Asymptotics for the spectral and walk dimension as fractals approach Euclidean space

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Abstract

We discuss the behaviour of the dynamic dimension exponents for families of fractals based on the Sierpinski gasket and carpet. As the length scale factor for the family tends to infinity the lattice approximations to the fractals look more like the tetrahedral or cubic lattice in Euclidean space and the fractal dimension converges to that of the embedding space. However, in the Sierpinski gasket case, the spectral dimension converges to two for all dimensions. In two dimensions we prove a conjecture made in the physics literature concerning the rate of convergence. On the other hand, for natural families of Sierpinski carpets, the spectral dimension converges to the dimension of the embedding Euclidean space. In general we demonstrate that for both cases of finitely and infinitely ramified fractals, a variety of asymptotic values for the spectral dimension can be achieved.

1 Introduction

In [10] a numerical study was made of the behaviour of the spectral dimension on a particular subclass of Sierpinski gaskets. By considering the gaskets as a parameterized family of fractal lattices that converges to the triangular lattice in two dimensions, they were interested in seeing how the fractal properties behave at the crossover from fractal lattice to regular lattice behaviour. In that paper the spectral dimension was calculated for several hundred parameter values leading to the conjecture that, asymptotically, the spectral dimension converges to two with a particular logarithmic error.

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In this paper we will consider the behaviour of the dimension exponents for this family of Sierpinski gaskets in all dimensions and give a rigorous proof of the conjecture. We will show that the finite ramification persists in that, for the gaskets and other nested fractals, the spectral dimension cannot exceed 2 whatever the dimension of the space in which the fractals are embedded. In contrast we consider families of Sierpinski carpets which converge to the Euclidean space in which they are embedded and discuss the asymptotic behaviour of their dimension exponents. In this infinitely ramified case there is typically convergence to the spectral dimension of the embedding space in all dimensions.

As the fractals that we consider are either nested fractals or Sierpinski carpets, they are self-similar sets and their fractal dimension is easily calculated. The family of fractals $F_l \subset \mathbf{R}^d$ will be parameterized by their length scale factor l. Throughout the paper, we assume that each F_l is connected and defined as the fixed point of a family of similitudes $\{\psi_i = \psi_i^{(d,l)}; i = 1, \ldots, N_{d,l}\}$, $\psi_i : \mathbf{R}^d \to \mathbf{R}^d$ where $N_{d,l}$ is the number of maps of contraction factor l^{-1} required to generate the fractal. Thus

$$F_l = \bigcup_{i=1}^{N_{d,l}} \psi_i(F_l),$$

and, under the open set condition (which holds for all our fractals), all notions of fractal dimension have common value $d_f(d,l) = \log N_{d,l} / \log l$.

We give general definitions for the dynamic exponents, the spectral and walk dimensions, and note that these can be made precise in our setting. Firstly the walk dimension determines the time to distance scaling in the fractal and can be defined using the mean square displacement of a diffusion. Let $\{X_t^l; t \in \mathbf{R}\}$ be a Brownian motion on the fractal, F_l and denote by E^x the expectation for the Brownian motion started at the point $x \in F_l$. If the set is isotropic, in that the behaviour of the Brownian motion will be independent of its starting point, we define the walk dimension

$$d_w(d,l)^{-1} = \frac{1}{2} \lim_{t \to 0} \frac{\log E^x |X_t^l - x|^2}{\log t}.$$

(This could also be defined through the long time mean square displacement of a random walker on the corresponding fractal lattice.)

The spectral (or fracton) dimension describes the asymptotic scaling in the eigenvalue counting function. It is well known, [7], that the Laplace operator associated with a nested fractal exists and is unique. It also has a discrete spectrum consisting of eigenvalues with its only accumulation point at infinity. Let $C_l(\lambda)$ denote the number of eigenvalues of the Laplace operator on F_l that are less than λ. The spectral dimension is defined to be

$$d_s(d,l) = 2 \lim_{\lambda\to\infty} \frac{\log C_l(\lambda)}{\log \lambda}.$$

It has been shown that for p.c.f. self-similar sets, which include nested fractals, and also for Sierpinski carpets, that these exponents are related by $d_s = 2d_f/d_w$.

Our aim is to study the asymptotic behaviour of $d_s(d,l), d_w(d,l)$ as $l \to \infty$ for families of finite and infinitely ramified fractals. The results show that for natural families of Sierpinski gaskets and Vicsek sets embedded in Euclidean space $\mathbf{R}^d$ for $d \geq 2$, the spectral dimension $d_s(d,l)$ will converge to 2, for all d. In this case the walk dimension $d_w(d,l)$ will converge to d, the dimension of the embedding space. For the Sierpinski carpet, we will see that typically the exponents converge to those of the embedding space, in that $d_s(d,l) \to d$ and $d_w(d,l) \to 2$. However we will show that there are constructions which give various limits for $d_s(d,l)$ between 1 and $\bar{d}$ where $\bar{d} = \lim_{l\to\infty} d_f(d,l)$.

