 v(t_+, \widehat{x}_0),$$

where $t_- = \frac{C_1 + C_2}{2} \phi(r)$ and $t_+ = \frac{C_3 + C_4}{2} \phi(r)$. It follows from (3.9) that

$$e^{-\lambda_0 t_-} \psi(\widehat{x}_0) \leq C_6 e^{-\lambda_0 t_+} \psi(\widehat{x}_0).$$

Since $\psi(\widehat{x}_0) > 0$, we arrive at

$$\lambda_0 \leq \frac{\log C_6}{t_+ - t_-} \leq \frac{1}{\phi(\kappa r)},$$

where $\kappa > 0$ is chosen so that

$$\frac{(C_3 + C_4) - (C_1 + C_2)}{2} \phi(r/2) \geq \phi(\kappa r) \log C_6$$

for all $r > 0$. This along with (3.8) further yields that for all $t > 0$,

$$\text{ess sup}_{x,y\in B} p^B(t,x,y) \geq \frac{1}{\mu(B)} e^{-\frac{t}{\phi(\kappa r)}}.$$

Following the arguments between (4.52) and (4.60) in [BGK, 1130–1131] line by line with small modifications, we obtain that there is a constant $c' > 0$ such that for all x, $y \in B(x_0, C_5 r) \setminus \mathcal{N}_1$ and $t \in (t_0 + C_3\phi(r), t_0 + C_4\phi(r))$ with $t_0 = (C_3 - C_1)\phi(r)$,

$$p^B(t,x,y) \geq \frac{c'}{V(x_0,r)}. \tag{3.10}$$

Note that, in order to get (3.10) we should change [BGK, (4.57)] into

$$\text{ess sup}_{x\in B'} p^B(s,x,z) \leq C_6 p^B(t,y,z), \quad y,z \in B' := B(x_0, C_5 r) \setminus \mathcal{N}_1.$$

Furthermore, using (3.10) instead of [BGK, (4.60)], one can verify that NDL(ϕ) holds for this case by the almost same argument between (4.60) and (4.63) in [BGK, 1131–1132]. □

We next prove that PHI(ϕ) implies UJS.

Proposition 3.3. *Under* VD *and* (1.7)*,* PHI(ϕ) *implies* UJS*.*

Proof. (i) Since $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M;\mu)$, for any relatively compact open sets U and V with $\bar{U} \subset V$, there is a function $\psi \in \mathcal{F} \cap C_c(M)$ such that $\psi = 1$ on U and $\psi = 0$ on V^c. Consequently,

$$\int_{U\times V^c} J(dx,dy) = \int_{U\times V^c} (\psi(x) - \psi(y))^2 \, J(dx,dy) \leq \mathcal{E}(\psi,\psi) < \infty. \tag{3.11}$$

Since U and V are arbitrarily, we get that for almost all $x \in M$ and each $r > 0$,

$$J(x, B(x,r)^c) < \infty. \tag{3.12}$$

(ii) Let D be an open set of M, and $f(t,z)$ be a bounded and non-negative function on $(0,\infty) \times D^c$. Then

$$u(t,z) := \begin{cases} \mathbb{E}^z \left[f(t - \tau_D, X_{\tau_D}); \tau_D \leq t \right], & t > 0, z \in M_0, \\ 0, & t > 0, z \in \mathcal{N} \end{cases}$$

is non-negative on $(0,\infty) \times M$ and caloric in $(0,\infty) \times D$. In the proof below, the constants C_i $(i = 1, \ldots, 6)$ are taken from the definition of PHI(ϕ). For any $x, y \in M_0$ and $0 < r \leq \frac{1}{2} d(x,y)$. For any $0 < \varepsilon < r$ and $0 < h < (C_1 + C_2)\phi(r)/2$, define

$$f_h(t,z) = \mathbf{1}_{((C_1+C_2)\phi(r)/2-h,(C_1+C_2)\phi(r)/2)}(t)\mathbf{1}_{B(y,\varepsilon)}(z), \quad t > 0, z \in M.$$

For $t \geq (C_1+C_2)\phi(r)/2$, define

$$\begin{aligned}u_h(t,z) =&\mathbb{E}^z\left[f_h(t-\tau_{B(x,r)}, X_{\tau_{B(x,r)}}); \tau_{B(x,r)} \leq t\right]\\ =&\mathbb{P}^z\Big(X_{\tau_{B(x,r)}} \in B(y,\varepsilon), t-(C_1+C_2)\phi(r)/2 < \tau_{B(x,r)} < t-(C_1+C_2)\phi(r)/2+h\Big)\end{aligned}$$

if $z \in M_0$, and $u_h(t,z) = 0$ if $z \in \mathcal{N}$.

According to Lemma 2.11, for any $z \in B(x,r) \cap M_0$ and $t \geq (C_1+C_2)\phi(r)/2$,

$$\begin{aligned}u_h(t,z) &= \mathbb{E}^z\left[\int_0^{\tau_{B(x,r)}} dv \int_{B(y,\varepsilon)} \mathbf{1}_{(t-(C_1+C_2)\phi(r)/2, t-(C_1+C_2)\phi(r)/2+h)}(v)\, J(X_v, du)\right]\\ &= \int_{t-(C_1+C_2)\phi(r)/2}^{t-(C_1+C_2)\phi(r)/2+h} \mathbb{E}^z\left[\mathbf{1}_{(0,\tau_{B(x,r)})}(v) \int_{B(y,\varepsilon)} J(X_v, du)\right] dv\\ &= \int_{t-(C_1+C_2)\phi(r)/2}^{t-(C_1+C_2)\phi(r)/2+h} P_v^{B(x,r)} H(z)\, dv,\end{aligned}$$

where $H(z) := \int_{B(y,\varepsilon)} J(z,du)$.

Applying PHI(ϕ) to u_h in $Q(0,x,C_4\phi(r),r)$, we obtain that for any $x_0 \in B(x,\varepsilon_1) \setminus (\mathcal{N}_{u_h} \cup \mathcal{N})$ with $\varepsilon_1 \leq C_5 r$,

$$u_h((C_1+C_2)\phi(r)/2, x_0) \leq C_6 u_h((C_3+C_4)\phi(r)/2, x).$$

Now, by the definition of u_h and Proposition 3.1,

$$\begin{aligned}u_h((C_3+C_4)\phi(r)/2, x) = &\int_{B(x,r)} p^{B(x,r)}\left(\frac{(C_3+C_4)-(C_1+C_2)}{2}\phi(r), x, z\right)\\ &\qquad \times u_h((C_1+C_2)\phi(r)/2, z)\, \mu(dz)\\ &\leq \frac{c_1}{V(x,r)} \int_{B(x,r)} u_h((C_1+C_2)\phi(r)/2, z)\, \mu(dz).\end{aligned}$$

Combining both inequalities above and integrating by $\frac{1}{V(x,\varepsilon_1)}\int_{B(x,\varepsilon_1)} \cdots \mu(dx_0)$, we have

$$\begin{aligned}&\frac{1}{V(x,\varepsilon_1)} \int_{B(x,\varepsilon_1)} u_h((C_1+C_2)\phi(r)/2, x_0)\, \mu(dx_0)\\ &\leq \frac{c_2}{V(x,r)} \int_{B(x,r)} u_h((C_1+C_2)\phi(r)/2, z)\, \mu(dz).\end{aligned} \tag{3.13}$$

According to (3.11), $H \in L^1(B(x,r);\mu)$. Then, as $h \to 0$,

$$\begin{aligned}&\Big|\int_{B(x,\varepsilon_1)} \Big(\frac{1}{h} u_h((C_1+C_2)\phi(r)/2, z) - H(z)\Big)\, \mu(dz)\Big|\\ &\leq \frac{1}{h}\int_0^h \int_{B(x,\varepsilon_1)} \Big|P_v^{B(x,r)} H(z) - H(z)\Big|\, \mu(dz)\, dv\\ &\leq \frac{1}{h}\int_0^h \|(P_v^{B(x,r)} H - H)\|_{L^1(B(x,r);\mu)}\, dv \to 0,\end{aligned}$$

thanks to the continuity of the semigroup $\{P_t^{B(x,r)}\}$ in $L^1(B(x,r);\mu)$. Similarly, we have

$$\lim_{h\to 0}\Big|\int_{B(x,r)}\Big(\frac{1}{h}u_h((C_1+C_2)\phi(r)/2,z)-H(z)\Big)\mu(dz)\Big|=0.$$

Thus dividing both sides of (3.13) by h and taking $h\to 0$, we have

$$\frac{1}{V(x,\varepsilon_1)}\int_{B(x,\varepsilon_1)}\int_{B(y,\varepsilon)}J(z,du)\,\mu(dz)\le\frac{c_2}{V(x,r)}\int_{B(x,r)}\int_{B(y,\varepsilon)}J(z,du)\,\mu(dz).$$

Letting $\varepsilon_1\to 0$, by (3.11), (3.12) and the Lebesgue differentiation theorem (e.g. see [H, Theorem 1.8]), we find that for μ-a.e $x\in M$,

$$J(x,B(y,\varepsilon))\le\frac{c_2}{V(x,r)}\int_{B(x,r)}\int_{B(y,\varepsilon)}J(z,du)\,\mu(dz)=\frac{c_2}{V(x,r)}\int_{B(y,\varepsilon)}\int_{B(x,r)}J(z,du)\,\mu(dz).$$

The above inequality implies that $J(x,dy)$ is absolutely continuous with respect to the measure $\mu(dy)$. So there is a non-negative function $J(x,y)$ so that $J(x,dy)=J(x,y)\,\mu(dy)$. Since $J(dx,dy)$ is a symmetric measure, we may modify the values of $J(x,y)$ so that it is symmetric in (x,y) for μ-a.e. $x,y\in M$. Dividing the above by $V(y,\varepsilon)$ and then sending $\varepsilon\to 0$, we have by the Lebesgue differentiation theorem again that for μ-a.e. $x,y\in M$ and $0<r<\frac{1}{2}d(x,y)$, we have

$$J(x,y)\le\frac{c_2}{V(x,r)}\int_{B(x,r)}J(z,y)\,\mu(dz),$$

proving UJS. □

Corollary 3.4. *If* VD, (1.7), UJS *and* NDL(ϕ) *are satisfied, then* $\mathrm{J}_{\phi,\le}$ *holds. In particular,* $\mathrm{J}_{\phi,\le}$ *holds under* VD, (1.7) *and* PHI(ϕ).

Proof. For any $x\in M_0$ and $r,t>0$, by Lemma 2.11,

$$\begin{aligned}1&\ge\mathbb{P}^x(X_{\tau_{B(x,r)}}\notin B(x,r),\tau_{B(x,r)}\le t\text{ and }\tau_{B(x,r)}\text{ is a jumping time})\\&=\int_0^t\int_{B(x,r)}p^{B(x,r)}(s,x,y)J(y,B(x,r)^c)\,\mu(dy)\,ds.\end{aligned}$$

By using NDL(ϕ) and taking $t=\phi(\varepsilon r)$ (where $\varepsilon\in(0,1)$ is the constant in the definition of NDL(ϕ)), we obtain that for any $x\in M_0$ and $r>0$,

$$\begin{aligned}1&\ge\int_{t/2}^t\int_{B(x,\varepsilon\phi^{-1}(t/2))}p^{B(x,r)}(s,x,y)J(y,B(x,r)^c)\,\mu(dy)\,ds\\&\ge\frac{t}{2}\text{ess inf}_{\,s\in[t/2,t],y\in B(x,\varepsilon\phi^{-1}(t/2))}p^{B(x,r)}(s,x,y)\int_{B(x,\varepsilon\phi^{-1}(t/2))}J(y,B(x,r)^c)\,\mu(dy)\\&\ge\frac{c_1t}{V(x,\phi^{-1}(t))}\int_{B(x,\varepsilon\phi^{-1}(t/2))}J(y,B(x,r)^c)\,\mu(dy).\end{aligned}$$

Thus, by VD and (1.7), there are constants $c_2, c_3 > 1$ such that

$$\int_{B(x,r)} J(y, B(x, c_2 r)^c)\, \mu(dy) \le \frac{c_3 V(x,r)}{\phi(r)}. \tag{3.14}$$

For fixed $x, y \in M$, set $r = \frac{d(x,y)}{1+c_2} \le \frac{d(x,y)}{2}$. Then, by (1.17) and (3.14),

$$\begin{aligned} J(x,y) \le& \frac{c_4}{V(x,r)} \int_{B(x,r)} J(z,y)\, \mu(dz) \\ \le& \frac{c_4^2}{V(x,r)V(y,r)} \int_{B(x,r)} \int_{B(y,r)} J(z,u)\, \mu(du)\, \mu(dz) \\ \le& \frac{c_4^2}{V(x,r)V(y,r)} \int_{B(x,r)} \int_{B(x,c_2 r)^c} J(z,u)\, \mu(du)\, \mu(dz) \\ \le& \frac{c_5}{V(x,r)V(x,r)} \int_{B(x,r)} J(z, B(x,c_2 r)^c)\, \mu(dz) \le \frac{c_6}{V(x,r)\phi(r)}, \end{aligned}$$

which completes the proof, thanks to VD and (1.7) again. □

We note that by Proposition 3.1, Corollary 3.4, Proposition 3.5 in the next subsection and Theorem 1.10, we have PHI(ϕ) $\Longrightarrow$ UHK(ϕ).

3.2 Consequences of NDL(ϕ)

In this subsection, we present some consequences of NDL(ϕ). Since PHI(ϕ) implies NDL(ϕ) by Proposition 3.2, this subsection can be regarded as a continuation of Subsection 3.1.

Proposition 3.5. *Assume that* VD, (1.7), *and* NDL(ϕ) *hold. Then*

(i) PI(ϕ) *holds. If furthermore* RVD *is satisfied, then* FK(ϕ) *also holds.*

(ii) $E_{\phi,\ge}$ *holds. If in addition* RVD *is satisfied, then we have* $E_{\phi,\le}$ *and so* E_ϕ.

In particular, if VD, RVD, (1.7) *and* PHI(ϕ) *hold, then so do* (i) *and* (ii).

Proof. (i) For any $x_0 \in M$ and $r > 0$, let $B = B(x_0, r)$. Define a bilinear form $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ on $L^2(B;\mu)$ by

$$\begin{aligned} \bar{\mathcal{E}}(u,v) &= \int_{B\times B} (u(x)-u(y))(v(x)-v(y)) J(x,y)\, \mu(dx)\, \mu(dy), \\ \bar{\mathcal{F}} &= \left\{ u \in L^2(B;\mu) : \bar{\mathcal{E}}(u,u) < \infty \right\}. \end{aligned}$$

One can easily check by using Fatou's lemma that $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ is closable and is a Dirichlet form on $L^2(B;\mu)$. Let $\{\bar{P}_t\}$ be the L^2-semigroup associated with $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$. Let $\bar{\mathcal{F}}_B$ be the closure of $\bar{\mathcal{F}} \cap C_c(B)$. Then $(\bar{\mathcal{E}}, \bar{\mathcal{F}}_B)$ is a regular Dirichlet form on $L^2(B;\mu)$, whose associated semigroup will be denoted as $\{\bar{P}_t^B\}$. By [CF, Theorem 5.2.17], $(\bar{\mathcal{E}}, \bar{\mathcal{F}}_B)$ is the resurrected Dirichlet form of $(\mathcal{E}, \mathcal{F}_B)$. In other words, if we denote by $\bar{X}^B = \{\bar{X}^B_t\}$ the Hunt process associated with the regular Dirichlet form $(\bar{\mathcal{E}}, \bar{\mathcal{F}}_B)$ on $L^2(B;\mu)$, then $\bar{X}^B$ is the resurrection of $X^B = \{X^B_t\}$ in B, and so $\bar{X}^B$ can be obtained from X^B by creation through a Feynman-Kac transform. Consequently, $\bar{X}^B$ has a transition density function $\bar{p}^B(t,x,y)$ with respect to μ and $\bar{p}^B(t,x,y) \geq p^B(t,x,y)$ for every $t > 0$ and $x, y \in B \cap M_0$. This together with NDL(ϕ) implies that there exist $\varepsilon \in (0,1)$ and $c_1 > 0$ such that for all $x_0 \in M$ and $x, y \in B(x_0, \varepsilon^2 r) \cap M_0$,

$$\bar{p}^B(\phi(\varepsilon r), x, y) \geq p^B(\phi(\varepsilon r), x, y) \geq \frac{c_1}{V(x_0, r)}.$$

On the other hand, we know from [CF, Section 6.2], $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ is the active reflected Dirichlet space for $(\bar{\mathcal{E}}, \bar{\mathcal{F}}_B)$. Although $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ may not be regular as a Dirichlet form on $L^2(B;\mu)$, by Silverstein [Si, Theorem 20.1], there is a locally compact separable metric space $\widetilde{B}$ (called regularizing space) so that $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ is regular on $L^2(\widetilde{B}; \widetilde{\mu})$ and B is intrisincally open in $\widetilde{B}$. Here $\widetilde{\mu}$ is an extension of μ to $\widetilde{B}$ by setting $\widetilde{\mu}(\widetilde{B} \setminus B) = 0$. Let $\widetilde{X} = \{\widetilde{X}_t\}$ denote the Hunt process on $\widetilde{B}$ associated with the regular Dirichlet form $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ on $L^2(\widetilde{B};\mu)$. Then the part process $\widetilde{X}^B = \{\widetilde{X}^B_t\}$ of $\widetilde{X}$ killed upon leaving B has the same distribution as $\bar{X}^B$. Now for $f \in \bar{\mathcal{F}}$, by the basic property of Dirichlet form (see, for example, [CF, (1.1.4)]),

$$\begin{aligned}
\bar{\mathcal{E}}(f,f) &\geq \frac{1}{\phi(\varepsilon r)} \int_B f(x)(f - \bar{P}_{\phi(\varepsilon r)} f)(x)\, \mu(dx) \\
&\geq \frac{1}{2\phi(\varepsilon r)} \mathbb{E}^{\widetilde{\mu}} \left[(f(\widetilde{X}_{\phi(\varepsilon r)}) - f(\widetilde{X}_0))^2 \right] \\
&\geq \frac{1}{2\phi(\varepsilon r)} \mathbb{E}^{\widetilde{\mu}} \left[(f(\widetilde{X}_{\phi(\varepsilon r)}) - f(\widetilde{X}_0))^2; \phi(\varepsilon r) < \tau_B \right] \\
&= \frac{1}{2\phi(\varepsilon r)} \int_{B \times B} \bar{p}^B(\phi(\varepsilon r), x, y)(f(x) - f(y))^2\, \mu(dx)\, \mu(dy) \\
&\geq \frac{c_2}{V(x_0, r)\phi(r)} \int_{B(x_0, \varepsilon^2 r)} \int_{B(x_0, \varepsilon^2 r)} (f(x) - f(y))^2\, \mu(dx)\, \mu(dy) \\
&\geq \frac{c_3}{\phi(r)} \int_{B(x_0, \varepsilon^2 r)} (f(x) - \bar{f}_{B(x_0, \varepsilon^2 r)})^2\, \mu(dx).
\end{aligned}$$

Recall that $\bar{f}_D := \frac{1}{\mu(D)} \int_D f\, d\mu$ for any open set D of M. In the last two inequalities above we have used VD, (1.7) and the fact that

$$\int_{B(x_0, \varepsilon^2 r)} \left(f(x) - \bar{f}_{B(x_0, \varepsilon^2 r)} \right)^2 \mu(dx) = \inf_{a \in \mathbb{R}} \int_{B(x_0, \varepsilon^2 r)} (f(x) - a)^2\, \mu(dx).$$

This establishes PI(ϕ).

