J. Math. Kyoto Univ. (JMKYAZ)
00-0 (2005), 000–000

Construction of diffusion processes on fractals, d-sets, and general metric measure spaces

By

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Abstract

We give a sufficient condition to construct non-trivial μ-symmetric diffusion processes on a locally compact separable metric measure space (M, ρ, μ). These processes are associated with local regular Dirichlet forms which are obtained as continuous parts of Γ-limits for approximating non-local Dirichlet forms. For various fractals, we can use existing estimates to verify our assumptions. This shows that our general method of constructing diffusions can be applied to these fractals.

1. Introduction

According to the Central Limit Theorem, Brownian motion on $\mathbb{R}^d$ or on a Riemannian manifold can be obtained as the scaling limit (as $r \to 0$) of canonical random walks: after an exponentially distributed time with mean r^2 the particles have to jump uniformly distributed into the ball of radius r centered at the previous location. By an analogous procedure, many elliptic diffusions on flat or curved spaces can be obtained. In [19], the basic idea of this procedure was used to construct diffusion processes on arbitrary metric measure spaces.

However, typical diffusions on fractals and also diffusions on $\mathbb{R}^d$ equipped with a Cantor like speed measure have a different space-time scaling. For these cases, the procedure in [19] leads to the constant (trivial) processes. In order to construct or approximate them by random walks as above, one has to take into account the specific time scale function $h(r)$ (replacing the usual diffusive scale r^2).

Given an arbitrary metric measure space (M, ρ, μ) and an arbitrary increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ ("time scale function") we define approximating

2000 *Mathematics Subject Classification(s).* Primary 60J60; Secondary 49Q20, 60J35
Keywords: Dirichlet forms; diffusion processes; Γ-convergence; Lipschitz spaces; heat kernels; fractals.
Communicated by N/A
Received Date
Revised Date

Dirichlet forms $\mathcal{E}^r$ on $L^2(M,\mu)$ by

$$\mathcal{E}^r(u) = \frac{1}{h(r)} \int_M \frac{1}{\mu(B(x,r))} \int_{B(x,r)} [u(x) - u(y)]^2 \, \mu(dy) \, \mu(dx).$$

In Chapter 2 we formulate basic conditions on $\mathcal{E}^r$ which will imply that there exists a diffusion process associated with the scaling limits of these forms. The crucial point here is to deduce locality of the limiting Dirichlet form (which is equivalent to continuity of the limiting Markov process). Actually, for this result we allow even more general frameworks. The main difficulty is the lack of appropriate cut-off functions.

Instead of studying the above approximating Dirichlet forms on the original metric space, in many cases it is easier to study discrete Dirichlet forms on graphs which approximate the metric space. Chapter 3 is devoted to introduce this concept of approximating graphs and to compare the Dirichlet forms on these graphs with the previous ones.

In Chapter 4, we prove that suitable two-sided heat kernel estimates for an arbitrary regular Dirichlet form on an metric measure space will imply that the form is comparable to a scaling limit of the previous approximating forms. This in turn allows us to verify part of our basic conditions. The main theorem in this chapter (Theorem 4.1) extends the results in [7] which require the volume growth $\mu(B(x,r))$ to be comparable to r^α and the time scale $h(r)$ to be comparable to $r^{2\beta}$ for some $\alpha \geq 1, \beta \geq 1$. But since the volume growth and the time scale are not necessarily polynomial growth (see for instance [10]), such an extension seems to be necessary.

In Chapter 5 we present two different classes of examples. The first ones are p.c.f. (=post critically finite) self-similar sets, e.g. the Sierpinski gaskets. By applying the known results, we characterize the domains of self-similar local regular Dirichlet forms on them (Proposition 5.1). The second ones are (generalized) Sierpinski carpets. For these classes, the domains of local regular Dirichlet forms are known (cf. [15], [18]), which we restate in Proposition 5.2. In both cases, we can verify all of our basic conditions, thus diffusions on them can be (re)constructed through the method in Chapter 2.