We begin by working on the Sierpinski gasket and establish a rigorous version of the result conjectured in [10]. In the final Section we will consider the other families of fractals and discuss the possible limits of the spectral dimension as the families approach the Euclidean space.

2 Asymptotics for Sierpinski gaskets

Let $l \geq 2$ be a natural number. We divide the sides of the unit equilateral triangle in $\mathbf{R}^2$ evenly into l and remove all the downward pointing triangles. Iterating this procedure indefinitely, we have a Sierpinski gasket which we will denote by $SG(2,l)$ (see Figure 1 for level 4 for $l = 2$ and level 2 for $l = 4$). This family of fractals was used to generate random fractals in [5, 6] in order to determine the effect of randomization on the analytic properties of fractals. We can follow the same procedure to construct a family of gaskets $SG(d,l)$ in $\mathbf{R}^d$, starting from the unit equilateral d-dimensional tetrahedron. Denote by $N_{d,l}$ the number of small upward pointing triangles obtained at the first step of the iteration; $N_{2,l} = l(l+1)/2$, $N_{3,l} = l(l+1)(2l+1)/6$ and $N_{d,l}$ can be calculated recursively using

$$N_{d,l} = \sum_{j=1}^{l} N_{d-1,j} \asymp l^d. \tag{2.1}$$

Here and in the following, we will write $a_n \asymp b_n$ ($\{a_n\}, \{b_n\}$ are positive sequences) if there exist positive constants c_1, c_2 such that $c_1 \le a_n/b_n \le c_2$. Note that the Hausdorff dimension of $SG(d,l)$ is $\log N_{d,l}/\log l$.

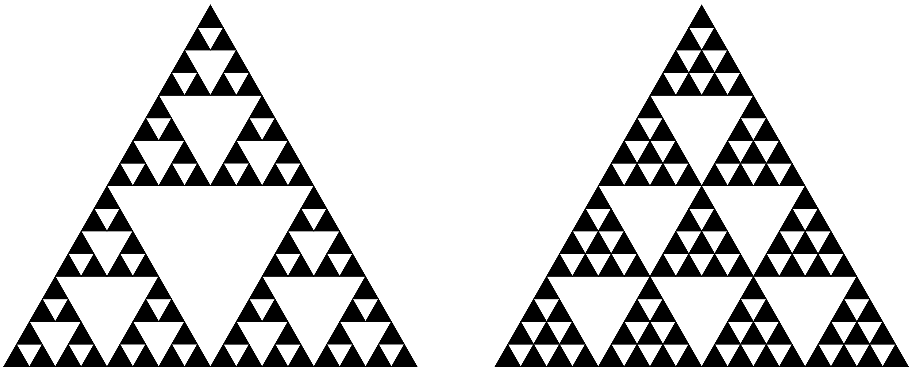

Figure 1: $l = 2$ $l = 4$

Now, let V_0 be the set of vertices of the unit equilateral d-dimensional tetrahedron and let V_1 be the set of vertices of the small tetrahedra obtained at the first step of the iteration. We denote by E_m ($m = 0,1$) the set of edges of length l^{-m} which are a subset of $SG(d,l)$ and connect the vertices of V_m. We can regard (V_m, E_m) as an electrical network by putting a unit resistor on each edge in E_m. For each potential f on V_m (i.e. $f \in S(V_m) \equiv \{h : h$ is a function on $V_m\}$), the energy of the network can then be defined as

$$B_m(f,f) = \frac{1}{2} \sum_{\substack{x,y \in V_m \\ (x,y) \in E_m}} (f(x) - f(y))^2.$$

We can then easily show that there exists a resistance scale factor $\rho_{d,l} > 1$ so that

$$\rho_{d,l} \cdot \inf\{B_1(f,f) : f \in S(V_1), f|_{V_0} = v\} = B_0(v,v) \qquad \text{for all } v \in S(V_0). \tag{2.2}$$

Note that l corresponds to b and $\rho_{2,l}$ corresponds to R_b in [10]. Using these scaling constants, the following formula is known.

Proposition 2.1 *(Theorem 8.18, [1])*

$$d_s(d,l) = \frac{2\log N_{d,l}}{\log(N_{d,l}\cdot\rho_{d,l})}, \qquad d_w(d,l) = \frac{\log(N_{d,l}\cdot\rho_{d,l})}{\log l}.$$

We now state our main theorem concerning the asymptotic behaviour of $\rho_{d,l}$.