That PI(ϕ) implies FK(ϕ) under additional assumption RVD is given in Proposition 2.9. (Note that, under additional assumption RVD, FK(ϕ) is also a direct consequence of PHI(ϕ), thanks to Propositions 3.1 and 2.9.)

(ii) By VD, (1.7) and NDL(ϕ), for some $\varepsilon \in (0,1)$,

$$\mathbb{P}^x(\tau_{B(x,r)} \geq \phi(\varepsilon r)) = \int_{B(x,r)} p^{B(x,r)}(\phi(\varepsilon r), x, y)\, \mu(dy)$$

$$\geq \int_{B(x,\varepsilon^2 r)} p^{B(x,r)}(\phi(\varepsilon r), x, y)\, \mu(dy) \geq c_6,$$

and thus $\mathbb{E}^{x_0}\tau_{B(x_0,r)} \geq c_6\phi(r)$. This proves $\mathrm{E}_{\phi,\geq}$.

Next, we assume that RVD is satisfied. Let $B = B(x_0, r)$ with $x_0 \in M_0$ and $r > 0$, and $B' = B(x_0, r/(2l_\mu))$, where $l_\mu > 1$ is the constant in (1.6). Then, VD, (1.7) and NDL(ϕ) give us that for $t = \phi(r/\varepsilon)$ with some $\varepsilon \in (0, 1)$,

$$p(t, x, y) \geq \frac{c_1}{V(x_0, r)}, \quad x, y \in B\backslash\mathcal{N}.$$

Fix $y_0 \in M$ with $(1+2l_\mu)r/(2l_\mu(1+l_\mu)) < d(x_0, y_0) < (1+2l_\mu)r/(2(1+l_\mu))$ (such a point y_0 indeed exists due to RVD), then for any $x \in B'\backslash\mathcal{N}$,

$$\begin{aligned}\mathbb{P}^x(X_t \notin B') \geq & \mathbb{P}^x(X_t \in B(y_0, r/(2(1+l_\mu)))) = \int_{B(y_0,r/(2(1+l_\mu)))} p(t, x, y)\, \mu(dy) \\ \geq & \frac{c_2 V(y_0, r/(2(1+l_\mu)))}{V(x_0, r)} \geq c_3,\end{aligned}$$

where VD is used in the last inequality. So, we have $\mathbb{P}^x(\tau_{B'} > t) \leq \mathbb{P}^x(X_t \in B') \leq 1 - c_3$ for all $x \in B'\backslash\mathcal{N}$. Hence, by the Markov property, $\mathbb{P}^x(\tau_{B'} > kt) \leq (1 - c_3)^k$, and thus $\mathbb{E}^x\tau_{B'} \leq c_4 t$. Since $\mathbb{E}^{x_0}\tau_{B(x_0,r/(2l_\mu))} = \mathbb{E}^{x_0}\tau_{B'}$, replacing $r/(2l_\mu)$ by r gives us that $\mathbb{E}^{x_0}\tau_{B(x_0,r)} \leq c_5\phi(r)$, where (1.7) is used in the inequality above. Therefore, E_ϕ holds. (Note that, by Lemma 2.8, under VD and (1.7), FK(ϕ) implies $\mathrm{E}_{\phi,\leq}$. Then, $\mathrm{E}_{\phi,\leq}$ can be also deduced from PHI(ϕ) directly under additional assumption RVD, thanks to Propositions 3.1 and 2.9.) □

Combining all the conclusions of this and previous subsections, we can obtain the following main result in this section.

Theorem 3.6. *Assume that μ and ϕ satisfy* VD, RVD *and* (1.7) *respectively. Then the following hold*

$$\begin{aligned}\mathrm{PHI}(\phi) & \Longrightarrow \mathrm{UHKD}(\phi) + \mathrm{NDL}(\phi) + \mathrm{UJS} + \mathrm{E}_\phi + \mathrm{J}_{\phi,\leq} \\ & \Longleftrightarrow \mathrm{UHKD}(\phi) + \mathrm{NDL}(\phi) + \mathrm{UJS} \\ & \Longleftrightarrow \mathrm{UHK}(\phi) + \mathrm{NDL}(\phi) + \mathrm{UJS}.\end{aligned}$$

Proof. Note that by Corollary 3.4, NDL(ϕ) + UJS $\Longrightarrow$ $\mathrm{J}_{\phi,\leq}$; and that by Proposition 3.5, NDL(ϕ) implies E_ϕ. According to Theorem 1.10, UHK(ϕ) + conservativeness $\Longleftrightarrow$ UHKD(ϕ) + $\mathrm{J}_{\phi,\leq}$ + E_ϕ. Then the required assertion now follows from all the previous propositions. (Here we note that both PHI(ϕ) and NDL(ϕ) imply the conservativeness of the process $\{X_t\}$, see Proposition 2.4 and Proposition 3.2.) □

3.3 Hölder regularity

Another consequence of NDL(ϕ) is that, it along with $E_{\phi,\le}$ and $J_{\phi,\le}$ implies the joint Hölder regularity of bounded caloric functions. In other words, NDL(ϕ) + $E_{\phi,\le}$ + $J_{\phi,\le}$ imply PHR(ϕ) and EHR. For our purpose, in the following lemma we use the definition of NDL(ϕ) with $\varepsilon\phi(r)$ and $\phi^{-1}(\varepsilon t)$ replaced by $\phi(\varepsilon r)$ and $\varepsilon\phi^{-1}(t)$, respectively.

Lemma 3.7. *Suppose that* VD, (1.7) *and* NDL(ϕ) *hold. For every $0 < \delta \le \varepsilon$ (where ε is the constant in the definition of* NDL(ϕ)*), there exists a constant $C_1 > 0$ such that for every $r > 0$, $x \in M_0$, $t \ge \delta\phi(r)$ and any compact set $A \subset [t - \delta\phi(r), t - \delta\phi(r)/2] \times B(x, \phi^{-1}(\varepsilon\delta\phi(r)/2))$,*

$$\mathbb{P}^{(t,x)}(\sigma_A < \tau_{[t-\delta\phi(r),t]\times B(x,r)}) \ge C_1 \frac{m \otimes \mu(A)}{V(x,r)\phi(r)}, \tag{3.15}$$

where $m \otimes \mu$ is a product of the Lebesgue measure on $\mathbb{R}_+$ and μ on M.

Proof. The proof is almost the same as that for [CKK2, Lemma 4.9(i)]. Let $\tau_r = \tau_{[t-\delta\phi(r),t]\times B(x,r)}$ and $A_s = \{y \in M : (s,y) \in A\}$. For any $t, r > 0$ and $x \in M_0$,

$$\begin{aligned}\delta\phi(r)\mathbb{P}^{(t,x)}(\sigma_A < \tau_r) &\ge \int_0^{\delta\phi(r)} \mathbb{P}^{(t,x)}\left(\int_0^{\tau_r} \mathbf{1}_A(t-s, X_s)\,ds > 0\right) du \\ &\ge \int_0^{\delta\phi(r)} \mathbb{P}^{(t,x)}\left(\int_0^{\tau_r} \mathbf{1}_A(t-s, X_s)\,ds > u\right) du \\ &= \mathbb{E}^{(t,x)}\left[\int_0^{\tau_r} \mathbf{1}_A(t-s, X_s)\,ds\right].\end{aligned}$$

Note that, for any $t \ge \delta\phi(r)$,

$$\begin{aligned}\mathbb{E}^{(t,x)}\left[\int_0^{\tau_r} \mathbf{1}_A(t-s, X_s)\,ds\right] &= \int_{\delta\phi(r)/2}^{\delta\phi(r)} \mathbb{P}^{(t,x)}\left(\left(t-s, X_s^{B(x,r)}\right) \in A\right) ds \\ &= \int_{\delta\phi(r)/2}^{\delta\phi(r)} \mathbb{P}^{x}\left(X_s^{B(x,r)} \in A_{t-s}\right) ds \\ &= \int_{\delta\phi(r)/2}^{\delta\phi(r)} ds \int_{A_{t-s}} p^{B(x,r)}(s,x,y)\,\mu(dy).\end{aligned}$$

By VD, (1.7) and NDL(ϕ), for any $s \in [\delta\phi(r)/2, \delta\phi(r)]$ and $y \in B(x, \phi^{-1}(\varepsilon\delta\phi(r)/2)) \setminus \mathcal{N}$,

$$p^{B(x,r)}(s,x,y) \ge \frac{c_1}{V(x,r)}.$$

Thus,

$$\mathbb{E}^{(t,x)}\left[\int_0^{\tau_r} \mathbf{1}_A(t-s, X_s)\,ds\right] \ge \frac{c_1}{V(x,r)} \int_{\delta\phi(r)/2}^{\delta\phi(r)} ds \int_{A_{t-s}} \mu(dy) = \frac{c_1 m \otimes \mu(A)}{V(x,r)}.$$

Combining all the conclusions above, we obtain the desired assertion. □

Proposition 3.8. *Assume that* VD, (1.7), NDL(ϕ), $\mathrm{E}_{\phi,\le}$ *and* $\mathrm{J}_{\phi,\le}$ *hold. For every* $\delta \in (0,1)$*, there exist positive constants* $C > 0$ *and* $\gamma \in (0,1]$*, where* γ *is independent of* δ*, so that for any bounded caloric function* u *in* $Q(t_0, x_0, \phi(r), r)$*,*

$$|u(s,x) - u(t,y)| \le C \left(\frac{\phi^{-1}(|s-t|) + d(x,y)}{r} \right)^{\gamma} \text{ess sup}_{[t_0, t_0+\phi(r)] \times M} |u|$$

for $dt \times \mu$*-almost all* (s,x) *and* $(t,y) \in Q(t_0, x_0, \phi(\delta r), \delta r)$*. In other words, under* VD *and* (1.7), $\mathrm{NDL}(\phi) + \mathrm{E}_{\phi,\le} + \mathrm{J}_{\phi,\le}$ *imply* PHR(ϕ) *and* EHR.

Proof. With estimate (3.15), the result can be proved in exactly the same way as that for [CK1, Theorem 4.14]. We omit the details here. □

The following two consequences of Hölder regularities will be used in Subsection 4.2.

Proposition 3.9. *Suppose* (1.7) *holds. Then* PHR(ϕ) *implies* $\mathrm{E}_{\phi,\ge}$.

Proof. Let $B = B(x_0, r)$ for $x_0 \in M_0$ and $r > 0$. Define $u(t,x) = \mathbb{P}^x(\tau_B > t)$, which is a bounded parabolic function in $(0,\infty) \times B$. Since $\lim_{t \to 0} u(t, x_0) = 1$, there is some $t_0 > 0$ so that $u(t_0, x_0) \ge 3/4$. By (1.14) of PHR(ϕ) and (1.7), there is a constant $\delta_0 > 0$ independent of x_0 and $r > 0$ so that $|u(s,x_0) - u(t_0,x_0)| \le 1/4$ for $s \in [t_0, t_0 + \delta_0 \phi(r)]$. It follows then

$$\mathbb{E}^{x_0}\left[\tau_{B(x_0,r)}\right] = \int_0^\infty u(s,x_0)\, ds \ge \int_{t_0}^{t_0+\delta_0\phi(r)} u(s,x_0)\, ds \ge \delta_0 \phi(r)/2.$$

That is, $\mathrm{E}_{\phi,\ge}$ holds. □

Lemma 3.10. *Suppose* EHR *holds. Let* $D \subset M$ *be an open set with* $\text{ess sup}_{y \in D \cap M_0} \mathbb{E}^y \tau_D < \infty$*. Fix a function* $f \in B_b(D)$ *and set* $u = G^D f$*. Then for any* $B(x_0, r) \subset D$ *and* $0 < r_1 \le r$*,*

$$\mathrm{osc}_{B(x_0,r_1)\cap M_0} u \le 2 \sup_{y \in B(x_0,r)\cap M_0} |f(y)| \sup_{y \in B(x_0,r)\cap M_0} \mathbb{E}^y \tau_{B(x_0,r)} + c\,(r_1/r)^{\theta} \sup_{z \in D \cap M_0} |u(z)|,$$

where $c > 0$ *and* $\theta \in (0,1]$ *only depend on the constants in* EHR*.*

Proof. Note that for any $x \in D \cap M_0$,

$$G^D|f|(x) = \mathbb{E}^x \left[\int_0^{\tau_D} |f(X_t)|\, dt \right] \le \sup_{y \in D \cap M_0} |f(y)|\, \mathbb{E}^x \tau_D.$$

Consequently, for any $r_1 \in (0, r)$,

$$\mathrm{osc}_{B(x,r_1)\cap M_0} G^{B(x_0,r)} f \le 2 \sup_{y \in B(x_0,r_1)\cap M_0} G^{B(x_0,r)}|f|(y)$$

$$\le 2 \sup_{y\in B(x_0,r)\cap M_0} |f(y)| \sup_{y\in B(x_0,r_1)\cap M_0} \mathbb{E}^y \tau_{B(x,r)}.$$

Since $G^D f(y) - G^{B(x_0,r)} f(y) = \mathbb{E}^y[G^D f(X_{\tau_{B(x_0,r)}})] = \mathbb{E}^y[u(X_{\tau_{B(x_0,r)}})]$ is harmonic in $B(x_0,r)$, and $u = 0$ outside D, we have by EHR and Remark 1.13(ii) that

$$\begin{aligned}
&\mathrm{osc}_{B(x,r_1)\cap M_0} u \\
&\le \mathrm{osc}_{B(x_0,r_1)\cap M_0} G^{B(x_0,r)} f + \mathrm{osc}_{B(x_0,r_1)\cap M_0} (G^D f - G^{B(x_0,r)} f) \\
&\le 2 \sup_{y\in B(x,r)\cap M_0} |f(y)| \sup_{y\in B(x_0,r_1)\cap M_0} \mathbb{E}^y \tau_{B(x_0,r)} + c(r_1/r)^\theta \sup_{y\in D\cap M_0} |\mathbb{E}^y[u(X_{\tau_{B(x,r)}})]| \\
&\le 2 \sup_{y\in B(x,r)\cap M_0} |f(y)| \sup_{y\in B(x,r)\cap M_0} \mathbb{E}^y \tau_{B(x,r)} + c\,(r_1/r)^\theta \sup_{z\in D\cap M_0} |u(z)|.
\end{aligned}$$

This proves the lemma. □

4 Equivalences of PHI(ϕ)

We have already given some part of the proof of Theorem 1.17 in Section 3. In this section, we will complete the proof. In Subsection 4.1, we prove (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4). (1) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) will be proved in Subsection 4.2, and (1) $\Longleftrightarrow$ (7) in Subsection 4.3.

4.1 PHI(ϕ) $\Longleftrightarrow$ PHI$^+$(ϕ) $\Longleftrightarrow$ UHK(ϕ)+NDL(ϕ)+UJS $\Longleftrightarrow$ NDL(ϕ)+ UJS

In this subsection, we will establish (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) in Theorem 1.17. Since (1) $\Longrightarrow$ (3) is already proved in Subsection 3.1, and (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) hold trivially, it remains to show that prove (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2).

