対称拡散過程の熱核評価、ハルナック不等式の安定性とその応用

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\( X \): locally compact separable metric space (diam \( X = \infty \))

(\( \mathcal{E}, \mathcal{F} \)): reg. local Dirichlet form on \( L^2(X, \mu) \)

\(-\Delta, \{X_t\}_t\): the corresponding non-neg. S.A. operator and the diffusion.

- Elliptic Harnack inequality (EHI): \( \exists c_3 > 0 \) s.t. \( \forall B(x, R) \), \( \forall u \): non-negative harmonic fu. on \( B(x, R) \) (i.e. \( \Delta u(x) = 0 \) for \( x \in B(x, R) \)), then

\[
\sup_{B(x, R/2)} u \leq c_3 \inf_{B(x, R/2)} u.
\] (EHI)

Let \( \beta \geq 2 \) and denote \( V(x, R) := \mu(B(x, r)) \).

- (Sub-)Gaussian heat kernel estimates:

\[
\frac{c_4}{\mu(B(x, t^{1/2}))} \exp\left(-\frac{d(x, y)^2}{c_4 t}\right) \leq p_t(x, y) \leq \frac{c_5}{\mu(B(x, t^{1/2}))} \exp\left(-\frac{d(x, y)^2}{c_5 t}\right). \quad \text{(HK(2))}
\]

\[
\frac{c_4}{\mu(B(x, t^{1/\beta}))} \exp\left(-\left(\frac{d(x, y)^\beta}{c_4 t}\right)^{\frac{1}{\beta-1}}\right) \leq p_t(x, y) \leq \frac{c_5}{\mu(B(x, t^{1/\beta}))} \exp\left(-\left(\frac{d(x, y)^\beta}{c_5 t}\right)^{\frac{1}{\beta-1}}\right). \quad \text{(HK(\beta))}
\]
• Let $Q = Q(x_0, T, R) = (0, 4T) \times B(x_0, 2R)$,

$$Q_-(T, 2T) \times B(x_0, R) \quad \text{and} \quad Q_+ = (3T, 4T) \times B(x_0, R).$$

Parabolic Harnack inequality (PHI($\beta$)): $\exists c_6 > 0$ s.t. the following holds.

Let $x_0 \in X$, $R > 0$, $T = R^\beta$, and $u = u(t, x) : Q \to \mathbb{R}_+$ satisfies $\frac{\partial u}{\partial t} = \Delta u$ in $Q$. Then,

$$\sup_{Q_-} u \leq c_6 \inf_{Q_+} u. \quad \text{(PHI($\beta$))}$$
$(HK(\beta)) \Leftrightarrow (PHI(\beta))$ から拡散過程の様々な性質が導き出せる

- $c_1 t^{1/\beta} \leq E^x[d(x, X_t)] \leq c_2 t^{1/\beta} \ (\beta > 2$: 劣拡散的$)$

- 重複対数の定理  （i.e. $\limsup_{t \to \infty} \frac{d(X_t, X_0)}{t^{1/\beta} (\log \log t)^{1-1/\beta}} = C$, $P^x$-a.s.）

- 熱方程式の解のHölder連続性

- 椎円型ハルナック不等式 （EHI）

- Liouville property (i.e. positive harm. fu. on $X$ is const.)

  Indeed, if $m_u := \inf_X u$, then by (EHI), $\sup_B (u - m_u) \leq c \inf_B (u - m_u) \to 0$ as $B \to \infty$. So $u \equiv m_u$, $\mu$-a.e.

- グリーン核の評価
Divergence form \( \mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) \) on \( \mathbb{R}^n \) satisfying the unif. ellip. cond. (i.e. \( \sigma^{-1} I \leq a(\cdot) \leq \sigma I \) for \( \exists \sigma \geq 1 \)).

- De Giorgi ('57), Nash ('58) [70]: Hölder cont. for elliptic/parabolic functions
- Moser ('61,'64,'71) [69,68,67]: Harnack ineq.
- Aronson ('67) [2]: (HK(2))
- Krylov-Safanov ('80): Prob. proof for Harnack
- Davies ('81, '87, '89) [32,31]: Off-diagonal upper estimates
- Fabes-Stroock ('86) [34]: A new proof of Moser’s PHI using the old idea of Nash.
- Carlen-Kusuoka-Stroock ('87) [25]: equiv. of the Nash inequalities
- Li-Yau ('86) [65]: smooth non-cpt compl. R-mfd, non-neg. Ricci, \( \Delta \implies \text{(HK}(2)) \)
• Grigor’yan (’92) [39], Saloff-Coste (’92) [73]: (HK(2)) ⇔ (VD) + (PI(2))

• Biroli-Mosco (’95) [20], Sturm (’95,’96) [75,76], Delmotte (’99) [32]: extension to Dirichlet forms on meas. met. spaces and graphs

(A) Volume doubling (VD): \( V(x, 2R) \leq c_1 V(x, R), \quad \forall x \in X, \ R \geq 0. \)

(B) Poincaré inequality (PI(\(\beta\))): \( \exists c_2 \) s.t. \( \forall B = B(x, R) \subset X \) and \( \forall f \in \mathcal{F}, \)
\[
\int_B (f(x) - \bar{f}_B)^2 d\mu(x) \leq c_2 R^\beta \mathcal{E}_B(f, f), \quad \text{where} \ \bar{f}_B = \frac{1}{\mu(B)} \int_B f(x) d\mu(x). \quad (\text{PI}(\beta))
\]

Sub-Gaussian case

• Grigor’yan-Telcs (’01,’02) [42,41], Barlow-Bass (’03) [9]

• Kigami (’04) [57], Grigor’yan (’05) [36]

• Barlow-Coulhon-K (’05) [15], Barlow-Bass-K (’05) [14]
講演プラン

1. ガウス型の場合の古典的手法

2. 測度付き距離空間、グラフ上のディリクレ形式：ハルナック不等式と熱核評価

3. 強再帰的な場合

4. 臨界点における確率モデルの熱核評価
2.1 The Nash inequality

$X$: locally compact separable metric space

$(\mathcal{E}, \mathcal{F})$: Dirichlet form on $L^2(X, \mu)$

$-\Delta, \{P_t\}$: the corresponding non-negative self-adjoint operator and the semigroup

**Theorem 2.1** (The Nash inequality, [25])

The following are equivalent for any $\delta > 0$.

1) There exist $c_1, \theta > 0$ such that for all $f \in \mathcal{F} \cap L^1$,

$$\|f\|_2^{2+4/\theta} \leq c_1(\mathcal{E}(f, f) + \delta \|f\|_2^2)\|f\|_1^{4/\theta},$$

*(Nash)*

where $\|f\|_p := (\int_X |f|^p d\mu)^{1/p}$.

2) $\forall t > 0$, $P_t(L^1) \subset L^\infty$ and it is a bounded operator. Moreover, $\exists c_2, \theta > 0$ s.t.

$$\|P_t\|_{1\to \infty} \leq c_2 e^{\delta t t^{-\theta/2}}, \quad \forall t > 0.$$
Proof of Theorem 2.1:

1) ⇒ 2): Let \( f \in L^2 \cap L^1 \) with \( \| f \|_1 = 1 \) and \( u(t) := (P_t f, P_t f)_2 \). Then,

\[
\frac{u(t+h) - u(t)}{h} = \frac{1}{h}(P_{t+h} f + P_t f, P_{t+h} f - P_t f)_2 = (P_{t+h} f + P_t f, \frac{(P_h - I)P_t f}{h})_2
\]

\[
\lim_{h \downarrow 0} 2(P_t f, \Delta P_t f)_2 = -2E(P_t f, P_t f).
\]

Hence \( u'(t) = -2E(P_t f, P_t f) \). Now by 1),

\[
2u(t)^{1+2/\theta} \leq c_1(-u'(t) + 2\delta u(t)) \| P_t f \|_1^{4/\theta} \leq c_1(-u'(t) + 2\delta u(t))
\]

because \( \| P_t f \|_1 \leq \| f \|_1 = 1 \). Thus,

\[
2(e^{-2\delta t}u(t))^{1+2/\theta} \leq 2e^{-2\delta t}u(t)^{1+2/\theta} \leq -c_1(e^{-2\delta t}u(t))'.
\]

Set \( v(t) = (e^{-2\delta t}u(t))^{-2/\theta} \), then \( v'(t) \geq 4/(c_1 \theta) \). Since \( \lim_{t \downarrow 0} v(t) = u(0)^{-2/\theta} > 0 \),
it follows that \( v(t) \geq 4t/(c_1 \theta) \). This means \( u(t) \leq c_2 e^{2\delta t} t^{-\theta/2} \) where \( c_2 = (c_1 \theta/4)^{\theta/2} \).
Hence
\[ \|Pt f\|_2 \leq c_3 e^{\delta t} t^{-\theta/4}\|f\|_1, \quad \forall f \in L^2 \cap L^1, \]
which implies \(\|P_t\|_{1 \to 2} \leq c_3 e^{\delta t} t^{-\theta/4}\). Since \(P_t = P_{t/2} \circ P_{t/2}\) and \(\|P_{t/2}\|_{1 \to 2} = \|P_{t/2}\|_{2 \to \infty}\), we obtain 2).