Theorem 2.2 $\rho_{2,l} \asymp \log l$. *For* $d \geq 3$, $\rho_{d,l} \asymp 1$.

Using Proposition 2.1 and (2.1), we have the following corollary of the theorem.

Corollary 2.3 *For* $d \geq 2$, *we have the following as* $l \to \infty$.

$$d_s(d,l) = \begin{cases} 2 - \frac{\log\log l}{\log l} + \frac{O(1)}{\log l} & d=2, \\ 2 - \frac{O(1)}{\log l} & d>2, \end{cases} \tag{2.3}$$

$$d_w(d,l) = \begin{cases} 2 + \frac{\log\log l}{\log l} + \frac{O(1)}{\log l} & d=2, \\ d + \frac{O(1)}{\log l} & d>2. \end{cases} \tag{2.4}$$

(2.3) for $d=2$ proves $(2b)$ in [10].

3 Proof of Theorem 2.2

The key to proving the results in this paper are the shorting and cutting laws for electrical networks. They are originally due to Rayleigh and some discussion can be found in [3] Section 6.2.

The shorting law: *Shorting sets of vertices in a graph can only decrease the effective resistance between two given nodes in the graph.*

The cutting law: *Cutting certain edges of the graph can only increase the effective resistance between two given nodes in the graph.*

For the proof of Theorem 2.2 we will first concentrate on the 2-dimensional case. Note that by taking $v(a)=0, v(b)=v(c)=1$ in (2.2),

$$\rho_{2,l} = (\inf\{B_1(f,f) : f \in S(V_1), f(a)=0, f(b)=f(c)=1\})^{-1},$$

where a,b,c are the vertices of V_0 labelled as in Figure 2, (which corresponds to the case $l=3$). By shorting the vertices that belong to each horizontal line (the thick lines in Figure 2), we see from the shorting law that $\rho_{2,l}$ is bigger than (or equal to) the effective resistance between a and b in a linear network of the type shown on the right hand side of Figure 2.

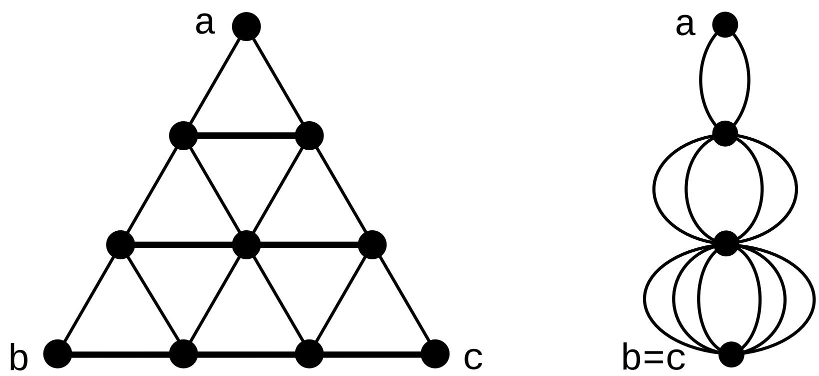

Figure 2: Shorting the gasket

Therefore,

$$\rho_{2,l} \geq \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots \frac{1}{2l} \sim \frac{1}{2} \log l.$$

Here $a_n \sim b_n$ ($\{a_n\}, \{b_n\}$ are positive sequences) means $\lim_{n\to\infty} a_n/b_n = 1$. Thus the lower bound is obtained.

The proof of the upper bound is divided into several steps. First, define

$$\begin{aligned} R_l^1 (= R_{2,l}^1) &= (\inf\{B_1(f,f) : f \in S(V_1), f(a) = 0, f(b) = 1\})^{-1} \\ R_l^2 (= R_{2,l}^2) &= (\inf\{B_1(f,f) : f \in S(V_1), f(a) = 0, f(z) = 1 \ \ \forall z \in \overline{bc}\})^{-1}. \end{aligned}$$

Then R_l^1 is the effective resistances between a and b, while R_l^2 is the effective resistance between a and $\overline{bc}$. Note that in general, the effective resistance $R(A, B)$ between two disjoint sets A and B ($\emptyset \neq A, B \subset V_1, \ A \cap B = \emptyset$) can be defined as

$$R(A, B) = (\inf\{B_1(f,f) : f \in S(V_1), f(x) = 0 \ \ \forall x \in A, \ f(y) = 1 \ \ \forall y \in B\})^{-1}.$$

By definition, the following is clear

$$R_l^1 \geq \rho_{2,l} \geq R_l^2. \tag{3.1}$$

We now prove the key lemma.