2. General Construction of Diffusion Processes

Let us fix a metric measure space (M,ρ,μ) where (M,ρ) is a locally compact separable metric space and μ is a Radon measure with $\mu(B(x,r)) > 0$ for each $r > 0$ and $x \in M$. Here and in the sequel, $B(x,r)$ denotes the open ball of radius r centered at x. Moreover, for each $r > 0$ we fix a measurable nonnegative function $(x,y) \mapsto k_r(x,y)$ which vanishes if $\rho(x,y) > r$ and satisfies $\sup_{x\in K} \int_M [k_r(x,y) + k_r(y,x)] \, \mu(dy) < \infty$ for each compact $K \subset M$. In terms of these quantities we define the *approximating Dirichlet forms* $\mathcal{E}^r$ with $\mathcal{C}_0(M) \subset \mathcal{D}(\mathcal{E}^r) \subset L^2(M,\mu)$ as follows

$$\mathcal{E}^r(u) := \int_M \int_M |u(x) - u(y)|^2 k_r(x,y) \, \mu(dy) \, \mu(dx),$$

where $\mathcal{C}_0(M)$ is a space of continuous compactly supported functions on M and $\mathcal{D}(\mathcal{E}^r) = \{u \in L^2(M,\mu) : \mathcal{E}^r(u) < \infty\}$. Finally, we fix throughout this chapter a sequence $(r_n)_n$ of positive numbers decreasing to 0 and put

$$\mathcal{E}^*(u) := \limsup_{n\to\infty} \mathcal{E}^{r_n}(u)$$

and

$$\mathcal{F}^* := \{u \in \mathcal{C}_0(M) : \ \mathcal{E}^*(u) < \infty\}.$$

We will discuss limit points of the forms $\mathcal{E}^r$ under (some of) the following:

Assumptions 1.

(A1) $\mathcal{F}^*$ is dense in $\mathcal{C}_0(M)$.

(A2) There exists $\delta > 0$ such that

$$\liminf_{n\to\infty} \mathcal{E}^{r_n}(u_n) \geq \delta \cdot \mathcal{E}^*(u)$$

for all $u \in L^2(M,\mu)$ and all $(u_n)_n \subset L^2(M,\mu)$ with $u_n \to u$ in $L^2(M,\mu)$.

Occasionally, we impose instead of (A2) the stronger assumption:

(A2*) There exists $\delta > 0$ such that

$$\liminf_{n\to\infty} \mathcal{E}^{r_n}(u_n) \geq \delta \cdot \sup_n \mathcal{E}^{r_n}(u)$$

for all $u \in L^2(M,\mu)$ and all $(u_n)_n \subset L^2(M,\mu)$ with $u_n \to u$ in $L^2(M,\mu)$.

Given the sequence $(\mathcal{E}^{r_n})_{n\in\mathbb{N}}$ we will define a form $\mathcal{E}^0$ as a Γ-limit in the sense of De Giorgi. Recall that for any $u \in L^2(M,\mu)$ we define

$$\Gamma\text{-}\limsup_{n\to\infty} \mathcal{E}^{r_n}(u) = \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{v \in L^2 \\ \|u-v\| \leq \alpha}} \mathcal{E}^{r_n}(v)$$

and

$$\Gamma\text{-}\liminf_{n\to\infty} \mathcal{E}^{r_n}(u) = \lim_{\alpha\to 0} \liminf_{n\to\infty} \inf_{\substack{v \in L^2 \\ \|u-v\| \leq \alpha}} \mathcal{E}^{r_n}(v).$$

Here $\|.\|$ denotes the norm in $L^2 = L^2(M,\mu)$. We say that the sequence $(\mathcal{E}^{r_n})_n$ is Γ-*convergent* if $\Gamma\text{-}\limsup_{n\to\infty} \mathcal{E}^{r_n}(u) = \Gamma\text{-}\liminf_{n\to\infty} \mathcal{E}^{r_n}(u)$ for each $u \in L^2(M,\mu)$. In this case, we write $\Gamma\text{-}\lim_{n\to\infty} \mathcal{E}^{r_n}(u)$ for the common value of $\Gamma\text{-}\limsup_{n\to\infty} \mathcal{E}^{r_n}(u)$ and $\Gamma\text{-}\liminf_{n\to\infty} \mathcal{E}^{r_n}(u)$. The functional $\Gamma\text{-}\lim_{n\to\infty} \mathcal{E}^{r_n}$ on $L^2(X,m)$ is then called Γ-*limit* of the functionals $\mathcal{E}^{r_n}$, $n \in \mathbb{N}$. See [5] for more details.