\[ \Box \]

**Remark.** Generalization of Theorem 2.1: by Coulhon [28], Tomisaki [77] etc.

See subsection 8.1.
2.2 The Davies method

\[ \hat{\mathcal{F}} := \{ h + c : h \in \mathcal{F}_b, c \in \mathbb{R} \} \]

\[ \hat{\mathcal{F}}_\infty := \{ \psi \in \hat{\mathcal{F}} : e^{-2\psi} \Gamma(e^\psi, e^\psi) \ll \mu, e^{2\psi} \Gamma(e^{-\psi}, e^{-\psi}) \ll \mu \}. \]

**Theorem 2.4** (Carlen-Kusuoka-Stroock [25], Theorem 3.25)

Assume (Nash). Then, \( \exists c > 0 \text{ s.t. } \forall \rho \in (0,1], \)

\[ p_t(x, y) \leq c (\rho t)^{-\theta/2} e^{-E((1+\rho)t,x,y)+\delta \rho t} \quad \text{for } t > 0 \text{ and } x, y \in X, \quad (2.4) \]

where

\[ E(t, x, y) := \sup \{ |\psi(x) - \psi(y)| - t\Lambda(\psi)^2 : \Lambda(\psi) < \infty \} \]

with

\[ \Lambda(\psi)^2 := \max \left\{ \left\| \frac{d e^{-2\psi} \Gamma(e^\psi, e^\psi)}{d\mu} \right\|_\infty, \left\| \frac{d e^{2\psi} \Gamma(e^{-\psi}, e^{-\psi})}{d\mu} \right\|_\infty \right\}. \]
証明の方針： Step I 以下の式を $\forall f \in \mathcal{F}, \forall p \in [1, \infty)$ で示す ([25], Theorem 3.9)。

$$\mathcal{E}(e^{\psi} f^{2p-1}, e^{-\psi} f) \geq p^{-1} \mathcal{E}(f^p, f^p) - 9p \Lambda(\psi)^2 \|f\|_{2p}^2$$

Step II: $f_t(x) := e^{\psi(x)}[P_t(e^{-\psi} f)](x)$ とし、上の不等式と (Nash) を以下の式に用いる。

$$\frac{\partial}{\partial t} \|f_t\|_{2p}^{2p} = -2p \mathcal{E}(e^{\psi} f_t^{2p-1}, e^{-\psi} f_t).$$

Step III: 得られた微分不等式を評価する ([25], Lemma 3.21)。

Upper bound の出し方 \quad $\mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j})$ on $\mathbb{R}^n$ satisfying $\sigma^{-1} I \leq a(\cdot) \leq \sigma I$ for $\exists \sigma \geq 1$. In this case, (Nash) holds with $\theta = n$, $\delta = 0$ and

$$\Lambda(\psi)^2 = \sup_{x} (\nabla \psi(x), a(x) \nabla \psi(x)).$$

Let $\rho = 1$. Taking $\psi(x) = \theta \cdot x$ for some $\theta \in \mathbb{R}^n$ in (2.4), we get

$$p_t(x, y) \leq c_1 t^{-d/2} \exp(\theta \cdot (x - y) + 2\|\theta\|^2 \sigma t).$$
Optimize: \( \theta = (y - x)/(4\sigma t) \), we obtain

\[
p_t(x, y) \leq c_1 t^{-d/2} \exp(-\frac{|y - x|^2}{8\sigma t}),
\]

and the Gaussian upper bound is obtained.

実はもっと良い評価が出せる

\[
d_{\mathcal{E}}(x, y) := \sup\{\psi(x) - \psi(y) : \psi \in \hat{\mathcal{F}}_\infty \cap C(X), \Lambda(\psi) \leq 1\}.
\]

This is a metric and sometimes called an intrinsic metric. By a simple computation,

\[
E((1 + \varrho)t, x, y) = \frac{d_{\mathcal{E}}(x, y)^2}{4(1 + \varrho)t}.
\]

So, we conclude

\[
p_t(x, y) \leq c_1(\varrho t)^{-d/2} \exp(-\frac{d_{\mathcal{E}}(x, y)^2}{4(1 + \varrho)t}).
\]

**Remark.** For \( \beta > 2 \), this method does not work! Indeed, for diffusions on ‘typical’ fractals, the energy meas. is singular to the Hdiff. measure ([47,61]) so \( d_{\mathcal{E}}(x, y) \equiv 0 \).
2.3 Moser’s arguments

$X$: Riemannian manifold

$\Delta$: the Laplace-Beltrami operator satisfying $(\text{PI}(\beta))$.

$\mu$ the Riemannian measure satisfying $c_1 r^\alpha \leq \mu(B(x, r)) \leq c_2 r^\alpha$, $\forall x \in X, r \geq 1$.

$$\int_B f = \mu(B)^{-1} \int_B f d\mu.$$  

$(\text{PI}(\beta)) \Rightarrow (2.5)$: the Sobolev inequality

$$(\int_B |f|^{2\kappa})^{1/\kappa} \leq c_1 R^\beta \int_B |\nabla f|^2, \quad f \in C_0^\infty(B).$$  

(2.5)

Here $\kappa = \bar{\alpha}/(\bar{\alpha} - 2)$, $\bar{\alpha} = 3 \vee \alpha$.

$$d\Gamma(f, f) = |\nabla f|^2 d\mu \text{ for } f \in \mathcal{F}.$$  

Let $u > 0$ be harmonic on $B$, $v = u^p$ for $p > 0$, $1/2 < a_2 < a_1 < 1$, $B_i := B(x_0, a_i R)$.

$\varphi \in C_0^\infty(B_1)$: a cut-off function for $B_2 \subset B_1$. 

\[ \]
By “converse to the Poincaré inequality” (see Lemma 4.6 below),
\[ \int_{B_1} |\varphi \nabla v|^2 \leq c_2 \|
abla \varphi\|_\infty^2 \int_{B_1} v^2. \] (2.6)

Using (2.5) with \( f = v \) and (2.6),
\[ \left( \int_{B_2} u^{2\kappa p} \right)^{1/\kappa} \leq c_3 R^\beta \int_{B_2} |\nabla v|^2 \leq c_3 R^\beta \int_{B_1} \varphi^2 |\nabla v|^2 \leq c_4 R^\beta \|
abla \varphi\|_\infty^2 \int_{B_1} v^2. \]

Take “classical” cut-off function \( \varphi(x) = \frac{d(x,B^c)}{R(a_1-a_2)} \Rightarrow \|
abla \varphi\|_\infty^2 \leq \frac{c_5}{(a_1-a_2)^2 R^2}. \) Thus
\[ \left( \int_{B_2} u^{2\kappa p} \right)^{1/\kappa} \leq c_6 R^{\beta-2}(a_1-a_2)^{-2} \int_{B_1} u^{2p}. \] (2.7)

Let \( a_k := (1 + 2^{-k})/2, p_k := p\kappa^k \) and \( B_k := B(x_0, a_k R) \). (Then \( a_k - a_{k+1} = 2^{-k-2} \).)

Set \( I_k := \left( \int_{B_{k+1}} u^{2p_k} \right)^{1/(2p_k)} \). Then, by (2.7) we have
\[ I_{k+1} \leq (c_7 R^{\beta-2} 2^{2k})^{1/(2p_k)} I_k. \]

By iteration (this part is the first part of Moser’s argument), we have
\[ I_k \leq \prod_{l=0}^{k-1} (c_7 R^{\beta-2} 2^{2l})^{1/(2p_l)} I_0 \leq c_8 R^{c' \beta - 2} I_0. \]
The last inequality is due to $\sum_{l} \kappa^{-l} < \infty$ and $\sum_{l} l \kappa^{-l} < \infty$, because $\kappa > 1$.

Take $k \to \infty$. Since $p_k \to \infty$ and $u$ is continuous, we have

$$\sup_{y \in B(x_0, R/2)} u(y) \leq c_8 R^{c'(\beta-2)} (\int_B u^{2p})^{1/(2p)} =: c_8 R^{c'(\beta-2)} \Phi(2p, B).$$

Taking $u^{-1}$ instead of $u$, we have

$$\inf_{y \in B(x_0, R/2)} u(y) \geq c'_8 R^{-c'(\beta-2)} \Phi(-2p, B).$$

Now, let $\beta = 2$. (The second part of Moser’s argument; comparison between $\Phi(2p, B)$ and $\Phi(-2p, B)$.) Let $w := \log u$.

A) $\int_Q |\nabla w|^2 \leq c \mu(Q)/R^2$ (Prop 4.9 (a)).

B) (The John-Nirenberg ineq. (Exp. integrability of BMO functions).)

