6 Application: RW on critical branching processes

RW on the percolation cluster on $\mathbb{Z}^d$ ($d \geq 2$)

**Supercritical**

De Masi, Ferrari, Goldstein and Wick (1989 [33]): Inv. principle for the annealed case
Sidoravicius and A.-S. Sznitman (2004 [74]): Inv. principle for the quenched case
Mathieu and Remy (2004): Isoperimetric ineq. and heat kernel decay
Barlow (2004 [5]): Detailed Gaussian heat kernel estimates

**Critical** Unknown!!

Kesten (1986 [55]): $d = 2$ ‘subdiffusive behaviour’

cf. $d = 2$: Smirnov, Lawler, Schramn and Werner

$\Rightarrow$ Shape of the cont. limit etc. (Very Active)
6.2 The model and main results

\( \mathcal{G} \): random tree. We could regard this in two ways.

- Critical percolation on the \( n_0 \)-ary tree \( \mathcal{B} \), condi. the cluster containing 0 being infinite
- Critical branching process with \( \text{Bin}(n_0, 1/n_0) \) offspring distrib., condi. on non-extinction.

\( \mathcal{B} \): \( n_0 \)-ary tree, 0: the root, \( E(\mathcal{B}) \): edge set.

\( \mathcal{B}_n \): the set of \( n_0^n \) points in the \( n \)th generation, \( \mathcal{B}_{\leq n} = \bigcup_{i=0}^{n} \mathcal{B}_i \).

\( \eta_e, e \in E(\mathcal{B}) \), be i.i.d. Bernoulli \( 1/n_0 \) r.v. \( (\eta_e = 1 \iff e \text{ is open}) \)

\[ \mathcal{C}(0) := \{ x \in \mathcal{B} : \text{there exists an } \eta \text{–open path from 0 to } x \} \]

Clearly, \( Z_n = |\mathcal{C}(0) \cap \mathcal{B}_n| \) is a critical GW process with \( \text{Bin}(n_0, 1/n_0) \) offspring distri.

As \( Z \) has extinction probability 1, the cluster \( \mathcal{C}(0) \) is \( P \)-a.s. finite.
Incipient infinite cluster (IIC') on $\mathbb{B}$. Two constructions.

**Lemma 6.1** ([55], Lemma 1.14) Let $A \subset \mathbb{B}_{\leq k}$. Then

$$\lim_{n \to \infty} P(C(0) \cap \mathbb{B}_{\leq k} = A | Z_n \neq 0) = |A \cap \mathbb{B}_k| P(C(0) \cap \mathbb{B}_{\leq k} = A) =: \mathbb{P}_0(A).$$

$\exists 1 \mathbb{P}$: extension of $\mathbb{P}_0$ to a prob. on the set of $\infty$-con. subsets of $\mathbb{B}$ containing 0.

$G'$: rooted labeled tree with the distri. $\mathbb{P} \Rightarrow$ IIC on $\mathbb{B}$. $\exists 1 H$ backbone of $G'$.

(Another construction) $\{\xi_i\}_{i \geq 1}$: i.i.d., unif. distri. on $\{1, 2, \cdots, n_0\}$, indep. of $(\eta_e)$.

For $n \geq 0$ let $\Xi_n = (0, \xi_1, \ldots, \xi_n)$, and let

$$\widetilde{\eta}_e := \begin{cases} 1 & \text{if } e = \{\Xi_n, \Xi_{n+1}\} \text{ for some } n \geq 0, \\ \eta_e & \text{otherwise}, \end{cases}$$

$$G := \{x \in \mathbb{B} : \text{there exists a } \widetilde{\eta}-\text{open path from } 0 \text{ to } x\},$$

$G$ has law $\mathbb{P}$. $H = \{\Xi_n, n \geq 0\}$: backbone of $G$
Let \( \mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | x \in \mathcal{G}) \), \( \mathbb{P}_{xy}(\cdot) = \mathbb{P}(\cdot | x, y \in \mathcal{G}) \),
\[ \mathbb{P}_{x,y,b}(\cdot) = \mathbb{P}(\cdot | x, y \in \mathcal{G}, H = b). \]

For each fixed \( \mathcal{G} = \mathcal{G}(\omega) \), \( \{Y_t\} \): cont. time S.R.W. on \( \mathcal{G} \),
\( P^x_\omega \): law of \( \{Y_t\} \) starting at \( x \in \mathcal{G}(\omega) \)
\( E^x_\omega \): its average
\( q^\omega_t(x, y) := \mathbb{P}^x(Y_t = y)/\mu_y. \)