Theorem 2.1. *(i) Assume (A1). Then for a suitable subsequence (r'_n) of (r_n) the Γ-limit*

$$\mathcal{E}^0 := \Gamma\text{-}\lim_{n\to\infty} \mathcal{E}^{r'_n}$$

exists, is dominated by $\mathcal{E}^*$ *on* $L^2(M,\mu)$ *and it can be extended to a regular Dirichlet form* $(\mathcal{E},\mathcal{F})$ *on* $L^2(M,\mu)$ *with core* $\mathcal{F}^*$.
Hence, there exists a μ*-reversible strong Markov process associated with* $(\mathcal{E},\mathcal{F})$.

(ii) Assume in addition (A2) and let $\mathcal{E}^{(c)}$ *denote the diffusion part of* $\mathcal{E}$. *Then* $\mathcal{F}^* = \mathcal{F}\cap\mathcal{C}_0(M)$,

$$\mathcal{E}^* \geq \mathcal{E} \geq \mathcal{E}^{(c)} \geq \delta\cdot\mathcal{E}^*$$

on $\mathcal{F}$ *and* $(\mathcal{E}^{(c)},\mathcal{F})$ *is a strongly local regular Dirichlet form.*
Hence, there exists a diffusion process associated with $(\mathcal{E}^{(c)},\mathcal{F})$. *It is* μ*-reversible and strongly Markovian.*

Remark 1. a) Typical examples are $k_r(x,y) = r^{-2-d}\cdot 1_{B(x,r)}(y)$ provided $\mu(B(x,r)) \leq C_M\cdot r^d$ for all $r>0$ and $x\in M$ or $k_r(x,y) = r^{-2}\cdot \mu(B(x,r))^{-1}\cdot 1_{B(x,r)}(y)$ provided μ satisfies the so-called doubling property (see Assumptions 2). In these cases, (A1) is always satisfied since $\mathcal{C}_0^{\mathrm{Lip}}(M)\subset\mathcal{F}^*$ and the Dirichlet form $(\mathcal{E},\mathcal{F})$ can always be shown to be strongly local (see [19]).

A detailed discussion of conditions on (M,ρ,μ) (mainly, the so-called *measure contraction property*) which imply (A2*) for the latter choice of k_r can be found in [20].

b) We may weaken Assumption (A1) replacing the condition $\limsup\mathcal{E}^{r_n}<\infty$ by $\Gamma\text{-}\limsup\mathcal{E}^{r_n}<\infty$. However, in general it is not possible to replace it by $\liminf\mathcal{E}^{r_n}<\infty$ or $\Gamma\text{-}\liminf\mathcal{E}^{r_n}<\infty$ since these conditions are not preserved under passing to subsequences.

c) Under (A2), obviously $\delta\cdot\mathcal{E}^*\leq\Gamma\text{-}\liminf\mathcal{E}^{r_n}\leq\liminf\mathcal{E}^{r_n}\leq\mathcal{E}^*$ which for instance in the definition of $\mathcal{F}^*$ allows to replace $\mathcal{E}^*$ by $\mathcal{E}_* := \liminf\mathcal{E}^{r_n}$ or by $\Gamma\text{-}\liminf\mathcal{E}^{r_n}$.

d) If we choose another subsequence $(\tilde{r}_n)$ to define the Γ-limit then the resulting forms $\tilde{\mathcal{E}}$, $\tilde{\mathcal{E}}^{(c)}$ are equivalent to $\mathcal{E}$, $\mathcal{E}^{(c)}$. However, it might happen that e.g. $\tilde{\mathcal{E}}$ is local whereas $\mathcal{E}$ is not local (or vice versa).

e) If $\delta=1$ (i.e. if the sequence $(\mathcal{E}^{r_n})$ is convergent in a suitable strong sense) then the Γ-limit coincides with the pointwise limit and is already strongly local:

$$\mathcal{E}^* = \lim_{n\to\infty}\mathcal{E}^{r_n} = \Gamma\text{-}\lim_{n\to\infty}\mathcal{E}^{r_n} = \mathcal{E} = \mathcal{E}^{(c)}.$$

f) A slight modification of the following proof will show that under (A1, A2):

$$\begin{aligned}&\int_M \varphi^2(x)\,\mu^{(c)}_{\langle u\rangle}(dx)\\ &\geq \delta\cdot\limsup_{n\to\infty}\int_M\int_M |u(x)-u(y)|^2\; k_{r_n}(x,y)\varphi(x)\varphi(y)\,\mu(dy)\,\mu(dx)\end{aligned}$$

for each $u\in\mathcal{F}\cap\mathcal{C}_0(M)$ and $\varphi\in\mathcal{C}_0(M)$.