$Q_0$: a cube. If $f \in L^1(Q_0)$ satisfies $\fint_Q |f - f_Q| \leq 1$, $\forall Q \subset Q_0$ (such functions are called BMO fu.), then $\exists c, c' > 0$ s.t. $\fint_{Q_0} \exp(cf) \leq c'$. 


Using Schwarz, (PI(2)) and (A),

\[
\left( \int_Q |w - w_Q| \right)^2 \leq \int Q |w - w_Q|^2 \leq c\left( \frac{R^2}{\mu(Q)} \right) \int Q |\nabla w|^2 \leq C.
\]

So, applying (B), we obtain

\[
\int_B u^{q_0} = \int_B \exp(q_0 w) \leq c, \quad \int_B u^{-q_0} = \int_B \exp(-q_0 w) \leq c',
\]

for some \( q_0 > 0 \). Taking \( p = q_0/2 \), we conclude

\[
\sup_{B(x_0, R/2)} u \leq c_1 \Phi(q_0, B) \leq c_2 \Phi(-q_0, B) \leq c_3 \inf_{B(x_0, R/2)} u \Rightarrow \text{(EHI)}. \quad \Box
\]

**Remark.** If \( \beta > 2 \), one still obtains an \( L^\infty \) bound on \( u \) in \( B(x, R/2) \), but the constant now depends on \( R \), so that the final constant in the (EHI) will also depend on \( R \)!

As we see, the problem arises in the first (‘easy’) part of Moser’s argument. Instead of the linear cut-off functions, one needs cut-off functions such that the term \( R^{\beta-2} \) in the right hand side of (2.7) disappears.
3 Framework and main theorem

3.1 Framework

Metric measure spaces (MM)

\((X, d)\): connected loc. cpt compl. sep. metric space \((d: \text{geodesic})\)

\(\mu\): Borel measure on \(X\) s.t. \(0 < \mu(B) < \infty, \forall B \neq \emptyset\)

\(B(x, r) = \{y : d(x, y) < r\}, V(x, r) = \mu(B(x, r))\).

For simplicity, assume \(\text{diam } X = \infty\).

Metric measure Dirichlet spaces (MMD)

\((X, d, \mu)\): MM space, \(\mathcal{E}, \mathcal{F}\): regular, strong local Dirichlet form on \(L^2(X, \mu)\)

\(\Delta\): corresponding (non-positive) self-adjoint operator \(\mathcal{E}(h, g) = -\int \Delta h g \, d\mu\)

\(\{P_t\}\): corresponding semigroup

Assume that \(\mathcal{E}, \mathcal{F}\) is conservative (i.e. \(P_t1 = 1, \forall t > 0\)).
\[ \Gamma(f, g) \text{: signed measure} \]

\( \forall f \in \mathcal{F}_b, \exists 1 \Gamma(f, f) \text{: Borel measure (the energy measure) satisfying} \]

\[ \int_X gd\Gamma(f, f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \quad g \in \mathcal{F}_b. \]

(Rem: We take the quasi-continuous modification of \( g \in \mathcal{F}_b \) without writing \( \tilde{g} \).)

\[ \Gamma(f, g) := \frac{1}{2}(\Gamma(f + g, f + g) - \Gamma(f, f) - \Gamma(g, g)), \quad f, g \in \mathcal{F}. \]

Leibniz and chain rules: if \( f_1, \ldots, f_m, g, \varphi(f_1, \ldots, f_m) \in \mathcal{F}_b, \)

\[ d\Gamma(fg, h) = fd\Gamma(g, h) + gd\Gamma(f, h), \]

\[ d\Gamma(\varphi(f_1, \ldots, f_m), g) = \sum_{i=1}^{m} \frac{\partial\varphi}{\partial x_i}(f_1, \ldots, f_m)d\Gamma(f_i, g). \]

\( \bullet \) \( Y = (Y_t, t \geq 0, \mathbb{P}^x, x \in X) \text{: diffusion process associated with } (\mathcal{E}, \mathcal{F}) \text{ on } L^2(X, \mu). \)
**Examples.** 1. $X$: Riemannian manifold, $d$: Riem. metric, $\mu$: Riem. measure.

$\mathcal{C}$: $C^\infty$ functions on $X$ with compact support,

$$\mathcal{E}(f, f) = \int_X |\nabla f|^2 d\mu, \quad f \in \mathcal{C}.$$

$\mathcal{E}$: completion of $\mathcal{C}$ with respect to the norm $||f||_2 + \mathcal{E}(f, f)^{1/2}$, $d\Gamma(f, g) = \nabla f \cdot \nabla g \, d\mu$.

2. Cable system of a graph. $(G, E, \nu)$: a weighted graph

Define the cable system $G_C$ by replacing each edge of $G$ by a copy of $(0, 1)$.

$\mu$: measure on $G_C$ given by $d\mu(t) = \nu_{xy} \, dt$

$\mathcal{C}$: the functions in $C(G_C)$ which have compact support and are $C^1$ on each cable

$$\mathcal{E}(f, f) = \int_{G_C} |f'(t)|^2 d\mu(t).$$
3. \( D \): a domain in \( \mathbb{R}^d \) with a smooth boundary

\[ C := C^2_0(\overline{D}), \mu: \text{Lebesgue measure}, \text{and} \]

\[ \mathcal{E}(f, f) = \frac{1}{2} \int_D |\nabla f|^2 d\mu. \]

The associated diffusion \( Y \) is Brownian motion on \( D \) with normal reflection on \( \partial D \).
4. Diffusions on fractals. \( F \subset \mathbb{R}^d \): connected set with diameter 1  

Suppose \( \exists d \) geodesic metric on \( F \). \( \mu \): Hausdorff \( \alpha \)-measure on \( F \) (with respect to \( d \))  

Suppose that \( \mu(B(x, r)) \asymp r^\alpha, \quad x \in F, \ r > 0 \). Let  

\[
N_{\sigma, \infty}(f) := \sup_{0 < r \leq 1} r^{-\alpha - 2\sigma} \int_F \int_F 1_{B(y, r)}(x) |f(x) - f(y)|^2 d\mu(x) d\mu(y),
\]

\[
\Lambda_{2, \infty}^\sigma(F) := \{ u \in L^2(F, \mu) : N_{\sigma, \infty}(u) < \infty \}.
\]

There exist many fractals satisfying the above with a Dirichlet form \( \mathcal{E} \) on \( L^2(F, \mu) \) for which the domain \( \mathcal{F} \) of \( \mathcal{E} \) is given by \( \Lambda_{2, \infty}^{\beta/2}(F) \), and \( \mathcal{E}(f, f) \asymp N_{\sigma, \infty}(f) \).  

\( F = F_{SG} \): (compact) Sierpinski gasket, \( F_n \): set of vertices of triangles of side \( 2^{-n} \); \( x \sim y \iff x \) and \( y \) are in some triangle of side \( 2^{-n} \). Then, with \( \beta = \log 5/\log 2 \),

\[
\mathcal{E}(f, f) = c \lim_{n \to \infty} (5/3)^n \sum_{x \sim y} (f(x) - f(y))^2, \quad f \in \Lambda_{2, \infty}^{\beta/2}(F).
\]
Weighted graphs \((G, E)\): an infinite locally finite connected graph, \(x \sim y \iff (x, y) \in E\).

\(\{\mu_{xy}\}_{x,y \in G}\): edge weights (conductances) \(\mu_{xy} = \mu_{yx} \geq 0, \mu_{xy} > 0 \iff x \sim y\).

\(\mu\): \(\mu(A) := \sum_{x \in A} \mu_x\), where \(\mu_x := \sum_y \mu_{xy}\), \(d\): graph distance

\((G, \mu)\) has controlled weights \((p_0\)-condition) if there exists \(p_0 > 0\) such that

\[
\frac{\mu_{xy}}{\mu_x} \geq p_0, \quad \forall x \sim y \in G.
\]
The Laplacian and the Dirichlet form are defined on \((G, \mu)\) by

\[
\Delta f(x) = \frac{1}{\mu_x} \sum_y \mu_{xy}(f(y) - f(x)).
\]

\[
\mathcal{E}(f, g) = \frac{1}{2} \sum_x \sum_y (f(x) - f(y))(g(x) - g(y))\mu_{xy}, \quad f, g \in \mathcal{F} := L^2(G, \mu).
\]

If \(f \in \mathcal{F}\) we define the measure \(\Gamma_G(f, f)\) on \(G\) by setting

\[
\Gamma_G(f, f)(x) = \sum_{y \sim x} (f(x) - f(y))^2 \mu_{xy}.
\]

\(\bullet \ Y = \{Y_t\}_{t \geq 0}\): continuous time RW on \(G\) associated with \(\mathcal{E}\) and the measure \(\mu\).

\(Y\) is called the simple random walk on \(G\) if \(\mu_{xy} \equiv 1\) for \(x \sim y\).

\(Y\) waits at \(x\) for an exponential mean 1 random time and then moves to a neighbour \(y\) of \(x\) with probability proportional to \(\mu_{xy}\).