Kesten (1986 [55]): \( \mathbb{P} \)-distri. of \( n^{-1/3}d(0, Y_n) \) converges.
Theorem 6.2 (a) $\exists c_0, c_1, c_2, S(x) \text{ s.t. } \mathbb{P}_x(S(x) \geq m) \leq c_0(\log m)^{-1}, \ \forall x$ and

$$c_1t^{-2/3}(\log \log t)^{-17} \leq q_t^\omega(x, x) \leq c_2t^{-2/3}(\log \log t)^3 \quad \forall t \geq S(x), x \in \mathcal{G}(\omega).$$

(b) $d_s(\mathcal{G}) := -2 \lim_{t \to \infty} \frac{\log q_t^\omega(x, x)}{\log t} = 4/3 \ \mathbb{P} - a.s.$

(c) $c_1t^{-2/3} \leq \mathbb{E}_x[q_t(x, x)] \leq c_2t^{-2/3}.$

$q_t(x, x)$ does have oscillations of order $(\log \log t)^a$ as $t \to \infty$.

Proposition 6.3 $\lim \inf_{t \to \infty}(\log \log t)^{1/6}t^{2/3}q_2^\omega(0, 0) \leq 2, \quad P_\omega^0 - a.s.$

Theorem 6.4 (a) $c_1t^{1/3} \leq \mathbb{E}_x E_\omega^xd(x, Y_t) \leq \mathbb{E}_x E_\omega^x \sup_{0 \leq s \leq t} d(x, Y_s) \leq c_2t^{1/3}.$

(b) $\exists T(x)$ with $\mathbb{P}_x(T(x) < \infty) = 1 \ \text{s.t.}$

$$c_3t^{1/3}(\log \log t)^{-12} \leq E_\omega^x[d(x, Y_t)] \leq c_4t^{1/3} \log t \quad \forall t \geq T(x).$$
Quenched off-diagonal bounds for $q_t^\omega(x, y)$.

**Theorem 6.5** (1) Let $x, y \in \mathcal{G}$, $t > 0$ be s.t. $N := \lceil \sqrt{d(x, y)^3/t} \rceil \geq 8$.

Then, $\exists F_* = F_*(x, y, t)$ with $\mathbb{P}_{x_0,y_0,b}(F_*(x, y, t)) \geq 1 - c_1 \exp(-c_2 N)$, s.t.

$$q_t^\omega(x, y) \leq c_3 t^{-2/3} \exp(-c_4 N), \quad \forall \omega \in F_*.$$

(2) Let $x, y \in \mathcal{G}$, $m \geq 1$, $\kappa \geq 1$ and let $T = d(x, y)^3 \kappa/m^2$.

Then, $\exists G_* = G_*(x, y, m, \kappa)$ with $\mathbb{P}_{x,y,b}(G_*(x, y, m, \kappa) \text{ holds}) \geq 1 - c_1 \kappa^{-1}$, s.t.

$$q_{2T}(x, y) \geq c_2 T^{-2/3} e^{-c_3(\kappa+c_4)m}, \quad \forall \omega \in G_*.$$

Annealed off-diagonal bounds for $q_t^\omega(x, y)$.

**Theorem 6.6** Let $x, y \in \mathbb{B}$. Then

$$c_4 t^{-2/3} \exp(-c_5 \left(\frac{d(x, y)^3}{t}\right)^{1/2}) \leq \mathbb{E}_{x,y} q_t^\omega(x, y) \leq c_1 t^{-2/3} \exp(- c_2 \left(\frac{d(x, y)^3}{t}\right)^{1/2}),$$

where the lower bound is for $c_3 d(x, y) \leq t$. 
Rescaled height process: \[ \tilde{Z}_t^{(n)} = n^{-1/3} d(0, Y_{nt}), \quad t \geq 0. \]

\{Z^{(n)}\} are tight w.r.t. the annealed law \( \mathbb{P}^* = \mathbb{P} \times P^0_\omega \). (Theorem 6.4 (a) or Kesten [55])

However, the large scale fluctuations in \( \mathcal{G} \) mean that we do not have quenched tightness.

**Theorem 6.7** \( \mathbb{P} \)-a.s., the processes \( (\tilde{Z}^{(n)}, n \geq 1) \) are not tight with respect to \( P^0_\omega \).