Proof. (i) The existence of $\mathcal{E}^0 := \Gamma\text{-}\lim_{n\to\infty}\mathcal{E}^{r'_n}$ as well as the fact that it is a Dirichlet form follow from general results on Γ-convergence. Moreover, due to (A1) its domain $\mathcal{D}(\mathcal{E}^0) := \{u \in L^2(M,\mu) : \mathcal{E}^0(u) < \infty\}$ contains $\mathcal{F}^*$ since $\mathcal{E}^0 \leq \mathcal{E}^*$. Hence, $(\mathcal{E}^0, \mathcal{F}^*)$ is a closable Markovian form and its closure $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form (with core $\mathcal{F}^*$). See [5, 17, 19].

(ii) Assumption (A2) implies that $\mathcal{E}^0 \geq \delta \cdot \mathcal{E}^*$. Hence,

$$\mathcal{E}^* \geq \mathcal{E}^0 \geq \delta \cdot \mathcal{E}^*$$

on $L^2(M,\mu)$. Moreover, we know $\infty > \mathcal{E}^0 = \mathcal{E} \geq \mathcal{E}^{(c)}$ on $\mathcal{F}$. This already implies $\mathcal{F}^* = \mathcal{F} \cap \mathcal{C}_0(M)$.

Now note that for each $u \in \mathcal{F} \cap \mathcal{C}_0(M)$

$$\mathcal{E}^{(c)}(u) = \lim_{\lambda\to\infty} \lambda^{-2} \cdot \left(\mathcal{E}(\cos[\lambda u]) + \mathcal{E}(\sin[\lambda u])\right) \tag{2.1}$$

(which holds for each regular Dirichlet form, see [17] page 389) and that for each $u \in L^2(M,\mu)$

$$\mathcal{E}^r(\cos[\lambda u]) + \mathcal{E}^r(\sin[\lambda u]) = \int_M \int_M 2\left(1 - \cos[\lambda|u(x) - u(y)|]\right) k_r(x,y)\, \mu(dy)\, \mu(dx) \tag{2.2}$$

(simple calculation). Hence, for each $u \in \mathcal{F} \cap \mathcal{C}_0(M)$

$$\begin{aligned}
&\mathcal{E}^{(c)}(u)\\
&= \lim_{\lambda\to\infty} \lambda^{-2} \cdot \left(\mathcal{E}(\cos[\lambda u]) + \mathcal{E}(\sin[\lambda u])\right)\\
&\geq \delta \cdot \limsup_{\lambda\to\infty} \limsup_{n\to\infty} \lambda^{-2} \cdot \left(\mathcal{E}^{r_n}(\cos[\lambda u]) + \mathcal{E}^{r_n}(\sin[\lambda u])\right)\\
&= \delta \cdot \limsup_{\lambda\to\infty} \limsup_{n\to\infty} \int_M \int_M \frac{2}{\lambda^2}\left(1 - \cos[\lambda|u(x) - u(y)|]\right) k_{r_n}(x,y)\, \mu(dy)\, \mu(dx)\\
&\overset{(*)}{=} \delta \cdot \limsup_{n\to\infty} \int_M \int_M |u(x) - u(y)|^2\; k_{r_n}(x,y)\, \mu(dy)\, \mu(dx)\\
&= \delta \cdot \mathcal{E}^*(u)
\end{aligned}$$

where $(*)$ is due to the uniform continuity of u and the fact that $k_r(x,y) = 0$ if $\rho(x,y) > r$. This proves

$$\mathcal{E}^*(u) \geq \mathcal{E}(u) \geq \mathcal{E}^{(c)}(u) \geq \delta \cdot \mathcal{E}^*(u)$$

for all $u \in \mathcal{F} \cap \mathcal{C}_0(M)$ which in turn (by density) implies the same inequality for all $u \in \mathcal{F}$.