\(q_t(\cdot, \cdot)\): the transition density (heat kernel density) of \(Y\) with respect to \(\mu\);

\[
q_t(x, y) = \mathbb{P}^x(Y_t = y)/\mu_y. \quad (3.1)
\]
3.2 Inequalities

\((X, d, \mu, E)\): MMD space

Let \(\beta, \bar{\beta} \geq 2\) and

\[
\Psi(s) = \Psi_{\bar{\beta}, \beta}(s) = \begin{cases} 
  s^{\bar{\beta}} & \text{if } s \leq 1 \\
  s^{\beta} & \text{if } s > 1.
\end{cases}
\]

(3.1)

\(\Psi(s)\) will give the space/time scaling on the space \(X\).

(I) Volume doubling (VD):

\[
V(x, 2R) \leq c_1 V(x, R), \quad \forall x \in X, R \geq 0.
\]

(VD)

(VD) implies that \(\exists c_1, \alpha > 0\) s.t. if \(x, y \in X\) and \(0 < r < R\), then

\[
\frac{V(x, R)}{V(y, r)} \leq c_1 \left( \frac{d(x, y) + R}{r} \right)_{\alpha}.
\]

(9.1)

See subsection 9.1 for other consequences of (VD).
(II) Poincaré inequality (PI(Ψ)): \( \exists c_2 \) s.t. \( \forall B = B(x, R) \subset X \) and \( \forall f \in \mathcal{F} \),

\[
\int_B (f(x) - \overline{f}_B)^2 d\mu(x) \leq c_2 \Psi(R) \int_B d\Gamma(f, f),
\]

(PI(Ψ))

where \( \overline{f}_B = \mu(B)^{-1} \int_B f(x) d\mu(x) \).

(III) \( u \) is harmonic on a domain \( D \) if \( u \in \mathcal{F}_{loc} \) and \( \mathcal{E}(u, g) = 0 \ \forall g \in \mathcal{F} \) with support in \( D \).

Elliptic Harnack inequality (EHI): \( \exists c_3 > 0 \) s.t. \( \forall B(x, R), \forall u: \) non-negative harmonic function on \( B(x, R), \exists \) a quasi-continuous modification \( \tilde{u} \) of \( u \) that satisfies

\[
\sup_{B(x, R/2)} \tilde{u} \leq c_3 \inf_{B(x, R/2)} \tilde{u}.
\]

(EHI)

Remark. A standard argument (see subsec. 9.3), (EHI) implies \( \tilde{u} \) is Hölder continuous.
(IV) Let $Q = Q(x_0, T, R) = (0, 4T) \times B(x_0, 2R)$, 

$$Q_-(T, 2T) \times B(x_0, R) \quad \text{and} \quad Q_+ = (3T, 4T) \times B(x_0, R).$$

Parabolic Harnack inequality (\text{PHI}(\Psi)): \exists c_4 > 0 \text{ s.t. the following holds.}

Let $x_0 \in X$, $R > 0$, $T = \Psi(R)$, and $u = u(t, x) : Q \to \mathbb{R}_+$ satisfies $\frac{\partial u}{\partial t} = \Delta u$ in $Q$.

\exists a quasi-continuous modification $\tilde{u}$ of $u$ (for each $t$) that satisfies

$$\sup_{Q_-} \tilde{u} \leq c_4 \inf_{Q_+} \tilde{u}. \quad (\text{PHI}(\Psi))$$
(V) $A, B$: disjoint subsets of $X$. We define the effective resistance $R(A, B)$ by

$$R(A, B)^{-1} = \inf \left\{ \int_X d\Gamma(f, f) : f = 0 \text{ on } A \text{ and } f = 1 \text{ on } B, \ f \in \mathcal{F} \right\}. \quad (3.4)$$

(RES($\Psi$)): $\exists c_1, c_2 > 0$ s.t. $\forall x_0 \in X, \forall R \geq 0$,

$$c_1 \frac{\Psi(R)}{V(x_0, R)} \leq R(B(x_0, R), B(x_0, 2R^c)) \leq c_2 \frac{\Psi(R)}{V(x_0, R)}. \quad (\text{RES}(\Psi))$$

(VI) (CS($\Psi$)): $\exists \theta \in (0, 1], \exists c_1, c_2 > 0$ s.t. the following holds.

$\forall x_0 \in X, \forall R > 0, \exists$ a cut-off function $\varphi(= \varphi_{x_0, R})$ with the properties:

(a) $\varphi(x) \geq 1$ for $x \in B(x_0, R/2)$. \quad (b) $\varphi(x) = 0$ for $x \in B(x_0, R^c)$.

(c) $|\varphi(x) - \varphi(y)| \leq c_1 (d(x, y)/R)^\theta, \forall x, y \in X$.

(d) For any ball $B(x, s)$ with $0 < s \leq R$ and $f \in \mathcal{F}$,

$$\int_{B(x, s)} f^2 d\Gamma(\varphi, \varphi) \leq c_2 (s/R)^{2\theta} \left( \int_{B(x, 2s)} d\Gamma(f, f) + \Psi(s)^{-1} \int_{B(x, 2s)} f^2 d\mu \right). \quad (3.5)$$
(VII) For \( (t, r) \in (0, \infty) \times [0, \infty) \), let

\[
\Lambda_1 = \{(t, r) : t \leq 1 \lor r\}, \quad \Lambda_2 = \{(t, r) : t \geq 1 \lor r\}, \quad g_{\beta}(r, t) = \exp\left(-\left(\frac{r^{\beta}}{t}\right)^{1/(\beta-1)}\right).
\]

\((\text{HK}(\Psi))\): the heat kernel \( p_t(x, y) \), \( x, y \in X \) and \( t \in (0, \infty) \), exists and satisfies

\[
\frac{c_1g_{\beta}(c_2d(x, y), t)}{V(x, t^{1/\beta})} \leq p_t(x, y) \leq \frac{c_3g_{\beta}(c_4d(x, y), t)}{V(x, t^{1/\beta})}, \quad \forall (t, d(x, y)) \in \Lambda_1,
\]

\[
\frac{c_1g_{\beta}(c_2d(x, y), t)}{V(x, t^{1/\beta})} \leq p_t(x, y) \leq \frac{c_3g_{\beta}(c_4d(x, y), t)}{V(x, t^{1/\beta})}, \quad \forall (t, d(x, y)) \in \Lambda_2.
\]

Let \( h(r) := \Psi(r)/r \). Then, \((\text{HK}(\Psi))\) is equivalent to

\[
\frac{c_1}{V(x, \Psi^{-1}(t))}\exp\left(-\frac{c_2d(x, y)}{h^{-1}(t/d(x, y))}\right) \leq p_t(x, y) \leq \frac{c_3}{V(x, \Psi^{-1}(t))}\exp\left(-\frac{c_4d(x, y)}{h^{-1}(t/d(x, y))}\right),
\]

\( \forall x, y \in X \) and \( t \in (0, \infty) \) where we let \( d(x, y)/h^{-1}(t/d(x, y)) = 0 \) if \( d(x, y) = 0 \).

\((LHK(\Psi))\): the first inequality of (3.8), \((UHK(\Psi))\): the second inequality of (3.8).
(VIII) (VD)$_{\text{loc}}$: (VD) holds for $x \in X$, $0 < R \leq 1$.

(PI($\bar{\beta}$))$_{\text{loc}}$, (EHI)$_{\text{loc}}$, (CS($\bar{\beta}$))$_{\text{loc}}$, (PHI($\bar{\beta}$))$_{\text{loc}}$ – define similarly.

(HK($\bar{\beta}$))$_{\text{loc}}$: We require the bounds only for $t \in (0, 1)$ – so only (3.6) is involved.

(IX) (a) We call $\varphi$ a cut-off function for $A_1 \subset A_2$ if $\varphi = 1$ on $A_1$ and is zero on $A_2^c$.

(b) (PI)$_{\text{loc}}$: $\forall c_1 > 0$, $\exists c_2 > 0$ s.t.

$$\int_B (f(x) - \bar{f}_B)^2 d\mu(x) \leq c_2 \int_B d\Gamma(f, f)$$

for any ball $B = B(x, c_1) \subset X$ and $f \in \mathcal{F}$.

(c) (CC)$_{\text{loc}}$: $\forall x_0 \in X$, $\exists$ a cut-off function $\varphi(= \varphi_{x_0})$ for $B(x_0, 1/2) \subset B(x_0, 1)$ s.t.

$$\int_{B(x_0, 1)} d\Gamma(\varphi, \varphi) \leq c_3 V(x_0, 1).$$

Remark. (PI($\bar{\beta}$))$_{\text{loc}}$ for $\bar{\beta} \geq 2 \Rightarrow$ (PI)$_{\text{loc}}$, (CS($\bar{\beta}$))$_{\text{loc}}$ for $\bar{\beta} > 0 \Rightarrow$ (CC)$_{\text{loc}}$.