Lemma 3.1 $R_l^2 \asymp \log l$.

PROOF. We first prepare two electrical networks and consider two effective resistances as in Figure 3. One is the effective resistance between a and L_1 (the

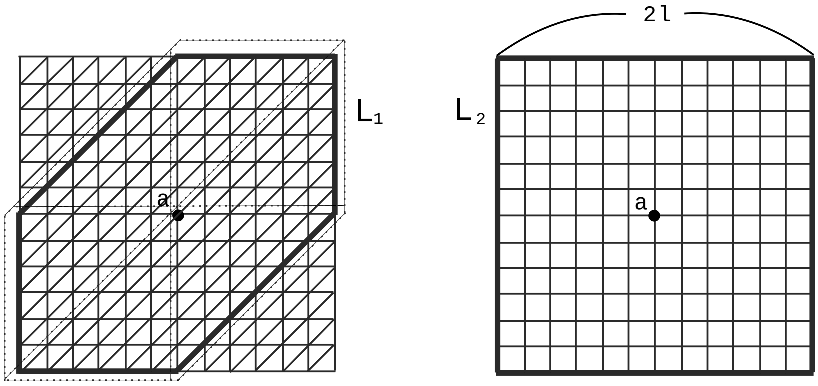

Figure 3: Triangular lattice Regular lattice

thick lines) on the triangular lattice with unit resistors, which we denote by $\tilde{R}_l$. The other is the effective resistance between a and L_2 (the thick lines) on the regular lattice with unit resistors, which we denote by $\bar{R}_l$. Using shorting and cutting law arguments, we see that $\tilde{R}_l \asymp \bar{R}_l$. Further, we can embed 6 $SG(2,l)$s into the triangular lattice as indicated by the broken lines in Figure 3. Using this fact and the symmetry of the triangle, it is easy to see $R_l^2 \asymp \tilde{R}_l$. Thus, it is enough to prove $\bar{R}_l \asymp \log l$, but this is a well-known fact. Indeed, let $G_l(a,x)$ be the Green function of the electrical network corresponding to the regular lattice (in terms of probability, it is the expected number of visit to x for the corresponding Markov chain starting from a before arriving at L_2). For the 2-dimensional regular lattice, it is a standard fact that $G_l(a,a) \asymp \log l$. Let $g_l(a,x) = 4G_l(a,x)/\{\mu_x G_l(a,a)\}$ where μ_x is the number of bonds that connect to x. Then $g_l(a,a) = 1$ and $g_l(a,y) = 0$ for all $y \in L_2$. As $g_l(a,\cdot)$ is harmonic inside $L_2 \cup \{a\}$, this function attains the infimum in the definition of the effective resistance so that

$$(\bar{R}_l)^{-1} = B(g_l(a,\cdot), g_l(a,\cdot)) = 4/G_l(a,a) \asymp (\log l)^{-1},$$

where $B(\cdot,\cdot)$ is the energy of the network on the regular lattice and we have used the reproducing kernel property of $u(x) \equiv G_l(a,x)/\mu_x$ (i.e., $B(u,f) = f(a)$ for all f such that $f(z) = 0$ on $z \in L_2$) in the second equality. This completes the proof. ∎

We next define another graph (V_1', E_1') as in Figure 4, i.e. we add vertices on the perpendicular bisector L of $\overline{ab}$. When an edge is divided into two edges by

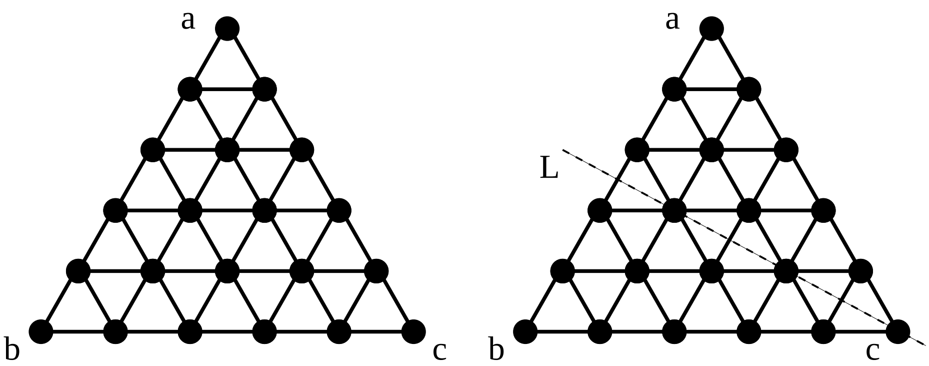

Figure 4: (V_1, E_1) (V_1', E_1')

L, we put a 1/2 resistor on each small edge. This allows us to regard (V_1', E_1') as an electrical network whose energy we denote by $B_1'(\cdot,\cdot)$. Define ${R'}_l^1, {R'}_l^2$ in the same way as R_l^1, R_l^2 by changing B_1, V_1 to B_1', V_1'.

