Heat Kernel Estimates for Stable-like Processes on d-Sets

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Abstract

The notion of d-set arises in the theory of function spaces and in fractal geometry. Geometrically self-similar sets are typical examples of d-sets. In this paper stable-like processes on d-sets are investigated, which include reflected stable processes in Euclidean domains as a special case. More precisely, we establish parabolic Harnack principle and derive sharp two-sided heat kernel estimate for such stable-like processes. Results on the exact Hausdorff dimensions for the range of stable-like processes are also obtained.

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1 Introduction

Let F be a closed d-set in $\mathbf{R}^n$ with $n \geq 2$ and $0 < d \leq n$. That is, there is a positive Borel measure μ on F such that there exist $C_2 > C_1 > 0$ so that

$$C_1 r^d \leq \mu(B(x,r)) \leq C_2 r^d \qquad \text{for all } \ x \in F, \ 0 < r \leq 1. \tag{1.1}$$

Here $B(x,r) := \{y \in F : |x-y| < r\}$ and $|\cdot|$ is the Euclidean metric in $\mathbf{R}^n$. Such a measure μ is called a d-measure (which is also called Ahlfors regular in some literatures) on F. The notion of d-set arises in the theory of function spaces and in fractal geometry. Geometrically self-similar sets are typical examples of d-sets. It is known (cf. Jonsson and Wallin [22]) that F is a d-set if and only if (1.1) holds with μ being the d-dimensional Hausdorff measure m_d restricted to F, and that if F is a d-set and μ_1 and μ_2 are two d-measures on F then there are positive constants c_1 and c_2

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such that $c_1\mu_2 \leq \mu_1 \leq c_2\mu_2$ on F. On F, one can define a symmetric bilinear form $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ as follows. Fix a d-measure μ on F and $0 < \alpha < 2$. Define

$$\mathcal{F}^{(\alpha)} = \left\{ u \in \mathbf{L}^2(F, \mu) : \int_{F\times F} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mu(dx)\mu(dy) < \infty \right\} \tag{1.2}$$

$$\mathcal{E}^{(\alpha)}(u,v) = \frac{1}{2} \int_{F\times F} \frac{c(x,y)(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}} \mu(dx)\mu(dy) \tag{1.3}$$

for $u,\ v \in \mathcal{F}^{(\alpha)}$, where $c(x,y)$ is a symmetric function on $F \times F$ that is bounded between two strictly positive constants $C_4 > C_3 > 0$, that is,

$$C_3 \leq c(x,y) \leq C_4 \qquad \text{for } \mu\text{-a.e. } x, y \in F. \tag{1.4}$$

It is easy to check that $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ is a regular Dirichlet form on $\mathbf{L}^2(F,\mu)$ and therefore there is an associated μ-symmetric Hunt process Y on F starting from every point in F except for an exceptional set that has zero capacity. We call such kind of process a stable-like process on F. (Note that some authors use the terminology "stable-like process" for a jump process on $\mathbf{R}^n$ whose generator at $x \in \mathbf{R}^n$ is $-(-\Delta)^{\alpha(x)}$ where $\delta < \alpha(x) \leq 1$. See [20] and the references therein.)

The study of stable-like processes on d-sets is motivated by the work of function space theory on d-sets. Indeed, there have been intensive study of Besov spaces by Triebel [32], Jonsson and Wallin [22], and others on d-sets. They have extended trace and restriction theorems (such as Theorem 2.1) and Sobolev-type embedding theorems to such disordered spaces. This enables them to obtain results on the asymptotic behavior of eigenvalues for the corresponding self-adjoint operators on these fractal-like spaces. Recently, it has become clear that there is natural correspondence between Besov spaces on d-sets and non-local regular Dirichlet spaces (1.2)-(1.3) on these sets. For example, the reflected stable processes in a Euclidean domain studied in Bogdan, Burdzy and Chen [5] and the subordination of reflecting Brownian motion in a Euclidean domains studied in Farkas and Jacob [14] and in Jacob and Schilling [19] are examples of processes associated with Dirichlet space of the form (1.2)-(1.3).

Various results of Dirichlet forms have been shown on d-sets. Fukushima and Uemura [16] obtained capacitary inequalities on contractive Besov spaces including $\mathcal{F}^{(\alpha)}$. Stós [29] and Kumagai [25] defined several stable-like processes on a d-set F as a subordination of some nice diffusions on F or as the trace of the n-dimensional Brownian motion on F and proved that the corresponding Dirichlet forms are comparable. As a related work of Besov spaces on d-sets, Caetano [8] and Farkas and Jacob [14] study the question whether the space of smooth functions with compact support in a Euclidean domain is dense in the Besov or Triebel-Lizorkin type spaces. In the fractal contexts, the domain of local regular Dirichlet forms which correspond to Brownian motion are shown to be function spaces of Besov and Lipschitz types (see [21, 24, 27, 17]). There are also related work by

Strichartz [30] on relations between various function spaces of Hölder-Zygmond, Besov and Sobolev types and by Zähle [35] studying pseudo-differential operators through Riesz potentials on fractals.

The purpose of this paper is to establish a parabolic Harnack principle and to give two-sided sharp estimate for the transition density function of Y. As a consequence, we show that Y can be refined to be a Feller process starting from every point in F. Here is the main result of this paper.

Theorem 1.1 *Suppose that F is a closed d-set satisfying*

$$\mu(B(x,r)) \leq C_2\, r^d \qquad \text{for every } x \in F \text{ and } r > 0. \tag{1.5}$$

Then, for $0<\alpha<2$, there is a Feller process Y on F associated with the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ on $\mathbf{L}^2(F,\mu)$ and Y has a Hölder continuous transition density function $p(t,x,y)$. Furthermore, there are constants $c_2>c_1>0$ that depend only on n, d, α, and the constants C_1 and C_2 in (1.1). such that

$$c_1 \min\left\{(t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}}\right\} \leq p(t,x,y) \leq c_2 \min\left\{t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}}\right\}, \tag{1.6}$$

for all $x,y\in F$ and $0<t\leq 1$. Moreover, except for the case of $0<\alpha=d<2$, the constants c_1 and c_2 can be chosen to depend only on (n,d,α) and on the constants (C_1,C_2,C_3,C_4) in (1.1), (1.5) and (1.4) respectively.

It is known (cf. Jonsson and Wallin [22]) that if F is a closed set satisfying (1.1) then for every finite $r_0>0$, inequality (1.1) holds for every $r\leq r_0$, possibly with different values of C_1 and C_2. Hence any bounded d-set satisfies condition (1.5).

As an application of Theorem 1.1, we show the following.

Theorem 1.2 *Under the assumption of Theorem 1.1, for every $x\in F$, $\mathbf{P}^x$-a.s., the Hausdorff dimension of $Y[0,1] := \{Y_t : 0\leq t\leq 1\}$ is $\min\{\alpha, d\}$.*

We remark that when F is the Euclidean closure of an open n-set in $\mathbf{R}^n$, $c(x,y)$ is constant and μ is the Lebesgue measure on $\mathbf{R}^n$, the corresponding process Y is the reflected α-stable process on F studied in Bogdan, Burdzy and Chen [5]. In particular, when $F=\mathbf{R}^n$ and μ is the Lebesgue measure on $\mathbf{R}^n$, Y is a symmetric α-stable process in $\mathbf{R}^n$. Note that our results are applicable to Riemannian manifolds setting, since any compact Riemannian manifold F can be embedded into a higher dimensional Euclidean space $\mathbf{R}^n$ as a d-set.

The program of this paper is to study the two-sided heat kernel estimates for the processes given by Dirichlet forms (1.2)-(1.3). For this, we first derive the on-diagonal heat kernel estimates by establishing the Nash's inequality in Section 3. We next establish various estimates on hitting probabilities and prove the parabolic Harnack inequality in Section 4. The heat kernel estimates (1.6) then follows from these hitting probability estimates and the parabolic Harnack inequality. We point out that if we assume *a priori* that the transition density function $p(t,x,y)$ of a symmetric Markov process Y on F has estimate (1.6), then it is quite standard to show that the parabolic Harnack inequality holds for Y and that parabolic harmonic functions of Y are Hölder continuous (see the approach in Fabes and Stroock [12]. Alternatively, using the two-sided estimate (1.6), one can easily check that the tightness results in Section 4 holds and thus yield the parabolic Harnack inequality.

As mentioned previously, stable-like processes on a d-set also arise from diffusion processes on fractal sets through subordination. For this connection, assume that there exists a μ-symmetric conservative Feller diffusion $\{B_t\}_{t\geq 0}$ on F which has a jointly continuous symmetric transition density $p(t,x,y)$ with the following estimates: there are constants $c_i > 0$ for $i = 1,2,3,4$, $d_s > 0$ and $d_w \geq 2$ such that

$$\begin{aligned} c_1\, t^{-d_s/2} \exp\left(-c_2 \left(|x-y|^{d_w}/t\right)^{1/(d_w-1)}\right) &\leq p(t,x,y) \qquad (1.7) \\ \leq c_3\, t^{-d_s/2} \exp\left(-c_4 \left(|x-y|^{d_w}/t\right)^{1/(d_w-1)}\right) &\quad \text{for all } 0<t\leq 1 \text{ and } x,y\in F. \end{aligned}$$

It is known that certain fractals such as Sierpinski gaskets and Sierpinski carpets admit such diffusion processes. For $0 < s < 2/d_w$, let $\{\xi_t\}_{t>0}$ be the strictly s-stable subordinator, i.e., it is a one dimensional non-negative Lévy process independent of $\{B_t\}_{t\geq 0}$ with $\xi_0 = 0$ and $\mathbf{E}[\exp(-\lambda(\xi_{t+s} - \xi_s)] = \exp(-t\lambda^s)$ for all positive λ, t and s. Then, as it is shown in Stós [29] and Kumagai [25], the s-subordinated process $Y_t := B_{\xi_t}$ of B is a μ-symmetric strong Markov process on F whose Dirichlet form has the form of (1.2)-(1.3) with $\alpha = s\, d_w$.

Harnack inequalities and two-sided heat kernel estimates for general stable-like processes in $\mathbf{R}^d$ have only been studied very recently. In [23], Kolokoltsov obtained (1.6) for certain stable-like processes in $\mathbf{R}^d$. Bass and Levin [4] used a completely different approach to obtain a similar estimate and a parabolic Harnack inequality for discrete time Markov chain on $\mathbf{Z}^d$ where the conductance between x and y is comparable to $|x-y|^{-(d+\alpha)}$ for $0<\alpha<2$. In a closely related work [3], Bass and Levin established an elliptic Harnack inequality for stable-like processes in $\mathbf{R}^d$. Later Song and Vondraček [28] applied their technique to extend the Harnack inequality to certain class of jump-type processes in $\mathbf{R}^d$.

The approach of this paper is very much influenced by those in Bass and Levin [3, 4]. However there are some new twists for stable-like processes on d-sets, which also yield some new results even in the $\mathbf{R}^n$ case. For example, in general even with $F = \mathbf{R}^n$, the space $C_c^\infty(\mathbf{R}^n)$ of smooth functions with compact support in $\mathbf{R}^n$ may not be contained in the domains of the infinitesimal generators of the processes considered in this paper and the processes may not be a semimartingale. In this sense, the parabolic Harnack inequality established in Proposition 4.3 below extends the Harnack inequality derived in Bass and Levin [3], whose result when applies to symmetric process Y requires that $c(x,y)$ in the Dirichlet form (1.3) of Y with $F=\mathbf{R}^n$ and $c(x,y)$ be of the form of $f(x,y-x)$, where $f(x,h)$ is an even function in h. That extra assumption in $c(x,y)$ enables them to write down the infinitesimal generator of Y in terms of the principle value of an integral. In this paper, no such condition is imposed on $c(x,y)$.

In this paper, we use ":=" as a way of definition, which is read as "is defined to be" and "=:" means "is denoted by". For two real numbers a and b, $a\wedge b := \min\{a,\, b\}$. For functions f and g, notation "$f\approx g$" means that there exist constants $c_2>c_1>0$ such that $c_1\, g\le f\le c_2\, g$. We will use c, with or without subscripts, to denote strictly positive constants whose values are insignificant and may change from line to line. For a set $A\subset \mathbf{R}^n$, we will use $\mathrm{diam}(A)$ and $\dim_H A$ to denote the diameter and the Hausdorff dimension of A, respectively.

2 Besov Spaces on d-Sets

In this section we recall the definitions and some basic properties of Besov spaces on F. Throughout this section, F is a closed d-set in $\mathbf{R}^n$, which can be unbounded.

For $0<s<1$, the Besov space $B^{2,2}_s(F)$ on F is defined by

$$B^{2,2}_s(F)=\left\{u: u \text{ is measurable on } F, \quad \|u|B^{2,2}_s(F)\|<\infty\right\}, \tag{2.1}$$

where

$$\|u|B^{2,2}_s(F)\| := \|u\|_{\mathbf{L}^2(F,\mu)}+\left(\int_{F\times F}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mu(dx)\mu(dy)\right)^{1/2}. \tag{2.2}$$

For each $f\in \mathbf{L}^1_{loc}(\mathbf{R}^n)$ and $x\in\mathbf{R}^n$, define

$$Rf(x)=\lim_{r\downarrow 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}f(y)dy,$$

if the limit exists, where m is the Lebesgue measure in $\mathbf{R}^n$. It is well-known that the limit exists quasi-everywhere in $\mathbf{R}^n$ with respect to the Newtonian capacity if $n\ge 3$ or logarithmic capacity if $n=2$ and coincides with $f(x)$ almost everywhere in $\mathbf{R}^n$. For each $\beta>0$, denote by $B^{2,2}_\beta(\mathbf{R}^n)$ the classical Besov space on $\mathbf{R}^n$ (see Remark 2.2 below for its definition). The following trace theorem plays an important role in the study of Besov spaces on d-sets (see, for instance, Chapters V and VI in [22] or Section 20 in [32]).

Theorem 2.1 *For $0 < s < 1$, the trace operator $Tr_F : f \mapsto Rf$ is a bounded linear surjection from $B^{2,2}_{s+(n-d)/2}(\mathbf{R}^n)$ onto $B^{2,2}_s(F)$ and it has a bounded linear right inverse operator E_F (which is called the extension operator in literature) so that $Tr_F \circ E_F$ is the identity map on $B^{2,2}_s(F)$. The operator norm of the extension operator E_F is bounded by a constant that depends only on the constants C_1 and C_2 in (1.1).*

Remark 2.2 (i) The last assertion of Theorem 2.1 is not explicitly stated in [22]. However if following the proof there in section V.1.3 carefully, one sees easily that the operator norm of the extension operator E_F is bounded by a constant that depends only on the constants C_1 and C_2 in (1.1).

(ii) Note that for $\beta > 0$ with integer $k < \beta \le k+1$, the classical Besov space $B^{2,2}_\beta(\mathbf{R}^n)$ is defined to be

$$B^{2,2}_\beta(\mathbf{R}^n) = \left\{ u \in C^k(\mathbf{R}^n) : \ \|u\|_{B^{2,2}_\beta} := \sum_{0\le|j|\le k} \|D^j u\|_2 + \sum_{|j|=k} \left(\int_{\mathbf{R}^n} \frac{\|\Delta_h D^j f\|_2^2}{|h|^{n+2(\beta-k)}} dh \right)^{1/2} < \infty \right\},$$

where for $j = (j_1, j_2, \cdots, j_n) \in \mathbf{Z}^n_+$, $|j| = \sum_{k=1}^n j_k$ and $D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$, Δ_h is the difference operator so that for $h \in \mathbf{R}^n$, $(\Delta_h f)(x) = f(x+h) - f(x)$, and $\| \cdot \|_2$ denotes the $\mathbf{L}^2$-norm in $\mathbf{L}^2(\mathbf{R}^n, m)$ (see, for instance, section I.1.5 in [22].) It is known (cf. Section V.1.1 in [22]) that when $0 < \beta < 1$, norm $\|u\|_{B^{2,2}_\beta}$ is equivalent to $\left\| u \,\middle|\, B^{2,2}_\beta(\mathbf{R}^n) \right\|$ defined by (2.2) with $F = \mathbf{R}^n$, and therefore $B^{2,2}_\beta(\mathbf{R}^n)$ is the same as the space defined by (2.1) with $F = \mathbf{R}^n$. Furthermore, space $B^{2,2}_\beta(\mathbf{R}^n)$ coincides with the classical Bessel potential space on $\mathbf{R}^n$ (also called the fractional Sobolev space or the Liouville space); see, for instance, p. 8 in Section I.1.5 of [22]. □

Clearly $\mathcal{F}^{(\alpha)}$ in (1.2) is just $B^{2,2}_{\alpha/2}(F)$ and by Theorem 2.1, it is easy to check that $(\mathcal{E}^{(\alpha)}, B^{2,2}_{\alpha/2}(F))$ is a regular Dirichlet space on $\mathbf{L}^2(F, \mu)$ (a detailed proof for this is given, for instance, in Theorem 3 of [29]). We note that when F is the Euclidean closure of an open n-set in $\mathbf{R}^n$, $c(x,y)$ is constant and μ is the Lebesgue measure in $\mathbf{R}^n$, the associated process Y is a reflected α-stable process on F studied in Bogdan, Burdzy and Chen [5]. When $F = \mathbf{R}^n$ and μ is the Lebesgue measure on $\mathbf{R}^n$, Y is a symmetric α-stable process in $\mathbf{R}^n$.