Lemma 4.1. *Assume that* VD, (1.7), UHK(ϕ), NDL(ϕ) *and* UJS. *Let* $\delta \le \varepsilon$ *(where* $\varepsilon \in (0,1)$ *is the constant in the definition of* NDL(ϕ)*), and* $\theta \ge 1/2$. *Let* $0 < \delta_0 < \delta$ *and* $0 < \delta_1 < \delta_2 < \delta_3 < \delta_4$ *such that* $(\delta_3 - \delta_2)\phi(r) \ge \phi(\delta_0 r)$ *and* $\delta_4\phi(r) \le \phi(\delta r)$ *for all* $r > 0$. *Set*

$$Q_1 = (t_0, t_0 + \delta_4\phi(r)) \times B(x_0, \delta_0^2 r), \quad Q_2 = (t_0, t_0 + \delta_4\phi(r)) \times B(x_0, r)$$

for $x_0 \in M$, $t_0 \ge 0$ *and* $r > 0$. *Define*

$$Q_3 = [t_0 + \delta_1\phi(r), t_0 + \delta_2\phi(r)] \times B(x_0, \delta_0^2 r/2) \setminus \mathcal{N}$$

and

$$Q_4 = [t_0 + \delta_3\phi(r), t_0 + \delta_4\phi(r)] \times B(x_0, \delta_0^2 r/2) \setminus \mathcal{N}.$$

Let $f : (t_0, \infty) \times M \to \mathbb{R}_+$ *be bounded and supported in* $(t_0, \infty) \times B(x_0, (1+\theta)r)^c$. *Then there is a constant* $C_2 > 0$ *such that the following holds:*

$$\mathbb{E}^{(t_1,y_1)} f(Z_{\tau_{Q_1}}) \le C_2 \mathbb{E}^{(t_2,y_2)} f(Z_{\tau_{Q_2}}) \quad \textit{for every } (t_1,y_1) \in Q_3 \textit{ and } (t_2,y_2) \in Q_4.$$

Proof. The proof is the same as that of [CKK1, Lemma 5.3]. We present the proof here for the sake of completeness.

Without loss of generality, we may and do assume that $t_0 = 0$. For $x_0 \in M$ and $s > 0$, set $B_s = B(x_0, s)$. By Lemma 2.11, for any $(t_2, y_2) \in Q_4$,

$$\begin{aligned}
\mathbb{E}^{(t_2,y_2)} f(Z_{\tau_{Q_2}}) &= \mathbb{E}^{(t_2,y_2)} f(t_2 - (\tau_{B_r} \wedge t_2), X_{\tau_{B_r} \wedge t_2}) \\
&= \mathbb{E}^{(t_2,y_2)} \left[\int_0^{t_2} \mathbf{1}_{\{t \le \tau_{B_r}\}} \, dt \int_{B^c_{(1+\theta)r}} f(t_2 - t, v) J(X_t, v) \, \mu(dv) \right] \\
&= \int_0^{t_2} dt \int_{B^c_{(1+\theta)r}} f(t_2 - t, v) \mathbb{E}^{(t_2,y_2)} \left[\mathbf{1}_{\{t \le \tau_{B_r}\}} J(X_t, v) \right] \mu(dv) \\
&= \int_0^{t_2} ds \int_{B^c_{(1+\theta)r}} f(s, v) \mathbb{E}^{(t_2,y_2)} \left[\mathbf{1}_{\{t_2 - s \le \tau_{B_r}\}} J(X_{t_2 - s}, v) \right] \mu(dv) \\
&= \int_0^{t_2} ds \int_{B^c_{(1+\theta)r}} f(s, v) \, \mu(dv) \int_{B_r} p^{B_r}(t_2 - s, y_2, z) J(z, v) \, \mu(dz) \qquad (4.1) \\
&\ge \int_0^{t_1} ds \int_{B^c_{(1+\theta)r}} f(s, v) \, \mu(dv) \int_{B_{\delta_0^2 r}} p^{B_r}(t_2 - s, y_2, z) J(z, v) \, \mu(dz).
\end{aligned}$$

Since for $s \in [0, t_1]$, $\phi(\delta_0 r) \le t_2 - t_1 \le t_2 - s \le \phi(\delta r)$, by VD, (1.7) and NDL($\phi$), we know that the right hand side of the inequality above is greater than or equal to

$$\frac{c_1}{V(x_0, r)} \int_0^{t_1} ds \int_{B^c_{(1+\theta)r}} f(s, v) \, \mu(dv) \int_{B_{\delta_0^2 r}} J(z, v) \, \mu(dz).$$

So the proof is complete, once we can obtain that for every $(t_1, y_1) \in Q_3$,

$$\mathbb{E}^{(t_1,y_1)} f(Z_{\tau_{Q_1}}) \le \frac{c_2}{V(x_0, r)} \int_0^{t_1} ds \int_{B^c_{(1+\theta)r}} f(s, v) \, \mu(dv) \int_{B_{\delta_0^2 r}} J(z, v) \, \mu(dz). \qquad (4.2)$$

Similar to the argument for (4.1), we have by using Lemma 2.11,

$$\begin{aligned}
\mathbb{E}^{(t_1,y_1)} f(Z_{\tau_{Q_1}}) &= \int_0^{t_1} ds \int_{B^c_{(1+\theta)r}} f(s, v) \, \mu(dv) \int_{B_{\delta_0^2 r}} p^{B_{\delta_0^2 r}}(t_1 - s, y_1, z) J(z, v) \, \mu(dz) \\
&= \int_0^{t_1} ds \int_{B_{\delta_0^2 r}} p^{B_{\delta_0^2 r}}(t_1 - s, y_1, z) \, \mu(dz) \int_{B^c_{(1+\theta)r}} f(s, v) J(z, v) \, \mu(dv).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\int_{B_{\delta_0^2 r}} p^{B_{\delta_0^2 r}}(t_1 - s, y_1, z) \, \mu(dz) \int_{B^c_{(1+\theta)r}} f(s, v) J(z, v) \, \mu(dv) \\
&= \int_{B_{\delta_0^2 r} \setminus B_{3\delta_0^2 r/4}} p^{B_{\delta_0^2 r}}(t_1 - s, y_1, z) \, \mu(dz) \int_{B^c_{(1+\theta)r}} f(s, v) J(z, v) \, \mu(dv)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_{3\delta_0^2 r/4}} p^{B_{\delta_0^2 r}}(t_1-s,y_1,z)\,\mu(dz)\int_{B^c_{(1+\theta)r}} f(s,v)J(z,v)\,\mu(dv)\\
&=: I_1+I_2.
\end{aligned}
$$

On the one hand, when $z\in(B_{\delta_0^2 r}\backslash B_{3\delta_0^2 r/4})\cap M_0$, we have $\delta_0^2 r/4\le d(y_1,z)\le 3\delta_0^2 r/2$, and so by UHK($\phi$), VD and (1.7),

$$
p^{B_{\delta_0^2 r}}(t_1-s,y_1,z)\le \frac{c_3 t_1}{V(y_1,d(y_1,z))\phi(d(y_1,z))}\le \frac{c_4}{V(x_0,r)}
$$

for some constants $c_3,c_4>0$. Hence, $\int_0^{t_1} I_1\,ds$ is less than or equal to the right hand side of (4.2). On the other hand, for $z\in B_{3\delta_0^2 r/4}$, by UJS and VD,

$$
\begin{aligned}
\int_{B^c_{(1+\theta)r}} J(z,v)f(s,v)\,\mu(dv)\le&\frac{c_5}{V(x_0,r)}\int_{B(z,\delta_0^2 r/4)} J(w,v)\,\mu(dw)\int_{B^c_{(1+\theta)r}} f(s,v)\,\mu(dv)\\
\le&\frac{c_5}{V(x_0,r)}\int_{B_{\delta_0^2 r}} J(w,v)\,\mu(dw)\int_{B^c_{(1+\theta)r}} f(s,v)\,\mu(dv).
\end{aligned}
$$

Note that the right hand side of the above inequality does not depend on z. Multiplying both sides by $p^{B_{\delta_0^2 r}}(t_1-s,y_1,z)$ and integrating over $z\in B_{3\delta_0^2 r/4}$ and then over $s\in[0,t_1]$, we obtain that $\int_0^{t_1} I_2\,ds$ is also less than or equal to the right hand side of (4.2). This proves the lemma. □

Once again, in the following lemma we use the definition of NDL(ϕ) with $\varepsilon\phi(r)$ and $\phi^{-1}(\varepsilon t)$ replaced by $\phi(\varepsilon r)$ and $\varepsilon\phi^{-1}(t)$, respectively.

Lemma 4.2. *Suppose that* VD, (1.7) *and* NDL(ϕ) *hold. Let* $0<\delta\le\varepsilon/4$ *such that* $4\delta\phi(2r)\le\varepsilon\phi(r)$ *for all* $r>0$*, where* $\varepsilon\in(0,1)$ *is the constant in the definition of* NDL(ϕ)*. Then there exists a constant* $C_3>0$ *such that for every* $R>0$*,* $r\in(0,\phi^{-1}(\varepsilon\delta\phi(R)/2)/2]$*,* $x_0\in M$*,* $\delta\phi(R)/2\le t-s\le 4\delta\phi(2R)$*,* $x\in B(x_0,\phi^{-1}(\varepsilon\delta\phi(R)/2)/2)\setminus\mathcal{N}$*, and* $z\in B(x_0,\phi^{-1}(\varepsilon\delta\phi(R)/2))\setminus\mathcal{N}$

$$
\mathbb{P}^{(t,z)}(\sigma_{U(s,x,r)}\le\tau_{[s,t]\times B(x_0,R)})\ge C_3\frac{V(x,r)}{V(x,R)},
$$

where $U(s,x,r)=\{s\}\times B(x,r)$.

Proof. The left hand side of the desired estimate is equal to

$$
\mathbb{P}^z(X^{B(x_0,R)}_{t-s}\in B(x,r))=\int_{B(x,r)} p^{B(x_0,R)}(t-s,z,y)\,\mu(dy). \tag{4.3}
$$

By VD, (1.7), NDL(ϕ), and the facts that $\delta\phi(R)/2\le t-s\le 4\delta\phi(2R)$ and $B(x,r)\subset B(x_0,\phi^{-1}(\varepsilon\delta\phi(R)/2))$, (4.3) is greater than or equal to

$$
c_1\frac{V(x,r)}{V(z,R)}\ge c_2\frac{V(x,r)}{V(x,R)}.
$$

This proves the desired assertion. □

Having these two lemmas as well as Lemma 3.7 at hand, one can obtain the following form of $\mathrm{PHI}^+(\phi)$.

Theorem 4.3. *Suppose that* VD *and* (1.7) *hold. Under* $\mathrm{UHK}(\phi)$, $\mathrm{NDL}(\phi)$ *and* UJS, *the following* $\mathrm{PHI}^+(\phi)$ *holds: there exist constants* $\delta > 0$, $C > 1$ *and* $K \geq 1$ *such that for every* $x_0 \in M \setminus \mathcal{N}$, $t_0 \geq 0$, $R > 0$ *and every non-negative function* u *on* $[0, \infty) \times M$ *that is parabolic on* $Q := (t_0, t_0 + 4\delta\phi(CR)) \times B(x_0, CR)$, *we have*

$$\text{ess sup}_{(t_1,y_1)\in Q_-} u(t_1, y_1) \leq K \text{ ess inf}_{(t_2,y_2)\in Q_+} u(t_2, y_2), \tag{4.4}$$

where $Q_- = [t_0 + \delta\phi(CR), t_0 + 2\delta\phi(CR)] \times B(x_0, R)$ *and* $Q_+ = [t_0 + 3\delta\phi(CR), t_0 + 4\delta\phi(CR)) \times B(x_0, R)$.

Proof. Let $\varepsilon \in (0, 1)$ be the constant in $\mathrm{NDL}(\phi)$. Take and fix some $\delta \in (0, \varepsilon/4]$ so that $\delta\phi(2r) \leq \phi(\varepsilon r)/4$ for all $r > 0$ and take $\delta_0 \in (0, \delta)$ so that $\phi(\delta_0 r) \leq \delta\phi(r)$ for all $r > 0$. The existence of such δ and δ_0 is guaranteed by the assumption (1.7). We choose δ and δ_0 in such a way so that Lemma 4.1 holds by taking δ_j to be $j\delta$ for $j = 1, 2, 3, 4$ there. Condition (1.7) ensures that there is a constant $c_0 \in (0, 1/2)$ so that $\phi^{-1}(\delta\varepsilon\phi(r)/2) \geq c_0 r$ for every $r > 0$. Take

$$C = (2/c_0) + 2 \quad \text{and} \quad C_0 = C - 2 = 2/c_0. \tag{4.5}$$

The reason of defining such C_0 is that the conclusion of Lemma 4.2 holds for any $x, z \in B(x_0, R/C_0)$.

Let u be a non-negative function on $[0, \infty) \times M$ that is parabolic on $Q := (t_0, t_0 + 4\delta\phi(CR)) \times B(x_0, CR)$. We will show (4.4) holds.

The proof below is mainly based on that of [CKK1, Theorem 5.2] with some non-trivial modifications; see also the proof of [CK1, Proposition 4.3] or of [CK2, Theorem 4.12]. Truncating u by n outside Q and then passing $n \to \infty$ if needed, without loss of generality, we may and do assume that $t_0 = 0$, and that the function u is bounded on Q, see Step 3 in the proof of [CKK1, Theorem 5.2] (e.g. page 1085 in [CKK1]). Furthermore, by looking at $au + b$ for suitable constants a and b, we may and do assume that $\inf_{(t,y)\in Q_+} u(t, y) = 1/2$. Let $(t_*, y_*) \in Q_+$ be such that $u(t_*, y_*) \leq 1$. It is enough to show that $u(t, x)$ is bounded from above in Q_- by a constant that is independent of the function u.

For any $t \geq \delta\phi(r)$, set $Q^{\downarrow}(t, \delta, x, r) = [t - \delta\phi(r), t] \times B(x, r)$. Note that

$$m \otimes \mu(Q^{\downarrow}(t, \delta, x, r)) = \delta\phi(r)V(x, r).$$

By Lemma 3.7, there exists a constant $c_1 \in (0, 1/2)$ so that for any $r \leq R/2$ and any compact set D satisfying that

$$D \subset \left[t - \delta\phi(r), t - \frac{1}{2}\delta\phi(r)\right] \times B(x, c_0 r) \subset Q^{\downarrow}(t, \delta, x, r)$$

and

$$m \otimes \mu(D)/m \otimes \mu(Q^{\downarrow}(t, \delta, x, r)) \geq \frac{c_0^{d_2}}{4\widetilde{C}_\mu},$$

we have

$$\mathbb{P}^{(t,x)}(\sigma_D < \tau_{Q^{\downarrow}(t,\delta,x,r)}) \geq c_1,$$

where $\widetilde{C}_\mu$ and d_2 are the constants in (3.1). Let C_2 be the constant C_2 in Lemma 4.1 with $\delta_j = j\delta$ and $\theta = 1/2$. Define

$$\eta = \frac{c_1}{3}, \quad \xi = \frac{1}{3} \wedge (C_2^{-1}\eta).$$

We claim that there is a universal constant $K \geq 2$ to be determined later, which is independent of R and the function u, such that $u \leq K$ on Q_-. We are going to prove this by contradiction.

Suppose this is not true. Then there is some point $(t_1, x_1) \in Q_-$ such that $u(t_1, x_1) \geq K$. We will show that there are a constant $\beta > 0$ and a sequence of points $\{(t_k, x_k)\}$ in $[t_0 + \delta\phi(CR)/2, t_0 + 2\delta\phi(CR)) \times B(x_0, 2R) \subset Q$ so that $u(t_k, x_k) \geq (1+\beta)^{k-1}K$, which contradicts to the assumption that u is bounded on Q.