Lemma 3.2 *For $i = 1, 2$, $R_l^i = {R'}_l^i$.*

PROOF. Take an edge $\{x, y\} \in E_1$ such that its middle point x' is in $V_1' \setminus V_1$. Clearly,

$$(f(x) - f(y))^2 \le 2\{(f(x) - f(x'))^2 + (f(x') - f(y))^2\},$$

so that $(R_l^i)^{-1} \le ({R'}_l^i)^{-1}$. On the other hand, by taking $(f(x) + f(y))/2$ as $f(x')$, we have

$$\begin{aligned}(f(x) - f(y))^2 &= 2\{(f(x) - \frac{f(x)+f(y)}{2})^2 + (\frac{f(x)+f(y)}{2} - f(y))^2\} \\ &= 2\{(f(x) - f(x'))^2 + (f(x') - f(y))^2\},\end{aligned}$$

which guarantees $(R_l^i)^{-1} = ({R'}_l^i)^{-1}$. ∎

Lemma 3.3 $R_l^1 \le 2R_l^2$ *for all $l \ge 2$.*

PROOF. We first note that the potential $f \in S(V_1')$ which attains the infimum in the definition of ${R'}_l^1$ enjoys the property that $f(z) = 1/2$ for all $z \in L \cap V_1'$ by the symmetry of the gasket. Thus, when we short the network (V_1', E_1') so that all the vertices below L (the vertices in the right-angled triangle bL) are the same voltage 1 and the voltage of a is 0, the corresponding effective resistance between a and L is $R'^1_l/2$. To see this note that the effective resistance from a to L is

$$R(a,L)^{-1} = \inf\{B_{aL}(f,f) : f(a) = 0, f|_L = 1\},$$

where $B_{aL}(.,.)$ is the energy form restricted to the triangle aL. Let g be the function that attains the infimum. By symmetry $B(2f,2f) = 2B_{aL}(g,g)$ and hence

$$R(a,L)^{-1} = B_{aL}(g,g) = 2B(f,f) = 2(R'^1_l)^{-1}.$$

By the shorting law, we have

$$R'^2_l \geq R'^1_l/2.$$

Using Lemma 3.2, we obtain the result. ∎

Combining (3.1), Lemma 3.1 and Lemma 3.3, we obtain Theorem 2.2 for $d = 2$.

We now give a proof of Theorem 2.2 for $d \geq 3$. Note firstly that $\rho_{d,l} > 1$ for all $d, l \geq 2$. The fact that this holds for the class of nested fractals introduced in [9] can be found in [1] Corollary 6.28. Further, $\{\rho_{d,l}\}_{d,l}$ is non-increasing w.r.t. d (this is easily proved by cutting arguments). Thus $\rho_{d,l} \leq \rho_{3,l}$. Therefore we only need to prove that $\rho_{3,l}$ is bounded from above. The proof goes though in essentially the same way as that for $d = 2$ by replacing the 2-dimensional arguments with suitable 3-dimensional ones. The statement of the key Lemma 3.1 should be replaced by $R^2_l \asymp 1$. Again it is a standard fact that for the 3-dimensional regular lattice, $G_l(a,a) - 1 \asymp 1/l$ so that $B(g_l(a,\cdot), g_l(a,\cdot)) = 6/G_(a,a) \asymp 1$.

4 Other fractals

In this section, we give a discussion of other families of fractals based on cubes. There are very natural classes of Sierpinski carpets and Vicsek sets whose lattice approximations converge to the cubic lattice as the length scale is refined. We will establish results for these families which show the difference between finite and infinite ramification.

Note that for nested fractals we have $\rho_{d,l} > 1$ and hence there are the following bounds on their spectral and walk dimensions,

$$d_s(d,l) \leq 2, \quad d_w(d,l) \geq d_f(d,l).$$

4.1 Vicsek sets

Firstly we mention another class of finitely ramified fractals, the Vicsek sets. These are fractals constructed as $(2n+1)\times(2n+1)$ checkerboards, level 2 of the $n=3$ case is shown in Figure 5, and we can follow the same procedures to study the asymptotics of their spectral dimension. Firstly we compute the asymptotic