By the trace theorem stated in Theorem 2.1 above and the Adams' embedding theorem for Besov space $B^{2,2}_\beta(\mathbf{R}^n)$ (see Theorem on p. 8 and Lemma 1 on p.214 in Jonsson and Wallin [22]), the following Sobolev inequality holds on a d-set F when $0 < \alpha < d \wedge 2$:

$$\|u\|_{\mathbf{L}^{2d/(d-\alpha)}(F,\mu)} \le c \, \|u | B^{2,2}_{\alpha/2}(F)\| \qquad \text{for all } u \in B^{2,2}_\alpha(F), \tag{2.3}$$

where c depends only on d, α, and the constants C_1 and C_2 in (1.1). Using this and the Varopoulos theorem (see [33] or Corollary 2.4.3 in [10]), one concludes that for $0<\alpha<d\wedge 2$, there is a Borel set $N\subset F$ having zero capacity with respect to $(\mathcal{E}^{(\alpha)}, B^{2,2}_{\alpha/2}(F))$ so that the symmetric strong Markov process $Y^{(\alpha)}$ associated with $(\mathcal{E}^{(\alpha)}, B^{2,2}_{\alpha/2}(F))$ in (1.2)-(1.3) has a density functions $p_t^{(\alpha)}(x,y)$ with respect to the measure μ for every $x\in F\setminus N$ and that the following upper bound holds:

$$e^{-t}\,p_t^{(\alpha)}(x,y)\le c\,t^{-d/\alpha} \qquad \text{for every } t>0,\ x\in F\setminus N, \text{ and for } \mu\text{-a.e. } x,\in F. \tag{2.4}$$

3 Nash Inequality for Besov Spaces on d-Sets

Throughout this section, F is a closed d-set in $\mathbf{R}^n$ with $0<d\le n$. We will show that the upper bound estimate (2.4) holds for the transition density function of $Y^{(\alpha)}$ with $0<d\le\alpha<2$ as well. Note that the equivalence between the heat kernel on-diagonal upper bound estimate (2.4) and the Sobolev inequality (2.3) holds only for $\alpha<d\wedge 2$, while it is known that (2.4) is equivalent to the Nash inequality 3.1 below for every $\alpha\in(0,\,2)$ and $d\in(0,\,n]$ (see Theorem 2.1 in Carlen, Kusuoka and Stroock [9]). So we will establish a Nash's inequality for Besov space $B^{2,2}_{\alpha/2}(F)$. In the case of $\alpha>d$, the Nash's inequality will be obtained by using Sobolev embedding theorem for Besov space on $\mathbf{R}^n$, while for the case of $\alpha=d$, it will be derived by using a combination of complex interpolation method and probabilistic subordination technique.

Proposition 3.1 *Suppose that $d<\alpha<2$. There exists a constant $c>0$ that depends only on n, d, α, and the constants C_1 and C_2 in (1.1) such that*

$$\|u\|_{\mathbf{L}^2}^{2+\frac{2\alpha}{d}}\le c\,\left\|u|B^{2,2}_{\alpha/2}(F)\right\|^2\cdot\|u\|_{\mathbf{L}^1}^{\frac{2\alpha}{d}} \qquad \textit{for all }\ u\in B^{2,2}_{\alpha/2}(F), \tag{3.1}$$

where $\|\cdot\|_{\mathbf{L}^p}$ is the $\mathbf{L}^p$-norm in $\mathbf{L}^p(F,\mu)$.

Proof. First we claim that, with $d<\alpha<2$, every function u in $B^{2,2}_{\alpha/2}(F)$ is a continuous function on F that vanishes at infinity. Indeed, by Theorem 2.1, there is a bounded extension operator from $B^{2,2}_{\alpha/2}(F)$ into $B^{2,2}_{(n+\alpha-d)/2}(\mathbf{R}^n)$, while according to Theorem 2.8.1 in [31], the following embedding holds:

$$B^{2,2}_{(n+\alpha-d)/2}(\mathbf{R}^n)\subset C_H^{(\alpha-d)/2}(\mathbf{R}^n). \tag{3.2}$$

Here for $0<\beta<1$,

$$C_H^\beta(\mathbf{R}^n)=\left\{u\in C(\mathbf{R}^n):\ \|u\|_{C^\beta}:=\sup_{x\in\mathbf{R}^n}|u(x)|+\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\beta}<\infty\right\}.$$

So $B^{2,2}_{\alpha/2}(F)$ can be continuously embedded into the Hölder space $C_H^{(\alpha-d)/2}(\mathbf{R}^n)$. As $C_c^\infty(\mathbf{R}^n)$ is dense in $B^{2,2}_{(n+\alpha-d)/2}(\mathbf{R}^n)$ (see, e.g., Theorem 4.1.3 in [1]) and the latter space can be continuously embedded into the Hölder space $C_H^{(\alpha-d)/2}(\mathbf{R}^n)$ by (3.2), we have $B^{2,2}_{(n+\alpha-d)/2}(\mathbf{R}^n) \subset C_\infty(\mathbf{R}^n)$, the space of continuous functions vanishing at infinity. It now follows from Theorem 2.1 that

$$B^{2,2}_{\alpha/2}(F) \subset C_\infty(F),$$

where $C_\infty(F)$ is the space of continuous functions on F vanishing at infinity.

Next we show that the Nash's inequality holds. To simplify notations, set $Q_1(u) = \|u|B^{2,2}_{\alpha/2}(F)\|^2$ for $u \in B^{2,2}_{\alpha/2}(F)$. By (3.2) and Theorem 2.1, there is $c_1 > 0$ that depends only on n, d, α, and the constants C_1 and C_2 in (1.1) such that

$$|u(x) - u(y)| \le c_1 \sqrt{Q_1(u)} |x-y|^\beta \qquad \text{for all } u \in B^{2,2}_{\alpha/2}(F) \text{ and } x, y \in F, \tag{3.3}$$

where $\beta = (\alpha - d)/2$. Note that for $u \in B^{2,2}_{\alpha/2}(F) \subset C_\infty(F)$, there is $x_u \in F$ such that

$$|u(x_u)| = \|u\|_\infty := \max_{x\in F} |u(x)|. \tag{3.4}$$

Without loss of generality, we may and do assume that $\|u\|_\infty = 1$. From (3.3), we have

$$|u(x_u)| - |u(y)| \le c_1 |x_u - y|^\beta \sqrt{Q_1(u)}.$$

So $|u(y)| \ge 1/2$ on the set $\{y \in F : \, |y - x_u| < (2c_1\sqrt{Q_1(u)})^{-1/\beta}\}$. Let $r_0 = \min\left\{(2c_1\sqrt{Q_1(u)})^{-1/\beta}, \, 1\right\}$. Then

$$Q_1(u) \ge \int_{B(x_u, r_0)} u(x)^2 \mu(dx) \ge \frac{1}{4}\, \mu(B(x_u, r_0)) \ge \frac{1}{4}\, C_1 r_0^d.$$

It follows that $Q_1(u) \ge c_0$, where $c_0 > 0$ is a constant that depends only on (c_1, β, d), which in turn depend only on (n, d, α). Hence it follows from the lower bound in (1.1),

$$\|u\|^2_{\mathbf{L}^2} \ge \frac{1}{4}\mu\left(\left\{y \in F : |y - x_u|^\beta < (2c_1\sqrt{Q_1(u)})^{-1}\right\}\right) \ge c_2 Q_1(u)^{-d/(2\beta)}, \tag{3.5}$$

where $c_2 = \frac{1}{4} C_1\, (2c_1)^{-d/\beta}$. Now, with $\mu = 2d/\alpha$,

$$\begin{aligned} \|u\|^{2+4/\mu}_{\mathbf{L}^2} &= \|u\|^{2-4/\mu}_{\mathbf{L}^2} \|u\|^{8/\mu}_{\mathbf{L}^2} \le \|u\|^{2-4/\mu}_{\mathbf{L}^2} \|u\|^{4/\mu}_\infty \|u\|^{4/\mu}_{\mathbf{L}^1} \\ &\le (c_2 Q_1(u)^{-d/(2\beta)})^{1-2/\mu} \|u\|^{4/\mu}_{\mathbf{L}^1} = c_3 Q_1(u) \|u\|^{4/\mu}_{\mathbf{L}^1}. \end{aligned}$$

Clearly $c_3 = c_2^{1-2/\mu}$ is a constant that depends only on n, d, α, and the constants C_1 and C_2 in (1.1). This proves the proposition. □

By Theorem 2.1 of [9], Proposition 3.1 implies that the upper bound estimate (2.4) holds when $d < \alpha < 2$. However the proof in Proposition 3.1 breaks down when $d = \alpha$. In Proposition 3.3, we will show that the Nash's inequality (3.1) holds for the critical case $d = \alpha < 2$ as well.

Recall that a Borel set $N \subset F$ is called properly exceptional (with respect to Y) if $\mu(N) = 0$ and for every $x \in F \setminus N$,

$$\mathbf{P}^x(\text{there is some } t > 0 \text{ such that } Y_t \in N \text{ or } Y_{t-} \in N) = 0.$$

Theorem 3.2 *For every $0 < \alpha < 2$ and $\alpha \neq d$, there exists a properly exceptional set N of Y and a jointly measurable symmetric function $p(t, x, y)$ on $[0, \infty) \times \widetilde{F} \times \widetilde{F}$ such that*

$$\mathbf{E}^x[f(Y_t)] = \int_{\widetilde{F}} p(t, x, y) f(y) \mu(dy) \qquad \text{for every } x \in \widetilde{F} \text{ and } f \geq 0,$$

and satisfies the Chapman-Kolmogorov equation

$$p(t+s, x, y) = \int_{\widetilde{F}} p(s, x, z) p(t, z, y) \mu(dz) \qquad \text{for every } s,\, t \geq 0 \text{ and } x, y \in \widetilde{F},$$

and that

$$e^{-t} p(t, x, y) \leq c\, t^{-d/\alpha} \qquad \text{for every } t > 0 \text{ and } x, y \in \widetilde{F}, \tag{3.6}$$

where $\widetilde{F} := F \setminus N$ and $c > 0$ is a constant that depends only on n, d, α, and the constants C_1, C_2, C_3 and C_4 in (1.1) and (1.4) respectively. So when the process Y is restricted on $\widetilde{F}$, $p(t, x, y)$ is its transition density function.

Proof. Using the Sobolev inequality (2.3) when $\alpha < d$ and the Nash's inequality when $d < \alpha < 2$ and by the same argument as those on page 52 in Barlow [2] (while using Theorem 2 of [34]), one concludes that process $Y^{(\alpha)}$ admits a properly exceptional set N and such a jointly measurable symmetric function $p(t, x, y)$ on $[0, \infty) \times \widetilde{F} \times \widetilde{F}$. □

Now we show that the Nash's inequality (3.1) and therefore (3.6) holds for the critical case $d = \alpha < 2$ as well, by using an interpolation method and a probabilistic subordination technique. We will suppress $\sim$ from $\widetilde{F}$ and write $\widetilde{F}$ as F.

Proposition 3.3 *For the case of $0 < d = \alpha < 2$, there exists $c > 0$ such that*

$$\|u\|_{\mathbf{L}^2}^4 \leq c \left\| u | B^{2,2}_{\alpha/2}(F) \right\|^2 \cdot \|u\|_{\mathbf{L}^1}^2 \qquad \text{for every } u \in B^{2,2}_{\alpha}(F). \tag{3.7}$$

Proof. Fix some $\beta \in (\alpha, 2)$. It follows from the complex interpolation (see [22]),

$$B^{2,2}_{\alpha/2}(F) = \Big[L^2(F), B^{2,2}_{\beta/2}(F)\Big]_\theta ,$$

where $\theta = \alpha/\beta$. Let $\overline{Y}^{(\beta)}$ be the 1-subprocess of $Y^{(\beta)}$ on F associated with $(\mathcal{E}^{(\beta)}, B^{2,2}_{\beta/2})(F))$ in (1.2)-(1.3) with β in place of α there. As $d < \beta < 2$, it follows from Proposition 3.1 and Theorem 3.2 that $\overline{Y}^{(\beta)}$ has transition density function $\overline{p}^{(\beta)}_t(x, y)$ with $\overline{p}^{(\beta)}_t(x, y) \le c_1 t^{-d/\beta}$ for $t > 0$. If we use $\overline{Y}$ to denote the process obtained from $Y^{(\beta)}$ through θ-subordination, then the domain of the Dirichlet space of $\overline{Y}$ is $B^{2,2}_{\alpha/2}(F)$ and its $\mathcal{E}_1$-norm is equivalent to that of $B^{2,2}_{\alpha/2}(F)$ (cf. Farkas and Jacob [14]). Hence to establish (3.7), it suffices to show (cf. Theorem 2.1 in Carlen, Kusuoka and Stroock [9]) that the heat kernel $\overline{p}_t(x, y)$ of $\overline{Y}$ has the upper bound $c_2 t^{-1}$ for $t \le 1$.

Let $\eta(t, z)$ be the density function for the θ-subordinator, i.e.

$$\int_0^\infty e^{-\lambda z}\eta(t, z)dz = e^{-t\lambda^\theta} \quad \text{for any } \lambda > 0.$$

Function η has the scaling property $\eta(\lambda t, z) = \lambda^{-1/\theta}\eta(t, \lambda^{-1/\theta}z)$. It follows that for $0 < t \le 1$,

$$\begin{aligned}
\overline{p}_t(x, y) &= \int_0^1 \overline{p}^{(\beta)}_s(x, y)\eta(t, s)ds + \int_1^\infty \overline{p}^{(\beta)}_s(x, y)\eta(t, s)ds \\
&\le \int_0^1 \overline{p}^{(\beta)}_s(x, y)t^{-1/\theta}\eta(1, t^{-1/\theta}s)ds + c_3 \\
&= \int_0^{t^{-1/\theta}} \overline{p}^{(\beta)}_{t^{1/\theta}r}(x, y)\eta(1, r)dr + c_3 \\
&\le c_1 \int_0^{t^{-1/\theta}} (t^{1/\theta}r)^{-d/\beta}\eta(1, r)dr + c_3 \\
&\le c_1\, t^{-1} \int_0^\infty r^{-d/\beta}\eta(1, r)dr + c_3.
\end{aligned}$$

Here in the last inequality we used the fact that $\theta = \alpha/\beta$ and that $d = \alpha$. As $d < \beta < 2$, $\eta(1, r)$ is continuous and $\eta(1, r) \le c_4 r^{-1-\theta}$ for large r,

$$\int_0^\infty r^{-d/\beta}\eta(1, r)dr < \infty.$$

Thus we have proved that there is a constant $c_2 > 0$ such that $\overline{p}_t(x, y) \le c_2\, t^{-1}$ for $t \le 1$. $\square$

Since the Nash's inequality holds for the critical case $0 < d = \alpha < 2$, we see that, by exactly the same argument, Theorem 3.2 holds for $0 < d = \alpha < 2$ as well. In other words, Theorem 3.2 holds for every $0 < \alpha < 2$.

4 Heat Kernel Estimates

Throughout the remaining of this paper, we assume that F is a closed d-set in $\mathbf{R}^n$ satisfying condition (1.5).

In this section, we will prove Theorem 1.1, the main theorem of this paper. For notational convenience we will suppress the superscript (α) from $Y^{(\alpha)}$ and $\mathcal{E}^{(\alpha)}$ when there is no danger of confusions.

Our approach is motivated by the work of Bass and Levin [3, 4] on stable-like processes on $\mathbf{Z}^n$ and on $\mathbf{R}^n$. However there are some new twists for processes on d-sets, as paper [3] deals with stable-like processes on $\mathbf{R}^n$, when restricted to the symmetric processes case, requiring $c(x,y) = f(x, y-x)$ and $f(x,h)$ be an even function in h, while paper [4] is concerned about the transition density function estimates for discrete time stable-like Markov chains on $\mathbf{Z}^n$.

To keep the exposition as transparent as possible, we first state three key propositions that are needed to prove Theorem 1.1, followed by the proof of Theorem 1.1, and then give proofs for these three propositions. Let $p(t,x,y)$ be the transition density function of Y on $\widetilde{F} := F \setminus N$, where N is the properly exceptional set in Theorem 3.2. Propositions 4.1–4.3 will imply the heat kernel estimate (1.6) of $p(t,x,y)$ for $x,\ y \in \widetilde{F}$. In Theorem 4.14, we will show that such $p(t,x,y)$ is jointly Hölder continuous in (t,x,y) and therefore it can be extended continuously to $[0,\infty)\times F\times F$, which proves Theorem 1.1. In the remaining of this section, we will suppress $\widetilde{\ }$ from $\widetilde{F}$ and write $\widetilde{F}$ as F.

