Random walks on graphs and applications to random media III

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11 Supercritical percolation cluster and Random conductance model

- Bond percolation on $\mathbb{Z}^d$ ($d \geq 2$)

$(\Omega, \mathcal{F}, \mathbb{P})$: prob. space for the randomness of the space, $p_n^\omega(x, y) := \mathbb{P}^x(Y_n = y)/\mu_y$.

$p > p_c$: No anomalous behavior for long time.

(a) [Gaussian heat kernel estimates] (Barlow ’04)

$$\frac{c_1}{t^{d/2}} \exp\left(-c_2 \frac{d(x, y)^2}{t}\right) \leq p_t^\omega(x, y) \leq \frac{c_3}{t^{d/2}} \exp\left(-c_4 \frac{d(x, y)^2}{t}\right),$$

$\mathbb{P}$-a.s. $\omega$ for $t \geq d(x, y) \vee \exists U_x, x, y \in \mathcal{G}$.

(b) [Quenched invariance principle]

(Sidoravicius-Sznitman ’04, Berger-Biskup ’07, Mathieu-Piatnitski. ’07)

$$n^{-1} Y_{n^2 t}^\omega \rightarrow B_{\sigma t} \quad \mathbb{P}\text{-a.s. } \omega \text{ for some } \sigma > 0$$

cf. ”Annealed” invariance principle: known since 80’s
Random conductance model (symmetric (reversible) RWRE)

Consider \((\mathbb{Z}^d, E_d)\), \(d \geq 2\) where \(E_d\) is the set of non-oriented n.n. bonds.

Let the conductance \(\{\mu_e : e \in E_d\}\) be i.i.d. (more generally stat. ergo.) on \((\Omega, \mathcal{F}, \mathbb{P})\).

Two natural MCs

Trans. prob. \(P(x, y) = \mu_{xy}/\mu_x\).

1. Constant speed random walk (CSRW): holding time is \(\exp(1)\) for each point

2. Variable speed random walk (VSRW): holding time at \(x\) is \(\exp\) distri. with mean \(\mu_x^{-1}\)

The corresponding discrete Laplace operators are

\[
\mathcal{L}_C f(x) = \frac{1}{\mu_x} \sum_y (f(y) - f(x)) \mu_{xy}, \quad \mathcal{L}_V f(x) = \sum_y (f(y) - f(x)) \mu_{xy}.
\]

Let \(\nu\) be s.t. \(\nu(x) = 1, \forall x \in \mathbb{Z}^d\). Then, for each finite supported \(f, g\),

\[
\mathcal{E}(f, g) = -(\mathcal{L}_V f, g)_\nu = -(\mathcal{L}_C f, g)_\mu.
\]
RW on supercrit. perco. is a special case \( (\mathbb{P}(\mu_e = 1) = p, \mathbb{P}(\mu_e = 0) = 1 - p) \)

Assume \( \mathbb{P}(\mu_e > 0) > p_c(\mathbb{Z}^d) \). Then \( \exists \mathcal{C} \) infinite cluster. We consider \( \mathbb{P}(\cdot | 0 \in \mathcal{C}) \).

Let \( \{Y_t\}_{t \geq 0}, \{P^x_\omega\}_{x \in \mathbb{Z}^d} \) be either the CSRW or VSRW and define

\[
q^\omega_t(x, y) = P^x_\omega(Y_t = y)/\theta_y
\]

be the heat kernel of \( \{Y_t\}_{t \geq 0} \) where \( \theta \) is either \( \nu \) or \( \mu \).

(Q1) Heat kernel estimates? (Q2) Invariance principle?

Let \( T > 0, F: \text{bdd. cont. on } D([0, T], \mathbb{R}^d) \). Set \( \Psi_\varepsilon := E^0_\omega F(\varepsilon Y_{\cdot/\varepsilon^2}), \Psi_0 := E_{BM} F(\sigma W_{\cdot}) \).