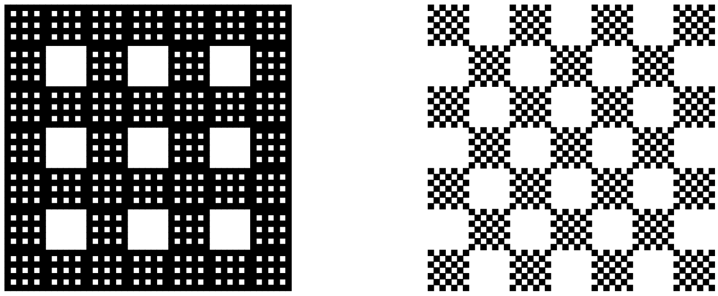

Figure 5: The 7×7 grid-like Sierpinski carpet and Vicsek set

behaviour of the fractal dimension. Let $l = 2n+1$. A simple induction formula shows that if $N_{d,l}$ denotes the number of maps for the d-dimensional Vicsek set with side length l^{-1}, then

$$N_{d,l} = N_{d-1,l} + n(2n+1)^{d-1}.$$

Thus we have $n(2n+1)^{d-1} \le N_{d,l} \le dn(2n+1)^{d-1}$ and

$$d_f(d,l) = \frac{\log N_{d,l}}{\log l} \to d, \text{ as } l \to \infty.$$

For the the spectral dimension we observe that as they are nested fractals we have immediately that $\rho_{d,l} > 1$, for all l, d. In two dimensions we can use an argument from the shorting law to produce a similar network to that on the right hand side of Figure 2, to get

$$\begin{aligned} \rho_{2,l} &\ge \frac{1}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{10} + \dots + \frac{1}{2(2n-1)} + \frac{1}{2(2n+1)} \\ &= 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} + \frac{1}{2(2n+1)} \\ &= \sum_{i=1}^{n} \frac{1}{2i-1} + \frac{1}{2(2n+1)} \sim \frac{1}{2}\log l. \end{aligned}$$

For the upper bound on the resistance we are already on the cubic lattice and hence the usual cutting argument for the cubic lattice gives (with a little work) that in two dimensions $\rho_{2,l} \asymp \log l$, while in higher dimensions $\rho_{d,l} \leq \rho_{3,l} \asymp 1$. Putting these results together with the dimension formulae give the same asymptotics as in Corollary 2.3 for $d_s(d,l)$ and $d_w(d,l)$ as $l \to \infty$. Hence the behaviour is similar to that of the Sierpinski gasket as opposed to the Sierpinski carpet that we consider next.

Note that, if we consider the Vicsek set in which the checkerboard pattern has enough centre cubes removed to leave only a strip two cubes wide around each face, we will be able to prove that $d_s(d,l) \to 1$ as $l \to \infty$ using the same arguments. (In this case, $d_f(d,l) \to 1$ so that the fractal dimension does not converge to the dimension of the embedding space.)

4.2 Grid-like Sierpinski carpets

We construct a family of Sierpinski carpets which have a grid like structure. Consider the d-dimensional unit cube and divide the sides in $l = 2n+1$. Now remove n^d cubes to form a grid consisting of $n+1$ full slabs of cubes alternating with n slabs with the $(d-1)$-dimensional pattern. We iterate this process of removing n^d cubes from each remaining cube to form a sequence of closed sets F_n converging to a Sierpinski carpet. The pattern is shown to level 2 in Figure 5 for the case $n = 3$ in two dimensions. We write L_x for the hyperplane $x_1 = x$.

The fractal dimension of this family of Sierpinski carpets is easily seen to be $d_f(d,l) = \log((2n+1)^d - n^d)/\log(2n+1)$ and as $n \to \infty$ we have $d_f(d,l) \to d$. In order to determine the spectral dimension we need to find the scaling in the resistance. To define the resistance scale factor we consider the scaling in the resistance between opposite faces of the carpet. Let

$$R_m^{-1} = \inf\{\int_{l^m F_m} |\nabla f|^2 dx : f|_{L_0} = 0, f|_{L_{l^m}} = 1\}.$$

In [2] it has been shown that there is a constant ρ_F such that for further constants $c_1, c_2 > 0$, $c_1 \rho_F^m \leq R_m \leq c_2 \rho_F^m$. Note that the scale factor ρ_F cannot be expressed as the solution to a simple variational problem over one step, unlike the finitely ramified case, (2.2).

The key to understanding the resistance scaling is the shorting and cutting argument of [2] Section 5, where it is shown in (5.4) and (5.5) that the resistance scale factor is controlled as follows

$$\sum_{i=1}^{l} a_i^{-1} \leq \rho_F \leq \frac{l}{|\text{disjoint paths of length } l \text{ across F}|}, \tag{4.1}$$

where a_i is the number of cells in the cross section at $x_1 = i/l$.