The first proposition is a tightness result for Y_t. For a subset $K \subset F$, we let $\sigma_K := \inf\{t \geq 0 : Y_t \in K\}$ and $\tau_K := \inf\{t \geq 0 : Y_t \notin K\}$ to denote the first entering and exiting time of K by Y. We will use $B(x,r)$ to denote a ball centered at x with Euclidean radius r.

Proposition 4.1 *For each $r_0 > 0$, $A > 0$ and $0 < B < 1$, there exists $0 < \gamma < 1$ such that for every $0 < r \leq r_0$,*

$$\mathbf{P}^x \left(\tau_{B(x,\, Ar)} < \gamma\, r^\alpha \right) \leq B.$$

Moreover, except for the case of $0 < \alpha = d < 2$, the constant γ can be chosen to depend only on $(r_0, A, B, n, d, \alpha)$ and the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) respectively.

The next proposition is an analogy of two inequalities in Proposition 4.7 and in the proof of Theorem 5.2 in Bass and Levin [4].

Proposition 4.2 *(i) For each $a > 0$, there exists $c_1 > 0$ such that*

$$\mathbf{P}^x \left(\sigma_{B(y,\, ar)} < r^\alpha \right) \leq c_1 \left(\frac{r}{|x-y|} \right)^{d+\alpha} \qquad \text{for every } r \in (0,\ 2^{1/\alpha}]. \tag{4.1}$$

Moreover, except for the case of $0 < \alpha = d < 2$, *the constant* c_1 *above can be chosen to depend only on* (a, n, d, α) *and on the constants* (C_1, C_2, C_3, C_4) *in (1.1), (1.5) and (1.4) respectively.*
(ii) For each $a, b > 0$, *there exists* $c_2 > 0$ *such that*

$$\mathbf{P}^x\left(\sigma_{B(y,ar)} < r^\alpha\right) \geq c_2 \left(\frac{r}{|x-y|}\right)^{d+\alpha}, \tag{4.2}$$

for every $r \in (0,\, 2^{1/\alpha}]$ *and such that* $|x-y| \geq b\,r$. *Moreover, except for the case of* $0 < \alpha = d < 2$, *the constant* c_2 *above can be chosen to depend only on* (a, b, n, d, α) *and on the constants* (C_1, C_2, C_3, C_4) *in (1.1), (1.5) and (1.4) respectively.*

The last key proposition is a parabolic Harnack inequality. For this we need to introduce space-time process $Z_s := (V_s, Y_s)$, where $V_s = V_0 + s$. The filtration generated by Z satisfying the usual condition will be denoted as $\{\widetilde{\mathcal{F}}_s;\ s \geq 0\}$. The law of the space-time process $s \mapsto Z_s$ starting from (t, x) will be denoted as $\mathbf{P}^{(t,x)}$. We say that a non-negative Borel measurable function $q(t, x)$ on $[0, \infty) \times F$ is *parabolic* in a relatively open subset D of $[0, \infty) \times F$ if for every relatively compact open subset D_1 of D, $q(t, x) = \mathbf{E}^{(t,x)}\left[q(Z_{\tau_{D_1}})\right]$ for every $(t, x) \in D_1$, where $\tau_{D_1} = \inf\{s > 0 :\ Z_s \notin D_1\}$.

For each $R_0 > 0$, we denote $\gamma_{R_0} := \gamma(R_0, 1/2, 1/2) < 1$ the constant in Proposition 4.1 corresponding to $r_0 = R_0$ and $A = B = 1/2$. For $t \leq 1$ and $r \leq R_0$, we define

$$Q_{R_0}(t, x, r) := [t, t + \gamma_{R_0} r^\alpha] \times B(x, r).$$

Proposition 4.3 *For every* $R_0 > 0$, $0 < \delta \leq \gamma_{R_0}$, *there exists* $c > 0$ *such that for every* $z \in F$, $0 < R \leq R_0$ *and every non-negative function* q *on* $[0, \infty) \times F$ *that is parabolic and bounded on* $[0, 3\gamma_{R_0} R^\alpha] \times B(z, R)$,

$$\sup_{(t,y) \in Q_{R_0}(\delta R^\alpha, z, R/3)} q(t, y) \leq c \inf_{y \in B(z, R/3)} q(0, y).$$

Moreover, except for the case of $0 < \alpha = d < 2$, *the constant* c *above can be chosen to depend only on* $(R_0, \delta, n, d, \alpha)$ *and on the constants* (C_1, C_2, C_3, C_4) *in (1.1), (1.5) and (1.4) respectively.*

Remark 4.4 Note that the parabolic Harnack inequality implies the elliptic Harnack inequality.
□

It is easy to see the following.

Lemma 4.5 *For each* $t_0 > 0$ *and* $x_0 \in F$, $q(t, x) := p(t_0 - t, x, x_0)$ *is parabolic on* $[0, t_0) \times F$.

Proof. For every $(t,x) \in [0,t_0) \times F$ and $0 < r < s < t_0 - t$, we have by Markov property of Y,

$$\begin{aligned}\mathbf{E}^{(t,x)}\left[q(Z_s)|\widetilde{\mathcal{F}}_r\right] &= \mathbf{E}^x\left[p(t_0-t-s,\, Y_s,\, x_0)|\mathcal{F}_r\right] = \mathbf{E}^{Y_r}\left[p(t_0-t-s,\, Y_{s-r},\, x_0)\right] \\ &= \int_F p(s-r,\, Y_r,\, z)p(t_0-t-s,\, z,\, x_0)d\mu(z) = p(t_0-t-r,\, Y_r,\, x_0) = q(Z_r),\end{aligned}$$

where $Z_s = (t+s, Y_s)$. So $\left\{q(Z_s), \widetilde{\mathcal{F}}_s;\, 0 \le s < t_0 - t\right\}$ is a $\mathbf{P}^{(t,x)}$-martingale for every $(t,x) \in [0,t_0) \times F$ and hence it is parabolic on $[0,t_0) \times F$ as it is shown in the last section that $q(s,y) \le c\,(t_0-s)^{-d/\alpha}$. □

We now give the proof of Theorem 1.1 by assuming Propositions 4.1-4.3.

Proof of Theorem 1.1 (Upper bound). Let $\lambda = |x-y|$. By Theorem 3.2, we only need to consider the case that $\lambda > t^{1/\alpha}$. Let $t_0 = (1+\gamma_1)t$. By Proposition 4.2(i),

$$\int_{B(y,\frac{1}{3}t^{1/\alpha})} p(t_0,\, x,\, z)\,\mu(dz) \le \mathbf{P}^x\left(Y_s \text{ hits ball } B\left(y,\, \frac{t_0^{1/\alpha}}{3(1+\gamma_1)^{1/\alpha}}\right) \text{ by time } t_0\right) \le c_1 \frac{t^{1+d/\alpha}}{\lambda^{d+\alpha}}. \tag{4.3}$$

Set $q(s,z) := p(t_0-s, z, x) = p(t_0-s, x, z)$. By Lemma 4.5, q is parabolic in $[0,t_0) \times F$. It follows from (4.3) and the fact that $\mu(B(y, t^{1/\alpha}/3)) \ge c_2\, t^{d/\alpha}$,

$$\inf_{z \in B(y,\frac{1}{3}t^{1/\alpha})} q(0,z) = \inf_{z \in B(y,\frac{1}{3}t^{1/\alpha})} p(t_0,x,z) \le \frac{c_3\, t}{\lambda^{d+\alpha}}.$$

Now by Proposition 4.3 with $R_0 = 1$ and $\delta = \gamma_1$,

$$p(t,x,y) = q(\gamma_1 t, y) \le \sup_{(s,z) \in Q_1(\gamma_1 t, y, \frac{1}{3}t^{1/\alpha})} q(s,z) \le c_4 \inf_{z \in B(y,\, \frac{1}{3}t^{1/\alpha})} q(0,z) \le \frac{c_5 t}{\lambda^{d+\alpha}},$$

and the proof is completed. □

To prove the lower bound, we need the following lemma, which corresponds to Proposition 5.1 of Bass and Levin [4].

Lemma 4.6 *There exist c_1, $c_2 > 0$ such that*

$$p(t,x,y) \ge c_2\, t^{-d/\alpha}$$

for all $0 < t \le 1$ and $x, y \in F$ with $|x-y| \le c_1\, t^{1/\alpha}$. Moreover, except for the case of $0 < \alpha = d < 2$, the constants c_1 and c_2 can be chosen to depend only on (n,d,α) and on the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) respectively.

Proof. By Proposition 4.1, there exists $c_3 > 1$ such that

$$\mathbf{P}^x(\sup_{s\le t} |Y_s - x| > c_3 t^{1/\alpha}) \le 1/2 \qquad \text{for all } 0 < t \le 1.$$

On the other hand, using the upper bound in Theorem 3.2, there exist $0 < c_4 < c_3/2$ such that

$$\mathbf{P}^x(Y_t \in B(x, c_4 t^{1/\alpha})) \le 1/4 \qquad \text{for all } 0 < t \le 1.$$

Thus, if we set $E(t) = B(x, c_3 t^{1/\alpha}) \setminus B(x, c_4 t^{1/\alpha})$, then

$$\mathbf{P}^x(Y_t \in E) \ge 1/4 \qquad \text{for all } 0 < t \le 1.$$

Now for an arbitrary but fixed $t \in (0, 1]$, define $t_0 = (1 - \gamma_1)t$. Applying to above with t_0 in place of t yields $p(t_0, x, z) \ge c_6 t_0^{-d/\alpha}$ for some $z \in E(t_0)$, since $\mu(E(t_0)) \le c_5 {t_0}^{d/\alpha}$. Now applying Proposition 4.3 with $R_0 = 1$ and $\delta = \gamma_1$ to the parabolic function $q(s, \cdot) := p(t - s, x, \cdot)$, we have that for all y with $|y - x| < c_1 t_0^{1/\alpha}$, where $c_1 := (\gamma_1/4)^{1/\alpha}$,

$$p(t, x, y) \ge \inf_{w\in B(x, c_1 {t_0}^{1/\alpha})} q(0, w) \ge c_5 \sup_{(s,w)\in Q_1(\gamma_1 t_0, x, c_1 {t_0}^{1/\alpha})} q(s, w) \ge c_5 p(t_0, x, z) \ge c_2 t^{-d/\alpha},$$

where $c_2 := c_4\, c_5\, (1 - \gamma_1)^{-d/\alpha}$. □

Proof of Theorem 1.1 (Lower bound). Due to Lemma 4.6, it is enough to prove the theorem for $|x - y| \ge c_2\, t^{1/\alpha}$. By Proposition 4.1, starting at $z \in B(y, t^{1/\alpha})$, there is a positive probability (independent of z and t) such that the process Y does not move more than $c_1 t^{1/\alpha}$ by time t. Thus, by Proposition 4.2 (ii) and the strong Markov property of Y,

$$\mathbf{P}^x(Y_t \in B(y,\, c_3 t^{1/\alpha})) \ge c_4 \frac{t^{1+d/\alpha}}{|x - y|^{d+\alpha}}.$$

Applying above with $t_0 = (1 - \gamma_1)t$ in place of t, we have

$$\mathbf{P}^x(Y_{t_0} \in B(y,\, c_5 t^{1/\alpha})) \ge c_6 \frac{t^{1+d/\alpha}}{|x - y|^{d+\alpha}}.$$

As $\mu(B(y,\, c_5 t^{1/\alpha})) \le c_7 t^{d/\alpha}$, the above implies $p(t_0, x, z) \ge c_8\, t/|x-y|^{d+\alpha}$ for some $z \in B(y,\, c_5 t^{1/\alpha})$. By applying Proposition 4.3 as in the proof of Lemma 4.6, we obtain the desired result.

Thus we have proved estimate (1.6) for $x, y \in \widetilde{F}$. The Hölder continuity of $p(t, x, y)$ will be proved in Theorem 4.14. □

In the remaining of this section, we are going to prove Propositions 4.1–4.3.

Recall that since Y is a symmetric strong Markov process on F, its jump behavior is described by a pair (N, H), *a Lévy system* of Y, in which N is a kernel from $(F, \mathcal{B}(F))$ to itself satisfying $N(x, \{x\}) = 0$ for any $x \in F$, and H a positive continuous additive functional of Y with bounded 1-potential, such that for any positive $\mathcal{B}(E \times E)$-measurable function F, the dual predictable projection (or compensator) of the homogeneous random measure

$$\eta(\omega, dt) := \sum_{s>0} F(Y_{s-}(\omega), Y_s(\omega)) 1_{\{Y_{s-}(\omega) \neq Y_s(\omega)\}} \varepsilon_s(dt) \tag{4.4}$$

is $\int_0^\cdot NF(Y_s) dH_s$, where $NF(x) := \int_{E_\Delta} N(x, dy) F(x, y)$. [Here ε_s is the unit point mass at s.] Let μ_H denote the Revuz measure of H. Then it is known (see, for example, Appendix A.3 of [15]) that $J(dx, dy) := \frac{1}{2} \mu_H(dx) N(x, dy)$ is the jump measure in the Beurling-Deny decomposition of the Dirichlet form for Y and therefore

$$\mu_H(dx) N(x, dy) = \frac{c(x, y)}{|x - y|^{d+\alpha}} \mu(dx) \mu(dy).$$

So one can take $N(x, dy) = \frac{c(x,y)}{|x-y|^{d+\alpha}} \mu(dy)$ and $H_t = t$ as a Lévy system for Y.

Lemma 4.7 *Let f be a non-negative measurable function on $\mathbf{R}_+ \times F \times F$, vanishing on the diagonal. Then for every $t \geq 0$, $x \in F$ and predictable stopping time T of $\{\mathcal{F}_t\}_{t \geq 0}$,*

$$\mathbf{E}^x \left[\sum_{s \leq T} f((s, Y_{s-}, Y_s)) \right] = \mathbf{E}^x \left[\int_0^T \int_F \frac{c(Y_s, y) f((s, Y_s, y))}{|Y_s - y|^{d+\alpha}} d\mu(y) ds \right]$$

Proof. By the Lévy system, the lemma holds for $f(t, x, y)$ of the form $1_{[a,b)} f_0(x, y)$ for some $0 \leq a \leq b$ and measurable $f_0 \geq 0$. Using monotone class theorem, the lemma holds for general measurable f on $\mathbf{R}_+ \times F \times F$. □

Proof of Proposition 4.1. We divide this long proof into several steps.

Step 1. First of all, note that it is enough to consider the case $t_0 \geq 1$. For $\delta \in (0, 1]$, define measure $\mu^{(\delta)}$ on $\delta^{-1} F := \{\delta^{-1} x : x \in F\}$ by

$$\mu^{(\delta)}(A) = \delta^{-d} \mu(\delta A) \qquad \text{for every } A \subset \delta^{-1} F. \tag{4.5}$$

Clearly $\mu^{(\delta)}$ is a d-measure on $\delta^{-1} F$ satisfying (1.1) and (1.5) with the same constants $C_2 > C_1 > 0$ since $\delta \leq 1$. By somewhat abusing the notations a little bit, it is easy to check that $Y^{(\delta)} := \{\delta^{-1} Y_{\delta^\alpha t};\, t \geq 0\}$ is an $\mu^{(\delta)}$-symmetric Hunt process on $\delta^{-1} F$ whose Dirichlet form is $(\mathcal{E}^{(\delta)}, B^{2,2}_{\alpha/2}(\delta^{-1} F))$, where

$$\mathcal{E}^{(\delta)}(u, u) = \frac{1}{2} \int_{\delta^{-1} F \times \delta^{-1} F} \frac{c(\delta x, \delta y)(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \mu^{(\delta)}(dx) \, \mu^{(\delta)}(dy) \quad \text{for } u \in B^{2,2}_{\alpha/2}(\delta^{-1} F). \tag{4.6}$$

Note that for $u \in B^{2,2}_{\alpha/2}(\delta^{-1}F)$, by a change of variable, $f(x) := u(\delta^{-1}x) \in B^{2,2}_{\alpha/2}(F)$ and

$$\mathcal{E}^{(\delta)}(u,u) = \delta^{\alpha-d}\,\mathcal{E}(f,f), \tag{4.7}$$

$$\|u\|_{\mathbf{L}^1(\delta^{-1}F,\,\mu^{(\delta)})} = \delta^{-d}\|f\|_{\mathbf{L}^1(F,\,\mu)} \quad \text{and} \quad \|u\|_{\mathbf{L}^2(\delta^{-1}F,\,\mu^{(\delta)})} = \delta^{-d/2}\|f\|_{\mathbf{L}^2(F,\,\mu)}. \tag{4.8}$$

Recall that for each $s > 0$, $\mathcal{E}^{(\delta)}_s(u,u) := \mathcal{E}^{(\delta)}(u,u) + s\int_{\delta^{-1}F}|u(y)|^2\,\mu^{(\delta)}(dy)$. Note that by the equivalence of the heat kernel upper bound estimate in Theorem 3.2 with the Nash's inequality (3.1) (see Theorem 2.1 in [9]),

$$\|f\|^{2+\frac{2\alpha}{d}}_{\mathbf{L}^2(F,\,\mu)} \le c_1\,\mathcal{E}_1(f,f)\,\|f\|^{\frac{2\alpha}{d}}_{\mathbf{L}^1(F,\,\mu)} \qquad \text{for every } f \in B^{2,2}_{\alpha/2}(F).$$

Thus we have by (4.7)-(4.8) that for every $\delta > 0$ and $u \in B^{2,2}_{\alpha/2}(\delta^{-1}F)$,

$$\begin{aligned}
\|u\|^{2+\frac{2\alpha}{d}}_{\mathbf{L}^2(\delta^{-1}F,\,\mu^{(\delta)})} &\le c_1\left(\mathcal{E}^{(\delta)}(u,u) + \delta^{\alpha}\|u\|^2_{\mathbf{L}^2(\delta^{-1}F,\,\mu^{(\delta)})}\right)\|u\|^{\frac{2\alpha}{d}}_{\mathbf{L}^1(\delta^{-1}F,\,\mu^{(\delta)})} \\
&\le c_1\,\mathcal{E}^{(\delta)}_{\delta^\alpha}(u,u)\,\|u\|^{\frac{2\alpha}{d}}_{\mathbf{L}^1(\delta^{-1}F,\,\mu^{(\delta)})}.
\end{aligned} \tag{4.9}$$

In other words, the Nash inequality (4.9) holds for $(\mathcal{E}^{(\delta)},\, B^{2,2}_{\alpha/2}(\delta^{-1}F))$ with a universal constant c_1 for every $\delta > 0$.