**Weak FCLT (“Annealed IP”)** \( \Psi_\varepsilon \rightarrow \Psi_0 \) in \( \mathbb{P} \)-prob.

**QFCLT (Quenched IP)** \( \Psi_\varepsilon \rightarrow \Psi_0 \) in \( \mathbb{P} \)-a.s.

**Note:** Our interest is QFCLT. When \( \mathbb{E}\mu_e < \infty \), weak FCLT was obtained in 80’s

(Kipnis-Varadhan ’86, De Masi-Ferrari-Goldstein-Wick ’89 (\( \sigma > 0 \)))
For (Q1) Heat kernel estimates:

**Theorem 11.1** (Barlow-Deuschel ’10)

If $\mathbb{P}(1 \leq \mu_e < \infty) = 1$, then (1) holds for VSRW.

- Anomalous behavior for $\mathbb{P}(0 < \mu_e \leq 1) = 1$

(Fontes-Mathieu ’06) Annealed result: VSRW on $\mathbb{Z}^d$ with $\mu_{xy} = \omega(x) \wedge \omega(y)$

where $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. with $\omega(x) \leq 1$ for all $x$ and $\exists \gamma > 0$ s.t.

$$
\mathbb{P}(\omega(0) \leq s) \asymp s^\gamma \text{ as } s \downarrow 0,
$$

$$
\Rightarrow \lim_{t \to \infty} \frac{\log \mathbb{E}[P_0^0(Y_t = 0)]}{\log t} = -\left(\frac{d}{2} \wedge \gamma\right).
$$

(Berger-Biskup-Hoffman-Kozma ’08) Quenched HK estimates for discrete time MC:
Theorem 11.2 Assume $\mathbb{P}(0 < \mu_e \leq 1) = 1$. (i) For $\mathbb{P}$-a.e. $\omega$, $\exists C_1(\omega) > 0$ s.t.

$$
P^n_{\omega}(0, 0) \leq C_1(\omega) \begin{cases} 
n^{-d/2}, & d = 2, 3, \\
n^{-2 \log n}, & d = 4, \\
n^{-2}, & d \geq 5. 
\end{cases}
$$

Further, (a) $\lim_{n \to \infty} n^2 P^n_{\omega}(0, 0) = 0$ $\mathbb{P}$-a.s. for $d \geq 5$,

(b) $\lim_{n \to \infty} n^2 \frac{P^n_{\omega}(0, 0)}{\log n} = 0$ $\mathbb{P}$-a.s. for $d = 4$.

(ii) For any incr. seq. $\{\lambda_n\}_{n \in \mathbb{N}}$, $\lambda_n \to \infty$, $\exists$ i.i.d. law $\mathbb{P}$ with $\mathbb{P}(0 < \mu_e \leq 1) = 1$ and $C_2(\omega), C_3(\omega) > 0$ s.t. for a.e. $\omega \in \{|C(0)| = \infty\}$,

$$
P^{2n}_{\omega}(0, 0) \geq C_3(\omega) n^{-2} \lambda_n^{-1} \quad \text{for } d \geq 5
$$

$$
P^{2n}_{\omega}(0, 0) \geq C_3(\omega) n^{-2} \log n \lambda_n^{-1} \quad \text{for } d = 4.
$$

along a subsequence that does not depend on $\omega$.

((i)-(b) Biskup-Loudor-Rozinov-Vandenberg-Rodes ’11, (ii)-(b) Biskup-Boukhadra ’11)
Why $n^{-2}$?

Suppose $\forall$ large $n$, the above config. occur w.h.p.

Strategry for RW to come back to origin in $2n$ steps (w.p. $\geq n^{-2}$)

(i) RW goes directly towards the trap (costs $e^{O(\ell_n)}$),

(ii) it crosses the weak bond (costs $1/n$), spends time $n - 2\ell_n$ on the strong bond (costs $O(1)$), and crosses a weak bond again (costs $1/n$),

(iii) it goes back to the origin on time (cost $e^{O(\ell_n)}$ term).