Recall that β_1, β_2, c_3 and c_4 are the constants in (1.7). Then, by (3.1) and (1.7), we have for all $x \in M$ and all $0 < r_1 < r_2 \wedge r_3 < \infty$:

$$\frac{V(x,r_1)\phi(r_1)}{V(x,r_2)\phi(r_3)} \geq \frac{1}{c_4\widetilde{C}_\mu}\left(\frac{r_1}{r_2}\right)^{d_2}\left(\frac{r_1}{r_3}\right)^{\beta_2}. \tag{4.6}$$

Let C_3 be the constant in Lemma 4.2, and set $r := RK^{-1/(2(d_2+\beta_2))}$. We take $K \geq 2$ large enough so that $K \geq (2\widetilde{C}_\mu/(C_3\xi\delta_0^{2d_2}))^2(2C_0)^{d_2}$ and that, in view of (1.7),

$$r < R/8 \;\text{ and }\; \phi(r) < \frac{1}{8}\phi(R) \qquad \text{for all } R > 0 \text{ and } r = RK^{-1/(2(d_2+\beta_2))}.$$

With such r, we have by (4.6)

$$\frac{m \otimes \mu(Q^{\downarrow}(t,\delta,x,r))}{\phi(R)V(x,C_0R)} = \frac{\delta\phi(r)V(x,r)}{\phi(R)V(x,C_0R)} \geq \frac{\delta}{c_4\widetilde{C}_\mu C_0^{d_2}\sqrt{K}}. \tag{4.7}$$

Take $\tilde{t} = t_1 + (5/2)\delta\phi(r)$ and define $\tilde{U} = \{\tilde{t}\} \times B(x_1, \delta_0^2 r/2)$. Observe that $t_* - \tilde{t} \geq \frac{1}{2}\delta\phi(CR)$ since $t_* - t_1 \geq \delta\phi(CR)$. If the parabolic function $u \geq \xi K$ on $\tilde{U}$, we would have by (4.5) and Lemma 4.2 that

$$\begin{aligned}
1 \geq u(t_*, y_*) &= \mathbb{E}^{(t_*,y_*)}u(Z_{\sigma_{\tilde{U}} \wedge \tau_{Q_*}}) \geq \xi K \mathbb{P}^{(t_*,y_*)}(\sigma_{\tilde{U}} \leq \tau_{Q_*}) \geq \xi K \frac{C_3 V(x_1, \delta_0^2 r/2)}{V(x_1, C_0R)} \\
&\geq \frac{C_3\xi K}{\widetilde{C}_\mu}(\delta_0^2 r/(2C_0R))^{d_2} \geq \frac{C_3\xi\delta_0^{2d_2}\sqrt{K}}{(2C_0)^{d_2}\widetilde{C}_\mu} \geq 2,
\end{aligned}$$

where $Q_* = [t_1 - \delta\phi(r), t_*] \times B(x_0, C_0R)$. This contradiction yields that

$$\text{there is some } y_1 \in B(x_1, \delta_0^2 r/2) \text{ so that } u(\tilde{t}, y_1) < \xi K.$$

We next show that

$$\mathbb{E}^{(t_1,x_1)}\left[u(Z_{\tau_r}) : X_{\tau_r} \notin B(x_1, 3r/2)\right] \leq \eta K, \tag{4.8}$$

where $\tau_r := \tau_{(t_1-\delta\phi(r),t_1+3\delta\phi(r))\times B(x_1,\delta_0^2 r)}$. If it is not true, then we would have by Lemma 4.1 with $\delta_j = j\delta$ $(j = 1, 2, 3, 4)$ and $\theta = 1/2$ that

$$\begin{aligned}\xi K > u(\tilde t, y_1) \geq & \mathbb{E}^{(\tilde t, y_1)}\left[u(Z_{\tau_{[t_1-\delta\phi(r),t_1+3\delta\phi(r)]\times B(x_1,r)}}) : X_{\tau_{[t_1-\delta\phi(r),t_1+3\delta\phi(r)]\times B(x_1,r)}} \notin B(x_1, 3r/2)\right] \\ \geq & C_2^{-1}\mathbb{E}^{(t_1,x_1)}\left[u(Z_{\tau_r}) : X_{\tau_r} \notin B(x_1, 3r/2))\right] \\ > & C_2^{-1}\eta K \geq \xi K,\end{aligned}$$

which is a contradiction. This establishes (4.8).

Let A be any compact subset of

$$\tilde A := \left\{(s, y) \in \left[t_1 - \delta\phi(r), t_1 - \frac{1}{2}\delta\phi(r)\right] \times B(x_1, c_0 r) : u(s, y) \geq \xi K\right\},$$

and define $U_1 = \{t_1\} \times B(x_1, \delta_0^2 r)$. By Lemmas 3.7 and 4.2 and the strong Markov property,

$$\begin{aligned}1 \geq u(t_*, y_*) &\geq \mathbb{E}^{(t_*,y_*)}[u(Z_{\sigma_A}) : \sigma_A \leq \tau_{Q_*}] \\ &\geq \mathbb{E}^{(t_*,y_*)}[u(Z_{\sigma_A}) : \sigma_{U_1} < \tau_{Q_*}, \sigma_A < \tau_{[t_1-\delta\phi(r),t_*]\times B(x_1,2r)}] \\ &\geq \mathbb{P}^{(t_*,y_*)}(\sigma_{U_1} < \tau_{Q_*}) \inf_{z\in B(x_1,r/2)} \mathbb{E}^{(t_1,z)}[u(Z_{\sigma_A}) : \sigma_A < \tau_{[t_1-\delta\phi(r),t_*]\times B(z,r)}] \\ &\geq C_3\frac{V(x_1, \delta_0^2 r)}{V(x_1, C_0 R)} \cdot \xi K C_1 \inf_{z\in B(x_1,r/2)} \frac{m \otimes \mu(A)}{V(z, r)\phi(r)} \\ &\geq \frac{C_1C_3\xi K}{c_4\widetilde C_\mu}\Big(\frac{\delta_0^2}{2}\Big)^{d_2}\frac{m \otimes \mu(A)}{V(x_1, C_0R)\phi(R)},\end{aligned} \tag{4.9}$$

where in the third inequality we used the fact that $\tau_{[t_1-\delta\phi(r),t_*]\times B(x_1,2r)} \leq \tau_{Q_*}$. Since A is an arbitrary compact subset of $\widetilde A$, we have by (4.9) that

$$\frac{m \otimes \mu(\widetilde A)}{V(x_1, C_0R)\phi(R)} \leq \frac{c_4\widetilde C_\mu}{C_1C_3\xi K}\left(\frac{2}{\delta_0^2}\right)^{d_2}.$$

Thus by (4.7),

$$\frac{m \otimes \mu(\widetilde A)}{m \otimes \mu(Q^{\downarrow}(t_1, \delta, x_1, r))} \leq \frac{c_4^2\widetilde C_\mu^2 C_0^{d_2}}{\delta C_1C_3\xi\sqrt{K}}\left(\frac{2}{\delta_0^2}\right)^{d_2},$$

which is no larger than $\frac{c_0^{d_2}}{4\widetilde C_\mu}$ by taking K sufficiently large. Let

$$D = \left[t_1 - \delta\phi(r), t_1 - \frac{1}{2}\delta\phi(r)\right] \times B(x_1, c_0r) \setminus \tilde A$$

and $M = \sup_{(s,y)\in Q^{\downarrow}(t_1,\delta,x_1,3r/2)} u(s, y)$. Note that

$$\frac{m \otimes \mu(\widetilde D)}{m \otimes \mu(Q^{\downarrow}(t_1, \delta, x_1, r))} = \frac{\delta\phi(r)V(x_1, c_0r)}{2\delta\phi(r)V(x_1, r)} - \frac{m \otimes \mu(\widetilde A)}{m \otimes \mu(Q^{\downarrow}(t_1, \delta, x_1, r))} \geq \frac{c_0^{d_2}}{4\widetilde C_\mu}.$$

We have by (4.8),

$$\begin{aligned}K &\le u(t_1,x_1) = \mathbb{E}^{(t_1,x_1)}[u(Z_{\sigma_D\wedge\tau_r})]\\ &= \mathbb{E}^{(t_1,x_1)}[u(Z_{\sigma_D\wedge\tau_r}) : \sigma_D < \tau_r] + \mathbb{E}^{(t_1,x_1)}[u(Z_{\sigma_D\wedge\tau_r}) : \sigma_D \ge \tau_r, X_{\tau_r} \notin B(x_1,3r/2)]\\ &\quad + \mathbb{E}^{(t_1,x_1)}[u(Z_{\sigma_D\wedge\tau_r}) : \sigma_D \ge \tau_r, X_{\tau_r} \in B(x_1,3r/2)]\\ &\le \xi K\mathbb{P}^{(t_1,x_1)}(\sigma_D < \tau_r) + \eta K + M\mathbb{P}^{(t_1,x_1)}(\sigma_D \ge \tau_r).\end{aligned}$$

Therefore,

$$M/K \ge \frac{1-\eta-\xi\mathbb{P}^{(t_1,x_1)}(\sigma_D<\tau_r)}{\mathbb{P}^{(t_1,x_1)}(\sigma_D\ge\tau_r)} \ge \frac{1-\eta-\xi c_1}{1-c_1} \ge \frac{1-(2c_1)/3}{1-c_1} =: 1+2\beta,$$

where $\beta = c_1/(6(1-c_1))$. Consequently, there exists a point $(t_2,x_2) \in Q^{\downarrow}(t_1,\delta,x_1,2r) \subset Q$ such that $u(t_2,x_2) \ge (1+\beta)K =: K_2$.

Iterating the procedure above, we can find a sequence of points $\{(t_k,x_k)\}_{k=1}^{\infty}$ in $[t_0 + \delta\phi(CR)/2, t_0+2\delta\phi(CR)) \times B(x_0,2R)$ in the following way. Following the above argument with (t_2,x_2) and K_2 in place of (t_1,x_1) and K respectively, we obtain that there exists a point $(t_3,x_3) \in Q^{\downarrow}(t_2,\delta,x_2,2r_2)$ such that

$$r_2 = RK_2^{-1/(d_2+\beta_2)} = (1+\beta)^{-1/(d_2+\beta_2)}RK^{-1/(d_2+\beta_2)}$$

and

$$u(t_3,x_3) \ge (1+\beta)K_2 = (1+\beta)^2K =: K_3.$$

We continue this procedure to obtain a sequence of points $\{(t_k,x_k)\}$ such that $(t_{k+1},x_{k+1}) \in Q^{\downarrow}(t_k,\delta,x_k,2r_k)$ with

$$r_k := RK_k^{-1/(d_2+\beta_2)} = (1+\beta)^{-(k-1)/(d_2+\beta_2)}RK^{-1/(d_2+\beta_2)},$$

and

$$u(t_{k+1},x_{k+1}) \ge (1+\beta)^kK =: K_{k+1}.$$

As $0 \le t_k - t_{k+1} \le \delta\phi(2r_k)$ and $d(x_k,x_{k+1}) \le 2r_k$, we can take K large enough (independent of R and u) so that $(t_k,x_k) \in [t_0 + \delta\phi(CR)/2, t_0 + 2\delta\phi(CR)) \times B(x_0,2R)$ for all k. This is a contradiction because $u(t_k,x_k) \ge (1+\beta)^{k-1}K$ goes to infinity as $k \to \infty$, while u is bounded on Q. We conclude that u is bounded by K in Q_-. The proof is complete. □

Finally, we prove that under NDL(ϕ), $\mathrm{J}_{\phi,\le}$ is equivalent to UHK(ϕ), which immediately yields that NDL(ϕ) + UJS $\Longleftrightarrow$ PHI$^+$(ϕ).

Proposition 4.4. *Assume that* VD, (1.7) *and* RVD *hold. Then,*

$$\mathrm{NDL}(\phi) + \mathrm{J}_{\phi,\le} \Longleftrightarrow \mathrm{NDL}(\phi) + \mathrm{UHK}(\phi) \tag{4.10}$$

and so

$$\mathrm{NDL}(\phi) + \mathrm{UJS} \Longleftrightarrow \mathrm{PHI}^+(\phi) \Longleftrightarrow \mathrm{PHI}(\phi). \tag{4.11}$$

Proof. First, note that the process $\{X_t\}$ is conservative due to NDL(ϕ) (see Proposition 2.4). On the one hand, by Theorem 1.10, UHK(ϕ) implies $\mathrm{J}_{\phi,\le}$. On the other hand, according to Proposition 3.5, under VD, (1.7) and RVD, NDL(ϕ) implies FK(ϕ) and E_ϕ. In particular, the process $\{X_t\}$ possesses a heat kernel. Thus we have by [CKW, Theorem 4.25] that NDL(ϕ)+$\mathrm{J}_{\phi,\le}$ imply UHKD(ϕ). Furthermore, by Theorem 1.10, NDL(ϕ)+$\mathrm{J}_{\phi,\le}$ imply UHK(ϕ). This proves (4.10).

By Corollary 3.4, NDL(ϕ) + UJS $\Longrightarrow$ $\mathrm{J}_{\phi,\le}$, which along with (4.10) gives us

$$\mathrm{NDL}(\phi) + \mathrm{UJS} \Longleftrightarrow \mathrm{UHK}(\phi) + \mathrm{NDL}(\phi) + \mathrm{UJS}.$$

It now follows from Propositions 3.2 and 3.3, and Theorem 4.3 that

$$\mathrm{PHI}(\phi) \Longrightarrow \mathrm{NDL}(\phi) + \mathrm{UJS} \Longrightarrow \mathrm{PHI}^+(\phi).$$

This establishes assertion (4.11) as $\mathrm{PHI}^+(\phi) \Longrightarrow \mathrm{PHI}(\phi)$. □

4.2 $\mathrm{PHI}(\phi) \Longleftrightarrow \mathrm{PHR}(\phi) + \mathrm{E}_{\phi,\le} + \mathrm{UJS} \Longleftrightarrow \mathrm{EHR} + \mathrm{E}_\phi + \mathrm{UJS}$

The main contribution of this subsection is the following relations among PHI(ϕ), PHR(ϕ) and EHR, which establish the equivalences among (1), (5) and (6) of Theorem 1.17.

Theorem 4.5. *Assume that μ and ϕ satisfy* VD*,* RVD *and* (1.7) *respectively. Then*

$$\mathrm{PHI}(\phi) \Longleftrightarrow \mathrm{PHR}(\phi) + \mathrm{E}_{\phi,\le} + \mathrm{UJS} \Longleftrightarrow \mathrm{EHR} + \mathrm{E}_\phi + \mathrm{UJS}.$$

We start with the following key lemma.

Lemma 4.6. *Under* VD *and* (1.7)*,* EHR *and* $\mathrm{E}_{\phi,\le}$ *imply* FK(ϕ)*.*

Proof. According to Remark 1.13(ii), throughout this subsection we may and do assume that the constant $\varepsilon = 1/2$ in the definition of EHR.

For any open subset D of M, let G^D be the associated Green operator. Recall that for any open set D, it holds that

$$\lambda_1(D)^{-1} \le \sup_{x\in D\cap M_0} \mathbb{E}^x \tau_D = \sup_{x\in D\cap M_0} G^D \mathbf{1}(x). \tag{4.12}$$

For any ball $B = B(x,R) \subset M$ with $x \in M$ and $R > 0$, and any open set $D \subset B$, we will verify that

$$\sup_{x\in D\cap M_0} \mathbb{E}^x \tau_D \le c\phi(R) \left(\frac{\mu(D)}{V(x,R)}\right)^\nu, \tag{4.13}$$

where $c > 0$ and $\nu \in (0,1)$ are two constants independent of D and B. Once this is proved, FK(ϕ) immediately follows from (4.12) and (4.13).

Fix an arbitrary $x_0 \in D\cap M_0$. Let $R_k = 2\delta^k R$ for $k \ge 0$, where $\delta \in (0,1/2]$ is a constant to be determined later. Set $B_k = B(x_0,R_k)$ for $k \ge 0$. Clearly $D \subset B_0 = B(x_0,2R)$. Since $(G^{B_k} - G^{B_{k+1}})\mathbf{1}_D$ is a bounded non-negative function that is harmonic in B_{k+1}, we have by EHR and the μ-symmetry of the Green operator G^{B_k} that for any positive integers $n > k \geq 0$,

$$\begin{aligned}
&\sup_{y\in B_{n+1}\cap M_0}(G^{B_k}-G^{B_{k+1}})\mathbf{1}_D(y)\\
&\leq \inf_{y\in B_{n+1}\cap M_0}(G^{B_k}-G^{B_{k+1}})\mathbf{1}_D(y)+c_1\,\delta^{(n-k)\theta}\sup_{y\in B_{n+1}\cap M_0}|(G^{B_k}-G^{B_{k+1}})\mathbf{1}_D(y)|\\
&\leq \frac{1}{\mu(B_{n+1})}\int_{B_{n+1}}(G^{B_k}-G^{B_{k+1}})\mathbf{1}_D(y)\,\mu(dy)+c_1\,\delta^{(n-k)\theta}\sup_{y\in B_k\cap M_0}|G^{B_k}\mathbf{1}_D(y)| \qquad (4.14)\\
&\leq \frac{1}{\mu(B_{n+1})}\int \mathbf{1}_{B_k\cap D}(y)G^{B_k}\mathbf{1}_{B_{n+1}}(y)\,\mu(dy)+c_1\,\delta^{(n-k)\theta}\sup_{y\in B_k\cap M_0}|G^{B_k}\mathbf{1}(y)|\\
&\leq \frac{1}{V(x_0,R_{n+1})}\mu(D)\|G^{B_k}\mathbf{1}\|_\infty+c_1\,\delta^{(n-k)\theta}\sup_{y\in B_k\cap M_0}|G^{B_k}\mathbf{1}(y)|,
\end{aligned}$$

where $c_1 = 2^\theta c > 0$, and c and $\theta \in (0, 1]$ are the constants in EHR. On the other hand, we have by $\mathrm{E}_{\phi,\leq}$ that

$$\sup_{y\in B_k\cap M_0}|G^{B_k}\mathbf{1}(y)| \leq \sup_{y\in B_k\cap M_0}\mathbb{E}^y\tau_{B(y,2R_k)} \leq c_2\phi(2R_k). \qquad (4.15)$$

Taking $k = 0$ and $n = 1$ in (4.14) and $k = 1$ in (4.15), we find by (1.5) from VD and (1.7) that

$$\begin{aligned}
\mathbb{E}^{x_0}\tau_D &\leq \sup_{y\in B_2\cap M_0} G^{B_0}\mathbf{1}_D(y)\\
&\leq \sup_{y\in B_2\cap M_0}(G^{B_0}-G^{B_1})\mathbf{1}_D(y)+\sup_{y\in B_2\cap M_0}G^{B_1}\mathbf{1}(y)\\
&\leq c_3\left(\frac{\mu(D)}{V(x_0,R_2)}+\delta^\theta\right)\phi(2R_0)+c_2\phi(2R_1)\\
&\leq c_4\left(\frac{\mu(D)}{V(x_0,2R)}\delta^{-2d_2}+\delta^\theta\right)\phi(R)+c_4\phi(R)\delta^{\beta_1}\\
&\leq c_5\phi(R)\left(\frac{\mu(D)}{V(x,R)}\delta^{-2d_2}+\delta^{\theta\wedge\beta_1}\right). \qquad (4.16)
\end{aligned}$$

Define $\nu = \frac{\theta\wedge\beta_1}{2d_2+\theta\wedge\beta_1}$. If $\frac{\mu(D)}{V(x,R)} \leq (1/2)^{2d_2+\theta\wedge\beta_1}$, we take $\delta = \left(\frac{\mu(D)}{V(x,R)}\right)^{1/(2d_2+\theta\wedge\beta_1)}$, which is no larger than 1/2, in (4.16) to deduce

$$\mathbb{E}^{x_0}\tau_D \leq 2c_5\phi(R)\left(\frac{\mu(D)}{V(x,R)}\right)^\nu.$$

If $\frac{\mu(D)}{V(x,R)} > (1/2)^{2d_2+\theta\wedge\beta_1}$, we get from $\mathrm{E}_{\phi,\leq}$ that

$$\mathbb{E}^{x_0}\tau_D \leq c_6\phi(R)\left(\frac{\mu(D)}{V(x,R)}\right)^\nu.$$

Since $x_0 \in D \cap M_0$ is arbitrary, this establishes (4.13) and hence completes the proof. □

By VD, (1.7) and [CKW, Proposition 7.3], FK(ϕ) implies the existence of the Dirichlet heat kernel $p^D(t,\cdot,\cdot)$ for any bounded open subset $D \subset M$, and that there is a constant $C_\nu > 0$ such that for every $x_0 \in D$ and $t > 0$

$$\text{ess sup}_{\,x,y\in D} p^D(t,x,y) \le \frac{C_\nu}{V(x_0,r)} \left(\frac{\phi(r)}{t}\right)^{1/\nu}, \tag{4.17}$$

where $r = \text{diam}(D)$, the diameter of D.