For our grid-like carpets we can easily see that there exist constants c_3, c_4, such that

$$c_3 l^{2-d} \leq \rho_F \leq c_4 l^{2-d}.$$

Using this in the dimension formulae we obtain for $d \geq 2$,

$$d_s(d,l) = d - \frac{O(1)}{\log l}, \quad d_w(d,l) = 2 + \frac{O(1)}{\log l},$$

for large l. Thus, for this family of Sierpinski carpets, we have convergence of the exponents to those of the embedding space in all dimensions.

4.3 SC(l, b, d)

We can also consider the sets SC(l, b, d) as defined in [2] Remark 5.4. These are formed by removing a central block of b^d cubes from the unit square subdivided into l^d cubes of side l^{-1}. The estimates, derived from (4.1) and given in [2] (5.9), are

$$\frac{l-b}{l^{d-1}} + \frac{b}{l^{d-1} - b^{d-1}} \leq \rho_F \leq \frac{l}{l^{d-1} - b^{d-1}}.$$

Thus if we fix the size of b or let it grow as a proportion strictly less than one of l we find that, as before $\rho_F \asymp l^{2-d}$ and the dimension results are the same as in the previous carpet. However, if we let $b = l - 2$, in that we remove all but the boundary cells, we see that $\rho_F \asymp l^{3-d}$. Thus for the family of 'thin strip' carpets we have, as $l \to \infty$,

$$d_f(d,l) = d - 1 + \frac{O(1)}{\log l}, \quad d_s(d,l) = d - 1 - \frac{O(1)}{\log l}, \quad d_w(d,l) = 2 + \frac{O(1)}{\log l}.$$

4.4 Climbing frames

We can obtain even lower integer values for the spectral dimension by removing more of the borders of the fractals in the family. For example in $d = 3$ we can take a 'climbing frame', where we remove all the cubes except for the cubes which touch at least two faces of the unit cube. Using the same shorting and cutting arguments as before we have that $d_s(3,l) \to 1$ as $l \to \infty$.

This idea can be extended to d-dimensions to produce families which have as limits for their fractal and spectral dimension all integer values between 1 and d. Consider the unit cube in d-dimensions and remove all the cubes that touch at most k faces (where $0 \leq k \leq d-2$). The case $k = 0$ corresponds to the SC$(l, l-2, d)$ case, at $k = d-2$ we have a climbing frame. The number of remaining cells is seen to be $N_{d,l} \asymp l^{d-1-k}$.

In order to estimate the resistance scale factor we use (4.1). For the upper bound we consider the disjoint crossings of the cube. With careful thought we see that there are of order l^{d-2-k} cubes in the front face which connect by a length l path to the back.

For the lower bound we consider one edge and take $d-1$ dimensional cross-sections of the cube. At the two ends there are l^{d-1-k} cubes in the face, while in the middle we see that there are l^{d-2-k}. Putting the two estimates into (4.1) we have

$$2l^{k+1-d} + (l-2)l^{k+2-d} \leq \rho_F \leq c_5 l l^{k+2-d}.$$

Thus $\rho_F \asymp l^{k+3-d}$ and we have that as $l \to \infty$,

$$d_f(d,l) = d-1-k+\frac{O(1)}{\log l}, \quad d_s(d,l) = d-1-k-\frac{O(1)}{\log l}, \quad d_w(d,l) = 2+\frac{O(1)}{\log l}.$$

4.5 Possible limits

Our results so far have the spectral and fractal dimensions of the families converging to the same limit. We now show that this does not necessary occur. Consider the following family of carpets, one member of which is shown on the left side of Figure 6. The only connections between faces are around the edges of the carpet and each internal square is connected to the frame only at the centre of each face. Thus, by using the shorting and cutting arguments again, the resistance scales linearly with l as $l \to \infty$. However due to the internal structure we see that the fractal dimension of the set will converge to 2 as $l \to \infty$ and hence we see that $d_s(2,l) \to 4/3$ and $d_w(2,l) \to 3$ as $l \to \infty$. If we take sequences of

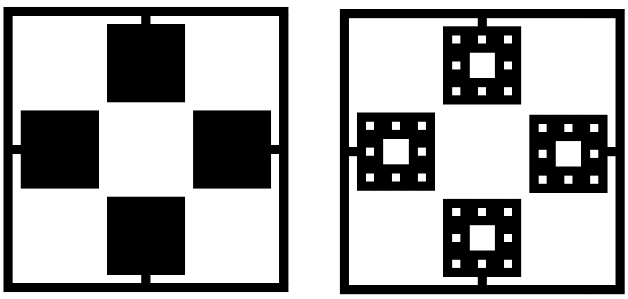

Figure 6: Sierpinski carpets with low spectral dimension

different structures on each face we can make families of carpets which, in the limit, take various (at least a countable infinity) fractal dimensions between 1 and 2. For instance, if we take $l = 3 \cdot 3^n + 2$ and place a $3^n \times 3^n$ approximation of the standard 3×3 Sierpinski carpet in the centre of each face, we get a family of fractals which will converge to an object with the same fractal dimension $\log 8/\log 3$ as the standard 3×3 Sierpinski carpet. The $n = 2$ case is shown on the right hand side of Figure 6. Now, by the construction, the resistance will scale linearly with the length scale factor giving $d_s \to 2\log 8/\log 24$ and $d_w \to \log 24/\log 3$.