Step 2. Consider the $\mu^{(\delta)}$-symmetric Hunt process $X^{(\delta)}$ on $\delta^{-1}F$ associated with the Dirichlet form $(\mathcal{C}^{(\delta)}_1,\, B^{2,2}_{\alpha/2}(\delta^{-1}F))$, where

$$\mathcal{C}^{(\delta)}(u,u) = \frac{1}{2}\int_{\{(x,y)\in\delta^{-1}F\times\delta^{-1}F:\,|x-y|\le 1\}} \frac{c(\delta x,\,\delta y)(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mu^{(\delta)}(dx)\,\mu^{(\delta)}(dy). \tag{4.10}$$

Note that for $u \in B^{2,2}_{\alpha/2}(\delta^{-1}F)$,

$$\begin{aligned}
0 &\le \mathcal{E}^{(\delta)}(u,u) - \mathcal{C}^{(\delta)}(u,u) \\
&= \frac{1}{2}\int_{\{(x,y)\in\delta^{-1}F\times\delta^{-1}F:\,|x-y|>1\}} \frac{c(\delta x,\,\delta y)(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mu^{(\delta)}(dx)\,\mu^{(\delta)}(dy) \\
&\le c\int_{\delta^{-1}F} u(x)^2\left(\int_{\{y\in\delta^{-1}F:\,|y-x|>1\}}|x-y|^{-d-\alpha}\mu^{(\delta)}(dy)\right)\mu^{(\delta)}(dx) \\
&\le c\int_{\delta^{-1}F} u(x)^2\mu^{(\delta)}(dx),
\end{aligned} \tag{4.11}$$

where $c > 0$ is a constant independent of $\delta > 0$. This, together with (4.9), implies that there is a constant $c_2 > 0$ such that for every $\delta > 0$ and $u \in B^{2,2}_{\alpha/2}(\delta^{-1}F)$,

$$\|u\|^{2+\frac{2\alpha}{d}}_{\mathbf{L}^2(\delta^{-1}F,\,\mu^{(\delta)})} \le c_2\,\mathcal{C}^{(\delta)}_{\delta^\alpha+c}(u,u)\,\|u\|^{\frac{2\alpha}{d}}_{\mathbf{L}^1(\delta^{-1}F,\,\mu^{(\delta)})}.$$

So by Theorem 2.1 in Carlen, Kusuoka and Stroock [9], there is a constant $c > 0$ such that for every $\delta > 0$, the transition density function $p^{(\delta)}(t, x, y)$ of $X^{(\delta)}$ with respect to measure $\mu^{(\delta)}$ has an upper bound

$$p^{(\delta)}(t, x, y) \leq c\, t^{-d/\alpha} e^{(\delta^\alpha + c)t} \qquad \text{for every } x, y \in \delta^{-1}F \text{ and } t > 0.$$

By Theorem 3.25 of [9], we can get the following off-diagonal estimate of $p^{(\delta)}(x, y)$: there is a constant $c_3 > 0$, independent of $\delta > 0$, such that

$$p^{(\delta)}(t, x, y) \leq c_3\, t^{-d/\alpha} e^{-E(2t,x,y)+(\delta^\alpha + c)t} \qquad \text{for } t > 0 \text{ and } x, y \in \delta^{-1}F, \tag{4.12}$$

where

$$E(t, x, y) := \sup\{|\psi(x) - \psi(y)| - t\Lambda(\psi)^2 : \Lambda(\psi) < \infty\}$$

with

$$\begin{aligned} \Lambda(\psi)^2 &:= \max\left\{ \|e^{-2\psi}\Gamma(e^\psi,\, e^\psi)\|_\infty,\ \|e^{2\psi}\Gamma(e^{-\psi},\, e^{-\psi})\|_\infty \right\}, \\ \Gamma(e^\psi,\, e^\psi)(x) &:= \frac{1}{2}\int_{\{y\in\delta^{-1}F:\, |y-x|\leq 1\}} \frac{c(\delta x,\, \delta y)(e^{\psi(x)} - e^{\psi(y)})^2}{|x-y|^{d+\alpha}}\, \mu^{(\delta)}(dy). \end{aligned}$$

Now for any $x \neq y$ in $\delta^{-1}F$, take $\psi \in C_c^1(\delta^{-1}F) \subset B^{2,2}_{\alpha/2}(\delta^{-1}F)$ with $\|\nabla\psi\|_\infty \leq 1$ such that $\psi(\xi) = B \cdot (\xi - x)$ for $|\xi - x| \leq 2|y - x|$ with $B = (y - x)/|y - x|$. Then

$$\begin{aligned} e^{-2\psi(\xi)}\Gamma(e^\psi,\, e^\psi)(\xi) &= \frac{1}{2}\int_{\{\eta\in\delta^{-1}F:\, |\eta-\xi|\leq 1\}} \frac{c(\delta\xi,\, \delta\eta)(e^{\psi(\eta)-\psi(\xi)} - 1)^2}{|\xi-\eta|^{d+\alpha}}\, \mu^{(\delta)}(d\eta) \\ &\leq \int_{\{\eta\in\delta^{-1}F:\, |\eta-\xi|\leq 1\}} \frac{c}{|\xi-\eta|^{d+\alpha-2}}\, \mu^{(\delta)}(d\eta) \\ &\leq c, \end{aligned}$$

where $c \in (0, \infty)$ is a constant that is independent of $\delta > 0$ and $x,\, y \in \delta^{-1}F$. Similarly, the same bound holds with $-\psi$ in place of ψ. On the other hand,

$$\psi(y) - \psi(x) = |x - y|.$$

Hence it follows from (4.12) that there are constants $c_4,\, c_5 > 0$ independent of $\delta > 0$ such that

$$p^{(\delta)}(t, x, y) \leq c_4\, t^{-d/\alpha} e^{-|x-y|+(c_5+\delta^\alpha)t} \qquad \text{for } t > 0 \text{ and } x, y \in \delta^{-1}F. \tag{4.13}$$

Step 3. It follows from (4.13) that for $t \in [1/4,\, t_0]$ and $\lambda > 0$,

$$\mathbf{P}^x\left(|X_t^{(\delta)} - x| \geq \lambda\right) = \int_{\{y\in\delta^{-1}F:\, |y-x|>\lambda\}} p^{(\delta)}(t, x, y)\mu^{(\delta)}(dy) \leq c\, e^{-(\lambda/2)+(c_5+\delta^\alpha)t_0}. \tag{4.14}$$

Define $\sigma_\lambda := \inf\{t \geq 0 : |X_t^{(\delta)} - X_0^{(\delta)}| \geq \lambda\}$. Then by (4.14) and the strong Markov property of $X^{(\delta)}$,

$$\begin{aligned}
\mathbf{P}^x(\sigma_\lambda \leq t_0/2) &\leq \mathbf{P}^x\left(\sigma_\lambda \leq t_0/2 \text{ and } |X_{t_0}^{(\delta)} - x| \leq \lambda/2\right) + \mathbf{P}^x\left(|X_{t_0}^{(\delta)} - x| > \lambda/2\right) \\
&\leq \mathbf{P}^x\left(\sigma_\lambda \leq t_0/2 \text{ and } |X_{t_0}^{(\delta)} - X_{\sigma_\lambda}^{(\delta)}| > \lambda/2\right) + c\, e^{-(\lambda/4)+(c_5+\delta^\alpha)t_0} \\
&\leq \int_0^{t_0/2} \mathbf{E}^x\left[\mathbf{P}^{X_s^{(\delta)}}\left(|X_{t_0-s}^{(\delta)} - X_0^{(\delta)}| > \lambda/2\right); \sigma_\lambda \in ds\right] + c\, e^{-(\lambda/4)+(c_5+\delta^\alpha)t_0} \\
&\leq c\, e^{-(\lambda/4)+(c_5+\delta^\alpha)t_0}.
\end{aligned}$$

Here in the second and the last inequalities, we used (4.14). By the strong Markov property of $X^{(\delta)}$, for every $\lambda > 0$,

$$\begin{aligned}
\mathbf{P}^x\left(\sup_{s \leq t_0}\left|X_s^{(\delta)} - X_0^{(\delta)}\right| > \lambda\right) &\leq \mathbf{P}^x(\sigma_\lambda \leq t_0/2) + \mathbf{P}^x(t_0/2 < \sigma_\lambda \leq t_0) \\
&\leq c\, e^{-(\lambda/4)+(c_5+\delta^\alpha)t_0} + \mathbf{P}^x(\sigma_{\lambda/2} \leq t_0/2) + \mathbf{E}^x\left[\mathbf{P}^{X_{t_0/2}^{(\delta)}}(\sigma_{\lambda/2} \leq t_0/2)\right] \\
&\leq c\, e^{-(\lambda/8)+(c_5+\delta^\alpha)t_0}. \qquad (4.15)
\end{aligned}$$

The constant $c > 0$ above is independent of $\delta > 0$, $x \in \delta^{-1}F$ and $\lambda > 0$.

Step 4. We now transfer the tightness result obtained in (4.15) for $X^{(\delta)}$ to process $Y^{(\delta)}$. Note that for $u,\, v \in B^{2,2}_{\alpha/2}(\delta^{-1}F)$,

$$\begin{aligned}
&\mathcal{E}^{(\delta)}(u,v) \\
=\;& \mathcal{C}^{(\delta)}(u,v) + \frac{1}{2}\int_{\{(x,y)\in\delta^{-1}F\times\delta^{-1}F:\,|x-y|>1\}} \frac{c(\delta x,\, \delta y)(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\, \mu^{(\delta)}(dx)\, \mu^{(\delta)}(dy) \\
=\;& \mathcal{C}^{(\delta)}(u,v) - \int_{\delta^{-1}F} \mathcal{B}^{(\delta)}u(x)\, v(x)\, \mu^{(\delta)}(dx),
\end{aligned}$$

where

$$\mathcal{B}^{(\delta)}u(x) := \int_{\{y\in\delta^{-1}F:\,|x-y|>1\}} (u(y)-u(x))\, \frac{c(\delta x,\, \delta y)}{|x-y|^{d+\alpha}}\, \mu^{(\delta)}(dy). \qquad (4.16)$$

It follows that, if we use $\mathcal{L}^{(\delta)}$ and $\mathcal{A}^{(\delta)}$ to denote the L^2-infinitesimal generator of $Y^{(\delta)}$ and $X^{(\delta)}$ respectively, then (cf. [15]) $\mathrm{Dom}(\mathcal{L}^{(\delta)}) = \mathrm{Dom}(\mathcal{A}^{(\delta)})$ and

$$\mathcal{L}^{(\delta)} = \mathcal{A}^{(\delta)} + \mathcal{B}^{(\delta)}.$$

For $u \in \mathbf{L}^2(\delta^{-1}F,\, \mu^{(\delta)})$, by Cauchy-Schwarz inequality,

$$\begin{aligned}
\|\mathcal{B}^{(\delta)}u\|^2_{\mathbf{L}^2(\delta^{-1}F,\mu^{(\delta)})} &\leq \int_{\delta^{-1}F}\left(\int_{\{y\in\delta^{-1}F:\,|x-y|>1\}}\frac{c(\delta x,\,\delta y)(u(y)-u(x))^2}{|x-y|^{d+\alpha}}\,\mu^{(\delta)}(dy)\right)\\
&\qquad\cdot\left(\sup_{x\in\delta^{-1}F}\int_{\{y\in\delta^{-1}F:\,|x-y|>1\}}\frac{c(\delta x,\,\delta y)}{|x-y|^{d+\alpha}}\,\mu^{(\delta)}(dy)\right)\,\mu^{(\delta)}(dx)\\
&\leq c\int_{\{(x,y)\in\delta^{-1}F\times\delta^{-1}F:\,|x-y|>1\}}\frac{c(\delta x,\,\delta y)(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mu^{(\delta)}(dx)\,\mu^{(\delta)}(dy)\\
&\leq c_6\int_{\delta^{-1}F}u(x)^2\,\mu^{(\delta)}(dx),
\end{aligned}$$

where in the last inequality we used (4.11). Here $c_6>0$ is a constant that is independent of $\delta>0$ and $u\in\mathbf{L}^2(\delta^{-1}F,\,\mu^{(\delta)})$. Denote by $\{Q_t^{(\delta)};\,t\geq 0\}$ the transition semigroup of $X^{(\delta)}$ and define

$$S_0(t)=Q_t^{(\delta)}\quad\text{ and }\quad S_k(t)=\int_0^t S_{k-1}(s)\mathcal{B}^{(\delta)}Q_{t-s}^{(\delta)}\,ds\quad\text{for } k\geq 1. \tag{4.17}$$

It is easy to see by using induction that each $S_k(t)$ is a bounded linear operator on $\mathbf{L}^2(\delta^{-1}F,\,\mu^{(\delta)})$ with operator norm $\|S_k(t)\|_{2,2}\leq (c_6\,t)^k/k!$. Hence $\sum_{k=0}^\infty S_k(t)=:P_t^{(\delta)}$ defines a bounded linear operator on $\mathbf{L}^2(\delta^{-1}F,\,\mu^{(\delta)})$. In fact by the proofs for Lemma 2.1, Theorems 3.2 and 3.6 in Leviatan [26], $\{P_t^{(\delta)};\,t\geq 0\}$ is a strongly continuous semigroup on $\mathbf{L}^2(\delta^{-1}F,\,\mu^{(\delta)})$ whose infinitesimal generator is $\mathcal{A}^{(\delta)}+\mathcal{B}^{(\delta)}=\mathcal{L}^{(\delta)}$. (Note that although the framework in [26] is for semigroups on space of continuous functions vanishing at infinite, its proof works for L^2-semigroups. See also Theorems 4.10.2 and 4.10.3 in Ethier and Kurtz [11].) Hence $\{P_t^{(\delta)};\,t\geq 0\}$ is the semigroup for process $Y^{(\delta)}$.