Lemma 4.7. *Assume that* (1.7), EHR *and* $\text{E}_{\phi,\le}$ *are satisfied. Let D be a bounded open subset of M. Let $t > 0$, $x \in D \setminus \mathcal{N}$ and $0 < r_1 < \phi^{-1}(t)$ such that $0 < r_1 \le r/2$ and $B(x,r) \subset D$, where $r = \left(\phi^{-1}(t)^{\beta_1} r_1^\theta\right)^{1/(\beta_1+\theta)}$, β_1 is the constant in* (1.7) *and θ is the Hölder exponent in* EHR*. Then,*

$$\text{ess osc}_{\,y\in B(x,r_1)} p^D(t,x,y) \le C \left(\frac{r_1}{\phi^{-1}(t)}\right)^\kappa \text{ess sup}_{\,y\in D}\, p^D(t/2,y,y),$$

where $\kappa = \beta_1\theta/(\beta_1+\theta)$, and C is a constant depending on the constants in (1.7) *and* $\text{E}_{\phi,\le}$.

Proof. The proof uses some ideas from but is more direct than that of [GT, Lemma 5.10]. For fixed $x \in D \setminus \mathcal{N}$ and $s > 0$, set $u(s,y) = p^D(s,x,y)$. According to Lemma 4.6 and (4.17),

$$\int_D u(s,y)^2 \,\mu(dy) = p^D(2s,x,x) < \infty.$$

Since, by the symmetry of $p^D(t/2,z,x) = p^D(t/2,x,z)$,

$$u(t,y) = \int_D p^D(t/2,y,z) p^D(t/2,z,x)\,\mu(dz) = P^D_{t/2} u(t/2,\cdot)(y),$$

we have $u(t,\cdot) \in \text{Dom}(\mathcal{L}^D) \subset \mathcal{F}^D$ for every $t > 0$. Thus for μ-a.e. $y \in D$,

$$\begin{aligned}
\partial_t u(t,y) &= \mathcal{L}^D P^D_{t/2} u(t/2,\cdot)(y) = P^D_{t/2} \mathcal{L}^D u(t/2,\cdot)(y) \\
&= \int_D p^D(t/2,y,z) \mathcal{L}^D u(t/2,\cdot)(z)\,\mu(dz) = -\mathcal{E}(p^D(t/2,y,\cdot), u(t/2,\cdot)).
\end{aligned}$$

Hence, by the Cauchy-Swarchz inequality and the spectral representation,

$$\begin{aligned}
|\partial_t u(t,y)| &\le \sqrt{\mathcal{E}(p^D(t/2,y,\cdot), p^D(t/2,y,\cdot))}\sqrt{\mathcal{E}(u(t/2,\cdot), u(t/2,\cdot))} \\
&= \sqrt{\mathcal{E}(P^D_{t/4} p^D(t/4,y,\cdot), P^D_{t/4} p^D(t/4,y,\cdot))}\sqrt{\mathcal{E}(P^D_{t/4} u(t/4,\cdot), P^D_{t/4} u(t/4,\cdot))} \\
&\le \sqrt{(2/t)\,\|p^D(t/4,y,\cdot)\|^2_{L^2(D;\mu)}}\sqrt{(2/t)\,\|u(t/4,\cdot)\|^2_{L^2(D;\mu)}} \\
&= \frac{2}{t}\sqrt{p^D(t/2,y,y) p^D(t/2,x,x)} \le \frac{2}{t} \text{ess sup}_{\,D\setminus\mathcal{N}}\, p^D(t/2,y,y).
\end{aligned}$$

In particular, by (4.17), $f(t,y) := \partial_t u(t,y)$ is a bounded function on D for every $t > 0$. Note that $\lim_{s\to\infty} p^D(s,x,y) = 0$ for every $y \in D \setminus \mathcal{N}$, also thanks to (4.17). Then we have

$$\begin{aligned} u(t,y) &= -\int_t^\infty \partial_s p^D(s,x,y)\,ds = -\int_0^\infty \partial_t p^D(t+r,x,y)\,dr \\ &= -\int_0^\infty \int_D p^D(r,y,z)\partial_t p^D(t,x,z)\,\mu(dz)\,dr = -G^D f(t,\cdot)(y). \end{aligned}$$

Hence, by EHR, Lemma 3.10 and $E_{\phi,\le}$, for any $0 < r_1 \le r/2$,

$$\begin{aligned} \text{ess osc}_{\,B(x,r_1)} u(t,\cdot) \le& 2 \sup_{y\in B(x,r)\setminus\mathcal{N}} |f(t,y)| \sup_{y\in B(x,r)\setminus\mathcal{N}} \mathbb{E}^y \tau_{B(x,r)} + c_1 \left(\frac{r_1}{r}\right)^\theta \sup_{y\in D\setminus\mathcal{N}} |u(t,y)| \\ \le& c_2 \left[\phi(r)\frac{A}{t} + \left(\frac{r_1}{r}\right)^\theta A\right], \end{aligned}$$

where $A = \sup_{z\in D\setminus\mathcal{N}} p^D(t/2,z,z)$. In the last inequality above, we also used the facts that $\sup_{y,z\in D\setminus\mathcal{N}} p^D(t,y,z) = \sup_{z\in D\setminus\mathcal{N}} p^D(t,z,z)$ and $t \mapsto \sup_{z\in D\setminus\mathcal{N}} p^D(t,z,z)$ is a decreasing function, see e.g., the proof of Lemma [CKW, Lemma 7.9].

For any $0 < r < \phi^{-1}(t)$, by (1.7),

$$\frac{\phi(r)}{t} \le c_3 \left(\frac{r}{\phi^{-1}(t)}\right)^{\beta_1},$$

whence it follows that for any $0 < r_1 \le r/2$ and $0 < r < \phi^{-1}(t)$,

$$\text{ess osc}_{\,B(x,r_1)} u \le C \left[\left(\frac{r}{\phi^{-1}(t)}\right)^{\beta_1} + \left(\frac{r_1}{r}\right)^\theta\right] A.$$

By choosing $r = \left(\phi^{-1}(t)^{\beta_1} r_1^\theta\right)^{1/(\beta_1+\theta)}$ in the inequality above, we proved the desired assertion. □

Lemma 4.8. *Suppose that* VD, (1.7), EHR *and* $E_{\phi,\le}$ *hold. Then for any $x \in M_0$, $t > 0$ and $0 < r \le 2^{-(\beta_1+\theta)/\beta_1}\phi^{-1}(t)$ the following estimate holds*

$$|p^{B(x,\phi^{-1}(t))}(t,x,x) - p^{B(x,\phi^{-1}(t))}(t,x,y)| \le \left(\frac{r}{\phi^{-1}(t)}\right)^\kappa \frac{C}{V(x,\phi^{-1}(t))}, \quad y \in B(x,r)\setminus\mathcal{N},$$

where β_1 is the constant in (1.7)*, θ is the Hölder exponent in* EHR*, and κ is the constant in Lemma* 4.7*.*

Proof. Fix $x \in M_0$ and $t, r_1 > 0$ with $0 < r_1 \le 2^{-(\beta_1+\theta)/\beta_1}\phi^{-1}(t)$. We choose $r = \left(\phi^{-1}(t)^{\beta_1} r_1^\theta\right)^{1/(\beta_1+\theta)}$ as in Lemma 4.7. Then, $0 < r_1 \le r/2$. By applying Lemma 4.7 with $D = B(x,\phi^{-1}(t))$, we get

$$\text{ess osc}_{\,y\in B(x,r_1)} p^{B(x,\phi^{-1}(t))}(t,x,y) \le C \left(\frac{r_1}{\phi^{-1}(t)}\right)^\kappa \text{ess sup}_{\,y\in B(x,\phi^{-1}(t))} p^{B(x,\phi^{-1}(t))}(t/2,y,y).$$

This along with (4.17) yields the desired assertion. $\square$

Having all the lemmas at hand, we can obtain the following result.

Proposition 4.9. *Let* VD, (1.7), EHR *and* E_ϕ *be satisfied. Then for any open subset* $D \subset M$*, the semigroup* $\{P_t^D\}$ *possesses the heat kernel* $p^D(t,x,y)$*, and moreover* NDL(ϕ) *holds true.*

Proof. The existence of heat kernel $p^D(t,x,y)$ associated with the semigroup $\{P_t^D\}$ for any open subset $D \subset M$ has been stated in the remark below Lemma 4.6, and so we only need to verify NDL(ϕ).

According to E_ϕ and [CKW, Lemma 4.17], there are constants $\varepsilon \in (0,1)$ and $\delta \in (0,1/2)$ such that for all $x \in M_0$ and for any $t, r > 0$ with $t \le \delta\phi(r)$, $\mathbb{P}^x(\tau_{B(x,r)} \le t) \le \varepsilon$. In the following, let $B = B(x,r)$ and $0 < t \le \delta\phi(r)$. Then for any $x \in B\backslash\mathcal{N}$, since the process $\{X_t\}$ has no killings inside M,

$$\int_B p^B(t,x,y)\,\mu(dy) = \mathbb{P}^x(\tau_B > t) \ge 1-\varepsilon.$$

Therefore,

$$p^B(2t,x,x) = \int_B p^B(t,x,y)^2\,\mu(dy) \ge \frac{1}{\mu(B)}\left(\int_B p^B(t,x,y)\,\mu(dy)\right)^2 \ge \frac{c_1}{V(x,r)}.$$

In particular, taking $r = \phi^{-1}(t/\delta) > 0$ in the inequality above, we arrive at

$$p^{B(x,\phi^{-1}(t/(2\delta)))}(t,x,x) \ge \frac{c_2}{V(x,\phi^{-1}(t))}.$$

Furthermore, according to Lemma 4.8, VD and (1.7), there exists a constant $c_3 > 0$ such that for any $0 < r \le 2^{-(\beta_1+\theta)/\beta_1}\phi^{-1}(t/(2\delta))$, we have

$$|p^{B(x,\phi^{-1}(t/(2\delta)))}(t,x,x) - p^{B(x,\phi^{-1}(t/(2\delta)))}(t,x,y)| \le \left(\frac{r}{\phi^{-1}(t)}\right)^\kappa \frac{c_3}{V(x,\phi^{-1}(t))}, \quad y \in B(x,r)\backslash\mathcal{N},$$

where β_1 is the constant in (1.7), θ is the Hölder exponent in EHR, and κ is the constant in Lemma 4.7.

Combining with both inequalities above and choosing $\eta \in (0,1)$ small enough such that $\eta^\kappa c_3 \le \frac{1}{2}c_2$ and $\eta\phi^{-1}(t) \le 2^{-(\beta_1+\theta)/\beta_1}\phi^{-1}(t/(2\delta))$ for all $t > 0$, one can get that for any $x \in M_0$ and $y \in B(x,\eta\phi^{-1}(t))\backslash\mathcal{N}$,

$$\begin{aligned} &p^{B(x,\phi^{-1}(t/(2\delta)))}(t,x,y) \\ &\ge p^{B(x,\phi^{-1}(t/(2\delta)))}(t,x,x) - |p^{B(x,\phi^{-1}(t/(2\delta)))}(t,x,x) - p^{B(x,\phi^{-1}(t/(2\delta)))}(t,x,y)| \\ &\ge \frac{c_2}{2V(x,\phi^{-1}(t))}. \end{aligned}$$

That is, thanks to VD and (1.7) again, there are constants $c_i > 0$ $(i = 4, 5, 6)$ such that $0 < 2c_4 \le c_5$ and for any $x \in M_0$ and $y \in B(x, 2c_4\phi^{-1}(t))\backslash\mathcal{N}$,

$$p^{B(x,c_5\phi^{-1}(t))}(t, x, y) \ge \frac{c_6}{V(x, \phi^{-1}(t))}.$$

Now, for any $x_0 \in M$ and $r, t > 0$ such that $(c_4+c_5)\phi^{-1}(t) \le r$, we have $B(x, c_5\phi^{-1}(t)) \subset B(x_0, r)$ for all $x \in B(x_0, c_4\phi^{-1}(t))$, and so

$$p^{B(x_0,r)}(t, x, y) \ge p^{B(x,c_5\phi^{-1}(t))}(t, x, y) \ge \frac{c_6}{V(x, \phi^{-1}(t))}, \quad x, y \in B(x_0, c_4\phi^{-1}(t))\backslash\mathcal{N}.$$

This proves that NDL(ϕ) holds true with $\varepsilon = c_4 \wedge \frac{1}{c_4+c_5}$. □

Note that by Proposition 4.9 and Proposition 2.4, EHR+E_ϕ imply the conservativeness of the process $\{X_t\}$ (see Proposition 2.4).

Next, we present the proof of Theorem 4.5.

Proof of Theorem 4.5. That PHI(ϕ) $\Longrightarrow$ NDL(ϕ) + E_ϕ + UJS + $\mathrm{J}_{\phi,\le}$ has been established in Subsection 3.1, where RVD is used. Since NDL + $\mathrm{E}_{\phi,\le}$ + $\mathrm{J}_{\phi,\le}$ $\Longrightarrow$ PHR(ϕ) by Proposition 3.8, we have PHI(ϕ) implies PHR(ϕ) + E_ϕ + UJS.

On the other hand, by Proposition 4.9 and (4.11) (where RVD is used too), we have

$$\mathrm{EHR} + \mathrm{E}_\phi + \mathrm{UJS} \Longrightarrow \mathrm{NDL}(\phi) + \mathrm{UJS} \Longleftrightarrow \mathrm{PHI}(\phi),$$

which together with Proposition 3.9 completes the proof of the theorem. □

4.3 PI(ϕ) + $\mathrm{J}_{\phi,\le}$ + CSJ(ϕ) + UJS $\Longleftrightarrow$ PHI(ϕ)

In this subsection, we will prove the above mentioned equivalence in Theorem 1.17. Note that, under VD, (1.7) and RVD, PHI(ϕ) $\Longrightarrow$ PI(ϕ) + $\mathrm{J}_{\phi,\le}$ + CSJ(ϕ) + UJS has already been proved by combining the results in Subsection 3.1, Propositions 3.5 and Theorem 1.10. So all we need is to prove the following theorem.

Theorem 4.10. *Assume that μ and ϕ satisfy* VD, RVD *and* (1.7) *respectively. Then*

$$\mathrm{PI}(\phi) + \mathrm{J}_{\phi,\le} + \mathrm{CSJ}(\phi) + \mathrm{UJS} \Longrightarrow \mathrm{PHI}(\phi).$$

First of all, note that PI(ϕ) + $\mathrm{J}_{\phi,\le}$ + CSJ(ϕ) imply the conservativeness of the process. Indeed, PI(ϕ) + RVD imply FK(ϕ) by Proposition 2.9, and FK(ϕ) + $\mathrm{J}_{\phi,\le}$ + CSJ(ϕ) imply E_ϕ by Proposition 2.7. Furthermore, $\mathrm{J}_{\phi,\le}$ + E_ϕ imply the conservativeness of the process (see [CKW, Lemma 4.21]).

To prove the theorem, we begin with the following logarithmic lemma, which plays the key role in the proof of Hölder continuity of harmonic functions. The proof below is motivated by that of [CKP1, Lemma 1.3].