It is not difficult to extend this type of construction to higher dimensions. Let the carpets be obtained by removing all hypercubes which do not intersect with any face of the unit cube (i.e. the $(l-2)^d$ subcube in the middle) and now, in the centre of each of the 2^d faces of dimension $d-1$ place a hypercubic structure, connected by a single hypercube, which does not intersect any hypercubes from the other faces. This gives a family of fractals with $d_f(d,l) \to d$. Now observe that the resistance scaling is the same as the case of $\mathrm{SC}(l,b,d)$, so that $\rho_F \asymp l^{3-d}$. Thus the spectral dimension $d_s(d,l) \to 2d/3$ as $l \to \infty$.

In order to push the spectral dimension further down we work with the k-climbing frame. On the centre of each of the faces of dimension $k+1$ place a 'fat' structure of dimension d, connected by a single hypercube, which does not intersect any of the other 'fat' structures from the other faces. Thus the carpet will have full dimension but these 'fat' structures will not affect the resistance scaling. In this case we will have $\rho_F \asymp l^{k+3-d}$ and hence $d_s(d,l) \to 2d/(k+3)$. Thus at the maximum value of $k = d-2$, we get a lower bound for the spectral dimension of $2d_f/(d_f+1)$.

Note that the following relationship holds between d_f and d_w with respect to the Euclidean metric (see [1] Theorem 3.20; where the results are given with respect to the shortest path metric),

$$2 \le d_w \le d_f + d_c. \tag{4.2}$$

Here $d_c \ge 1$ is called the chemical exponent. For the class of symmetric fractals we consider (i.e., nested fractals and Sierpinski carpets), d_c relates the Euclidean metric $\|\cdot\|$ to the shortest path (geodesic) metric $d(\cdot,\cdot)$ on the fractal, i.e. $d(x,y) \asymp \|x-y\|^{d_c}$ (see [4],[8]). For the examples we have introduced, $d_c = 1$. Using (4.2), we immediately have the following bounds on the spectral dimension,

$$\frac{2d_f(d,l)}{d_f(d,l)+d_c} \le d_s(d,l) \le d_f(d,l).$$

The above construction shows that the lower limit is obtainable in two dimensions when $d_c = 1$.

In order to obtain the worst possible behaviour in two dimensions for the case when the asymptotic fractal dimension is 2, we construct a family of carpets using a space filling curve. The generator for a member of such a family is shown in Figure 7. In this way we have $N_{2,l} \asymp l^2$ and $\rho_{d,l} \asymp l^2$ and hence $d_f(2,l) \to 2$

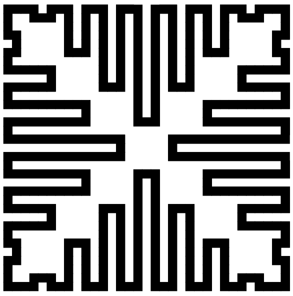

Figure 7: A member of a family which achieves the two dimensional worst case

while $d_s(2,l) \to 1$ and $d_w(2,l) \to 4$.

We conclude with the following question concerning the possible limits of the dimensional exponents for such families of fractals.

Question: Let F_l be a sequence of fractals in $\mathbf{R}^d$, indexed by their length scale factor l and chemical exponent d_c, with spectral dimension $d_s(d,l)$. Let $\bar{d}_s(d) = \lim_{l\to\infty} d_s(d,l)$ if the limit exists. Do there exist families of fractals such that $d_f(d,l) \to \bar{d}$ while all possible limits of the spectral dimension are attainable i.e. the full closed interval $[1,2]$ in the finitely ramified case and $[1,\bar{d}]$ in the infinitely ramified case?

We note that we can answer this question if we are allowed to use homogeneous random fractals, [5]. In that setting such a construction is relatively straightforward as we just select with appropriate probability between a worst case space filling curve fractal (Figure 7) and a grid-like fractal (LHS of Figure 5) and use the continuity of the scale factors with respect to the probability. However, if we can only take exactly self-similar fractals, the answer is not known.

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