Weighted graphs with contr. weights $\Rightarrow$ (PI)$_{\text{loc}}$, (CC)$_{\text{loc}}$, (PI($\bar{\beta}$))$_{\text{loc}}$, (CS($\bar{\beta}$))$_{\text{loc}}$ for $\bar{\beta} \geq 2$. 
(X) (E(Ψ)): \( \forall x_0 \in X, \forall R \geq 0, \)
\[
c_1 \Psi(R) \leq \mathbb{E}^{x_0}[\tau_{B(x_0,R)}] \leq c_2 \Psi(R),
\]
\( \text{(E(Ψ))} \)

where \( \tau_A = \inf\{t \geq 0 : Y_t \notin A\} \).

(E(Ψ)\(_\geq\)) : the first inequality in (E(Ψ)), (E(Ψ)\(_\leq\)) : the second.
We summarize the conditions we have introduced:

(VD) Volume doubling
(PI(Ψ)) Poincaré inequality
(EHI) Elliptic Harnack inequality
(PHI(Ψ)) Parabolic Harnack inequality
(RES(Ψ)) Resistance exponent
(CS(Ψ)) Cut-off Sobolev inequality
(CC) Controlled cut-off functions
(HK(Ψ)) Heat kernel estimates
(E(Ψ)) Walk dimension

When \( \bar{\beta} = \beta \), we would write \((\ldots(\beta))\) instead of \((\ldots(\Psi))\), for instance \((\text{PI}(\beta))\) instead of \((\text{PI}(\Psi))\).
3.3 Main Theorems

**Theorem 3.1** $X$: MMD space or infinite con. weighted graph with contr. weights.

The following are equivalent:

(a) $X$ satisfies ($\text{PHI}(\Psi)$).

(b) $X$ satisfies ($\text{HK}(\Psi)$).

(c) $X$ satisfies ($\text{VD}$), ($\text{PI}(\Psi)$) and ($\text{CS}(\Psi)$).

(d) $X$ satisfies ($\text{VD}$), ($\text{EHI}$) and ($\text{RES}(\Psi)$).

(e) $X$ satisfies ($\text{VD}$), ($\text{EHI}$) and ($\text{E}(\Psi)$).
Stability We discuss two kinds of stability of \((PHI(Ψ))\).

**Definition 3.2** A property \(P\) is **stable under bounded perturbation** if whenever \(P\) holds for \((\mathcal{E}^{(1)}, \mathcal{F})\), then it holds for \((\mathcal{E}^{(2)}, \mathcal{F})\), provided

\[
c_1 \mathcal{E}^{(1)}(f, f) \leq \mathcal{E}^{(2)}(f, f) \leq c_2 \mathcal{E}^{(1)}(f, f), \quad \text{for all } f \in \mathcal{F}. \tag{3.9}
\]

**Lemma 3.3** (Le Jan [64]) \(X: MMD\) space. Suppose \((\mathcal{E}^{(i)}, \mathcal{F}), i = 1, 2\) are str. loc. reg. \(D\)-forms that satisfy (3.9). Then the energy measures \(Γ^{(i)}\) satisfy

\[
c_1 dΓ^{(1)}(f, f) \leq dΓ^{(2)}(f, f) \leq c_2 dΓ^{(1)}(f, f), \quad \text{for all } f \in \mathcal{F}.
\]

By this lemma, \(PI(Ψ)\) and \(CS(Ψ)\) are stable under bounded perturbations.

**Theorem 3.4** Let \(X\) be a \(MMD\) space. Then \((PHI(Ψ))\) and \((HK(Ψ))\) are stable under bounded perturbations.
Rough isometry (M. Kanai in [52.53])

Definition 3.5 \((X_i, d_i, \mu_i), i = 1, 2: a MM space or a weighted graph.\)

\(\varphi : X_1 \to X_2\) is a rough isometry if \(\exists c_1 > 0, c_2, c_3 > 1\) s.t.

\[
X_2 = \bigcup_{x \in X_1} B_{d_2}(\varphi(x), c_1),
\]

\[
c_2^{-1}(d_1(x, y) - c_1) \leq d_2(\varphi(x), \varphi(y)) \leq c_2(d_1(x, y) + c_1),
\]

\[
c_3^{-1}\mu_1(B_{d_1}(x, c_1)) \leq \mu_2(B_{d_2}(\varphi(x), c_1)) \leq c_3\mu_1(B_{d_1}(x, c_1)).
\]

If \(\exists\) a rough isometry between two spaces they are said to be roughly isometric.
Stability of \((\text{PHI}(\Psi))\) under rough isometries.

**Theorem 3.6** \(X_i\): a MMD space satisfying \((VD)_{\text{loc}} + (PI)_{\text{loc}}\) or a weighted graph with contr. weights. Suppose \(\exists \varphi : X_1 \rightarrow X_2\) rough isom. Let \(\Psi_i(s) = s \bar{\beta}_i 1_{\{s \leq 1\}} + s^\beta 1_{\{s \geq 1\}}\).

(a) Suppose that \(X_2\) satisfies \((PI(\bar{\beta}_2))_{\text{loc}}\).

If \(X_1\) satisfies \((VD)\), \((CC)_{\text{loc}}\) and \((PI(\Psi_1))\) then \(X_2\) satisfies \((VD)\) and \((PI(\Psi_2))\).

(b) Suppose that \(X_2\) satisfies \((CS(\bar{\beta}_2))_{\text{loc}}\).

If \(X_1\) satisfies \((VD)\) and \((CS(\Psi_1))\) then \(X_2\) satisfies \((VD)\) and \((CS(\Psi_2))\).

So, \((\text{PHI}(\Psi))\) is stable under rough isom., given suitable local reg. of the two spaces.

**Examples**

1) S.G. graphs in the last page satisfies \((\text{PHI}(\log 5/\log 2))\) for \(R \geq 1\).

2) Fractal-like manifold in P 21: 2-dimensional Riemannian manifold

\[ \mathcal{L} = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) \]

on the manifold which satisfies the uniform elliptic condition enjoys \((HK(2))\) for \(t \leq 1 \lor d(x, y)\) and \((HK(\log 5/\log 2))\) for \(t \geq 1 \lor d(x, y)\).
4 Proof of Theorem 3.1

Recall that $h(r) = \Psi(r)/r$. We give some inequalities.

\[
p_t(x, y) \leq \frac{C_1}{V(x, \Psi^{-1}(t))}, \quad \forall x, y \in X, t > 0. \quad (DUHK(\Psi))
\]

\[
P^x(\tau_{B(x,r)} \leq t) \leq C_2 \exp \left( - \frac{C_3 r}{h^{-1}(t/r)} \right), \quad \forall x \in X, r, t > 0. \quad (ELD(\Psi))
\]

\[
p_t(x, x) \geq \frac{C_4}{V(x, \Psi^{-1}(t))}, \quad \forall x \in X, t > 0. \quad (DLHK(\Psi))
\]

\[
p_t(x, y) \geq \frac{C_5}{V(x, \Psi^{-1}(t))}, \quad \forall x, y \in X, t > 0 \text{ with } \Psi(d(x,y)) \leq C_6 t. \quad (NLHK(\Psi))
\]

4.1 Proof of $(e) \Rightarrow (b)$

For simplicity, we assume the existence of the (cont.) heat kernel and prove the following;

\[
(VD) + (DUHK(\Psi)) + (EHI) + (E(\Psi)) \Rightarrow (HK(\Psi)).
\]
**Step 1: Proof of** $E(\Psi) \Rightarrow ELD(\Psi)$.  

**Lemma 4.2** (Barlow-Bass) \{\(\xi_i\): non-negative random variables. 

*Suppose \(\exists 0 < p < 1, a > 0 \text{ s.t.}

\[
P(\xi_i \leq t | \sigma(\xi_1, \cdots, \xi_{i-1})) \leq p + at, \quad \forall t > 0.
\]

\[
\Rightarrow \log P(\sum_{i=1}^{n} \xi_i \leq t) \leq 2\left(\frac{at}{p}\right)^{1/2} - n \log \frac{1}{p}.
\]

**Proof.** Let \(\eta\) be a r.v. with distri. \(P(\eta \leq t) = (p + at) \land 1\). Then,

\[
E(e^{-\lambda \xi_i} | \sigma(\xi_1, \cdots, \xi_{i-1})) \leq Ee^{-\lambda \eta} = p + \int_{0}^{(1-p)/a} e^{-\lambda t} \, dt \leq p + a \lambda^{-1}.
\]

So,

\[
P(\sum_{i=1}^{n} \xi_i \leq t) = P(e^{-\lambda \sum_{i=1}^{n} \xi_i} \geq e^{-\lambda t}) \leq e^{\lambda t} Ee^{-\lambda \sum_{i=1}^{n} \xi_i}
\]

\[
\leq e^{\lambda t}(p + a \lambda^{-1})^n \leq p^n \exp(\lambda t + \frac{an}{\lambda p}).
\]

The result follows on setting \(\lambda = (an/(pt))^{1/2}\). \(\Box\)
Proof of \((E(\Psi)) \Rightarrow (ELD(\Psi))\). We first prove that \(0 < \exists c_1 < 1, \exists c_2 > 0\) s.t.

\[
P^x(\tau_{B(x,r)} \leq s) \leq 1 - c_1 + c_2 s/h(r) \quad \text{for all} \ x \in X, \ s \geq 0. \quad (4.1)
\]

Indeed, by the Markov property, for each \(x \in X\) we have

\[
E^x \tau_{B(x,r)} \leq s + E^x[1_{\{\tau_{B(x,r)}>s\}}E^{X_s}\tau_{B(x,r)}] \leq s + E^x[1_{\{\tau_{B(x,r)}>s\}}E^{X_s}\tau_{B(X_s,2r)}]. \quad (4.2)
\]

Applying \((E(\Psi))\) and using the doubling property of \(h,\)

\[
c_3 h(r) \leq s + c_4 h(2r) P^x(\tau_{B(x,r)} > s) = s + c_5 h(r)(1 - P^x(\tau_{B(x,r)} \leq s)). \quad (4.3)
\]

Rearranging gives (4.1).