Proposition 4.11. *Let $B_r = B(x_0, r)$ for some $x_0 \in M$ and $r > 0$. Assume that $u \in \mathcal{F}^{loc}_{B_R}$ is a bounded and superharmonic function in a ball B_R such that $u \geq 0$ on B_R. If* VD, *(1.7),* CSJ(ϕ) *and* $\mathrm{J}_{\phi,\leq}$ *hold, then for any $l > 0$ and $0 < 2r \leq R$,*

$$\int_{B_r\times B_r}\left[\log\Big(\frac{u(x)+l}{u(y)+l}\Big)\right]^2 J(dx,dy) \leq \frac{c_1 V(x_0,r)}{\phi(r)}\left(1+\frac{\phi(r)}{\phi(R)}\frac{\mathrm{Tail}\,(u_-;x_0,R)}{l}\right),$$

where $\mathrm{Tail}\,(u_-;x_0,R)$ *is the nonlocal tail of u_- in $B(x_0,R)$ defined by (2.2), and c_1 is a constant independent of u, x_0, r, R and l.*

Proof. According to CSJ(ϕ), $\mathrm{J}_{\phi,\leq}$ and [CKW, Proposition 2.3(5)], we can choose $\varphi \in \mathcal{F}_{B_{3r/2}}$ related to $\mathrm{Cap}(B_r, B_{3r/2})$ such that

$$\mathcal{E}(\varphi,\varphi) \leq 2\mathrm{Cap}(B_r, B_{3r/2}) \leq \frac{c_1 V(x_0,r)}{\phi(r)}. \tag{4.18}$$

Since u is a bounded and superharmonic function in a ball B_R and $\frac{\varphi^2}{u+l} \in \mathcal{F}_{B_{3r/2}}$ for any $l > 0$, we have by Theorem 2.2 that

$$\begin{aligned}
0 \leq& \mathcal{E}\big(u, \frac{\varphi^2}{u+l}\big)\\
=& \int_{B_{2r}\times B_{2r}} (u(x)-u(y))\Big(\frac{\varphi^2(x)}{u(x)+l}-\frac{\varphi^2(y)}{u(y)+l}\Big)\, J(dx,dy)\\
&+2\int_{B_{2r}\times B_{2r}^c} (u(x)-u(y))\frac{\varphi^2(x)}{u(x)+l}\, J(dx,dy)\\
=& \int_{B_{2r}\times B_{2r}} \Big((u(x)+l)-(u(y)+l)\Big)\Big(\frac{\varphi^2(x)}{u(x)+l}-\frac{\varphi^2(y)}{u(y)+l}\Big)\, J(dx,dy)\\
&+2\int_{B_{2r}\times B_{2r}^c} (u(x)-u(y))\frac{\varphi^2(x)}{u(x)+l}\, J(dx,dy)\\
=& \int_{B_{2r}\times B_{2r}} \varphi(x)\varphi(y)\Big(\frac{\varphi(y)}{\varphi(x)}+\frac{\varphi(x)}{\varphi(y)}-\frac{\varphi(x)(u(y)+l)}{\varphi(y)(u(x)+l)}-\frac{\varphi(y)(u(x)+l)}{\varphi(x)(u(y)+l)}\Big)\, J(dx,dy)\\
&+2\int_{B_{2r}\times B_{2r}^c} (u(x)-u(y))\frac{\varphi^2(x)}{u(x)+l}\, J(dx,dy)\\
=:& I_1+I_2.
\end{aligned}$$

Applying the inequality

$$\frac{a}{b}+\frac{b}{a}-2 = (a-b)(b^{-1}-a^{-1}) \geq (\log a - \log b)^2, \quad a,b>0$$

with $a = \frac{u(y)+l}{\varphi(y)}$ and $b = \frac{u(x)+l}{\varphi(x)}$, we find that

$$\frac{\varphi(x)(u(y)+l)}{\varphi(y)(u(x)+l)}+\frac{\varphi(y)(u(x)+l)}{\varphi(x)(u(y)+l)}-\frac{\varphi(y)}{\varphi(x)}-\frac{\varphi(x)}{\varphi(y)}$$

$$\geq \left(\log \frac{u(y)+l}{\varphi(y)} - \log \frac{u(x)+l}{\varphi(x)}\right)^2 - \left(\frac{\varphi(y)}{\varphi(x)} + \frac{\varphi(x)}{\varphi(y)} - 2\right),$$

and so

$$\begin{aligned} I_1 \leq & - \int_{B_{2r}\times B_{2r}} \varphi(x)\varphi(y) \left(\log \frac{u(y)+l}{\varphi(y)} - \log \frac{u(x)+l}{\varphi(x)}\right)^2 J(dx,dy) \\ & + \int_{B_{2r}\times B_{2r}} (\varphi(x)-\varphi(y))^2 J(dx,dy). \end{aligned}$$

On the other hand, due to the fact that $u \geq 0$ on B_R, for all $x \in B_{2r}$ and $y \in B_R \setminus B_{2r}$,

$$\frac{u(x)-u(y)}{u(x)+l} \leq 1;$$

while for all $x \in B_{2r}$ and $y \in B_R^c$,

$$\frac{u(x)-u(y)}{u(x)+l} \leq \frac{(u(x)-u(y))_+}{u(x)+l} \leq \frac{u(x)+u_-(y)}{u(x)+l} \leq 1 + l^{-1}u_-(y).$$

Therefore,

$$I_2 \leq 2 \int_{B_{2r}\times B_{2r}^c} \varphi^2(x)\, J(dx,dy) + 2l^{-1} \int_{B_{2r}\times B_R^c} u_-(y)\varphi^2(x)\, J(dx,dy).$$

Combining all the estimates above and the fact that $\varphi = 1$ on B_r, we obtain

$$\begin{aligned} & \int_{B_r\times B_r} \left[\log\left(\frac{u(x)+l}{u(y)+l}\right)\right]^2 J(dx,dy) \\ & \leq \int_{B_{2r}\times B_{2r}} \varphi(x)\varphi(y) \left(\log \frac{u(y)+l}{\varphi(y)} - \log \frac{u(x)+l}{\varphi(x)}\right)^2 J(dx,dy) \\ & \leq \int_{B_{2r}\times B_{2r}} (\varphi(x)-\varphi(y))^2 J(dx,dy) + 2 \int_{B_{2r}\times B_{2r}^c} \varphi^2(x)\, J(dx,dy) \\ & \quad + 2l^{-1} \int_{B_{2r}\times B_R^c} u_-(y)\varphi^2(x)\, J(dx,dy) \\ & \leq \mathcal{E}(\varphi,\varphi) + \frac{c_2 V(x_0,r)}{\phi(R) l} \mathrm{Tail}\,(u_-; x_0, R), \end{aligned}$$

where the last inequality follows from $\mathrm{J}_{\phi,\leq}$ and the fact that for any $x \in B_{3r/2}$ and $y \in B_R^c$ with $R \geq 2r$,

$$\frac{V(x_0, d(x_0,y))\phi(d(x_0,y))}{V(x, d(x,y))\phi(d(x,y))} \leq c' \left(1 + \frac{d(x_0,x)}{d(x,y)}\right)^{\beta_2+\alpha_2} \leq c' \left(1 + \frac{3r/2}{R-3r/2}\right)^{\beta_2+\alpha_2} \leq c'',$$

thanks to VD and (1.7). Hence, the desired assertion follows from the inequality and (4.18). □

For the diffusion case, Proposition 4.11 was originally due to Moser. In that case, one can use the Leibniz rule, but for the jump case some more care is required. See [KZ, Corollary 7.7] for a related inequality. In the following we give another proof that is more robust.

Proof. (Another proof of Proposition 4.11) For a function v on M and for fixed $x, y \in M$, write

$$\bar{v}(t) = \bar{v}_{xy}(t) := tv(x) + (1-t)v(y), \qquad t \in [0,1].$$

Take $\varphi \in \mathcal{F}_{B_{3r/2}}$ as in (4.18) in the previous proof. For any $x, y \in M$ and $l > 0$, it holds that

$$\begin{aligned}
&(u(x) - u(y)) \left[\varphi(x)^2/(u(x)+l) - \varphi(y)^2/(u(y)+l)\right] \\
&= \int_0^1 \left[\frac{d}{dt}\frac{\bar{\varphi}^2}{(\bar{u}+l)}(s)\right] \frac{d}{dt}(\bar{u}(s)+l)\, ds \\
&= \int_0^1 \frac{2\bar{\varphi}(s)\frac{d}{dt}\bar{\varphi}(s)}{(\bar{u}(s)+l)} \frac{d}{dt}(\bar{u}(s)+l)\, ds - \int_0^1 \left[\frac{\bar{\varphi}}{(\bar{u}+l)}(s)\right]^2 \left[\frac{d}{dt}(\bar{u}(s)+l)\right]^2 ds \\
&= \int_0^1 2\left[\bar{\varphi}(s)\frac{d}{dt}\bar{\varphi}(s)\right]\left[\frac{d}{dt}\log(\bar{u}(s)+l)\right] ds - \int_0^1 \bar{\varphi}(s)^2 \left[\frac{d}{dt}\log(\bar{u}(s)+l)\right]^2 ds.
\end{aligned}$$

Multiplying $J(x,y)$ and integrating over $B_{2r} \times B_{2r}$ w.r.t. $\mu \times \mu$ in both sides of the equality above, we have

$$\begin{aligned}
&\int_{B_{2r}\times B_{2r}} \int_0^1 \bar{\varphi}(s)^2 \left[\frac{d}{dt}\log(\bar{u}(s)+l)\right]^2 ds\, J(dx,dy) \\
&\qquad + \mathcal{E}(u, \varphi^2/(u+l)) - 2\int_{B_{2r}\times B_{2r}^c} (u(x)-u(y))\frac{\varphi^2(x)}{u(x)+l}\, J(dx,dy) \\
&= 2\int_{B_{2r}\times B_{2r}} \int_0^1 \left[\bar{\varphi}(s)\frac{d}{dt}\bar{\varphi}(s)\right]\left[\frac{d}{dt}\log(\bar{u}(s)+l)\right] ds\, J(dx,dy) \\
&\le 2\left[\int_{B_{2r}\times B_{2r}} \int_0^1 \bar{\varphi}(s)^2 \Big(\frac{d}{dt}\log(\bar{u}(s)+l)\Big)^2 ds\, J(dx,dy)\right]^{1/2} \\
&\quad \times \left[\int_{B_{2r}\times B_{2r}} \int_0^1 \Big(\frac{d}{dt}\bar{\varphi}(s)\Big)^2 ds\, J(dx,dy)\right]^{1/2} \\
&\le 2\left[\int_{B_{2r}\times B_{2r}} \int_0^1 \bar{\varphi}(s)^2 \Big(\frac{d}{dt}\log(\bar{u}(s)+l)\Big)^2 ds\, J(dx,dy)\right]^{1/2} \mathcal{E}(\varphi,\varphi)^{1/2}.
\end{aligned} \tag{4.19}$$

In the following, we set

$$K := \int_{B_{2r}\times B_{2r}} \int_0^1 \bar{\varphi}(s)^2 \Big(\frac{d}{dt}\log(\bar{u}(s)+l)\Big)^2 ds\, J(dx,dy).$$

Now, as in the previous proof,

$$2\left|\int_{B_{2r}\times B_{2r}^c} (u(x)-u(y))\frac{\varphi^2(x)}{u(x)+l}\, J(dx,dy)\right| = |I_2| \le \mathcal{E}(\varphi,\varphi) + \frac{c_2 V(x_0,r)}{\phi(R)l}\mathrm{Tail}\,(u_-; x_0, R).$$

Further, noting that $u+l$ is bounded and superharmonic on $2B$, we have by Theorem 2.2 that $\mathcal{E}(u, \varphi^2/(u+l)) \geq 0$. Plugging these and (4.18) into (4.19), we have

$$K - \frac{c_1 V(x_0,r)}{\phi(r)} - \frac{c_2 V(x_0,r)}{\phi(R)l}\mathrm{Tail}\,(u_-; x_0, R) \leq 2K^{1/2}\left(\frac{c_1 V(x_0,r)}{\phi(r)}\right)^{1/2}.$$

We thus obtain

$$K \leq \frac{c_3 V(x_0,r)}{\phi(r)}\left[1 + \frac{\phi(r)}{\phi(R)}\frac{\mathrm{Tail}\,(u_-; x_0, R)}{l}\right].$$

On the other hand, since $\varphi = 1$ on B_r, using the Cauchy-Schwarz inequality we have

$$\begin{aligned} K &\geq \int_{B_{2r}\times B_{2r}} (\varphi(x)^2 \wedge \varphi(y)^2) \int_0^1 \left[\frac{d}{dt}\log(\bar{u}(s)+l)\right]^2 ds\, J(dx,dy) \\ &\geq \int_{B_{2r}\times B_{2r}} (\varphi(x)^2 \wedge \varphi(y)^2) \left[\int_0^1 \frac{d}{dt}\log(\bar{u}(s)+l)\, ds\right]^2 J(dx,dy) \\ &\geq \int_{B_r\times B_r} [\log(u(y)+l) - \log(u(x)+l)]^2\; J(dx,dy). \end{aligned}$$

We therefore prove the desired inequality, by combining all the inequalities above. □

As a consequence of Proposition 4.11, we have the following statement.

Corollary 4.12. *Let $B_r = B(x_0, r)$ for some $x_0 \in M$ and $r > 0$. Assume that $u \in \mathcal{F}^{loc}_{B_R}$ is a bounded and superharmonic function in a ball B_R such that $u \geq 0$ on B_R. For any $a, l > 0$ and $b > 1$, define*

$$v = \left[\log\left(\frac{a+l}{u+l}\right)\right]_+ \wedge \log b.$$

If VD*,* (1.7)*,* CSJ(ϕ)*,* $\mathrm{J}_{\phi,\leq}$ *and* PI(ϕ) *hold, then for any $l > 0$ and $0 < 2\kappa r \leq R$,*

$$\frac{1}{V(x_0,r)}\int_{B_r} (v - \overline{v}_{B_r})^2 d\mu \leq c_1\left(1 + \frac{\phi(r)}{\phi(R)}\frac{\mathrm{Tail}\,(u_-; x_0, R)}{l}\right),$$

where $\kappa \geq 1$ is the constant in PI(ϕ)*, $\overline{v}_{B_r} = \frac{1}{\mu(B_r)}\int_{B_r} v\, d\mu$ and c_1 is a constant independent of u, x_0, r, R and l.*

Proof. By PI(ϕ) and (1.7), we have

$$\int_{B_r} (v - \overline{v}_{B_r})^2 d\mu \leq c_2\phi(r)\int_{B_{\kappa r}\times B_{\kappa r}} (v(x) - v(y))^2\, J(dx,dy).$$

Observing that v is a truncation of the sum of a constant and $\log(u+l)$,

$$\int_{B_{\kappa r}\times B_{\kappa r}} (v(x)-v(y))^2\, J(dx,dy) \leq \int_{B_{\kappa r}\times B_{\kappa r}} \left(\log\left(\frac{u(x)+l}{u(y)+l}\right)\right)^2 J(dx,dy).$$

Hence, it suffices to apply Proposition 4.11 to conclude the assertion. □

Proposition 4.13. *Let $B_r = B(x_0, r)$ for some $x_0 \in M$ and $r > 0$. Assume that $u \in \mathcal{F}^{loc}_{B_R}$ is a bounded and harmonic function in a ball B_R. If* VD, RVD, (1.7), $\mathrm{CSJ}(\phi)$, $\mathrm{J}_{\phi,\le}$ *and* $\mathrm{PI}(\phi)$ *hold, there are constants $\gamma \in (0, \beta_1)$ and $c > 0$ such that*

$$\operatorname{ess\,osc}_{B_{r'}} u \le c\Big(\frac{r'}{r}\Big)^{\gamma} \left[\left(\frac{1}{V(x_0,2r)}\int_{B(x_0,2r)} u^2\, d\mu\right)^{1/2} + \mathrm{Tail}\,(u; x_0, r)\right], \tag{4.20}$$

where $0 < r' \le r < R/2$. In particular, suppose that VD, RVD *and* (1.7) *hold, then we have*

$$\mathrm{PI}(\phi) + \mathrm{J}_{\phi,\le} + \mathrm{CSJ}(\phi) \Longrightarrow \mathrm{EHR}.$$

Proof. (i) First, by $\mathrm{J}_{\phi,\le}$ and Lemma 2.3, it is easy to see that

$$\mathrm{Tail}\,(u; x_0, r) \le c'\|u\|_\infty, \quad r > 0.$$

Thus, assuming (4.20), there is a constant $c'' > 0$ such that for all $0 < r < R/2$,

$$\operatorname{ess\,osc}_{B_r} u \le c''\Big(\frac{r}{R}\Big)^{\gamma}\|u\|_\infty.$$

From this, we can easily see that, once (4.20) is proved, EHR is yielded.