On the other hand, there is a constant $c_7>0$ that is independent of $\delta>0$ such that

$$\|\mathcal{B}^{(\delta)}u\|_\infty\leq c\,\|u\|_\infty\sup_{x\in\delta^{-1}F}\int_{\{y\in\delta^{-1}F:\,|x-y|>1\}}\frac{1}{|x-y|^{d+\alpha}}\,\mu^{(\delta)}(dy)\leq c_7\,\|u\|_\infty$$

for all bounded function u. Hence by reduction it is easy to show that each $S_k(t)$ is a bounded linear operator on $\mathbf{L}^\infty(\delta^{-1}F,\,\mu^{(\delta)})$ with operator norm $\|S_k(t)\|_{\infty,\infty}\leq (c_7\,t)^k/k!$. Hence $P_t^{(\delta)}$ is also the limit of $\sum_{k=0}^m S_k(t)$ as $m\to\infty$ with respect to the operator norm in $\mathbf{L}^\infty(\delta^{-1}F,\,\mu^{(\delta)})$. In particular, for any bounded function f on $\delta^{-1}F$,

$$\|P_t^{(\delta)}f-Q_t^{(\delta)}f\|_\infty\leq\sum_{k=1}^\infty\frac{(c_7\,t)^k}{k!}\,\|f\|_\infty\leq c_7\,t\,e^{c_7t}\,\|f\|_\infty.$$

It follows that there is a constant $c_8>0$ that depends on t_0, but is independent of $\delta>0$ such that for $\mu^{(\delta)}$-a.e. x in $\delta^{-1}F$, every $t\leq t_0$ and $\delta>0$,

$$\mathbf{P}^x\left(|Y_t^{(\delta)}-x|>\lambda\right)\leq\mathbf{P}^x\left(|X_t^{(\delta)}-x|>\lambda\right)+c_8\,t. \tag{4.18}$$

As $x \mapsto \mathbf{P}^x\left(|Y_t^{(\delta)} - x| > \lambda\right)$ is a quasi-continuous function with respect to the process $Y^{(\delta)}$, by looking at rational $\lambda > 0$ and rational $t > 0$ first, we conclude that there is a properly exceptional set $N^{(\delta)}$ of $Y^{(\delta)}$ such that (4.18) holds for all $\lambda > 0$ and for $x \in \delta^{-1}F \setminus N^{(\delta)}$. Now, we take $\delta \in (0,1]$. Applying the same argument as that in Step 3, we conclude there are positive constants c_9, c_{10} that depend on t_0, but are independent of $\delta \in (0,1]$ such that for $x \in \delta^{-1}F \setminus N^{(\delta)}$,

$$\mathbf{P}^x\left(\sup_{s\le t} |Y_s^{(\delta)} - x| > \lambda\right) \le c_9 e^{-c_{10}\lambda} + c_9\, t \qquad \text{for every } \lambda \ge 1 \text{ and } t \le t_0. \tag{4.19}$$

As $Y_t^{(\delta)} = \delta^{-1}\, Y_{\delta^\alpha t}$, $\delta N^{(\delta)}$ is an exceptional set of Y and so by enlarge the properly exceptional set N in Theorem 3.2 if necessary, we may assume (cf. [15])

$$N \supset \bigcup_{\delta \in \mathbf{Q}\cap(0,1]} \delta N^{(\delta)}.$$

Recall that F here is taken to be $F \setminus N$, so (4.19) implies that for every $x \in F$, $\delta \in \mathbf{Q} \cap (0,1]$ and $\lambda > 1$,

$$\mathbf{P}^x\left(\sup_{s\le \delta^\alpha t} |Y_s - x| > \delta\,\lambda\right) \le c_9 e^{-c_{10}\lambda} + c_9\, t \qquad \text{for every } \lambda \ge 1 \text{ and } t \le t_0. \tag{4.20}$$

Clearly by the continuity in δ, the above holds for every $\delta \in (0,1]$ as well.

Step 5. For $r_0 > 0$, $A > 0$ and $B \in (0,1)$, we choose $\lambda > A\, r_0$ and $s_0 < (r_0^\alpha/2) \wedge r_0$ so that $c_9 e^{-c_{10}\lambda} + c_9\, s_0 < B$ and therefore by (4.20),

$$\mathbf{P}^x\left(\sup_{s\le \delta^\alpha s_0} |Y_s - x| \ge \delta\,\lambda\right) \le B \qquad \text{for every } 0 < \delta \le 1.$$

Now for $r \in (0, r_0]$, letting $\delta = A\, r/\lambda$, we have

$$\mathbf{P}^x\left(\sup_{s\le \gamma\, r^\alpha} |Y_s - x| > A\, r\right) \le B,$$

where $\gamma = s_0\, A^\alpha/\lambda^\alpha$. This proves Proposition 4.1. Keeping a carefully track of the constants appeared in the above argument, we see that, except for the case of $0 < \alpha = d < 2$, the constant γ above can be chosen to depend only on (A, B, n, d, α) and on the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) respectively. □

We now turn to the proof of Proposition 4.2. In order to bound $\mathbf{P}^x(Y_{t_0} \in B(y, a t_0^{1/\alpha}))$ for $0 < t_0 \le 2$, it is convenient to look at $W_t := Y_t^{(t_0^{1/\alpha})} = t_0^{-1/\alpha} Y_{t_0 t}$ and to estimate $\mathbf{P}^x(W_1 \in B(y, a))$

for $x, y \in t_0^{-1/\alpha}F$. Set $\delta_0 := t_0^{1/\alpha}$. The Dirichlet form corresponding to $\{W_t\}$ is $(\mathcal{E}^{(\delta_0)}, B^{2,2}_{\alpha/2}(\delta_0^{-1}F))$ defined by (4.6).

Fix $\lambda > 0$ and define

$$\mathcal{C}^{(\delta_0,\lambda)}(u,u) = \frac{1}{2}\int_{\{(x,y)\in\delta_0^{-1}F\times\delta_0^{-1}F:\,|x-y|\le\lambda^{1/6}\}} \frac{c(\delta_0 x,\,\delta_0 y)(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mu^{(\delta_0)}(dx)\,\mu^{(\delta_0)}(dy). \quad (4.21)$$

Let $Q_t := Q_t^{(\delta_0,\lambda)}$ be the transition semigroup corresponding to the Dirichlet form $(\mathcal{C}^{(\delta_0,\lambda)}, B^{2,2}_{\alpha/2}(\delta_0^{-1}F))$. Define operator $\mathcal{B} := \mathcal{B}_\lambda^{(\delta_0)}$ by

$$\mathcal{B}_\lambda^{(\delta)}u(x) := \int_{\{y\in\delta_0^{-1}F:\,|x-y|>\lambda^{1/6}\}} (u(y)-u(x))\,\frac{c(\delta_0 x,\,\delta_0 y)}{|x-y|^{d+\alpha}}\,\mu^{(\delta_0)}(dy). \quad (4.22)$$

If we use $\mathcal{A}_\lambda^{(\delta_0)}$ to denote the infinitesimal generator of $(\mathcal{C}^{(\delta_0,\lambda)}, B^{2,2}_{\alpha/2}(\delta_0^{-1}F))$ then the infinitesimal generator of $\{W_t\}$ is $\mathcal{A}_\lambda^{(\delta_0)} + \mathcal{B}_\lambda^{(\delta_0)}$ (see Step 4 in the proof of Proposition 4.1).

Let K be the smallest integer greater than $6(d+\alpha)/\alpha$ and set

$$a_n := \lambda^{\frac{1}{2}+\frac{n}{4K}}.$$

We say that a function $g \in \mathcal{L}(y,n,c)$ for some $y \in \delta_0^{-1}F$ if

$$|g(z)| \le c\left\{\frac{1}{\lambda^{d+\alpha}} + \frac{1}{|z-y|^{d+\alpha}}1_{B(y,\,a_n)^c}(z) + H(z)\right\} \quad \text{for every } z \in \delta_0^{-1}F,$$

where H is a non-negative function supported in $B(y,a_n)$ with $\|H\|_1 + \|H\|_\infty \le 1$. To prove Proposition 4.2(i), we prepare the following lemma, which is a d-set version of Proposition 4.4 and Lemma 4.5 in Bass and Levin [4]. Note that we change several exponents from the original proof of [4] (for instance, $\lambda^{1/6}$ in (4.21)-(4.22) corresponding to $E = D^{1/2}$ in [4]). This is because our estimates (4.13) and (4.15) are weaker than the corresponding results in Proposition 2.4 and 2.5 of [4].

Lemma 4.8 *(i) For each $t \le 1$,*

$$\|Q_t f\|_1 \le \|f\|_1, \qquad \|Q_t f\|_\infty \le \|f\|_\infty. \quad (4.23)$$

There is a constant $c > 0$ that depends only on the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) sucu that

$$\|\mathcal{B}f\|_1 \le \frac{c}{\lambda^{\alpha/6}}\|f\|_1, \qquad \|\mathcal{B}f\|_\infty \le \frac{c}{\lambda^{\alpha/6}}\|f\|_\infty. \quad (4.24)$$

(ii) Suppose that $\lambda > 4^{4K}$, where K is the smallest integer greater than $6(d+\alpha)/\alpha$. There exists $c_1 > 0$ such that if $g \in \mathcal{L}(y,n,c)$ for some $y \in \delta_0^{-1}F$, $c > 0$ and $n \ge 1$, then $\mathcal{B}g \in \mathcal{L}(y,n+1,c_1 c)$

and $Q_s g \in \mathcal{L}(y, n+1, c_1 c)$ *for each* $s \le 1$. *Moreover, except for the case of* $0 < \alpha = d < 2$, *the constant* c_1 *can be chosen to depend only on* (n, d, α) *and on the constants* (C_1, C_2, C_3, C_4) *in (1.1), (1.5) and (1.4) respectively.*

Proof. Since Q_t is a symmetric Markovian semigroup, it is clear that (4.23) holds. As

$$\int_{\{y \in \delta_0^{-1} F:\, |y-x| > \lambda^{1/6}\}} \frac{c(\delta_0 x, \delta_0 y)}{|x-y|^{d+\alpha}} \mu^{(\delta_0)}(dy) \le c\, \lambda^{-\alpha/6}, \tag{4.25}$$

the second inequality of (4.24) holds. To get the first inequality of (4.24), note that by the triangle inequality and the symmetry of function $c(x, y)$,

$$\begin{aligned} \int_{\delta_0^{-1} F} |\mathcal{B} f(x)| \mu^{(\delta_0)}(dx) &\le 2 \int_{\delta_0^{-1} F} |f(x)| \mu^{(\delta_0)}(dx) \int_{\{y \in \delta_0^{-1} F:\, |y-x| > \lambda^{1/6}\}} \frac{c(\delta_0 x, \delta_0 y)}{|x-y|^{d+\alpha}} \mu^{(\delta_0)}(dy) \\ &\le 2c\, \lambda^{-\alpha/6} \, \|f\|_1, \end{aligned}$$

where in the last inequality we used (4.25) and Fubini theorem. This proves the Part (i) of the lemma.

We now prove Part (ii) of this lemma. By (4.23)-(4.24), $\|\mathcal{B}(\lambda^{-(d+\alpha)})\|_\infty \le c_1 \, \lambda^{-(d+\alpha)}$ and the same bound holds with Q_t in place of $\mathcal{B}$. Set $v(z) = 1_{B(y, a_n)^c}(z)/|z-y|^{d+\alpha}$ for some $y \in \delta_0^{-1} F$. As $\lambda \ge 4^{4K}$, $\|v\|_1 + \|v\|_\infty \le c_2$ for some constant $c_2 > 0$ that is independent of n and λ. Define

$$J_0(z) := 1_{B(y,\, a_{n+1})}(z) |\mathcal{B}(v + H)(z)|$$

and $J(z) := J_0(z)/(\|J_0\|_1 + \|J_0\|_\infty)$. By (4.24), $\|J_0\|_1 + \|J_0\|_\infty$ is bounded by a constant independent of n and λ and J is a non-negative function supported on $B(y,\, a_{n+1})$ with $\|J\|_1 + \|J\|_\infty \le 1$. The same argument holds for Q_t in place of $\mathcal{B}$.

It remains to estimate $1_{B(y,\, a_{n+1})^c}(z)\, |\mathcal{B} v(z)|$ and $1_{B(y,\, a_{n+1})^c}(z)\, |\mathcal{B} H(z)|$. Note that for $|z-y| \ge a_{n+1}$,

$$|\mathcal{B} v(z)| \le \int_{\{w \in \delta_0^{-1} F:\, |w-z| > \lambda^{1/6}\}} \frac{c_3 v(w)}{|w-z|^{d+\alpha}} \mu^{(\delta_0)}(dw) + \int_{\{w \in \delta_0^{-1} F:\, |w-z| > \lambda^{1/6}\}} \frac{c_3 v(z)}{|w-z|^{d+\alpha}} \mu^{(\delta_0)}(dw). \tag{4.26}$$

The second term is clearly bounded by $c_4 v(z)$. We thus consider the first term. Let $C := \{w \in \delta_0^{-1} F :\, |w-z| \ge |w-y|\}$. If $w \in C$, it follows from planar geometry that $|w-z| \ge |y-z|/2$. Hence

$$\begin{aligned} \int_{\{w \in C, |w-z| > \lambda^{1/6}\}} \frac{v(w)}{|w-z|^{d+\alpha}} \mu^{(\delta_0)}(dw) &\le \int_{\{w \in C, |w-z| > \lambda^{1/6}\}} \frac{1}{|w-z|^{d+\alpha}} \frac{1}{|w-y|^{d+\alpha}} \mu^{(\delta_0)}(dw) \\ &\le \frac{c}{|y-z|^{d+\alpha}} \int_{\{w \in \delta_0^{-1} F:\, |w-y| > \lambda^{1/6}\}} \frac{1}{|w-y|^{d+\alpha}} \, \mu^{(\delta_0)}(dw) \\ &\le \frac{c}{|y-z|^{d+\alpha}}. \end{aligned}$$

If $w \in C^c$, then $|w-y| \geq |y-z|/2$ so we get a similar bound. Combining these results yields that

$$1_{B(y,\,a_{n+1})^c}(z)\,|\mathcal{B}v(z)| \leq c\,1_{B(y,\,a_{n+1})^c}(z)\,v(x).$$

Recall that as $\lambda \geq 4^{4K}$, $a_{n+1} \geq 4a_n$. For $|\mathcal{B}H(z)|$ with $|z-y| \geq a_{n+1}$, we have (4.26) with v replaced by H and the second term is 0 since H is supported in $B(y,\,a_n)$. For the first term, since $|w-y| \leq a_n \leq a_{n+1}/2 \leq |z-y|/2$, it follows that $|w-z| \geq |z-y| - |y-w| \geq |z-y|/2$ and so

$$\int_{B(y,\,a_n)} \frac{H(w)}{|w-z|^{d+\alpha}} \mu^{(\delta_0)}(dw) \leq \frac{c\|H\|_1}{|z-y|^{d+\alpha}} \leq \frac{c}{|z-y|^{d+\alpha}}.$$

This says that

$$1_{B(y,\,a_{n+1})^c}(z)\,|\mathcal{B}H(z)| \leq c\,1_{B(y,\,a_{n+1})^c}(z)\,v(x).$$

So we have proved that there is c_1, independent of λ, such that for any $g \in \mathcal{L}(y,n,c)$, $\mathcal{B}g \in \mathcal{L}(y,n+1,c_1\,c)$.

Finally, we examine $Q_t v(z)$ and $Q_t H(z)$ when $|z-y| \geq a_{n+1}$. Write

$$Q_t v(z) = \int_{\{w\in\delta_0^{-1}F:\,|z-w|\leq a_{n+1}/2\}} v(w)Q_t(z,dw) + \int_{\{w\in\delta_0^{-1}F:\,|z-w|>a_{n+1}/2\}} v(w)Q_t(z,dw), \quad (4.27)$$

where $Q_t(z,dw)$ is the transition measure for Q_t. For $|z-y| > a_{n+1}$ and $|z-w| \leq a_{n+1}/2$, we have $|w-y| \geq |z-y|/2$. For such w, $v(w) \leq c_5/|z-y|^{d+\alpha}$, so that the first term in (4.27) is bounded by

$$\frac{c_5}{|z-y|^{d+\alpha}} \int_{\delta_0^{-1}F} Q_t(z,dw) \leq \frac{c_5}{|z-y|^{d+\alpha}}.$$

Let $X^{(\delta_0,\lambda)}$ be the symmetric Markov process on $\delta_0^{-1}F$ associated with the Dirichlet form $(\mathcal{C}^{(\delta_0,\lambda)}, B^{2,2}_{\alpha/2}(\delta_0^{-1}(F))$. By definition, we see that $X_t^{(\delta_0,\lambda)}$ and $\lambda^{1/6}X^{(\delta_0\lambda^{1/6})}_{t/\lambda^{\alpha/6}}$ have the same distribution. Thus, using $v(w) \leq a_n^{-d-\alpha} \leq c_6$, the second term in (4.27) is less than or equal to

$$\begin{aligned}
c_6\int_{\{w\in\delta_0^{-1}F:\,|z-w|>a_{n+1}/2\}} Q_t(z,dw) &\leq c_6\mathbf{P}^z(|X_t^{(\delta_0,\lambda)} - z| \geq a_{n+1}/2) \\
&= c_6\mathbf{P}^{\lambda^{-1/6}z}\left(|X^{(\delta_0\lambda^{1/6})}_{t/\lambda^{\alpha/6}} - \lambda^{-1/6}z| \geq a_{n+1}/(2\lambda^{1/6})\right) \\
&\leq c_7\exp\left(-c_8(a_{n+1}/\lambda^{1/6}) + c + (\delta_0\lambda^{1/6})^\alpha\right) \\
&\leq c_9\exp\left(-c_8\lambda^{(1/3)+(n+1)/(4K)} + c_{10}\lambda^{\alpha/6}\right),
\end{aligned}$$

where we apply (4.15) in the second inequality (note that $t/\lambda^{\alpha/6} \leq 1$) and use the fact $\delta^\alpha = t_0 \leq 2$ in the last inequality. Since $0<\alpha<2$, the last term is less than $c_9\exp(-c_{11}\lambda^{1/3}) \leq c_{12}\lambda^{-d-\alpha}$ when λ is large. Combining these estimates proves the result for $Q_t v(z)$. The estimate for $Q_t H(z)$ when $|z-y| \geq a_{n+1}$ goes in the same way. We have (4.27) with v replaced by H and the first term is 0 since H is supported in $B(y, a_n)$ and $a_{n+1} \geq 4a_n$. The second term can be estimated in the same way as that of v and the result is obtained. □

Proof of Proposition 4.2. We first prove (i). Recall that $W_t = \delta_0^{-1} Y_{\delta_0^\alpha t}$. We claim that there exists $c_0 > 0$ such that

$$\mathbf{P}^x(W_1 \in B(y,1)) \leq c_0/|x-y|^{d+\alpha} \quad \text{for every} \quad x, y \in \delta_0^{-1} F. \tag{4.28}$$

To prove this, let $\lambda = |x - y|$. Since the result is clear for $\lambda < \lambda_0 := 4^{4K}$, we assume that $\lambda \geq \lambda_0$. Here K is the smallest integer that is greater than $6(d+\alpha)/\alpha$. Let $f = 1_{B(y,1)}$, then clearly there exists $c > 0$ such that $f \in \mathcal{L}(y, 1, c)$. Thus, by Lemma 4.8(ii), $Q_t f \in \mathcal{L}(y, 2, c_1 c)$ for all $t \leq 1$. Set, as in (4.17),

$$S_0(t) := Q_t, \quad S_1(t) := \int_0^t Q_s \, \mathcal{B} \, Q_{t-s} \, ds \quad \text{and} \quad S_{k+1}(t) := \int_0^t S_k(s) \, \mathcal{B} \, Q_{t-s} \, ds \quad \text{for } k \geq 1.$$

Since $Q_1 f \in \mathcal{L}(y, 2, c_1 c)$ and $|x - y| = \lambda > a_2$, we have $|S_0(1) f(x)| < c\, \lambda^{-d-\alpha}$. By Lemma 4.8(ii), $Q_s \mathcal{B} Q_{t-s} f \in \mathcal{L}(y, 4, c_1^3 c)$ for each $0 \leq s \leq 1$ so that $|Q_s \, \mathcal{B} \, Q_{t-s} f(x)| \leq c\, \lambda^{-d-\alpha}$ (note that $a_n \leq \lambda$ for all $n \leq 2K$).