Next, let \(l \geq 1, b = r/l\), and define stopping times \(\sigma_i, \ i \geq 0\) by

\[
\sigma_0 = 0, \quad \sigma_{i+1} = \inf\{t \geq \sigma_i : d(X_{\sigma_i}, X_t) \geq b\}.
\]

Let \(\xi_i := \sigma_i - \sigma_{i-1}, \mathcal{F}_t\): the filtration generated by \(\{X_s : s \leq t\}\), \(\mathcal{G}_m := \mathcal{F}_{\sigma_m}\).
We have by (4.1)

$$P^x(\xi_{i+1} \leq t|G_i) = P^{X_{\sigma_i}}(\tau_B(X_{\sigma_i}, b) \leq t) \leq p + c_2 t/h(b),$$

where $0 < p < 1$. As $d(X_{\sigma_i}, X_{\sigma_{i+1}}) = b$, we have $d(X_0, X_{\sigma_l}) \leq r$, so that

$$\sigma_l = \sum_{i=1}^{l} \xi_i \leq \tau_B(X_0, r).$$

So, by Lemma 4.2,

$$\log P^x(\tau_B(x,r) \leq t) \leq 2p^{-1/2}(\frac{c_2lt}{h(r/l)})^{1/2} - l \log(1/p) = c_6\left(\frac{lt}{h(r/l)}\right)^{1/2} - c_7 l.$$

Now take $l_0 \in \mathbb{N}$ the largest integer $l$ that satisfies

$$c_7 l/2 > c_6\left(\frac{lt}{h(r/l)}\right)^{1/2}.$$ (4.4)

This is equivalent to $r/l > h^{-1}(c_8 t/r)$ where $c_8 = 4c_6^2/c_7^2$. Note: if $r \leq h^{-1}(c_8 t/r)$, then (ELD($\Psi$)) holds by taking $c_1 > 0$ large. So assume (4.4) holds for small $l \in \mathbb{N}$. Then,

$$l_0 < \frac{r}{h^{-1}(c_8 t/r)} \leq l_0 + 1, \quad \text{and} \quad \log P^x(\tau_B(x,r) \leq t) \leq -c_7 l_0/2.$$

We thus obtain (ELD($\Psi$)).
\textbf{Step 2: Proof of} \((VD) + (DUHK(\Psi)) + (ELD(\Psi)) \Rightarrow (UHK(\Psi)).\)

Fix \(x \neq y\) and \(t\) and let \(r := d(x, y), \epsilon < r/6.\)

Let \(\bar{\mu}_x = \mu|_{B_\epsilon(x)}, A_1 = \{z \in X : d(z, x) \leq d(z, y)\}\) and \(A_2 = X - A_1.\) Then

\[
P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y)) = P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y), Y_{\frac{t}{2}} \in A_1)
+ P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y), Y_{\frac{t}{2}} \in A_2) \equiv I_1 + I_2.
\]

Now, \(I_2 \leq P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y), \tau < \frac{t}{2}) = E^{\bar{\mu}_x}(1_{\tau < \frac{t}{2}} \int_{B_\epsilon(y)} p_{t-\tau}(Y_\tau, w) d\mu(w))\)
\[ \leq P^{\bar{\mu}_x}(\tau < t/2) \sup_{z \in B(x,r/2) \cup B_\epsilon(y)} p_{t/2}(z, z) \mu(B_\epsilon(y)), \text{ where } \tau := \tau_{B(x,r/2)}. \]

By (\textit{ELD}(\Psi)), we obtain
\[
I_2 \leq c_1 \left( \sup_{z \in B(x,r/2) \cup B_\epsilon(y)} p_{t/2}(z, z) \right) \mu(B_\epsilon(x)) \mu(B_\epsilon(y)) \exp \left( - \frac{c_2 r}{h^{-1}(t/r)} \right).
\]

For \( I_1 \), by the symmetry of \( p_t(x, y) \),
\[
P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y), Y_{t/2} \in A_1) = P^{\bar{\mu}_y}(Y_t \in B_\epsilon(x), Y_{t/2} \in A_1)
\]
which is bounded in exactly the same way as \( I_2 \), where \( x \) and \( y \) are changed. So,
\[
P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y)) \leq c_1 \left( \sup_{z \in B(x,r/2) \cup B(y,r/2)} p_{t/2}(z, z) \right) \mu(B_\epsilon(x)) \mu(B_\epsilon(y)) \exp \left( - \frac{c_2 r}{h^{-1}(t/r)} \right).
\]

By (\textit{DUHK}(\Psi)) and (\textit{VD}),
\[
\sup_{z \in B(x,r/2) \cup B(y,r/2)} p_{t/2}(z, z) \leq \frac{c_3}{V(x, \Psi^{-1}(t))} \left( \frac{r + \Psi^{-1}(t)}{\Psi^{-1}(t)} \right)^\alpha.
\]

If \( \Psi(r) \leq t \), this is bounded by \( c_4 V(x, \Psi^{-1}(t))^{-1} \). If \( \Psi(r) > t \), then, \( \forall \epsilon > 0, \exists c_\epsilon > 0 \) s.t.
\[
\left( \frac{r + \Psi^{-1}(t)}{\Psi^{-1}(t)} \right)^\alpha \exp \left( - \frac{\epsilon r}{h^{-1}(t/r)} \right) \leq c_\epsilon.
\]
This is because $M = r/\Psi^{-1}(t) \Leftrightarrow h(r/M) = tM/r \Rightarrow M < r/h^{-1}(t/r)$. In any case,

$$P^{\mu_x}(Y_t \in B_{\epsilon}(y)) \leq \frac{c_5}{V(x, \Psi^{-1}(t))} \mu(B_{\epsilon}(x)) \mu(B_{\epsilon}(y)) \exp(-\frac{c_6 r}{h^{-1}(t/r)}).$$

Dividing both sides by $\mu(B_{\epsilon}(x)), \mu(B_{\epsilon}(y))$ and using cont. of $p_t(x, y)$ gives $(UHK(\Psi))$.\end{proof}

**Step 3: Proof of $(VD) + (ELD(\Psi)) \Rightarrow (DLHK(\Psi))$.** Using (4.1),

$$P^x(Y_t \notin B(x, r)) \leq P(\tau_{B(x,r)} \leq t) \leq c_1 \exp\left(-\frac{c_2 r}{h^{-1}(t/r)}\right).$$

Hence, by choosing $r$ s.t. $c_3 \Psi(r) < t < c_4 \Psi(r)$ for $\exists c_3, c_4 > 0$, we have

$$P^x(Y_t \notin B(x, r)) \leq c_5 < 1.$$ 

Thus $P^x(Y_t \in B(x, r)) \geq 1 - c_5 > 0$. By Cauchy-Schwarz,

$$(1 - c_5)^2 \leq P^x(Y_t \in B(x, r))^2 = \left(\int_{B(x,r)} p_t(x,z) d\mu(z)\right)^2 \leq V(x,r) p_{2t}(x,x).$$

Now, using the lower bound of our choice of $t$ and (VD), we obtain the result.\end{proof}
**Step 4: Proof of (VD) + (DUHK(Ψ)) + (EHI) + (E(Ψ)) ⇒ (NLHK(Ψ)).**

(Sketch) Fix $x \in X$, $t > 0$ and set $R := \Psi^{-1}(t/\varepsilon)$ ($\varepsilon > 0$ will be chosen later).