(ii) In the following, we mainly prove (4.20). We begin with the argument of [CKP1, Theorem 1.2]. Before starting, let us fix some notations. For any $j \ge 0$ and $0 < 2r < R$, let $r_j = r\sigma^j$ and $B_j = B_{r_j}$, where $\sigma \in (0, 1/(4\kappa)]$ and $\kappa \ge 1$ is the constant in $\mathrm{PI}(\phi)$. Let us define

$$w(r_0) = w(r) = 2C_0\left[\left(\frac{1}{V(x_0,2r)}\int_{B(x_0,2r)} u^2\, d\mu\right)^{1/2} + \mathrm{Tail}\,(u; x_0, r)\right]$$

with the constant C_0 given in (2.3) of Proposition 2.6, and

$$w(r_j) = \Big(\frac{r_j}{r_0}\Big)^{\gamma} w(r_0)$$

for some $\gamma \in (0, \beta_1)$. In order to prove the required assertion, it will suffice to verify that

$$\operatorname{ess\,osc}_{B_j} u \le w(r_j), \quad j \ge 0. \tag{4.21}$$

Indeed, for any $0 < r' \le r$, we can choose $j \ge 0$ such that $r_{j+1} < r' \le r_j$. Then, by (4.21), we have

$$\operatorname{ess\,osc}_{B_{r'}} u \le \operatorname{ess\,osc}_{B_j} u \le w(r_j) \le \sigma^{\gamma}\Big(\frac{r_{j+1}}{r}\Big)^{\gamma} w(r) \le \sigma^{\gamma}\Big(\frac{r'}{r}\Big)^{\gamma} w(r).$$

Thus, the required assertion holds with $c = 2C_0\sigma^{\gamma}$.

(iii) We will prove (4.21) by induction. For this, note that $\mathrm{PI}(\phi) + \mathrm{RVD}$ imply $\mathrm{FK}(\phi)$ by Proposition 2.9. Then, according to the definition of $w(r_0)$ and Proposition 2.6, (4.21) holds for $j = 0$, since both the functions u_+ and u_- bounded subharmonic in B_R.

Now, we make an induction assumption and assume that (4.21) is valid for all $0 \le i \le j$ for some $j \ge 0$, and then we prove it holds also for $j+1$. We have that either

$$\frac{\mu(2B_{j+1} \cap \{u \ge \text{ess inf}_{B_j} u + w(r_j)/2\})}{\mu(2B_{j+1})} \ge \frac{1}{2}, \tag{4.22}$$

or

$$\frac{\mu(2B_{j+1} \cap \{u \le \text{ess inf}_{B_j} u + w(r_j)/2\})}{\mu(2B_{j+1})} \ge \frac{1}{2} \tag{4.23}$$

must hold. If (4.22) holds, we set $u_j := u - \text{ess inf}_{B_j} u$, and if (4.23) holds, we set $u_j := w(r_j) - (u - \text{ess inf}_{B_j} u)$. In both cases we have $u_j \ge 0$ on B_j and

$$\frac{\mu(2B_{j+1} \cap \{u_j \ge w(r_j)/2\})}{\mu(2B_{j+1})} \ge \frac{1}{2} \tag{4.24}$$

holds. Clearly, u_j is bounded and harmonic in B_R satisfying that

$$\begin{aligned} \text{ess sup}_{B_i} |u_j| \le & w(r_j) + \text{ess sup}_{B_i} |u - \text{ess inf}_{B_j} u| \\ \le & w(r_i) + \text{ess sup}_{B_i} |u - \text{ess inf}_{B_i} u| + |\text{ess inf}_{B_i} u - \text{ess inf}_{B_j} u| \\ \le & 2w(r_i) + \text{ess sup}_{B_i} u - \text{ess inf}_{B_i} u \\ \le & 3w(r_i), \quad 0 \le i \le j. \end{aligned} \tag{4.25}$$

We now claim that under the induction assumption we have

$$\text{Tail}\,(u_j; x_0, r_j) \le c_0 \sigma^{-\gamma} w(r_j), \tag{4.26}$$

where $c_0 > 0$ is independent of u, x_0, r and σ. Indeed, we have

$$\begin{aligned} \text{Tail}\,(u_j; x_0, r_j) = & \phi(r_j) \sum_{i=1}^{j} \int_{B_{i-1} \setminus B_i} \frac{|u_j(x)|}{V(x_0, d(x_0,x)) \phi(d(x_0,x))} \,\mu(dx) \\ & + \phi(r_j) \int_{B_0^c} \frac{|u_j(x)|}{V(x_0, d(x_0,x)) \phi(d(x_0,x))} \,\mu(dx) \\ \le & \phi(r_j) \sum_{i=1}^{j} \text{ess sup}_{B_{i-1}} |u_j| \int_{B_i^c} \frac{1}{V(x_0, d(x_0,x)) \phi(d(x_0,x))} \,\mu(dx) \\ & + \phi(r_j) \int_{B_0^c} \frac{|u_j(x)|}{V(x_0, d(x_0,x)) \phi(d(x_0,x))} \,\mu(dx) \\ \le & c_1 \sum_{i=1}^{j} \frac{\phi(r_j)}{\phi(r_i)} w(r_{i-1}), \end{aligned}$$

where in the last inequality we have used (4.25), Lemma 2.3,

$$|u_j| \le w(r_0) + \text{ess sup}_{B_0} |u| + |u|, \qquad j \ge 0$$

and

$$\begin{aligned}
&\int_{B_0^c} \frac{|u_j(x)|}{V(x_0,d(x_0,x))\phi(d(x_0,x))}\,\mu(dx)\\
&\le c'\Big[\frac{1}{\phi(r_0)}\big(\text{ess sup}_{B_0}|u|+w(r_0)\big)+\int_{B_0^c}\frac{|u(x)|}{V(x_0,d(x_0,x))\phi(d(x_0,x))}\,\mu(dx)\Big]\\
&\le c''\frac{w(r_0)}{\phi(r_0)}\le c''\frac{w(r_0)}{\phi(r_1)}.
\end{aligned}$$

Note that, in the second inequality above we used the fact that

$$\text{ess sup}_{B_0}|u|\le \text{ess sup}_{B_0}u^+ + \text{ess sup}_{B_0}u^- \le w(r_0)$$

deduced from Proposition 2.6. Estimating further, we have

$$\begin{aligned}
\sum_{i=1}^{j}\frac{\phi(r_j)}{\phi(r_i)}w(r_{i-1}) &= w(r_0)\Big(\frac{r_j}{r_0}\Big)^{\gamma}\sum_{i=1}^{j}\frac{\phi(r_j)}{\phi(r_i)}\Big(\frac{r_{i-1}}{r_j}\Big)^{\gamma}\\
&\le c_2 w(r_0)\Big(\frac{r_j}{r_0}\Big)^{\gamma}\sum_{i=1}^{j}\Big(\frac{r_j}{r_i}\Big)^{\beta_1}\Big(\frac{r_{i-1}}{r_j}\Big)^{\gamma}\\
&= c_2 w(r_0)\Big(\frac{r_j}{r_0}\Big)^{\gamma}\sum_{i=1}^{j}\Big(\frac{r_{i-1}}{r_i}\Big)^{\gamma}\Big(\frac{r_j}{r_i}\Big)^{\beta_1-\gamma}\\
&\le \frac{c_2\sigma^{-\gamma}}{1-\sigma^{\beta_1-\gamma}}w(r_j)\le c_3\sigma^{-\gamma}w(r_j),
\end{aligned}$$

where we used (1.7) in the first inequality, and used $\sigma\in(0,1/(4\kappa)]$ and $\beta_1>\gamma$ in the second inequality. Hence, (4.26) is proved with c_0 independent of σ.

Next, consider the function v defined as follows

$$v := \Big[\log\Big(\frac{w(r_j)/2+l}{u_j+l}\Big)\Big]_+\wedge k,\quad k,l>0.$$

Using the fact $\sigma\in(0,1/(4\kappa)]$ again and applying Corollary 4.12, we get

$$\frac{1}{\mu(2B_{j+1})}\int_{2B_{j+1}}(v-\overline{v}_{2B_{j+1}})^2\,d\mu\le c_4\left(1+l^{-1}\frac{\phi(r_{j+1})}{\phi(r_j)}\text{Tail}\,(u_j;x_0,r_j)\right).$$

This, along with (4.26) and (1.7), yields that

$$\frac{1}{\mu(2B_{j+1})}\int_{2B_{j+1}}(v-\overline{v}_{2B_{j+1}})^2\,d\mu\le c_5\left(1+l^{-1}\sigma^{\beta_1-\gamma}w(r_j)\right).$$

Hence, choosing $l=\varepsilon w(r_j)$ with $\varepsilon=\sigma^{\beta_1-\gamma}$, we get that

$$\frac{1}{\mu(2B_{j+1})}\int_{2B_{j+1}}(v-\overline{v}_{2B_{j+1}})^2\,d\mu\le c_6. \tag{4.27}$$

To continue, denote in short $\tilde{B} = 2B_{j+1}$. We obtain from (4.24) that

$$\begin{aligned}
k &= \frac{1}{\mu(\tilde{B} \cap \{u_j \ge w(r_j)/2\})} \int_{\tilde{B} \cap \{u_j \ge w(r_j)/2\}} k \, d\mu \\
&= \frac{1}{\mu(\tilde{B} \cap \{u_j \ge w(r_j)/2\})} \int_{\tilde{B} \cap \{v=0\}} k \, d\mu \\
&\le \frac{2}{\mu(\tilde{B})} \int_{\tilde{B}} (k - v) \, d\mu = 2(k - \overline{v}_{\tilde{B}}).
\end{aligned}$$

By integrating the preceding inequality over the set $\tilde{B} \cap \{v = k\}$, we further obtain

$$\frac{\mu(\tilde{B} \cap \{v = k\})}{\mu(\tilde{B})} k \le \frac{2}{\mu(\tilde{B})} \int_{\tilde{B} \cap \{v=k\}} (k - \overline{v}_{\tilde{B}}) \, d\mu \le \frac{2}{\mu(\tilde{B})} \int_{\tilde{B}} |v - \overline{v}_{\tilde{B}}| \, d\mu \le c_7,$$

where (4.27) and the Cauchy-Schwarz inequality are used in the last inequality. Let us take

$$k = \log \left(\frac{w(r_j)/2 + \varepsilon w(r_j)}{3\varepsilon w(r_j)} \right) = \log \left(\frac{\frac{1}{2} + \varepsilon}{3\varepsilon} \right) \approx \log \left(\frac{1}{\varepsilon} \right),$$

and so we have

$$\frac{\mu(\tilde{B} \cap \{u_j \le 2\varepsilon w(r_j)\})}{\mu(\tilde{B})} \le \frac{c_7}{k} \le \frac{c_8}{-\log \sigma}. \tag{4.28}$$

(iv) We are now in a position to start a suitable iteration to deduce the desired oscillation reduction. From here we make essential changes of the argument in the proof of [CKP1, Theorem 1.2]. Note that, in the setting of [CKP1] the proof is heavily based on the fractional Poincaré inequalities (see [CKP1, (5.11)]), which however are not available in the present situation. To deal with this difficulty, we apply Lemma 2.5 instead. In the following, we fix $j \ge 0$. First, for any $i \ge 0$, we define

$$\varrho_i = (1 + 2^{-i}) r_{j+1}, \quad B^i = B_{\varrho_i}$$

and set

$$k_i = (1 + 2^{-i}) \varepsilon w(r_j), \quad w_i = (k_i - u_j)_+, \quad A_i = \frac{\mu(B^i \cap \{u_j \le k_i\})}{\mu(B^i)}.$$

Then, we have by VD and Lemma 2.5 that

$$\begin{aligned}
A_{i+2}(k_{i+1} - k_{i+2})^2 &= \frac{1}{\mu(B^{i+2})} \int_{B^{i+2} \cap \{u_j \le k_{i+2}\}} (k_{i+1} - k_{i+2})^2 \, d\mu \\
&\le \frac{1}{\mu(B^{i+2})} \int_{B^{i+2}} w_{i+1}^2 \, d\mu \\
&\le \frac{c_8}{(k_i - k_{i+1})^{2\nu}} \left(\frac{1}{\mu(B^{i+1})} \int_{B^{i+1}} w_i^2 \, d\mu \right)^{1+\nu} \left(\frac{\varrho_{i+2}}{\varrho_{i+1} - \varrho_{i+2}} \right)^{\beta_2} \\
&\quad \times \left[1 + \frac{1}{k_i - k_{i+1}} \left(\frac{\varrho_{i+2}}{\varrho_{i+1} - \varrho_{i+2}} \right)^{d_2 + \beta_2 - \beta_1} \mathrm{Tail}\,(w_i; x_0, \varrho_{i+1}) \right]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{c_9}{[(2^{-i}-2^{-i-1})\varepsilon w(r_j)]^{2\nu}} \left[(\varepsilon w(r_j))^2 A_i\right]^{1+\nu} \left(\frac{1}{2^{-i}-2^{-i-1}}\right)^{\beta_2} \\
&\quad \times \left[1+\frac{1}{(2^{-i}-2^{-i-1})\varepsilon w(r_j)}\left(\frac{1}{2^{-i}-2^{-i-1}}\right)^{d_2+\beta_2-\beta_1} \mathrm{Tail}\,(w_i;x_0,r_{j+1})\right] \\
&\le c_{10}[\varepsilon w(r_j)]^2 A_i^{1+\nu} 2^{(1+2\nu+d_2+2\beta_2-\beta_1)i}\left(1+\frac{1}{\varepsilon w(r_j)}\mathrm{Tail}\,(w_i;x_0,r_{j+1})\right),
\end{aligned}
$$

where ν is the constant in FK(ϕ), and in the third inequality we have used the facts that $u_j \ge 0$ on $B^{i+1} \subset B_j$ and

$$
\int_{B^{i+1}} w_i^2\, d\mu \le k_i^2 \mu(B^{i+1}\cap\{w_i \ge 0\}) \le c'(\varepsilon w(r_j))^2 \mu(B^i \cap \{u_j \le k_i\}).
$$

Hence,

$$
A_{i+2} \le c_{11} A_i^{1+\nu} 2^{(3+2\nu+d_2+2\beta_2-\beta_1)i}\left(1+\frac{1}{\varepsilon w(r_j)}\mathrm{Tail}\,(w_i;x_0,r_{j+1})\right).
$$

Note that, by the facts that $u_j \ge 0$, $w_i \le 2\varepsilon w(r_j)$ on B_j and $|w_i| \le |u_j| + 2\varepsilon w(r_j)$ on M,

$$
\begin{aligned}
\mathrm{Tail}\,(w_i;x_0,r_{j+1}) &= \phi(r_{j+1}) \int_{B_j\setminus B_{j+1}} \frac{|w_i(x)|}{V(x_0,d(x_0,x))\phi(d(x_0,x))}\,\mu(dx) \\
&\quad + \frac{\phi(r_{j+1})}{\phi(r_j)}\mathrm{Tail}\,(w_i;x_0,r_j) \\
&\le c_{12}\Bigg(2\varepsilon w(r_j)\phi(r_{j+1}) \int_{B_{j+1}^c} \frac{\mu(dx)}{V(x_0,d(x_0,x))\phi(d(x_0,x))} \\
&\qquad + 2\varepsilon w(r_j)\phi(r_{j+1}) \int_{B_j^c} \frac{\mu(dx)}{V(x_0,d(x_0,x))\phi(d(x_0,x))} \\
&\qquad + \frac{\phi(r_{j+1})}{\phi(r_j)}\mathrm{Tail}\,(u_j;x_0,r_j)\Bigg) \\
&\le c_{12}\left(\varepsilon w(r_j) + \sigma^{\beta_1}\mathrm{Tail}\,(u_j;x_0,r_j)\right) \\
&\le c_{13}\left(1+\frac{\sigma^{\beta_1-\gamma}}{\varepsilon}\right)\varepsilon w(r_j) \le 2c_{13}\varepsilon w(r_j),
\end{aligned}
$$

where the second and the third inequalities follow from Lemma 2.3 and (4.26), respectively, and the last inequality is due to $\varepsilon = \sigma^{\beta_1-\gamma}$. Combining with all the conclusions above, we arrive at

$$
A_{i+2} \le c_{14} A_i^{1+\nu} 2^{(3+2\nu+d_2+2\beta_2-\beta_1)i}.
$$

Let $c^* = c_{14}^{-1/\nu} 2^{-(3+2\nu+d_2+2\beta_2-\beta_1)/\nu^2}$ and choose the constant $\sigma \in \left(0, \frac{1}{4} \wedge \exp^{-\left(\frac{c_8}{c^*}\right)}\right)$. Then, by (4.28),

$$
A_0 \le c^* = c_{14}^{-1/\nu} 2^{-(3+2\nu+d_2+2\beta_2-\beta_1)/\nu^2}.
$$

According to Lemma 2.10, we can deduce that $\lim_{i\to\infty} A_i = 0$. Therefore, $u_j \ge \varepsilon w(r_j)$ on B_{j+1}, and then we can find that

$$
\text{ess osc}_{B_{j+1}} u = \text{ess sup}_{B_{j+1}} u_j - \text{ess inf}_{B_{j+1}} u_j \le (1-\varepsilon) w(r_j) = (1-\varepsilon)\sigma^{-\gamma} w(r_{j+1}),
$$

where the inequality above follows from the fact that $\operatorname{ess\,sup}_{B_{j+1}} u_j \le w(r_j)$, since under (4.22)

$$\operatorname{ess\,sup}_{B_{j+1}} u_j = \operatorname{ess\,sup}_{B_{j+1}} u - \operatorname{ess\,inf}_{B_j} u \le \operatorname{ess\,sup}_{B_j} u - \operatorname{ess\,inf}_{B_j} u \le w(r_j),$$

or under (4.23)

$$\operatorname{ess\,sup}_{B_{j+1}} u_j = w(r_j) - \operatorname{ess\,inf}_{B_{j+1}} (u - \operatorname{ess\,inf}_{B_j} u) \le w(r_j).$$

Taking finally $\gamma \in (0, \beta_1)$ small enough such that $\sigma^\gamma \ge 1 - \varepsilon = 1 - \sigma^{\beta_1 - \gamma}$, we obtain that

$$\operatorname{ess\,osc}_{B_{j+1}} u \le w(r_{j+1})$$

holds, proving the induction step and finishing the proof of (4.21). □

We are now in a position to present the proof of the main theorem in this subsection.