Integrating over $s \leq t$, we have $|S_1(t) f(x)| \leq c\, \lambda^{-d-\alpha}$. Again by Lemma 4.8(ii),

$$Q_r \, \mathcal{B} \, Q_{s-r} \, \mathcal{B} \, Q_{t-s} f \in \mathcal{L}(y, 6, c_1^5 c)$$

and therefore $|Q_r \, \mathcal{B} \, Q_{s-r} \, \mathcal{B} \, Q_{t-s} f(x)| \leq c\, \lambda^{-d-\alpha}$. Integrating over r and s, we have $|S_2(t) f(x)| \leq c\, \lambda^{-d-\alpha}$. Continuing in this way, we have

$$|S_n(1) f(x)| \leq c_2(n)\, \lambda^{-d-\alpha} \quad \text{for every} \quad n \leq K. \tag{4.29}$$

On the other hand, by Lemma 4.8 (i), $\|\mathcal{B}\|_{\infty,\infty} \leq c_3\, \lambda^{-\alpha/6}$. Increase the value of λ_0 if necessary so that $c_3\, \lambda_0^{-\alpha/2} < 1/2$. We have by induction that for $\lambda \geq \lambda_0$

$$\|S_n(1) f\|_\infty \leq (c_3/\lambda^{\alpha/6})^n.$$

Consequently,

$$\sum_{n=K}^{\infty} |S_n(1) f(x)| \leq c/\lambda^{\alpha K/6} \leq c_4 \lambda^{-d-\alpha}. \tag{4.30}$$

As mentioned in the Step 4 in the proof of Proposition 4.1, if we set $P_t^{(\delta_0)} = \sum_{k=0}^\infty S_k(t)$, then $\{P_t^{(\delta_0)}\}$ is the semigroup corresponding to $\{W_t\}$. By (4.29) and (4.30), we have

$$|P_1^{(\delta_0)} f(x)| \leq \left(\sum_{n=0}^{K} c_2(n) + c_4 \right) \lambda^{-d-\alpha} = c_5\, \lambda^{-d-\alpha},$$

which proves (4.28). Since $W_t = \delta_0^{-1} Y_{\delta_0^\alpha t}$ and $\delta_0 = t_0^{1/\alpha}$, (4.28) can be written as

$$\mathbf{P}^x \left(Y_{t_0} \in B(y, t_0^{1/\alpha}) \right) \le c_0 \left(\frac{t_0^{1/\alpha}}{|x-y|} \right)^{d+\alpha} \quad \text{for every} \quad x, y \in F. \tag{4.31}$$

Now we prove (4.1). Since we can cover $B(y, at_0^{1/\alpha})$ by a finite number (which depends only on a and the dimension d) of balls of the form $B(z, t_0^{1/\alpha})$, it is enough to prove it when $a = 1$. Further, there is nothing to prove unless $\lambda / t_0^{1/\alpha}$ is large where $\lambda = |x-y|$. Define for $t_0 \le 2$

$$A := \left\{ Y_t \text{ hits } B(y, t_0^{1/\alpha}) \text{ before } t_0 \right\} \quad \text{and} \quad C := \left\{ \sup_{t \le t_0} |Y_s - Y_0| \le c_6 t_0^{1/\alpha} \right\}.$$

By Proposition 4.1, $\mathbf{P}^x(C) \ge 1/2$ if c_6 is large enough (note that c_6 does not depend on t_0 as long as $t_0 \le 2$). By the strong Markov property,

$$\mathbf{P}^x \left(Y_{t_0} \in B(y, (1+c_6) t_0^{1/\alpha}) \right) \ge \mathbf{E}^x \left[\mathbf{P}^{Y_S}(C);\, A \right] \ge \frac{1}{2} \mathbf{P}^x(A), \tag{4.32}$$

where $S := \inf\{t \ge 0 : Y_t \in B(y, t_0^{1/\alpha})\}$. As before, ball $B\left(y, (1+c_6) t_0^{1/\alpha}\right)$ can be covered by a finite number (which depends only on c_6 and d) of balls of radii $t_0^{1/\alpha}$, so that by (4.31), the left hand side of (4.32) is bounded by $c_7 (t_0^{1/\alpha}/\lambda)^{d+\alpha}$. Letting $r = t_0^{1/\alpha}$ establishes (4.1). Keeping a carefully track of the constants appeared in the above argument, we see that, except for the case of $0 < \alpha = d < 2$, the constant c_1 in (4.1) can be chosen to depend only on (a, n, d, α) and on the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) respectively.

The proof of (ii) is relatively easier. It is sufficient to show

$$\mathbf{P}^x \left(Y_t \text{ hits ball } B(y, t_0^{1/\alpha}) \text{ before } t_0 \right) \ge c_8 \left(\frac{t_0^{1/\alpha}}{|x-y|} \right)^{d+\alpha}, \tag{4.33}$$

for all $t_0 \in (0, 2]$ and $|x-y| \ge 2 t_0^{1/\alpha}$. This is because, first we can take $b = 2$ by regarding $b^\alpha t_0 / 2^\alpha$ (note that here it is enough to consider the case $b \le 2$) as t_0. Secondly, when $a \ge 1$, (4.2) clearly follows from (4.33); when $a < 1$,

$$\mathbf{P}^x \left(Y_t \text{ hits ball } B(y,\, a\, t_0^{1/\alpha}) \text{ before } t_0 \right) \ge \mathbf{P}^x \left(Y_t \text{ hits ball } B(y,\, T_0^{1/\alpha}) \text{ before } T_0 \right),$$

where $T_0 = a^\alpha t_0$.

Now with $B_x := B(x,\, t_0^{1/\alpha})$, $B_y := B(y,\, t_0^{1/\alpha})$ and $\tau_x := \tau_{B_x}$, it follows from Proposition 4.1 (with $t_0 = 2^{1/\alpha}$ and $A = B = 1/2$),

$$\mathbf{E}^x \left[t_0 \wedge \tau_x \right] \ge \gamma_{2^{1/\alpha}} t_0 \, \mathbf{P}^x \left(\tau_x \ge \gamma_{2^{1/\alpha}} t_0 \right) \ge \gamma_{2^{1/\alpha}} t_0 / 2, \tag{4.34}$$

for each $t_0 \le 2$, where $\gamma_{2^{1/\alpha}} := \gamma(2^{1/\alpha}, 1/2, 1/2)$. Thus, from Lemma 4.7,

$$
\begin{aligned}
\mathbf{P}^x\left(Y_t \text{ hits ball } B(y,\, t_0^{1/\alpha}) \text{ before } t_0\right) &\ge \mathbf{P}^x(Y_{t_0\wedge\tau_x} \in B(y, t_0^{1/\alpha})) \\
&= \mathbf{E}^x\left[\int_0^{t_0\wedge\tau_x}\int_{B_y} \frac{c(Y_s,u)}{|Y_s-u|^{d+\alpha}}\,\mu(du)ds\right] \\
&\ge c\mathbf{E}^x\left[t_0\wedge\tau_x\right]\int_{B_y}\frac{\mu(du)}{|x-y|^{d+\alpha}} \\
&\ge c\,t_0\frac{\mu(B(y,t_0^{1/\alpha}))}{|x-y|^{d+\alpha}} \\
&\ge c_9\frac{t_0^{1+d/\alpha}}{|x-y|^{d+\alpha}}.
\end{aligned}
$$

Here in the second to the third inequality, (4.34) is used. This establishes (4.33). Again keeping a carefully track of the constants appeared in the above argument, we see that, except for the case of $0<\alpha=d<2$, the constant c_2 in (4.2) can be chosen to depend only on (a,b,n,d,α) and on the constants (C_1,C_2,C_3,C_4) in (1.1), (1.5) and (1.4) respectively. □

To prove Proposition 4.3, we need three lemmas.

Lemma 4.9 *Let $R_0>0$. There exists $C_5>0$ independent of R_0 such that for every $x\in F$, $r\le R_0$, $y\in B(x,r/3)$ and a bounded nonnegative function h on $[0,\infty)\times F$ that is supported in $[0,\infty)\times B(x,2r)^c$,*

$$
\mathbf{E}^{(0,x)}\left[h(\tau_r, Y_{\tau_r})\right] \le C_5\mathbf{E}^{(0,y)}\left[h(\tau_r,Y_{\tau_r})\right], \tag{4.35}
$$

where $\tau_r=\tau_{Q_{R_0}(0,x,r)}$. Moreover, the constant C_5 above can be chosen to depend only on (n,d,α) and on the constants (C_1,C_2,C_3,C_4) in (1.1), (1.5) and (1.4) respectively.

Proof. Note that under $\mathbf{P}^{(0,z)}$, $\tau_r := \inf\{t\ge 0:\ Y_t\notin B(x,r)\}\wedge\gamma_{R_0}r^\alpha = \tau_{B(x,r)}\wedge\gamma_{R_0}r^\alpha$. Recall that $\gamma_{R_0} := \gamma(R_0,1/2,1/2)$. Since h is supported on $(0,\infty)\times B(x,2r)^c$, for any $z\in B(x,\,r/3)$

$$
\mathbf{E}^{(0,z)}\left[h(\tau_r,Y_{\tau_r})\right] = \mathbf{E}^{(0,z)}\left[h(\tau_r,Y_{\tau_r});\ Y_{\tau_r-}\neq Y_{\tau_r}\right].
$$

So by Lemma 4.7 with $T=\tau_r$,

$$
\begin{aligned}
&\mathbf{E}^{(0,z)}\left[h(\tau_r,Y_{\tau_r})\right] \\
=\;& \mathbf{E}^{(0,z)}\left[\int_0^{\tau_r}\int_{B(x,2r)^c}\frac{c(Y_s,u)h(s,u)}{|Y_s-u|^{d+\alpha}}\,\mu(du)\,ds\right] \\
\approx\;& \mathbf{E}^{(0,z)}\left[\int_0^{\tau_r}\int_{B(x,2r)^c}\frac{h(s,u)}{|x-u|^{d+\alpha}}\,\mu(du)\,ds\right] \\
=\;& \int_0^{\gamma_{R_0}r^\alpha}\int_{B(x,2r)^c}\frac{h(s,u)\,\mathbf{P}^{(0,z)}(s<\tau_r)}{|x-u|^{d+\alpha}}\,\mu(du)\,ds.
\end{aligned}
$$

Here, as mentioned at the end of the introduction, two functions $f \approx g$ means that there is a constant $\lambda > 1$ such that $\lambda^{-1} g \le f \le \lambda g$. On the other hand, by Proposition 4.1, for all $s \le \gamma_{R_0} r^\alpha$, $r \le R_0$ and $z \in B(x, r/3)$,

$$1/2 \le \mathbf{P}^z(s \le \tau_{B(z,r/2)}) \le \mathbf{P}^{(0,z)}(s \le \tau_r) \le 1.$$

This implies that the values of the function $z \mapsto \mathbf{E}^z \left[h(\tau_r, Y_{\tau_r})\right]$ are all comparable with each other with a universal constant multiple for any $z \in B(x, r)$, and therefore proves the lemma. □

Remark 4.10 The above lemma is a continuous time version of Lemma 3.5 in Bass and Levin [4], which concerns about discrete time Markov chain on $\mathbf{Z}^d$. The proof of Lemma 3.5 in Bass and Levin [4] contains a minor gap, which can be easily remedied as follows. In line 2 and line 6 after (3.3) in [4], the term $\mathbf{E}^{(0,y)}[\tau_r] - 1$ and $\mathbf{E}^{(0,x)}[\tau_r]$ should be $\mathbf{P}^{(0,y)}(\tau_r > k-1)$ and $\mathbf{P}^{(0,x)}(\tau_r > k-1)$, respectively. Using the tightness result, Theorem 2.8 in [4], one concludes that values of $\mathbf{P}^{(0,y)}(k-1 < \tau_r)$ for $y \in B(x, r/3)$ are all comparable to each other. With this modification, the proof of Lemma 3.5 in [4] goes through. □

For each $A \subset [0, \infty) \times F$, denote $\sigma_A := \inf\{t > 0 : Z_t \in A\}$ and $A_s := \{y \in F : \ (s, y) \in A\}$.

Lemma 4.11 *Let $R_0 > 0$. There exists $C_6 > 0$ independent of R_0 such that for all $x \in F$, $r \le R_0$ and any compact subset $A \subset Q_{R_0}(0, x, r/2)$,*

$$\mathbf{P}^{(0,x)}(\sigma_A < \tau_r) \ge C_6 \frac{m \otimes \mu(A)}{r^{d+\alpha}},$$

where $\tau_r = \tau_{Q_{R_0}(0,x,r)}$ and $m \otimes \mu$ is a product measure of the Lebesgue measure m on $\mathbf{R}_+$ and the d-measure μ on F. Moreover, except for the case of $0 < \alpha = d < 2$, the constant C_4 above can be chosen to depend only on (n, d, α) and on the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) respectively.

Proof. The conclusion of the lemma holds if $\mathbf{P}^x(\sigma_A < \tau_r) > 1/4$. So we will assume $\mathbf{P}^x(\sigma_A < \tau_r) \le 1/4$. Let $T = \sigma_A \wedge \tau_r$. Then

$$\mathbf{P}^{(0,x)}(\sigma_A < \tau_r) = \mathbf{P}^{(0,x)}((T, Y_T) \in A) \ge \mathbf{P}^{(0,x)}((T, Y_T) \in A; \ Y_{T-} \ne Y_T).$$

Applying Lemma 4.7 with $f(s,x,y) = 1_{\{x\neq y\}}1_A(s,y)$ and $T = \sigma_A \wedge \tau_r$, we have, with $t_0 := \gamma_{R_0}(r/2)^\alpha$ and $r \le R_0$,

$$\begin{aligned}
\mathbf{P}^{(0,x)}(\sigma_A < \tau_r) &= \mathbf{E}^{(0,x)}\left[\int_0^T \int_{A_s} \frac{c(Y_s,u)}{|u-Y_s|^{d+\alpha}}\,\mu(du)\,ds\right] \\
&\ge c_1 \mathbf{E}^{(0,x)}\left[\int_0^{t_0} \int_{A_s} \frac{1}{r^{d+\alpha}}\,\mu(du)\,ds;\ \sigma_A \wedge \tau_r \ge t_0\right] \\
&= c_1 \frac{m\otimes\mu(A)}{r^{d+\alpha}} \mathbf{P}^{(0,x)}(\sigma_A \wedge \tau_r \ge t_0),
\end{aligned}$$

where in the second to the last inequality, we used the fact that $|u-y| \le 3r/2$ for $y \in B(x,r)$ and $u \in A_s \subset B(x,r/2)$. By Proposition 4.1,

$$\mathbf{P}^{(0,x)}(\tau_r < t_0) \le \mathbf{P}^x(\tau_{B(x,r/2)} \le t_0) \le 1/2,$$

and so

$$\mathbf{P}^{(0,x)}(\sigma_A \wedge \tau_r \ge t_0) \ge 1 - \mathbf{P}^{(0,x)}(\sigma_A < \tau_r) - \mathbf{P}^{(0,x)}(\tau_r < t_0) \ge 1/4,$$

which proves the lemma. □

Define $U(t,x,r) = \{t\} \times B(x,r)$.