- Similarly to Step 3, if $\varepsilon > c_2$, we obtain

  $$p_t^B(x, x) \geq \frac{c_1}{V(x, \Psi^{-1}(t))}, \quad \text{where } B := B(x, R). \quad (4.6)$$

- Set $f(y) := \partial_t p_t^B(x, y)$. Applying Proposition 9.9 (time derivative) to $p_t^B$,

  $$|f(y)| \leq \frac{2}{t} \sqrt{p_t^B(x, x)p_t^B(y, y)} \leq \frac{2}{t} \sqrt{p_{t/2}(x, x)p_{t/2}(y, y)}, \quad y \in B.$$

By (DUHK(Ψ)) and (VD), $\exists \alpha, \alpha' > 0$ s.t.

$$p_{t/2}(y, y) \leq \frac{c_1}{V(y, \Psi^{-1}(t))} \leq \frac{c_1}{V(x, \Psi^{-1}(t))}(1 + \frac{d(x, y)}{\Psi^{-1}(t)})^\alpha \leq \frac{c_1(1 + \varepsilon^{-\alpha'})^\alpha}{V(x, \Psi^{-1}(t))}, \quad \forall y \in B.$$

Hence, by (VD), we have

$$|f(y)| \leq \frac{c_2(1 + \varepsilon^{-\alpha'})^\alpha/2}{tV(x, \Psi^{-1}(t))}, \quad \forall y \in B. \quad (4.7)$$
• Define \( u(y) = p_t^B(x, y) \). Then, \( \partial_t u = \Delta_B u \), so \( u = -G^B(\partial_t u) = G^B f \), where \( G^B = (-\Delta_B)^{-1} \) is the Green operator. Let \( \gamma > \alpha\alpha' / 2 \) and apply Proposition 9.6 (Oscillation inequality, (EHI) is used here) with \( \varepsilon^{\gamma+1} \) instead of \( \varepsilon \). Then, \( \exists \delta > 0 \) s.t. \( 0 < \forall r < R \),

\[
\text{Osc}_{B(x, \delta r)} u \leq 2(\bar{E}(x, r) + \varepsilon^{\gamma+1} \bar{E}(x, R) \| f \|_{\infty}),
\]

where \( \bar{E}(x, r) := \sup_z E^z[\tau_{B(x, r)}] \). By (E(\( \Psi \))) and (4.7), we obtain

\[
\text{Osc}_{B(x, \delta r)} u \leq \frac{\Psi(r) + \varepsilon^{\gamma+1} \Psi(R)}{t} \cdot \frac{c_4(1 + \varepsilon^{-\alpha'})^{\alpha/2}}{V(x, \Psi^{-1}(t))}.
\]

• By definition of \( R \), \( \frac{\varepsilon^{\gamma+1} \Psi(R)}{t} = \varepsilon^{\gamma} \).

Choose \( r \) by the eq. \( \Psi(r) = \varepsilon^{\gamma+1} \Psi(R) \), (so \( r \geq \delta' R \) for \( \exists \delta' > 0 \)). Hence,

\[
\text{Osc}_{y \in B(x, \delta' R)} p_t^B(x, y) \leq \text{Osc}_{B(x, \delta r)} u \leq \frac{2c_4\varepsilon^{\gamma}(1 + \varepsilon^{-\alpha'})^{\alpha/2}}{V(x, \Psi^{-1}(t))} \to 0 \quad (\text{as } \varepsilon \to 0). \quad (4.8)
\]

Choosing \( \varepsilon \) small enough and combining (4.8) with (4.6), we conclude that

\[
p_t(x, y) \geq p_t^B(x, y) \geq \frac{c_1^{1/2}}{V(x, \Psi^{-1}(t))}, \quad \forall y \in B(x, \delta' R). \quad \Box
\]
Step 5: Proof of $(VD) + (NLHK(\Psi)) \Rightarrow (LHK(\Psi))$. Chain argument.

(Sketch) Let \( \varepsilon = \varepsilon(t, d(x, y)) > 0 \) be s.t.

\[
c_1 t \leq h(\varepsilon) d(x, y) \leq c_2 t.
\] (4.9)

Due to \((NLHK(\Psi))\), we should only consider the case \( \Psi(d(x, y)) > C_6 t \), which means \( \varepsilon < c_3 d(x, y) \) for \( \exists c_3 > 0 \). Take \( N \in \mathbb{N} \) s.t. \( N \asymp d(x, y)/\varepsilon \).

Let \( \{x_i\}_{i=0}^N \) be such that \( x_0 = x, x_N = y \) and \( d(x_i, x_{i+1}) \leq \varepsilon \) for \( i = 0, 1, \cdots, N - 1 \).
(Such a seq. exists by the choice of $N$ and by the fact that \( d \) is a geodesic met.) Then,

\[
p_t(x, y) = \int_X \cdots \int_X p_{t/N}(x, z_1)p_{t/N}(z_1, z_2) \cdots p_{t/N}(z_{N-1}, y) d\mu(z_1) \cdots d\mu(z_{N-1})
\]

\[
\geq \int_{B(x_1, \varepsilon)} \cdots \int_{B(x_{N-1}, \varepsilon)} p_{t/N}(x, z_1) \cdots p_{t/N}(z_{N-1}, y) d\mu(z_1) \cdots d\mu(z_{N-1}).
\]

Clearly \( d(z_i, z_{i+1}) \leq 3\varepsilon \). Now, by the choice of \( \varepsilon \) and \( N \), we have \( \varepsilon \simeq \Psi^{-1}(\frac{t}{N}). \)

This together with \((NLHK(\Psi))\) and \((VD)\) and \((4.12)\), we have

\[
p_{t/N}(z_i, z_{i+1}) \geq \frac{c_6}{V(z_i, \Psi^{-1}(t/N))} \geq \frac{c_7}{V(x_i, \Psi^{-1}(t/N))} \geq \frac{c_8}{V(x_i, \varepsilon)}.
\]

So,

\[
p_t(x, y) \geq \frac{c_8}{V(x, \Psi^{-1}(t/N))} \prod_{i=1}^{N-1} \frac{c_8 \cdot V(x_i, \varepsilon)}{V(x_i, \varepsilon)} \geq \frac{c_8^{-N}}{V(x, \Psi^{-1}(t/N))} \geq \frac{\exp(-c_9N)}{V(x, \Psi^{-1}(t))}.
\]

On the other hand, by \((4.9)\) we have \( h^{-1}(t/d(x, y)) \leq c_{11}\varepsilon \), so that

\[
N \simeq \frac{d(x, y)}{\varepsilon} \leq c_{11} \frac{d(x, y)}{h^{-1}(t/d(x, y))}.
\]

We thus obtain \((LHK(\Psi))\). \(\square\)
4.2 Proof of $(c) \Rightarrow (d)$

**Lemma 4.4**

$$(VD) + (PI(\Psi)) + (CS(\Psi)) \Rightarrow (RES(\Psi)).$$

**Proof.** $(VD) + (PI(\Psi)) \Rightarrow (RES(\Psi))$

$f$: attains the minimum in the variational formula of $R(B(x_0, R), B(x_0, 2R)^c)$.

$$\overline{f} := \int_{B(x_0,3R)} f d\mu / V(x_0, 3R).$$

Choose $y_0$ s.t. $d(x_0, y_0) = 5R/2$.

By (9.1) (due to (VD)), $V(y_0, R/2) \geq c_2 V(x_0, R)$.

Depending on $\overline{f} \geq 1/2$ or $\overline{f} < 1/2$, $|f - \overline{f}| \geq 1/2$ on either $B(x_0, R)$ or $B(y_0, R/2)$, and then using (PI(\Psi)) we have

$$V(x_0, R) \leq c_3 \int_{B(x_0,3R)} (f - \overline{f})^2 d\mu \leq c_4 \Psi(R) \int_{B(x_0,3R)} d\Gamma(f, f)$$

$$= c_4 \Psi(R) R(B(x_0, R), B(x_0, 2R)^c)^{-1}. \quad \Box$$
\[(VD) + (CS(\Psi)) \Rightarrow (RES(\Psi)) \leq \]

\[\varphi: \text{a cut-off function for } B(x_0, R) \text{ given by } (CS(\Psi)).\]

Taking \(f \equiv 1, I = B(x_0, R)\) and \(I^* = B(x_0, 2R)\) in (3.5), we obtain

\[R(B(x_0, R/2), B(x_0, R))^{-1} \leq \int_I d\Gamma(\varphi, \varphi) \leq c_6 \Psi(R)^{-1} \int_{I^*} d\mu \leq c_7 \frac{V(x_0, R)}{\Psi(R)}.\]

\(\square\)

The rest is to show \((VD) + (PI(\Psi)) + (CS(\Psi)) \Rightarrow (EHI)\).

Recall the Moser’s argument in subsection 2.4. The crucial loss for the case \(\beta \neq 2\) is in using the bound (2.6); one needs a cutoff function \(\varphi\) such that the final term in (2.7) can be controlled by a term of order \(R^{-\beta}\).

Fix \(x \in X, R > 0. \varphi = \varphi_{x, R}: \text{the cut-off function in } (CS(\Psi)).\)

Define the measure \(\gamma = \gamma_{x, R}\) by

\[d\gamma = d\mu + \Psi(R)d\Gamma(\varphi, \varphi).\]
The first step in the argument is to use \((CS(\Psi))\) to obtain a weighted Sobolev inequality.