### 6.3 Ideas of the proof

Proof: analytic and probabilistic parts. Note: We cannot expect (VD)!!

**Definition 6.8** Let \( x \in \mathcal{G}, r \geq 1 \). Let \( M(x, r) \) be the smallest number \( m \) s.t. \( \exists A = \{z_1, \ldots, z_m\} \) with \( d(x, z_i) \in [r/4, 3r/4] \), for each \( i \), so that any path \( \gamma \) from \( x \) to \( B(x, r)^c \) must pass through the set \( A \).
Analytic estimates \( B := B(x_0, r), M := M(x_0, r), V := V(x_0, r) \).

**Proposition 6.9** (a) \((G, \mu)\): weighted graph. Suppose that \( \mu_{xy} \geq 1 \ \forall x \sim y \). Then

\[
q_{2rV(x,r)}(x, x) \leq \frac{2}{V(x,r)}, \quad x \in G, \ r > 0.
\]

(b) \(G\): tree. Let \( V_1 = V_1(x_0, r) = V(x_0, r/(32M(x_0, r))) \). Then if \( x \in B(x_0, r/(32M)) \),

\[
P^x(\tau_B \leq t) \leq (1 - \frac{V_1}{64MV}) + \frac{t}{2rV},
\]

and

\[
q_{2t}(x, x) \geq \frac{c_1 V_1(x_0, r)^2}{V(x_0, r)^3 M(x_0, r)^2} \quad \text{for } t \leq \frac{rV_1(x_0, r)}{64M(x_0, r)}.
\]

**Proof.** (a): similarly to Step A in subsection 5.2.

(b): similar argument as in Step B in subsection 5.2 (using the tree property and \( M(x, r) \) instead of (VD)) gives the estimate of \( E^x_\omega[\tau_B(x,r)] \). Then the argument in Step 3 in the proof of Proposition 4.1 gives the desired result.
Probabilistic estimates On-diagonal estimates: Need information of $V(x, r)$ and $M(x, r)$!

The probability that $V(x, r)$ and $M(x, r)$ behave badly is ‘small’.

**Proposition 6.10** (a) Let $\lambda > 0$, $r \geq 1$, $x, y \in \mathbb{B}$, and $b$ be a backbone. Then

$$
P_{x,y,b}(V(x, r) > \lambda r^2) \leq c_0 \exp(-c_1 \lambda),$$

$$
P_{x,y,b}(V(x, r) < \lambda r^2) \leq c_2 \exp(-c_3/\sqrt{\lambda}).$$

(b) For any $\varepsilon > 0$

$$
\limsup_{n \to \infty} \frac{V(0, n)}{n^2(\log \log n)^{1-\varepsilon}} = \infty, \quad \mathbb{P} - a.s.
$$

(c) Let $r \geq 1$, $x, y \in \mathbb{B}$, and $b$ be a backbone. Then

$$
P_{x,y,b}(M(x, r) \geq m) \leq c_4 e^{-c_5 m}.$$

These can be obtained, basically through large deviation estimates of the total population size of the critical branching process.
Idea of the proof of Proposition 6.10:

For simplicity, let $x \in H, \ d(0,x) > r$.

$|B(x,r)| \leq V(x,r) \leq 2|B(x,r)|$, so consider $|B(x,r)|$.

$\{\tilde{X}_n\} : \tilde{X}_0 = 1, \ \tilde{X}_1 \overset{(d)}{=} Bin(n_0 - 1, 1/n_0)$, from the 2nd generation, $Bin(n_0, 1/n_0)$.

$\tilde{Y}_n := \sum_{k=0}^n \tilde{X}_k$. Then,

$$\tilde{Y}_{r/2}[r/2] \overset{(d)}{\leq} |B(x,r)| \overset{(d)}{\leq} \tilde{Y}_r[r] + \tilde{Y}_r'[r].$$

(Here, for r.v. $\xi$, $\xi[n] \overset{(d)}{=} \sum_{i=1}^n \xi_i$, where $\{\xi_i\}$ i.i.d. with $\xi_i \overset{(d)}{=} \xi$.)

Now let $Y_n := \sum_{k=0}^n X_k$: total population size up to generation $n$. Then,

$$P(Y_n[n] \geq \lambda n^2) \leq c \exp(-c' \lambda), \quad P(Y_n[n] \leq \lambda n^2) \leq c \exp(-c'/\sqrt{\lambda}).$$

Similar estimates hold for $\tilde{Y}_n[n]$.