The cost (prob.) is $O(1)e^{O(\ell_n)}n^{-2}$ so if can take $\ell_n = o(\log n)$ then we get $n^{-2}$. 
Quenched invariance principle For $t \geq 0$, let $\{Y_t\}_{t \geq 0}$ be either CSRW or VSRW and

$$Y_t^{(\varepsilon)} := \varepsilon Y_{t / \varepsilon^2}.$$  \hspace{1cm} (3)

**Theorem 11.3** ($\mu_e \leq 1$ case: Biskup-Prescott ’07, Mathieu ’08, $\mu_e \geq 1$ case: Barlow-Deuschel ’10, unified: Andres-Barlow-Deuschel-Hambly ’11)

(i) Let $\{Y_t\}_{t \geq 0}$ be the VSRW. Then $\mathbb{P}$-a.s. $Y^{(\varepsilon)}$ converges (under $P^0_\omega$) in law to Brownian motion on $\mathbb{R}^d$ with covariance $\sigma^2_V I$ where $\sigma_V > 0$ is non-random.

(ii) Let $\{Y_t\}_{t \geq 0}$ be the CSRW. Then $\mathbb{P}$-a.s. $Y^{(\varepsilon)}$ converges (under $P^0_\omega$) in law to Brownian motion on $\mathbb{R}^d$ with covariance $\sigma^2_C I$ where $\sigma^2_C = \sigma^2_V / (2d \mathbb{E} \mu_e)$ if $\mathbb{E} \mu_e < \infty$ and $\sigma^2_C = 0$ if $\mathbb{E} \mu_e = \infty$. 

**Local central limit theorem**

**Theorem 11.4** Assume $\mathbb{P}(1 \leq \mu_e < \infty) = 1$. Let $q^\omega_t(x, y)$ be the VSRW and write $k_t(x) = (2\pi t \sigma_V^2)^{-d/2} \exp(-|x|^2/(2\sigma_V^2 t))$ where $\sigma_V$ is as in Theorem 11.3 (i). Let $T > 0$, and for $x \in \mathbb{R}^d$, write $[x] = ([x_1], \cdots, [x_d])$. Then

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T} |n^{d/2} q^\omega_{nt}(0, [n^{1/2} x]) - k_t(x)| = 0, \quad \mathbb{P} - a.s.$$ 

In general when $\mu_e \leq c$, such local CLT does NOT hold because of the anomalous behavior of the heat kernel and QIP.

**More about CSRW with $\mathbb{E}\mu_e = \infty$:** Let $\{\mu_e\}$ satisfies

$$\mathbb{P}(\mu_e \geq c_1) = 1, \quad \mathbb{P}(\mu_e \geq u) = c_2 u^{-\alpha}(1 + o(1)) \text{ as } u \to \infty, \quad (4)$$

for some constants $c_1, c_2 > 0$ and $\alpha \in (0, 1]$.

In order to state the result, we first introduce the FK process and FIN diffusion.
Definition 11.5 (Fractional-kinetics process) \( \{B_d(t)\} \): \( d \)-dim BM, \( \alpha \in (0, 1) \)
\{\( V_{\alpha}(t) \)\}_{t \geq 0}: \( \alpha \)-stable subord. (indep. of \( \{B_d(t)\} \)), \( V_{\alpha}^{-1}(s) := \inf\{t : V_{\alpha}(t) > s\} \)
Fractional-kinetics process \( \text{FK}_{d,\alpha} \) is defined by

\[
\text{FK}_{d,\alpha}(s) := B_d(V_{\alpha}^{-1}(s)), \quad s \in [0, \infty).
\]

- Non-Markovian process
- Self-similar, i.e. \( \text{FK}_{d,\alpha}(\cdot) \overset{d}{=} \lambda^{-\alpha/2}\text{FK}_{d,\alpha}(\lambda \cdot), \quad \forall \lambda > 0. \)
- The density \( p(t, x) \) started at 0 satisfies:
  \[
  \frac{\partial^\alpha}{\partial t^\alpha} p(t, x) = \frac{1}{2} \Delta p(t, x) + \delta_0(x) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}.
  \]