**Proposition 4.5** Let \(s \leq R\) and \(J \subset B(x_0, R)\) be a finite union of balls of radius \(s\).

\[
\exists \kappa > 1, c_1 > 0 \text{ s.t.} \\
(\mu(J)^{-1} \int_J |f|^{2\kappa} d\gamma)^{1/\kappa} \leq c_1 (\Psi(R)\mu(J)^{-1} \int_J d\Gamma(f, f) + (s/R)^{-2\theta}\mu(J)^{-1} \int_J f^2 d\gamma),
\]

where \(J^s = \{y : d(y, J) \leq s\}\).

(Strategy of the proof): Prove weighted Poincaré ineq. first, and then prove the weighted Nash ineq., which deduce the desired inequality. See subsection 9.8 for details.

The next result is the generalization of Lemma 4 of [69] to the case of a MMD space.

**Lemma 4.6** Let \(D\) be a domain in \(X\), let \(u\) be positive and harmonic in \(D\), \(v = u^k\), where \(k \in \mathbb{R}, k \neq \frac{1}{2}\), and let \(\eta\) be supported in \(D\). Suppose \(\int_D d\Gamma(\eta, \eta) < \infty\), then

\[
\int_D \eta^2 d\Gamma(v, v) \leq \left(\frac{2k}{2k-1}\right)^2 \int_D v^2 d\Gamma(\eta, \eta).
\]
$u$: harmonic and nonnegative in $B(x_0, 4R)$. (W.l.o.g. suppose $u$ is strictly positive.)

**Remark.** We do not initially have any a priori continuity for $u$.

**Proposition 4.7** Let $v$ be either $u$ or $u^{-1}$.

$\exists c_1$ s.t. if $B(x, 2r) \subset B(x_0, 4R)$ and $0 < q < 2$, then

$$\text{ess sup}_{B(x, r/2)} v^{2q} \leq c_1 V(x, 2r)^{-1} \int_{B(x, 2r)} (\Psi(r)d\Gamma(v^q, v^q) + v^{2q}d\mu).$$

**Proof.** (Sketch) $\varphi_0$: cut-off function given by (CS($\Psi$)) for $B(x, r)$. $h_n := 1 - 2^{-n}$, and

$\varphi_k(x) := (\varphi_0(x) - h_k)^+$, $d\gamma_0 := d\mu + \Psi(r)d\Gamma(\varphi_0, \varphi_0)$, $A_k := \{x : \varphi_0(x) > h_k\}$.

Then, $\mu(A_k) \asymp V(x, r) =: V.$

[Hölder cond. on $\varphi_0$ by (CS($\Psi$))] $\Rightarrow$ [if $x \in A_{k+1}, y \in A_k^c$, then $d(x, y) \geq c_3 r 2^{-k/\theta}] \Rightarrow$

$[\varphi_k > c_4 2^{-k}$ on $A_{k+1}^{s_k} =: A_k' \text{ where } s_k = \frac{1}{2} c_3 r 2^{-k/\theta}].$ By Proposition 4.5 with $f = v^p$, 

$$(V^{-1} \int_{A_k^c} f^{2k} d\gamma_0)^{1/k} \leq c_6 V^{-1} \int_{A_{k+1}'} d\Gamma(f, f) + 2^{2k} \int_{A_k} f^2 d\gamma_0.$$

By Lemma 4.6, we have the ‘converse to the Poincaré inequality’ for $f = v^p$;

$$
\Psi(r) \int_{A_{k+1}'} d\Gamma(f, f) \leq \Psi(r)(c_7 2^{-k})^{-2} \int_{A_{k+1}'} \varphi_k^2 d\Gamma(f, f) \leq c_8 2^{2k} \Psi(r) \int_{A_k} \varphi_k^2 d\Gamma(f, f)
$$

$$
\leq c_9 2^{2k} \Psi(r) \left( \frac{2p}{2p-1} \right)^2 \int_{A_k} f^2 d\Gamma(\varphi_k, \varphi_k) \leq c_{10} 2^{2k} \left( \frac{2p}{2p-1} \right)^2 \int_{A_k} f^2 d\gamma_0.
$$

So,

$$
(V^{-1} \int_{A_{k+1}} f^{2\kappa} d\gamma_0)^{1/\kappa} \leq c_{11} \left( \frac{2p}{2p-1} \right)^2 2^{2k} V^{-1} \int_{A_k} f^2 d\gamma_0. \quad (4.21)
$$

Now, argument similar to the first part of Moser’s argument.

$q_0 := q' \kappa^{-i}$ for $\exists i$, $p_n := 2q_0 \kappa^n$, and $\Psi_k = [\mu(A_k)^{-1} \int_{A_k} v^{p_k} d\gamma_0]^{1/p_k}$. 
Note that \( p_{k+1}/2\kappa = p_k/2 \). Applying (4.21) to \( f = v^{p_{k+1}/(2\kappa)} = v^{p_k/2} \) we have

\[
\Psi_{p_{k+1}/\kappa} = (\mu(A_{k+1})^{-1} \int_{A_{k+1}} v^{p_{k+1}} d\gamma_0)^{1/\kappa} \leq c_{13} 2^{2k} (\mu(A_k)^{-1} \int_{A_k} v^{p_k} d\gamma_0) = c_{13} 2^{2k} \Psi_k^{p_k}.
\]

Hence,

\[
\log \Psi_{m} \leq \log \Psi_{0} + \sum_{k=1}^{m} p_k^{-1} \log(c_{13} 2^{2k}).
\]

As the sum in (4.22) converges and \( \text{ess sup} B(x,r/2) v \leq \lim sup_{m \to \infty} \Psi_m \),

\[
\text{ess sup} B(x,r/2) v \leq c_{14} \Psi_{0} \leq c_{15} (V^{-1} \int_{B(x,r)} v^{2q_0} d\gamma_0)^{1/(2q_0)}.
\]

Let \( q \in (0, 2) \); we can take \( q_0 = q'\kappa^{-i} < q \). By the weighted Poincaré ineq. (Prop 9.20),

\[
\left( \int_{B(x,r)} \frac{v^{2q_0}}{V} d\gamma_0 \right)^{q/q_0} \leq c_{16} \int_{B(x,r)} \frac{v^{2q}}{V} d\gamma_0 \leq c_{18} V^{-1} \int_{B(x,2r)} (\Psi(r) d\Gamma(v^q, v^q) + v^{2q} d\mu).
\]

So, we conclude

\[
\text{ess sup} B(x,r/2) v^{2q} \leq c_{18} V(x, 2r)^{-1} \int_{B(x,2r)} (\Psi(r) d\Gamma(v^q, v^q) + v^{2q} d\mu). \quad \square
\]
Recall that $\varphi$ is a cut-off function for $B(x_0, R)$ given by $(\text{CS}(\Psi))$. We define

$$Q(t) = \{ x : \varphi(x) > t \}, \quad 0 < t < 1.$$  

**Corollary 4.8** Let $1 > s > t > 0$. There exists $\zeta > 2$ such that if $0 < q < \frac{1}{3}$,

$$\text{ess sup}_{Q(s)} v^{2q} \leq c_1 (s - t)^{-\zeta} V(x_0, R)^{-1} \int_{Q(t)} v^{2q} d\gamma.$$

The following corresponds to the second part of Moser’s arguments.

**Proposition 4.9** Let $w = \log u$, and write $\bar{w} = V(x_0, R)^{-1} \int_{B(x_0, R)} w \, d\mu$.

1. \( \int_{B(x_0, 2R)} d\Gamma(w, w) \leq c_1 \frac{V(x_0, R)}{\Psi(R)}. \)

2. \( \int_{\{ |w - \bar{w}| > A \} \cap Q(s)} d\gamma \leq c_2 \frac{V(x_0, R)}{A^2}, \quad \text{for } 0 < t < s \leq 1. \)

To get the Harnack inequality.
• [68]: generalization of the John-Nirenberg inequality with a complicated proof.
• Bombieri [22]: avoid such an argument for elliptic second order diff. eqs.
• Moser ([67], Lemma 3) carried the idea over to the parabolic case
• Bombieri-Giusti ([23], Theorem 4): ineq. in an abstract setting ([72], Lemma 2.2.6)

Using these, we can show that Corollary 4.8 and Proposition 4.9 (b) give

$$\text{ess sup}_{B(x_0, R/2)} \log u \leq c_1.$$ (4.26)

Let \( v = u^{-1} \). The same argument implies

$$\text{ess sup}_{B(x_0, R/2)} \log v \leq c_1, \text{ or } \text{ess inf}_{B(x_0, R/2)} \log u \geq -c_1. \text{ Combining we deduce }$$

$$e^{-c_1} \leq \text{ess inf}_{B(x_0, R/2)} u \leq \text{ess sup}_{B(x_0, R/2)} u \leq e^{c_1}.$$ 

**Theorem 4.10** \( \exists c_1 \text{ s.t. if } u \text{ is nonneg. and harmonic in } B(x_0, 4R), \text{ then } \)

$$\text{ess sup}_{B(x_0, R/2)} u \leq c_1 \text{ess inf}_{B(x_0, R/2)} u.$$