$\Rightarrow$ (a) holds.
We now define a ‘good’ random set.

**Definition 6.11** Let $x \in \mathcal{B}$, $r \geq 1$, $\lambda \geq 64$. $B(x, r)$ is $\lambda$-good if

$$x \in \mathcal{G}, \quad r^2 \lambda^{-2} \leq V(x, r) \leq r^2 \lambda, \quad M(x, r) \leq \frac{1}{64} \lambda,$$

$$V(x, r/\lambda) \geq r^2 \lambda^{-4}, \quad \text{and} \quad V(x, r/\lambda^2) \geq r^2 \lambda^{-6}.$$  

By Proposition 6.10, we have the following.

**Corollary 6.12** For $x \in \mathcal{B}$ and any possible backbone $b$

$$\mathbb{P}_{x,b}(B(x, r) \text{ is not } \lambda\text{-good}) \leq c_1 e^{-c_2 \lambda}.$$  

By Prop 6.9, if $B(x, r)$ is $\lambda$-good, then

$$c_1' t^{-2/3} \lambda^{-17} \leq q_2 t(x, x) \leq c_2' t^{-2/3} \lambda^3, \quad \frac{r^3}{\lambda^6} \leq \forall t \leq \frac{r^3}{\lambda^5}. \quad (++)$$
Idea of the proof of Theorem 6.2. (a) Take $\lambda_n = e + (2/c_2) \log n$, $r_n : r_n^3/\lambda_n^6 = e^n$, and let $F_n := \{B(x, r_n) \text{ is } \lambda_n\text{-good}\}$. By Cor 6.12, $\mathbb{P}(F_n^c) \leq c/n^2$.

Let $N := \min\{m : F_n^c \text{ occurs } \exists n \geq m\}$. Then $\mathbb{P}(N \geq m) \leq \sum_{n=m}^\infty \mathbb{P}(F_n^c) \leq c/m$.

Let $S(x) := e^N$. By $(++)$, 
\[ c_1' t^{-2/3} \lambda_n^{-17} \leq q_2t(x, x) \leq c_2' t^{-2/3} \lambda_n^3, \quad \forall n \geq \log S(x) + 1, \quad e^n \leq t \leq \lambda_n e^n. \quad (*) \]

Take $n = n(t)$ s.t. $\log t \in [n(t) - 1, n(t)]$.

Then $(*)$ holds for $t \geq S(x)$ with $\lambda_{n(t)} \sim (2/c_2) \log \log t$. $\Rightarrow$ Thm 6.2 (a).

(b) $\lambda_n = n$, $r_n : r_n^3/\lambda_n^6 = t$. $F_n$ as above. $N(\omega) := \min\{n : \omega \in F_n\}$.

By Cor 6.12, $\mathbb{P}(N > n) = \mathbb{P}(F_n^c) \leq e^{-cn}$. Thus, using $(++)$,
\[ \mathbb{E}_x[q_t(x, x)] \leq ct^{-2/3} \mathbb{E}_x N^3 \leq c't^{-2/3}. \]

Lower bound is easy by $(++)$ and Cor 6.12.

To get off-diagonal estimates, we need to take more refined ‘good’ random sets.
7 Some open problems

• Simpler stable equivalence conditions for \((\text{PHI}(\Psi))\): It is not easy to check \((\text{CS}(\Psi))\) in examples. Quite recently, Barlow-Bass proved \((\text{PHI}(\beta)) \iff (\text{VD}) + (\text{PI}(\beta)) + (\text{E}(\beta))\) for weighted graphs. Conjecture: \((\text{PHI}(\beta)) \iff (\text{VD}) + (\text{PI}(\beta)) + (\text{RES}(\beta))\).

• Stability of \((\text{EHI})\): Is \((\text{EHI})\) stable under rough isometries?

• Stability of \((\text{UHK}(\Psi))\): Is \((\text{UHK}(\Psi))\) stable under rough isometries?

Related conjecture by Grigor’yan: \((\text{UHK}(\beta)) \iff (\text{FK}(\beta)) + (\text{Anti FK}(\beta))\), which guarantees the optimality of \((\text{FK}(\beta))\) for balls.

• \(\text{RW on IIC on } \mathbb{Z}^d\): HK estimates for RW on infinite incipient clusters on \(\mathbb{Z}^d\)?

\(d = 2\) and \(d\) large enough, RW on such IIC is in the framework of resistance forms. So we have reasonable analytic estimates. Probabilistic estimates??