Corollary 4.12 *Let $R_0 > 0$. For any $0 < \delta \le \gamma_{R_0}$, there exists $C_7 > 0$ such that for every $0 < R \le R_0$, $(t,x) \in Q_{R_0}(0,z,R/3)$, $r \le R/4$ and $t \ge \delta r^\alpha$*

$$\mathbf{P}^{(0,z)}\left(\sigma_{U(t,x,r)} < \tau_{Q_{R_0}(0,z,R)}\right) \ge C_7 \frac{r^{d+\alpha}}{R^{d+\alpha}}.$$

Moreover, except for the case of $0 < \alpha = d < 2$, the constant C_7 above can be chosen to depend only on (δ, n, d, α) and on the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) respectively.

Proof. Let $Q' := [t - \delta r^\alpha,\, t] \times B(x,r/2) \subset Q_{R_0}(0,z,R/2)$. By Lemma 4.11,

$$\mathbf{P}^{(0,z)}\left(\sigma_{Q'} < \tau_{Q_{R_0}(0,z,R)}\right) \ge c_1 r^{d+\alpha}/R^{d+\alpha}.$$

Starting from a point in Q', by Proposition 4.1 there is a probability at least $\varepsilon = \varepsilon(\delta)$ that the process Y stays in $B(x,r)$ for at least δr^α amount of time. Thus, by the strong Markov property, with probability at least $c_1\,\varepsilon\, r^{d+\alpha}/(2R^{d+\alpha})$, the process hits Q' before exiting $Q_{R_0}(0,z,R)$ and stays within $B(x,r)$ for an additional δr^α amount of time, and hence hits $U(t,x,r)$ before exiting $Q_{R_0}(0,z,R)$. □

Recall that $Z_s = (V_s, Y_s)$ is the space-time process of Y, where $V_s = V_0 + s$.

Lemma 4.13 *For any bounded Borel measurable function $q(t,x)$ that is parabolic in an open subset D of $\mathbf{R}_+ \times F$, $s \mapsto q(Z_{s\wedge\tau_D})$ is right continuous $\mathbf{P}^{(t,x)}$-a.s. for every $(t,x) \in D$. Here $\tau_D = \inf\{s > 0 : Z_s \notin D\}$.*

Proof. Note that according to the definition of parabolicity, for every relatively compact open subset D_1 of D, $q(t,x) = \mathbf{E}^{(t,x)}\left[q(Z_{\tau_{D_1}})\right]$ for every $(t,x) \in D_1$. By the strong Markov property of Z, for every stopping time S of $\{\widetilde{\mathcal{F}}_s, s \geq 0\}$,

$$q(Z_{S\wedge\tau_{D_1}}) = \mathbf{E}^{(t,x)}\left(q(Z_{\tau_{D_1}})\big|\widetilde{\mathcal{F}}_{S\wedge\tau_{D_1}}\right) \qquad \mathbf{P}^{(t,x)}\text{-a.s..}$$

Here the martingale $s \mapsto \mathbf{E}^{(t,x)}\left[q(Z_{\tau_{D_1}})\big|\widetilde{\mathcal{F}}_{s\wedge\tau_D}\right]$ is taken to be its right continuous version. As $s \mapsto Z_s$ is right continuous and having left limits and that q is Borel measurable, the process $s \mapsto q(Z_{s\wedge\tau_{D_1}})$ is optional. Hence by an application of the Optional Section Theorem (cf. Theorem 4.10 in [18]), one has

$$\mathbf{P}^{(t,x)}\left(q(Z_{s\wedge\tau_{D_1}}) = \mathbf{E}^{(t,x)}\left(q(Z_{\tau_{D_1}})\big|\widetilde{\mathcal{F}}_{s\wedge\tau_{D_1}}\right) \text{ for all } s \geq 0\right) = 1.$$

This shows that $s \mapsto q(Z_{s\wedge\tau_{D_1}})$ is right continuous $\mathbf{P}^{(t,x)}$-a.s. for every $(t,x) \in D_1$. Since the above holds for every relatively compact subset D_1 of D, the lemma is proved. □

Proof of Proposition 4.3. By Lemma 4.9 and its proof, we see that $\inf_{y\in B(z,R/3)} q(0,y) > 0$ unless q is identically zero. Taking a constant multiple of q if needed, we may assume that

$$\inf_{y\in B(z,R/3)} q(0,y) = 1/2.$$

Let $v \in B(z,R/3)$ be such that $q(0,v) \leq 1$. We want to show that $q(t,x)$ is bounded from above in $Q_{R_0}(\delta R^\alpha, z, R/3)$ by a constant that is independent of the function q. In this proof, we suppress the subscript R_0 from γ_{R_0} and $Q_{R_0}(\cdot,\cdot,\cdot)$.

By Lemma 4.11, there exists $c_1 < 1/2$ such that if $r \leq R/4$ and $C \subset Q(t,x,r/3)$ having $m \otimes \mu(C)/\{m \otimes \mu(Q(t,x,r/3))\} \geq 2/3$, then

$$\mathbf{P}^{(t,x)}(\sigma_C < \tau_r) \geq c_1, \tag{4.36}$$

where $\tau_r := \tau_{Q(t,x,r)}$. Define

$$\eta = \frac{c_1}{3} \quad \text{and} \quad \xi = \frac{1}{3} \wedge (C_5\eta), \tag{4.37}$$

where C_5 is the constant in Lemma 4.9. We claim that there is a universal constant $K = K(\delta)$ to be determined later, which is independent of R and function q, such that $q \leq K$ on $Q(\delta R^\alpha, z, R/3)$.

We are going to prove this by contradiction. Suppose it is not, then there is some point $(t,x) \in Q(\delta R^\alpha, z, R/3)$ such that $q(t,x) > K$. We will show that there is a constant $\beta > 0$ and there is a sequence of points $\{(t_k, x_k)\}$ in $\widehat{Q}(0,z,R) := [0, 3\gamma R^\alpha] \times B(z,R)$ so that $q(t_k,x_k) \geq (1+\beta)^k K$, which contradicts to the assumption that q is bounded on $\widehat{Q}(0,z,R)$.

Let $r > 0$ to be the smallest r such that

$$\frac{m \otimes \mu(Q(0,x,r/3))}{R^{d+\alpha}} \geq \frac{3}{C_6\, \xi K}, \qquad \frac{r^{d+\alpha}}{R^{d+\alpha}} \geq \frac{2 \cdot 3^{d+\alpha}}{C_7\, \xi K}, \tag{4.38}$$

where C_6 and C_7 are the constants in Lemma 4.11 and Corollary 4.12 respectively. With K being sufficiently large, such r exists and can be made less than $R/9$. In fact the following estimate holds:

$$c_2 K^{-1/(d+\alpha)} \leq r/R \leq c_3 K^{-1/(d+\alpha)} \tag{4.39}$$

with $c_2 = \max\left\{\frac{3^{1+1/(d+\alpha)}}{(\gamma \cdot C_2 \cdot C_6 \cdot \xi)^{1/(d+\alpha)}}, \frac{3 \cdot 2^{1/(d+\alpha)}}{(C_7 \cdot \xi)^{1/(d+\alpha)}}\right\}$ and $c_3 = \max\left\{\frac{3^{1+1/(d+\alpha)}}{(\gamma \cdot C_1 \cdot C_6 \cdot \xi)^{1/(d+\alpha)}}, \frac{3 \cdot 2^{1/(d+\alpha)}}{(C_7 \cdot \xi)^{1/(d+\alpha)}}\right\}$, where C_1 and C_2 are the constants in (1.1). Let $U = \{t\} \times B(x, r/3)$. Were function $q \geq \xi K$ on U, we would have by Corollary 4.12 that

$$1 \geq q(0,v) = \mathbf{E}^{(0,v)}\left[q(Z_{\sigma_U \wedge \tau_Q})\right] \geq \xi K\, \mathbf{P}^{(0,v)}\left(\sigma_U < \tau_Q\right) \geq \xi K\, \frac{C_7 (r/3)^{d+\alpha}}{R^{d+\alpha}},$$

where $Q := Q(0,x,R)$, which contradicts to our choice of r in (4.39). Thus, there must be at least one point in U on which q takes a value less than ξK.

We next claim that

$$\mathbf{E}^{(t,x)}\left[q(\tau_r, Y_{\tau_r}) : Y_{\tau_r} \notin B(x, 2r)\right] \leq \eta K, \tag{4.40}$$

$\tau_r := \tau_{Q(t,x,r)}$. If not, then by Lemma 4.9, we would have for all $y \in B(x, r/3)$,

$$\begin{aligned} q(t,y) &\geq \mathbf{E}^{(t,y)}[q(\tau_r, Y_{\tau_r}) : Y_{\tau_r} \notin B(x,2r)] \\ &\geq C_5^{-1}\mathbf{E}^{(t,x)}[q(\tau_r, Y_{\tau_r}) : Y_{\tau_r} \notin B(x,2r)] \\ &\geq C_5^{-1}\eta K \\ &\geq \xi K, \end{aligned}$$

a contradiction to the fact that obtained in the preceding paragraph. So (4.40) holds.

Let A be any compact subset of

$$\widetilde{A} := \{(s,y) \in Q(t,x,r/3) : q(s,y) \geq \xi K\}.$$

By Lemma 4.11

$$1 \geq q(0,v) \geq \mathbf{E}^{(0,v)}[q(Z_{\sigma_A}) : \sigma_A < \tau_Q] \geq \xi K \mathbf{P}^{(0,v)}(\sigma_A < \tau_Q) \geq \xi K\, \frac{C_6\, m \otimes \mu(A)}{R^{d+\alpha}}.$$

So by (4.38),

$$\frac{m\otimes\mu(A)}{m\otimes\mu(Q(t,x,r/3))}\leq\frac{R^{d+\alpha}}{C_6\cdot m\otimes\mu(Q(t,x,r/3))\xi K}\leq\frac{1}{3}. \tag{4.41}$$

Since (4.41) holds for every compact subset A of $\widetilde{A}$, it holds for $\widetilde{A}$ in place of A.

Let $C=Q(t,x,r/3)\setminus\widetilde{A}$ and $M=\sup_{(s,y)\in Q(t,x,2r)}q(s,y)$. We write

$$\begin{aligned} q(t,x) &= \mathbf{E}^{(t,x)}[q(\sigma_C,Y_{\sigma_C}):\sigma_C<\tau_r]+\mathbf{E}^{(t,x)}[q(\tau_r,Y_{\tau_r}):\tau_r\leq\sigma_C,Y_{\tau_r}\notin B(x,2r)] \\ &\quad +\mathbf{E}^{(t,x)}[q(\tau_r,Y_{\tau_r}):\tau_r\leq\sigma_C,Y_{\tau_r}\in B(x,2r)]. \end{aligned}$$

The first term on the right is bounded by $\xi K\,\mathbf{P}^{(t,x)}(\sigma_C<\tau_r)$ in view of Lemma 4.13, the second term is bounded by ηK according to (4.40), and the third term is clearly bounded by $M\mathbf{P}^{(t,x)}(\tau_r\leq\sigma_C)$. Therefore,

$$K\leq\xi K\,\mathbf{P}^{(t,x)}(\sigma_C<\tau_r)+\eta K+M\,\mathbf{P}^{(t,x)}(\sigma_C\geq\tau_r)).$$

It follows from (4.41) and (4.36)-(4.37)

$$M/K\geq\frac{1-\eta-\xi\,\mathbf{P}^{(t,x)}(\sigma_C<\tau_r)}{\mathbf{P}^{(t,x)}(\sigma_C\geq\tau_r)}\geq\frac{1-\eta-\xi\,c_1}{1-c_1}\geq\frac{1-(2c_1)/3}{1-c_1}:=1+2\beta,$$

where $\beta=\frac{c_1}{6(1-c_1)}$. Hence there exists a point $(t_1,x_1)\in Q(t,x,2r)\subset\widehat{Q}(0,z,R)$ such that $q(t_1,x_1)\geq(1+\beta)K=:K_1$.

We now iterate the above procedure to obtain a sequence of points $\{(t_k,x_k)\}$ in the following way. Using the above argument (with (t_1,x_1) and K_1 in place of (t,x) and K), there exists $(t_2,x_2)\in Q(t_1,x_1,2r_1)$ such that $q(t_2,x_2)\geq(1+\beta)K_1=:K_2$. We continue this procedure to obtain a sequence of points $\{(t_k,x_k)\}$ such that $(t_{k+1},x_{k+1})\in Q(t_k,x_k,2r_k)$ and $q(t_{k+1},x_{k+1})\geq(1+\beta)^{k+1}K=:K_{k+1}$. As $0\leq t_{k+1}-t_k\leq\gamma(2r_k)^\alpha$, $|x_{k+1}-x_k|\leq 2r_k$, and

$$r_k\leq c_3R\cdot K_k^{-1/(d+\alpha)}\leq c_3(1+\beta)^{-k/(d+\alpha)}K^{-1/(d+\alpha)}R$$

by (4.39), we can take K large enough (independent of R and q) so that $(t_k,x_k)\in\widehat{Q}(0,z,R)$ for all k. This is a contradiction because $q(t_k,x_k)\geq(1+\beta)^kK$ goes to infinity as $k\to\infty$. We conclude that q is bounded by K in $Q(\delta R^\alpha,z,R/3)$, which completes the proof of the Proposition. Keeping a carefully track of the constants appeared in the above argument, we see that, except for the case of $0<\alpha=d<2$, the constant c in this proposition can be chosen to depend only on (R_0,δ,n,d,α) and on the constants (C_1,C_2,C_3,C_4) in (1.1), (1.5) and (1.4) respectively. □

Now we prove that the heat kernel $p(t,x,y)$ is Hölder continuous in (t,x,y). Recall that for $(t,x)\in[0,\infty)\times F$ and $r>0$, $Q_{R_0}(t,x,r):=[t,\,t+\gamma_{R_0}r^\alpha]\times B(x,R)$, where $\gamma_{R_0}:=\gamma(R_0,1/2,1/2)<1$ is the constant in Proposition 4.1 corresponding to $t_0=R_0$ and $A=B=1/2$.

Theorem 4.14 *For every $R_0 > 0$, there is a constant $c = c(R_0) > 0$ such that for every $0 < R \le R_0$ and every bounded parabolic function q in $Q_{R_0}(0, x_0, \max\{4,\, 4^{1/\alpha}\}R)$,*

$$|q(s,x) - q(t,y)| \le c\, \|q\|_{\infty,R}\, R^{-\beta} \left(|t-s|^{1/\alpha} + |x-y| \right)^{\beta} \tag{4.42}$$

holds for (s,x), $(t,y) \in Q_{R_0}(0, x_0, R)$, where $\|q\|_{\infty,R} := \sup_{(t,y)\in[0,\,\gamma_{R_0} \max\{4,\, 4^{\alpha}\}R^{\alpha}]\times F} |q(t,y)|$. In particular, for the transition density function $p(t,x,y)$ of Y, there are constants $c > 0$ and $\beta > 0$ such that for any $0 < t_0 < 1$, $t,\, s \in [t_0,\, 2]$ and $(x_i, y_i) \in F \times F$ with $i = 1, 2$,

$$|p(s, x_1, y_1) - p(t, x_2, y_2)| \le c\, t_0^{-(d+\beta)/\alpha} \left(|t-s|^{1/\alpha} + |x_1 - x_2| + |y_1 - y_2| \right)^{\beta}. \tag{4.43}$$

Moreover, except for the case of $0 < \alpha = d < 2$, the constant c above can be chosen to depend only on (R_0, t_0, n, d, α) and on the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) respectively.

Proof. The proof is a modification to the parabolic case from that of Theorem 4.1 in Bass and Levin [3], where the Hölder continuity is established for (elliptic) harmonic functions of stable-like processes in $\mathbf{R}^n$. For the reader's convenience, we spell out the details here.