Definition 11.6 (FIN diffusion: Fontes-Isopi-Newman ’02) \( \rho := \sum_i \nu_i \delta_{x_i} \) where \( (x_i, \nu_i) \)
on \( \mathbb{R} \times \mathbb{R}_+ \) is distributed with inhomog. PPP with intensity \( dx\alpha\nu^{-1-\alpha}d\nu \).
\( \phi_{\rho}(t) := \int_{\mathbb{R}} \ell(t, y)\rho(dy) \) where \( \ell(\cdot, \cdot) \) is the local time of BM.

FIN diffusion is defined by

\[
Z(s) = BM(\phi_{\rho}^{-1}(s)), \quad s \in [0, \infty).
\]
Theorem 11.7 Let \( \{Y_t\}_{t \geq 0} \) be the CSRW of RCM that satisfies (4).

(i) (Barlow-Černý ’10) Let \( d \geq 3, \alpha \in (0, 1) \) in (4) and let \( Y_t^{(\varepsilon)} := \varepsilon Y_{t/\varepsilon^{2/\alpha}} \). Then

\[
Y^{(\varepsilon)} \xrightarrow{d} c \cdot FK_{d,\alpha} \quad \text{under } P_\omega^0, \mathbb{P}\text{-a.s. on } D([0, \infty), \mathbb{R}^d) \text{ with } J_1\text{-topology.} \tag{5}
\]

(ii) (Černý ’11) Let \( d = 2, \alpha \in (0, 1) \) in (4) and let \( Y_t^{(\varepsilon)} := \varepsilon Y_{t(\log(1/\varepsilon))^{1-1/\alpha/\varepsilon^{2/\alpha}}} \).

Then (5) holds.

(iii) (Černý ’11) Let \( d = 1, \alpha \in (0, 1) \) in (4) and let \( Y_t^{(\varepsilon)} := \varepsilon Y_{c_\varepsilon c_\varepsilon t/\varepsilon}, \) where \( c_* = \mathbb{E}[\mu_e^{-1}], \)

\[
c_\varepsilon := \inf\{t \geq 0 : \mathbb{P}(\mu_e > t) \leq \varepsilon\} = \varepsilon^{-1/\alpha}(1 + o(1)).
\]

Then \( Y^{(\varepsilon)} \xrightarrow{d} Z(t) \) under \( \mathbb{P} \times P_0^\mu. \)

(iv) (Barlow-Zheng ’10) Let \( d \geq 3, \alpha = 1 \) in (4) with \( c_1 = c_2 = 1 \) and

let \( Y_t^{(\varepsilon)} := \varepsilon Y_{t \log(1/\varepsilon^{2})/\varepsilon^{2}}. \) Then

\[
Y^{(\varepsilon)} \xrightarrow{d} BM \quad \text{with cov. } \frac{1}{2} \sigma_V^2 I \quad \text{under } P_\omega^0, \mathbb{P}\text{-a.s. on } D([0, \infty), \mathbb{R}^d).
\]
Remark 11.8 \textit{Bouchaud’s trap model (BTM)} \( \{\tau_x\}_{x \in \mathbb{Z}^d} \): pos. i.i.d., \( a \in [0,1] \)

Random conductance \( \mu_{xy} := \tau_x^a \tau_y^a \) if \( x \sim y \), \( \text{Measure } \mu_x := \tau_x \)

\textit{BTM is the CSRW with the trans. prob. } \mu_{xy}/\sum_y \mu_{xy} \textit{ and the measure } \mu_x. \textit{ (Scaling limit: } d \geq 2 \textit{ Ben Arous-Černý ’07, } d=1 \textit{ Fontes-Isopi-Newman ’02) }