Proof of Theorem 4.10. By Theorem 4.5, it suffices to prove that

$$\mathrm{PI}(\phi) + \mathrm{J}_{\phi,\le} + \mathrm{CSJ}(\phi) + \mathrm{UJS} \Longrightarrow \mathrm{EHR} + \mathrm{E}_\phi + \mathrm{UJS}.$$

As mentioned in the remark below Theorem 4.10, under VD, RVD and (1.7),

$$\mathrm{PI}(\phi) + \mathrm{J}_{\phi,\le} + \mathrm{CSJ}(\phi) \Longrightarrow \mathrm{E}_\phi.$$

On the other hand, according to Proposition 4.13 (where RVD is used again),

$$\mathrm{PI}(\phi) + \mathrm{J}_{\phi,\le} + \mathrm{CSJ}(\phi) \Longrightarrow \mathrm{EHR}.$$

The proof is complete. □

Remark 4.14. By the proof above and Propositions 4.9 and 3.5, under VD, RVD and (1.7), we have the following relations without using UJS:

$$\mathrm{PI}(\phi) + \mathrm{J}_{\phi,\le} + \mathrm{CSJ}(\phi) \Longrightarrow \mathrm{EHR} + \mathrm{E}_\phi \Longrightarrow \mathrm{NDL}(\phi) \Longrightarrow \mathrm{PI}(\phi) + \mathrm{E}_\phi.$$

Proof of Corollary 1.18. Assume PHI(ϕ) and $\mathrm{J}_{\phi,\ge}$ are satisfied. Then by Theorem 1.17(4), J_ϕ and CSJ(ϕ) hold. So by Theorem 1.9(4), HK(ϕ) also holds.

Conversely, assume HK(ϕ) holds. By Theorem 1.9, J_ϕ and CSJ(ϕ) are satisfied. Note that UJS holds trivially because of J_ϕ. Thus by Theorem 1.17 again, PHI(ϕ) holds. □

5 Applications and Examples

The stability results in Theorem 1.17 allow us to obtain PHI for a large class of symmetric jump processes using "transferring method"; that is, by first establishing PHI for a particular symmetric jump process with jumping kernel $J(x, y)$, we can use Theorem 1.17 to obtain PHI for other symmetric jump processes whose jumping kernels are comparable to $J(x, y)$. Examples are given in [CKW, Section 6.1] on fractals that support anomalous diffusions with two-sided heat kernel estimates. The subordination of these diffusion processes enjoy HK(ϕ) and hence PHI(ϕ) by Corollary 1.18, and so can be served as the base examples.

In the remainder of this section, we show that some conditions in the equivalence statements of Theorem 1.17 are necessary through some counterexamples.

Example 5.1. (PHI(ϕ) **does not imply** HK(ϕ)**.)** Let $M = \mathbb{R}^d$ and $0 < \alpha < 2$. For $0 < \theta < 1$ and $v \in \mathbb{R}^d$ with $|v| = 1$, define $A = \{h \in \mathbb{R}^d : |(h/|h|, v)| \geq \theta\}$ and

$$J(x, y) = \mathbf{1}_A(x - y)|x - y|^{-d-\alpha}.$$

Clearly $J_{\phi,\leq}$ and UJS hold. According to [DK, Example 3], it is easy to see that this example satisfies PI(ϕ) with $\phi(s) = s^\alpha$. By Remark 1.5, SCSJ(ϕ) holds. So by Theorem 1.17, PHI(ϕ) holds. However, since $J_{\phi,\geq}$ does not hold, HK(ϕ) does not hold either.

Example 5.2. (EHI **and** E_ϕ **do not imply** PHI(ϕ)**.)** Let $M = \mathbb{R}^2$ and $1 < \alpha < 2$. Consider a symmetric Lévy process $X = \{X_t\}$ on $\mathbb{R}^2$ with the Lévy measure of the form

$$\nu(dx) = h(x)\, dx := |x|^{-2-\alpha} f(x/|x|)\, dx,$$

where $f : \mathcal{S}^1 \to \mathbb{R}_+$ is bounded and symmetric. Then, it is proved in [BS, Corollary 13] that EHI holds for non-negative harmonic functions. In fact, [BS, Theorem 1] gives more general fact in $\mathbb{R}^d$ setting with $d \geq 1$ that EHI holds for non-negative harmonic functions on $B(0, 1)$ if and only if there is a constant $C > 0$ such that the following holds

$$\int_{B(y,1/2)} |y - v|^{\alpha-d} h(v)\, dv \leq C \int_{B(y,1/2)} h(v)\, dv, \quad |y| > 1.$$

Let us take a particular choice of f given as follows. For $i \in \mathbb{N}$, let $\theta_i = (3\pi/8)4^{-i}$ and $\theta_i' = (3\pi/8)2^{-i}$. Note that $\sum_{i=1}^\infty (\theta_i + \theta_i') = \pi/2$. Define

$$H = \Big\{ e^{\theta\sqrt{-1}}, e^{-\theta\sqrt{-1}}, -e^{\theta\sqrt{-1}}, -e^{-\theta\sqrt{-1}} : \theta \in A \Big\},$$

where

$$A = [0, \theta_1) \cup \Big(\bigcup_{n=1}^\infty \Big[\sum_{i=1}^n (\theta_i + \theta_i'), \sum_{i=1}^n (\theta_i + \theta_i') + \theta_{n+1} \Big) \Big).$$

Set $f(x) = \mathbf{1}_H(x)$. Then, writing $\xi_n = \sum_{i=1}^n (\theta_i + \theta_i') + \theta_{n+1}/2$ and $J(x, y) = h(x - y)$, we see that

$$J(e^{\xi_n\sqrt{-1}}, 0) = 1.$$

Setting $H_n = \{e^{\theta\sqrt{-1}} : \theta \in [\xi_n - \theta_{n+1}/2, \xi_n + \theta_{n+1}/2)\}$, we have for large n,

$$\begin{aligned} V(e^{\xi_n\sqrt{-1}}, 2^{-n-1})^{-1} \int_{B(e^{\xi_n\sqrt{-1}},2^{-n-1})} J(z,0)\,dz &\le c(2^{n+1})^2 \int_{B(e^{\xi_n\sqrt{-1}},2^{-n-1})} \mathbf{1}_{H_n}(z/|z|)\,dz \\ &\le c' 4^n 2^{-n-1} 4^{-n-1} \le c_0 2^{-n}, \end{aligned}$$

so UJS does not hold. Therefore, by Theorem 1.17, PHI(ϕ) can not hold in this case.

We will briefly explain why E_ϕ holds with $\phi(r) = r^\alpha$. Note that the corresponding generator can be written as follows

$$\mathcal{L}u(x) = \int_{\mathbb{R}^2} \big(u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_{\{|z|<1\}}\big)\,\nu(dz).$$

For $g \in C_b^2(\mathbb{R}^2)$ with $0 \le g \le 1$, let $g_r(y) = g(y/r)$ for $r > 0$. Then, by similar computations as in [KSV, Lemma 13.4.1], we have $|\mathcal{L}g_r| \le c_1 r^{-\alpha}$, and so $\mathbb{P}^0(\tau_{B(0,r)} \le t) \le c_2 t/r^\alpha$ for all $t, r > 0$. This implies

$$\mathbb{E}^0[\tau_{B(0,r)}] \ge \frac{r^\alpha}{2c_2} \mathbb{P}^0(\tau_{B(0,r)} \ge r^\alpha/(2c_2)) \ge \frac{r^\alpha}{4c_2},$$

so that (since the process is the Lévy process) $\mathrm{E}_{\phi,\ge}$ holds. Next we have by the Lévy system formula,

$$\begin{aligned} \mathbb{P}^0(\tau_{B(0,r)} \le r^\alpha) &\ge \mathbb{P}^0(X \text{ hits } B(0,6r) \setminus B(0,3r) \text{ by time } r^\alpha) \\ &\ge \mathbb{P}^0(X_{r^\alpha \wedge \tau_{B(0,r)}} \in B(0,6r) \setminus B(0,3r)) \\ &= \mathbb{E}^0\left[\int_0^{r^\alpha \wedge \tau_{B(0,r)}} \nu((B(0,6r) \setminus B(0,3r)) - X_s)\,ds\right] \\ &\ge \nu(B(0,5r) \setminus B(0,4r))\,\mathbb{E}^0[r^\alpha \wedge \tau_{B(0,r)}] \\ &\ge \frac{c_3}{r^\alpha} \mathbb{E}^0[r^\alpha \wedge \tau_{B(0,r)}] \\ &\ge \frac{c_3}{r^\alpha} \cdot \frac{r^\alpha}{2c_2} \mathbb{P}^0\left(\tau_{B(0,r)} \ge r^\alpha/(2c_2)\right) \ge \frac{c_3}{4c_2} =: c_4. \end{aligned}$$

It follows that $\mathbb{P}^0(\tau_{B(0,r)} > r^\alpha) \le 1 - c_4$. Iterating this as in the proof of Proposition 3.5(ii), we obtain $\mathrm{E}_{\phi,\le}$.

Though the following example is not in the framework of our paper since the Lévy measure is singular to the Lebesgue measure on $\mathbb{R}^d$, it illustrates that in the context of symmetric jump processes, EHI in general does not follow from EHR and E_ϕ alone.

Example 5.3. (EHR **and** E_ϕ **do not imply** EHI **nor** PHI(ϕ)**.)** Let $M = \mathbb{R}^3$ and $0 < \alpha < 2$. Consider a symmetric process $X_t = (Z_t^{(1)}, Z_t^{(2)}, Z_t^{(3)})$, where $Z_t^{(i)}$, $i = 1, 2, 3$, are independent 1-dimensional symmetric α-stable processes. In [BC], it is proved that $\{X_t\}$ satisfies EHR and E_ϕ with $\phi(r) = r^\alpha$, but EHI and, consequently PHI(ϕ), fails. In addition, in this case one can easily see that UJS does not hold. (We note that in [BC], the authors discussed more general processes on $\mathbb{R}^d$ that are expressed by a system of stochastic differential equations $dX_t = A(X_{t-})\,dZ_t$, where $Z_t^{(i)}$, $1 \le i \le d$, are independent 1-dimensional symmetric α-stable processes and A is a matrix-valued function which is bounded, continuous and non-degenerate.) We also note that for this example, PI(ϕ) and SCSJ(ϕ) are satisfied by [DK, Example 4] and Remark 1.5 respectively.

References

[BBCK] M.T. Barlow, R.F. Bass, Z.-Q. Chen and M. Kassmann. Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.* **361** (2009), 1963–1999.

[BBK1] M.T. Barlow, R.F. Bass and T. Kumagai. Stability of parabolic Harnack inequalities on metric measure spaces. *J. Math. Soc. Japan* **58** (2006), 485–519.

[BBK2] M.T. Barlow, R.F. Bass and T. Kumagai. Parabolic Harnack inequality and heat kernel estimates for random walks with long range jumps. *Math. Z.* **261** (2009), 297–320.

[BGK] M.T. Barlow, A. Grigor'yan and T. Kumagai. On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan* **64** (2012), 1091–1146.

[BC] R.F. Bass and Z.-Q. Chen. Regularity of harmonic functions for a class of singular stable-like processes. *Math. Z.* **266** (2010), 489–503.

[BL] R.F. Bass and D. Levin. Harnack inequalities for jump processes. *Potential Anal.* **17** (2002), 375–388.

[BS] K. Bogdan and P. Sztonyk. Harnack's inequality for stable Lévy processes. *Potential Anal.* **22** (2005), 133–150.

[CS] L. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.* **62** (2009), 597–638.

[CKP1] A.D. Castro, T. Kuusi and G. Palatucci. Local behavior of fractional p-minimizers. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* **33** (2016), 1279–1299.

[CKP2] A.D. Castro, T. Kuusi and G. Palatucci. Nonlocal Harnack inequalities. *J. Funct. Anal.* **267** (2014), 1807–1836.

[C] Z.-Q. Chen. On notions of harmonicity. *Proc. Amer. Math. Soc.* **137** (2009), 3497–3510.

[CF] Z.-Q. Chen and M. Fukushima. *Symmetric Markov Processes, Time Change, and Boundary Theory.* Princeton Univ. Press, Princeton 2012.

[CKK1] Z.-Q. Chen, P. Kim and T. Kumagai. On heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces. *Acta Math. Sin. (Engl. Ser.)* **25** (2009), 1067–1086.

[CKK2] Z.-Q. Chen, P. Kim and T. Kumagai. Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.* **363** (2011), 5021–5055.

[CK1] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for stable-like processes on d-sets. *Stochastic Process Appl.* **108** (2003), 27–62.

[CK2] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Relat. Fields* **140** (2008), 277–317.

[CKW] Z.-Q. Chen, T. Kumagai and J. Wang. Stability of heat kernel estimates for symmetric jump processes on metric measure spaces. available at `arXiv:1604.04035`

[ChK] Z.-Q. Chen and K. Kuwae. On subhamonicity for symmetric Markov processes. *J. Math. Soc. Japan* **64** (2012), 1181–1209.

[ChZ] Z.-Q. Chen and X. Zhang. Hölder estimates for nonlocal-diffusion equations with drifts. *Commun. Math. Stat.* **2** (2014), 331–348.

[De] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana* **15** (1999), 181–232.

[DK] B. Dyda and M. Kassmann. Regularity estimates for elliptic nonlocal operators. available at `arXiv:1509.08320`

[FOT] M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes*. de Gruyter, Berlin, 2nd rev. and ext. ed., 2011.

[G] E. Giusti. *Direct Methods in the Calculus of Variations*. World Scientific Publishing Co. Inc., River Edge 2003.

[Gr] A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. (in Russian) *Matem. Sbornik.* **182** (1991), 55–87. (English transl.) *Math. USSR Sbornik* **72** (1992), 47–77.

[GH] A. Grigor'yan and J. Hu. Upper bounds of heat kernels on doubling spaces. *Mosco Math. J.* **14** (2014), 505–563.

[GT] A. Grigor'yan and A. Telcs. Two-sided estimates of heat kernels on metric measure spaces. *Ann. Probab.* **40** (2012), 1212–1284.

[H] J. Heinonen. *Lectures on Analysis on Metric Spaces*. Springer-Verlag, New York 2001.

[K1] M. Kassmann. Harnack inequalities: An introduction. *Boundary Value Problems* **2007** (2007), Article ID 81415, 21 pages.

[K2] M. Kassmann. A priori estimates for integro-differential operators with measurable kernels. *Calc. Var. Partial Differ. Equ.* **34** (2009), 1–21.

[KSV] P. Kim, R. Song and Z. Vondraček. Potential theory of subordinate Brownian motions revisited. *Stochastic Analysis and Applications to Finance – Essays in Honour of Jia-an Yan.* World Scientific, 2012, pp. 243–290.

[Kom] T. Komatsu. Uniform estimates for fundamental solutions associated with non-local Dirichlet forms. *Osaka J. Math.* **32** (1995), 833–860.

[KZ] S. Kusuoka and X.Y. Zhou. Dirichlet forms on fractals: Poincaré constant and resistance. *Probab. Theory Relat. Fields* **93** (1992), 169–196.

[LS] J. Lierl and L. Saloff-Coste. Parabolic Harnack inequality for time-dependent non-symmetric Dirichlet forms. available at `arXiv:1205.6493`

[MK] A. Mimica and M. Kassmann. Intrinsic scaling properties for nonlocal operators. *J. Eur. Math. Soc.*, to appear.

[Sa1] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Inter. Math. Res. Notices* **2** (1992), 27–38.

[Sa2] L. Saloff-Coste. *Aspects of Sobolev-type Inequalities.* Lond. Math. Soc. Lect. Notes, vol. **289**, Cambridge Univerisity Press, Cambridge 2002.

[SU] R. Schilling and T. Uemura. On the Feller property of Dirichlet forms generated by pseudo differential operators. *Tohoku Math. J.* **59** (2007), 401–422.

[Si] M.L. Silverstein. *Symmetric Markov Processes.* Lecture Notes in Mathematics, vol. **426**, Springer-Verlag, Berlin Heidelberg 1974.

[Sil] L. Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana Univ. Math. J.* **55** (2006), 1155–1174.

[SV] R. Song and Z. Vondraček. Harnack inequality for some classes of Markov processes. *Math. Z.* **246** (2004), 177–202.

[St] K.-T. Sturm. Analysis on local Dirichlet spaces III. The parabolic Harnack inequality. *J. Math. Pures Appl.* **75** (1996), 273–297.

Zhen-Qing Chen

Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail: `zqchen@uw.edu`

Takashi Kumagai:

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email: `kumagai@kurims.kyoto-u.ac.jp`

Jian Wang:

School of Mathematics and Computer Science, Fujian Normal University, 350007, Fuzhou, P.R. China.
Email: `jianwang@fjnu.edu.cn`