Let $Z_s = (V_s, Y_s)$ be the space-time process of Y, where $V_s = V_0 + s$. In the following, we suppress the subscript R_0 from γ_{R_0} and $Q_{R_0}(\cdot,\cdot,\cdot)$. Without loss of generality, assume that $0 \le q(z) \le \|q\|_{\infty,R} = 1$ for $z \in [0,\, \gamma_{R_0} \max\{4,\, 4^{\alpha}\}R^{\alpha}]\times F$. By Lemma 4.11 there is a constant $0 < c_1 < 1$ such that if $x \in F$, $0 < r < 1$ and $A \subset Q(t,x,r/2)$ with $\frac{m\otimes\mu(A)}{m\otimes\mu(Q(t,x,r/2))} \ge 1/3$, then

$$\mathbf{P}^{(t,x)}(\sigma_A < \tau_r) \ge c_1, \tag{4.44}$$

where $\tau_r := \tau_{Q(t,x,r)}$. By Lemma 4.7 with $f(s,y,z) = 1_{B(x,r)}(y)\, 1_{F\setminus B(x,s)}(z)$ and $T = \tau_r$, there is a constant $c_2 > 0$ such that if $s \ge 2r$,

$$\mathbf{P}^{(t,x)}(Y_{\tau_r} \notin B(x,s)) = \mathbf{E}^{(t,x)} \left[\int_0^{\tau_r} \int_{F\setminus\overline{B(x,s)}} \frac{c(Y_v, u)}{|Y_v - u|^{d+\alpha}}\, \mu(du)\, dv \right] \le c_2 r^{\alpha}/s^{\alpha}. \tag{4.45}$$

The last inequality is due to Lemma 5.1, whose proof uses only a special case of Lemma 4.7. Let

$$\eta = 1 - \frac{c_1}{4} \quad \text{and} \quad \rho = \frac{1}{2} \wedge \left(\frac{\eta}{2}\right)^{1/\alpha} \wedge \left(\frac{c_1\, \eta}{8\, c_2}\right)^{1/\alpha}.$$

Note that for every $(t,x) \in Q(0, x_0, R)$, q is parabolic in $Q(t,x,R) \subset Q(0,\, x_0,\, R\max\{2,\, 2^{1/\alpha}\})$. We will show that

$$\sup_{Q(t,x,\rho^k R)} q - \inf_{Q(t,x,\rho^k R)} q \le \eta^k \qquad \text{for all } k. \tag{4.46}$$

For notational convenience, we write Q_i for $Q(t,x,\rho^i R)$ and τ_i for $\tau_{Q(t,x,\rho^i R)}$. Define

$$a_i = \inf_{Q_i} q \quad \text{and} \quad b_i = \sup_{Q_i} q.$$

Clearly $b_i - a_i \le 1 \le \eta^i$ for all $i \le 0$. Now suppose that $b_i - a_i \le \eta^i$ for all $i \le k$ and we are going to show that $b_{k+1} - a_{k+1} \le \eta^{k+1}$. Observe that $Q_{k+1} \subset Q_k$ and so $a_k \le q \le b_k$ on Q_{k+1}. Define

$$A' := \{z \in Q_{k+1} : q(z) \le (a_k + b_k)/2\}.$$

We may suppose $\frac{m\otimes\mu(A')}{m\otimes\mu(Q_{k+1})} \ge 1/2$, for if not we use $1-q$ instead of q. Let A be a compact subset of A' such that $\frac{m\otimes\mu(A)}{m\otimes\mu(Q_{k+1})} \ge 1/3$. For any given $\varepsilon > 0$, pick $z_1, z_2 \in Q_{k+1}$ so that $q(z_1) \ge b_{k+1} - \varepsilon$ and $q(z_2) \le a_{k+1} + \varepsilon$. Then by (4.44)-(4.46),

$$\begin{aligned}
& b_{k+1} - a_{k+1} - 2\varepsilon \\
\le\ & q(z_1) - q(z_2) \\
=\ & \mathbf{E}^{z_1}\left[q(Z_{\sigma_A \wedge \tau_{k+1}}) - q(z_2)\right] \\
=\ & \mathbf{E}^{z_1}\left[q(Z_{\sigma_A}) - q(z_2);\, \sigma_A < \tau_{k+1}\right] + \mathbf{E}^{z_1}\left[q(Z_{\tau_{k+1}}) - q(z_2);\, \sigma_A > \tau_{k+1} \text{ and } Z_{\tau_{k+1}} \in Q_k\right] \\
& \quad + \sum_{i=1}^{\infty} \mathbf{E}^{z_1}\left[q(Z_{\tau_{k+1}}) - q(z_2);\, \sigma_A > \tau_{k+1} \text{ and } Z_{\tau_{k+1}} \in Q_{k-i} \setminus Q_{k+1-i}\right] \\
\le\ & \left(\frac{a_k + b_k}{2} - a_k\right) \mathbf{P}^{z_1}(\sigma_A < \tau_{k+1}) + (b_k - a_k)\mathbf{P}^{z_1}(\sigma_A > \tau_{k+1}) \\
& \quad + \sum_{i=1}^{\infty} (b_{k-i} - a_{k-i})\mathbf{P}^{z_1}(Z_{\tau_{k+1}} \notin Q_{k+1-i}) \\
\le\ & (b_k - a_k)\left(1 - \frac{\mathbf{P}^{z_1}(\sigma_A < \tau_{k+1})}{2}\right) + \sum_{i=1}^{\infty} c_2\, \eta^k (\rho^\alpha/\eta)^i \\
\le\ & (1 - \frac{c_1}{2})\, \eta^k + 2c_2 \eta^{k-1} \rho^\alpha \\
\le\ & (1 - \frac{c_1}{2})\eta^k + \frac{c_1}{4}\eta^k \\
=\ & \eta^{k+1}.
\end{aligned}$$

Since ε is arbitrary, we have $b_{k+1} - a_{k+1} \le \eta^{k+1}$ and this proves (4.46).

For $z = (s,x)$ and $w = (t,y)$ in $Q(0,x_0,R)$ with $s \le t$, let k be the smallest integer such that $|z - w| := (\gamma^{-1}|t-s|)^{1/\alpha} + |x - y| \le \rho^k R$. Then $\log(|z-w|/R) \ge (k+1)\log\rho$, $w \in Q(s,x,\rho^k R)$ and

$$|q(z) - q(w)| \le \eta^k = e^{k\log\eta} \le c_3 \left(\frac{|z-w|}{R}\right)^{\log\eta/\log\rho}.$$

This proves (4.42) with $\beta = \log\eta/\log\rho$.

By Lemma 4.5 and Theorem 3.2, for every $0 < t_0 < 1$ and $y \in F$, $q(t,x) := p(2-t,x,y)$ is a parabolic function on $[0, 2-\frac{t_0}{2}] \times F$ bounded above by $c_4\, t_0^{-d/\alpha}$.

For each fixed $t_0 \in (0,1)$, take R such that $\gamma_{R_0} R^\alpha = t_0/2$. Let $s, t \in [t_0, 2]$ with $s > t$ and $x_1, x_2 \in F$. Assume first that

$$|s-t|^{1/\alpha} + |x_1 - x_2| < \gamma_{R_0}^{1/\alpha} R = (t_0/2)^{1/\alpha} \tag{4.47}$$

and so $(2-t, x_2) \in Q_{R_0}(2-s, x_1, R) \subset [0, 2-\frac{t_0}{2}) \times F$. Applying (4.42) to the parabolic function $q(t,x)$ with $(2-s, x_1)$, $(2-t, x_2)$ and $Q_{R_0}(2-s, x_1, R)$ in place of (s,x), (t,y) and $Q_{R_0}(0, x_0, R)$ there respectively, we have

$$|p(s, x_1, y) - p(t, x_2, y)| \le c\, t_0^{-(d+\beta)/\alpha} (|t-s|^{1/\alpha} + |x_1 - x_2|)^\beta. \tag{4.48}$$

By Theorem 3.2, the inequality (4.48) is true when (4.47) does not hold. So (4.48) holds for every $t, s \in [t_0, 2]$ and $x_1, x_2 \in F$. Inequality (4.43) now follows from (4.48) by the symmetry of $p(t,x,y)$ in x and y. Keeping a carefully track of the constants appeared in the above argument, we see that, except for the case of $0 < \alpha = d < 2$, the constant c in (4.42) and (4.43) can be chosen to depend only on (R_0, t_0, n, d, α) and on the constants (C_1, C_2, C_3, C_4) in (1.1), (1.5) and (1.4) respectively. □

5 Hausdorff Dimension for the Range of Stable-Like Processes

Throughout this section, F is a closed d-set in $\mathbf{R}^n$ satisfying condition (1.5). Fix $0 < \alpha < 2$ and let Y be the Feller process on a d-set F whose Dirichlet form is given by (1.2)-(1.3). We will prove Theorem 1.2 in this section.

First we need an estimate on the expected exit time from balls in F.

Lemma 5.1 *There exist $c_2 > c_1 > 0$ such that for every $x \in F$ and $0 < r \le 1$,*

$$c_1 r^\alpha \le \inf_{z \in B(x, r/2)} \mathbf{E}^z[\tau_{B(x,r)}] \le \sup_{z \in B(x,r)} \mathbf{E}^z[\tau_{B(x,r)}] \le c_2 r^\alpha. \tag{5.1}$$

Proof. By Proposition 4.1, for $z \in B(x, r/2)$,

$$\mathbf{P}^z(\tau_{B(x,r)} \le \gamma_1 r^\alpha) \le \mathbf{P}^z(\sup_{s \le \gamma_1 r^\alpha} |Y_s - z| \ge r/2) \le 1/2,$$

Thus,

$$\mathbf{E}^z[\tau_{B(x,r)}] \ge \gamma_1 r^\alpha\, \mathbf{P}^z(\tau_{B(x,r)} > \gamma_1 r^\alpha) \ge \gamma_1 r^\alpha/2.$$

On the other hand, it follows from Lemma 4.7 that

$$1 \geq \mathbf{P}^z(Y_{\tau_{B(x,r)}} \notin \overline{B(x,2r)}) = \mathbf{E}^z\left[\int_0^{\tau_{B(x,r)}} \int_{\overline{B(x,2r)}^c} \frac{c(Y_s,u)}{|Y_s-u|^{d+\alpha}}\mu(du)\,ds\right] \geq c\,r^{-\alpha}\,\mathbf{E}^z[\tau_{B(x,r)}],$$

for some constant $c = c(F,\alpha) > 0$ and so $\mathbf{E}^z[\tau_{B(x,r)}] \leq c^{-1}\,r^\alpha$. Here in the last inequality, we used the fact that for $y \in B(x,r)$ and $u \in \overline{B(x,r)}^c$, $|u-y| \leq |u-x| + |x-y| \leq 2|u-x|$ and that $\int_{\overline{B(x,r)}^c} |u-x|^{-d-\alpha}\mu(du) \geq c' r^{-\alpha}$. This completes the proof of the lemma. □

Proof of Theorem 1.2. For every $x \in F$ and $0 < r \leq 1$, define

$$\tau_1 = \tau_{B(x,r)} \quad \text{and} \quad \tau_{k+1} = \tau_{B(Y_{\tau_k},r)} \circ \theta_{\tau_k} + \tau_k \quad \text{for } k \geq 1,$$

where θ_t is the time-shift operator for process Y_t. It follows from Lemma 5.1 that for every $n \geq 1$,

$$n\,c_1\,r^\alpha \leq \mathbf{E}^x[\tau_n] \leq n\,c_2\,r^\alpha. \tag{5.2}$$

By Lemma 5.1 and the Markov property of Y,

$$\mathbf{E}^z\left[\left(\tau_{B(x,r)}\right)^2\right] = 2\mathbf{E}^z\left[\int_0^{\tau_{B(x,r)}}\int_s^{\tau_{B(x,r)}} 1\,dtds\right] = 2\mathbf{E}^x\left[\int_0^{\tau_{B(x,r)}} \mathbf{E}^{Y_s}\left[\tau_{B(x,r)}\right]ds\right] \leq 2c_2^2\,r^{2\alpha} \tag{5.3}$$

for x, $z \in F$. Define

$$u(x) = \mathbf{E}^x\left[\left(\tau_{B(x,r)} - \mathbf{E}^x[\tau_{B(x,r)}]\right)^2\right],$$

which by (5.3) is bounded by $2c_2^2 r^{2\alpha}$. By the strong Markov property of Y,

$$\mathbf{E}^x\left[(\tau_n - \mathbf{E}^x[\tau_n])^2\right] = \mathbf{E}^x\left[u(x) + \sum_{k=1}^{n-1} u(Y_{\tau_k})\right] \leq 2n\,c_2^2 r^{2\alpha}. \tag{5.4}$$

Taking $r = n^{-1/\alpha}$, we have by Chebyshev's inequality that for any $\varepsilon > 0$,

$$\mathbf{P}^x\left(|\tau_n - \mathbf{E}_x[\tau_n]| > \varepsilon\right) \leq \varepsilon^{-2}\,\mathbf{E}^x\left[(\tau_n - \mathbf{E}^x[\tau_n])^2\right] \leq 2\varepsilon^{-2}\,c_2^2 n^{-1},$$

which tends to zero as $n \to \infty$. Therefore there exists a subsequence $\{n_k\}$ such that

$$\lim_{k\to\infty}\left(\tau_{n_k} - \mathbf{E}_x[\tau_{n_k}]\right) = 0 \quad \mathbf{P}^x\text{-a.s.}$$

This together with (5.2) with $r = n^{-1/\alpha}$ there implies that

$$\liminf_{k\to\infty} \tau_{n_k} \geq c_1 \qquad \mathbf{P}^x\text{-a.s.} \tag{5.5}$$

Hence $\mathbf{P}^x$-a.s., there is $K(\omega) > 0$ such that for $k > K(\omega)$, $\tau_{n_k} > c_1/2$. In other words, $\mathbf{P}^x$-a.s., for $k > K(\omega)$, $Y[0,\,c_1/2]$ can be covered with n_k number of balls with radii $n_k^{-1/\alpha}$. As $n_k \cdot (n_k^{-1/\alpha})^\alpha = 1$, the Hausdorff dimension of $Y[0,\, c_1/2]$ can not be larger than α. The same conclusion holds for $Y[0,1]$ by the Markov property of Y. As the state space of Y is the d-set F, we have proved that

$$\dim_H Y[0,1] \le d \wedge \alpha \qquad \mathbf{P}^x\text{-a.s. for every } x \in F.$$

To get the lower bound estimate, we do estimation on

$$\mathbf{E}^x \left[\int_0^1 \int_0^1 |Y_t - Y_s|^{-p} ds\, dt \right].$$

By Theorem 1.1,

$$\begin{aligned}
\mathbf{E}^x \left[|Y_t - Y_0|^{-p} \right] &\approx \int_F |y-x|^{-p} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \mu(dy) \\
&= \int_0^\infty r^{-p} \left(t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \right) d\mu(B(x,r)) \\
&= \int_0^{t^{1/\alpha}} t^{-d/\alpha} r^{-p}\, d\mu(B(x,r)) + \int_{t^{1/\alpha}}^\infty t\, r^{-d-\alpha-p}\, d\mu(B(x,r)) \\
&= I + II.
\end{aligned}$$

For $p < d$, by (1.1)

$$\begin{aligned}
I &= t^{-d/\alpha} r^{-p} \mu(B(x,r)) \Big|_0^{t^{1/\alpha}} + p \int_0^{t^{1/\alpha}} t^{-d/\alpha} \mu(B(x,r)) r^{-p-1} dr \\
&\le c\, t^{-d/\alpha}\, t^{(d-p)/\alpha} + c \int_0^{t^{1/\alpha}} t^{-d/\alpha}\, r^{-p-1+d} dr \\
&\le c_3\, t^{-p/\alpha},
\end{aligned}$$

while for $t \le 1$,

$$\begin{aligned}
II &= t\, r^{-d-\alpha-p} \mu(B(x,r)) \Big|_{t^{1/\alpha}}^\infty + (d+\alpha+p) \int_{t^{1/\alpha}}^\infty t\, r^{-\alpha-p-1} \mu(B(x,r))\, dr \\
&\le c_4\, t^{-p/\alpha}.
\end{aligned}$$

Hence we conclude that for $t \le 1$ and $p < d$,

$$\sup_{x \in F} \mathbf{E}^x \left[|Y_t - Y_0|^{-p} \right] \le c_5\, t^{-p/\alpha}.$$

It follows then by using the Markov property of Y that for $p < d \wedge \alpha$,

$$\begin{aligned}
\mathbf{E}^x \left[\int_0^1 \int_0^1 |Y_t - Y_s|^{-p} ds\, dt \right] &= 2\mathbf{E}^x \left[\int_0^1 \left(\int_t^1 |Y_t - Y_s|^{-p} ds \right) dt \right] \\
&= 2\mathbf{E}^x \left[\int_0^1 \mathbf{E}^{Y_t} \left[\int_0^{1-t} |Y_r - Y_0|^{-p} dr \right] dt \right] \\
&\le c_6 \int_0^1 \int_0^{1-t} r^{-p/\alpha}\, dr\, dt \\
&< \infty.
\end{aligned}$$

So by a result of Frostman (see Theorem 4.13 in [13]), $\dim_H Y[0,1] \ge p$ $\mathbf{P}^x$-a.s. Since this is true for every $p < d \wedge \alpha$, we have

$$\dim_H Y[0,1] \ge d \wedge \alpha \qquad \mathbf{P}^x\text{-a.s. for every } x \in F.$$

This completes the proof of this theorem. □

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Note added in revision. After the manuscript was submitted, we were informed of the papers [6, 7]. Although these two papers are related to the general topic of this paper, where [6] is the announcement of [7], their contents and starting point are different from ours. The processes considered in [6, 7] are the subordinations of a fractional diffusion on a d-set that *is assumed* to have two-sided heat kernel estimate (1.7). The main purpose of [6, 7] is to obtain Harnack inequality for these subordinated processes. In fact, a stronger result can be established. Under the assumption of [6, 7], a direct calculation using subordination and (1.7) shows that the heat kernels for the subordinated processes have estimate (1.6) and so, by the second paragraph following Theorem 1.2 in Section 1, the parabolic Harnack inequality follows. Furthermore, it can be shown, as it is done in Theorem 4.14 that any parabolic functions are Hölder continuous.

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