12 \textit{Strategy of the proof of Theorem 11.3}

*** Abuse of notation: We sometimes write \( \mathbb{P} \) for \( \mathbb{P}(\cdot | 0 \in \mathcal{C}). ***

Time changed process

\( \mathcal{C}: \infty \)-cluter of \( \{e \in E_d : \mu_e > 0\} \). Choose \( K > 0 \) large enough so that

\[
q(K) := \mathbb{P}(0 < \mu_e < K^{-1}) + \mathbb{P}(\mu_e > K) < p - p_c(\mathbb{Z}^d),
\]

and let \( \mathcal{C}_2 \) be the \( \infty \)-cluter of \( \{e \in E_d : \mu_e \in [K^{-1}, K]\} \) (to be precise, remove also bonds that connect to ”bad” bonds).
Let $Y$ be VSRW on $C$ and define $Z$ as a trace of $Y$ on $C_2$. Namely, let $A_t := \int_0^t 1_{\{Y_s \in C_2\}} ds$,

$$Z_t := Y_{A_t^{-1}}, \quad t \geq 0,$$

where $A_t^{-1} := \inf\{s : A_s > t\}$.

Note $\{Z_t\}$ is a jump process in general.

We can then "induce" conductance $\mu'_{xy}$ which gives trans. prob. of $\{Z_t\}$.

**Part 1, Proof of FCLT for $\{Z_t\}$:**

This is a big step and we will discuss later except one thing.

Environment seen from the particle (Kipnis-Varadhan)

$\Omega = [K^{-1}, K]^{E_d}$. \{$\mu'_e : e \in E_d$\} are defined on $(\Omega, \mathbb{P})$ and we write $\mu'_{\{x,y\}}(\omega) = \omega_{x,y}$.

Let $T_x: \Omega \to \Omega$ denote the shift by $x$, namely $(T_z\omega)_{xy} := \omega_{x+z,y+z}$. Define

$$V_t = V_t(\omega) = T_{Z_t} (\omega), \quad \forall t \in [0, \infty),$$

where $\{Z_t\}_{t \geq 0}$ is the MC.  Note $\mathbb{P}$ is erg. w.r.t. $V_t(\omega) := T_{Z_t}(\omega)$ (De Masi et al. ’89)
Part 2, Proof of FCLT for \( \{Y_t\} \):

\( \mathcal{H} := \mathcal{C} \setminus \mathcal{C}_2 \), \( x \in \mathcal{C} \), let \( \mathcal{H}(x) \) be the connected component of \( \mathcal{C} \setminus \mathcal{C}_2 \) containing \( x \).

**Lemma 12.1** For \( K \) chosen large enough, the following holds.

(i) All the con. comp. of \( \mathcal{H} \) are finite. Further, \( \exists c_1, c_2 \) s.t. \( \forall x \in \mathbb{L} \),

\[
P(x \in \mathcal{C}_1 \text{ and } \text{diam} \mathcal{H}(x) \geq n) \leq c_1 e^{-c_2 n}.
\]

(ii) \( \mathbb{P} \)-a.s., for \( n \) large, the vol. of any hole intersecting \([-n, n]^d\) is bdd by \( (\log n)^{3\alpha} \).

- \( \lim_{t \to \infty} A_t/t = \mathbb{P}(0 \in \mathcal{C}_2) =: C_0 > 0 \), \( \mathbb{P} \times P_{\omega}^0 \)-a.s.

(\( \bigcirc \mathbb{P} \) is erg. w.r.t. \( V_t(\omega) = T_{Z_t}(\omega) \) and \( A_t = \int_0^t 1\{0 \in \mathcal{C}_2(V_s(\omega))\} ds \))

Since

\[
Y_t^{(\varepsilon)} = \varepsilon(Y_{t/\varepsilon^2} - Z_{A_t/\varepsilon^2}) + \varepsilon(Z_{A_t/\varepsilon^2} - Z_{C_0t/\varepsilon^2}) + Z_{C_0t}^{(\varepsilon)},
\]

Lemma 12.1 and tightness of \( Y \) (which can be proved by HK upper bound) imply that the first two terms go to 0. We thus obtain FCLT for \( Y_t^{(\varepsilon)} \) with \( \sigma_Y^2 = C_0 \sigma_Z^2 \).
Part 3, Proof of FCLT for CSRW:

Note VSRW and CSRW are time changes of each others. Set

$$\tilde{A}_t := \int_0^t \mu_Y s \, ds = \int_0^t \mu_0(T_Y s) \, ds.$$ 

Then CSRW \( \{X_t\} \) can be written as

$$X_t = Y_{\tilde{A}_t^{-1}}, \quad \text{where } \tilde{A}_t^{-1} := \inf\{s : \tilde{A}_s > t\}.$$ 

Again by the ergo. thm, \( \lim_{t \to \infty} \tilde{A}_t / t = \mathbb{E}\mu_0 = 2d\mathbb{E}\mu_e \quad \mathbb{P} \times P_0^\omega \text{-a.s.} \)

So \( X^{(\varepsilon)} \) converges to \( \sigma_C B_t \) where \( B_t \) is BM and \( \sigma_C^2 = \sigma_V^2 / (2d\mathbb{E}\mu_e) \) if \( \mathbb{E}\mu_e < \infty \)

and \( \sigma_C^2 = 0 \) if \( \mathbb{E}\mu_e = \infty \).
13 QIP for \( \{Z_t\} \)

13.1 HK estimates

Lemma 13.1 (Perco. est. (Antal-Pisztora ’96)) \( \exists c_1, c_2, c_3 > 0 \) s.t. \( \forall x, y \in \mathbb{L}, \)

\[
\mathbb{P}(x, y \in C_2 \text{ and } d(x, y) \leq c_1|x - y|) \leq c_2 e^{-c_3|x-y|},
\]

\[
\mathbb{P}(x, y \in C_2 \text{ and } d(x, y) \geq c_1^{-1}|x - y|) \leq c_2 e^{-c_3|x-y|},
\]

where \( |\cdot - \cdot| \) is the Euclidean dist. and \( d(\cdot, \cdot) \) is the graph dist.

- (Percolation est.) \( \Rightarrow \) (HK estimates (1))

"In principle" stability of HK est. helps. (Finding "good balls" inside which one has good control of volume and Poincaré exp. Then use the Borel-Cantelli to obtain a.s. results.
13.2 Correctors

Idea behind $\varphi = \varphi_\omega : \mathbb{Z}^d \to \mathbb{R}^d$ be a harmonic map (so $M_t = \varphi(Y_t)$ is a $P^0_\omega$-mart.)

Corrector $\chi(x) := (\varphi - I)(x) = \varphi(x) - x$.

For simplicity, let us consider CLT (instead of FCLT) for $Y$. We have

$$\frac{Y_t}{t^{1/2}} = \frac{M_t}{t^{1/2}} - \frac{\chi(Y_t)}{t^{1/2}}.$$

Martingale CLT gives $M_t/t^{1/2} \xrightarrow{w} \text{Normal distri.}$ So ETP $\chi(Y_t)/t^{1/2} \to 0$.

This can be done once we have

(a) $P_\omega^0(|Y_t| \geq At^{1/2})$ is small ($\iff$ Can be shown by HK upper bound)

(b) $|\chi(x)|/|x| \to 0$ as $|x| \to \infty$.

Note There maybe many global harm. fu., so we should chose one s.t. (b) holds.

Now let us overview how to construct such corrector.
Dirichlet form corresp. to $\{V_t\}$ (Recall $V_t = V_t(\omega) = T_{Z_t}(\omega)$.)

Let $\mathbb{L}^2 = \mathbb{L}^2(\Omega, \mathbb{P})$ and for $F \in \mathbb{L}^2$ define $\nabla F(\omega, x) := F(T_x\omega) - F(\omega)$. Set

$$\hat{\mathcal{E}}(F, G) := \mathbb{E}[\sum_{x \in \mathbb{Z}^d} \omega_{0,x} \nabla F(\cdot, x) \nabla G(\cdot, x)], \quad \forall F, G \in \mathbb{L}^2.$$

Construction of corrector $M$: meas. on $\Omega \times \mathbb{Z}^d$ defined as

$$\int_{\Omega \times \mathbb{Z}^d} G dM := \mathbb{E}[\sum_{x \in \mathbb{Z}^d} \omega_{0,x} G(\cdot, x)], \quad \text{for} \ G : \Omega \times \mathbb{Z}^d \to \mathbb{R}.$$

Let $\overline{L}^2 := \{G \in \mathbb{L}^2(\Omega \times \mathbb{Z}^d, M) : G(T_x\omega, y - x) = G(\omega, y) - G(\omega, x) \ \mathbb{P}-\text{a.s.}\}$

Note that for $F \in \mathbb{L}^2$, $\nabla F \in \overline{L}^2$ and $\|\nabla F\|_{L^2}^2 = \hat{\mathcal{E}}(F, F) < \infty$.

Let $L^2_{\text{pot}} := Cl \{\nabla F : F \in \mathbb{L}^2\}$ and consider the orth. decomp. $\overline{L}^2 = L^2_{\text{pot}} \oplus L^2_{\text{sol}}$.

Let $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ be the identity and define $\chi_j \in L^2_{\text{pot}}$, $\varphi_j \in (L^2_{\text{pot}})^\perp$ by

$$\Pi_j = (\chi_j) \oplus \varphi_j \in L^2_{\text{pot}} \oplus (L^2_{\text{pot}})^\perp.$$

This gives the definition of the corrector $\chi = (\chi_1, \cdots, \chi_d) : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$. 
Lemma 13.2 For $G \in (L^2_{\text{pot}})^\perp$, $\sum_{x \in \mathbb{Z}^d} \omega_{0,x} G(\omega, x) = 0$, $\mathbb{P} - a.s.$

Hence $M_n := G(\omega, Y_n)$ is a $P^0_\omega$-martingale for $\mathbb{P}$-a.e. $\omega$.

Theorem 13.3 For $\mathbb{P}_2$-a.e. $\omega$, 
$$\lim_{n \to \infty} \max_{x \in C_2, |x| \leq n} \frac{|\chi(x)|}{n} = 0.$$ 

—Proof uses HK upper bound, percolation properties etc.

This implies FCLT for $\{Z_t\}$.

14 Other results

(i) RW on supercrit. perco. cluster on half/square planes (Z.Q. Chen-Croydon-K ’12)

$\mathbb{L} := \{(x_1, \cdots, x_d) \in \mathbb{Z}^d : x_{j_1}, \cdots, x_{j_l} \geq 0\}$ for some $1 \leq j_1 < \cdots < j_l \leq d$, $l \leq d$.

Consider bond perco. on $\mathbb{L}$ and consider supercrit. case. Then

$$n^{-1}Y^{\omega}_{[n^2t]} \to B_{\sigma t}, \quad \mathbb{P} - a.e. \omega \text{ (for some } \sigma > 0) \tag{QIP}$$
**Difficulty**

All the previous results use corrector method, which requires transl. inv. of the space.

**Ideas**

- Full use of heat kernel estimates.
- Use information of QIV on the whole space and methods of Dirichlet forms.

(ii) RW on long-range percolation

On \( \mathbb{Z}^d \) \((d \geq 1)\), prob. that \( \{x, y\} \) are open \( \asymp |x - y|^{-s} \) for \( s \in (d, (d + 2) \wedge 2d) \).

**Note** When open, put conductance 1!

(Crawford-Sly ’11): HK upper bound \( t^{-d/(s-d)} \)

(Crawford-Sly ’12): Scaling of the SRW converges (in a weak sense) to

\( (s - d) \)-stable process for \( s \in (d, d + 1) \) \( \mathbb{P} \)-a.s.