In particular, the forms $\mathcal{E}$ and $\mathcal{E}^{(c)}$ are equivalent. Therefore, $(\mathcal{E}^{(c)}, \mathcal{F})$ is closed and thus a strongly local regular Dirichlet form. □

From now on, the kernel $k_r(x,y)$ will be chosen more concretely. We fix a *time scale* h, i.e. a strictly increasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$, and put

$$k_r(x,y) := \frac{1}{h(r)} \cdot \frac{1}{\mu(B(x,r))} 1_{B(x,r)}(y). \tag{2.3}$$

The approximating Dirichlet form then reads

$$\mathcal{E}^r(u) = \frac{1}{h(r)} \int_M \fint_{B(x,r)} [u(x) - u(y)]^2 \, \mu(dy) \, \mu(dx) \tag{2.4}$$

for all $u \in L^2(M, \mu)$. Here $\fint_A \ldots \mu(dy) := \mu(A)^{-1} \int_A \ldots \mu(dy)$ denotes the normalized integral. Moreover, we let V denote the *volume growth*, i.e.

$$V(x,r) := \mu(B(x,r))$$

for $x \in M$ and $r \geq 0$. Finally, we define

$$\mathrm{Lip}_\mu(h, 2, \infty)(M) = \left\{ u \in \; L^2(M, \mu) : \sup_n \mathcal{E}^{r_n}(u) < \infty \right\}.$$

Note that (A2*) implies $\mathcal{F}^* = \mathrm{Lip}_\mu(h, 2, \infty)(M) \cap C_0(M)$.

In the sequel, we always assume the following *doubling properties* for the volume growth V and the time scale h.

Assumptions 2.

(VD) There exists $C_1 > 0$ such that $V(x, 2R) \leq C_1 V(x, R)$ for all $x \in M$ and $R > 0$.

(TD) There exists $C_2 > 0$ such that $h(2R) \leq C_2 h(R)$ for all $R > 0$.

It is well-known and easy to see that (VD), (TD) imply the existence of constants $\eta_1, \eta_2 > 0$ and $C_3, C_4 > 0$ such that for $x, y \in M$ and $0 < r < R$,

$$\frac{V(x,R)}{V(y,r)} \leq C_3 \left(\frac{\rho(x,y) + R}{r} \right)^{\eta_1}, \qquad \frac{h(R)}{h(r)} \leq C_4 \left(\frac{R}{r} \right)^{\eta_2}. \tag{2.5}$$

Example 2.1. Assume that there exist positive constants C, C' and d such that

$$Cr^d \leq V(x,r) \leq C' r^d$$

for all $x \in M, r > 0$. In this case, the set M is called a *d-set* and the measure μ is called *Ahlfors regular*. Then (VD) is satisfied.

Moreover, put $h(r) = r^{2\beta}$ for some positive number β. Then (TD) is satisfied. When $\beta < 1$, our Lipschitz space $\mathrm{Lip}_\mu(h, 2, \infty)(M)$ coincides with the Lipschitz space $\mathrm{Lip}(\beta, 2, \infty; M)$ studied, for instance, in [12].

Lemma 2.1. (i) Assume (VD). Then the generator A^r of $\mathcal{E}^r$ can be written as

$$A^r v(x) = \frac{C}{h(r)} \int_{B(x,r)} \frac{v(x) - v(y)}{V(x,y,r)} \mu(dy) = \frac{C}{h(r)} [v(x) - q_r v(x)]$$

with $C = 1 + 2^{\eta_1} C_3$, $V(x,y,r) := C \left[V(x,r)^{-1} + V(y,r)^{-1} \right]^{-1}$ and a Markov kernel q_r defined by

$$q_r(x, dy) := \frac{1}{V(x,y,r)} 1_{B(x,r)}(y) \mu(dy) + \left(1 - \int_{B(x,r)} \frac{1}{V(x,z,r)} \mu(dz) \right) \cdot \delta_x(dy).$$

Hence, the transition semigroup for $\mathcal{E}^r$ is given by

$$p_t^r = e^{-Ct/h(r)} \cdot \sum_{k=0}^{\infty} \frac{(Ct/h(r))^k}{k!} q_r^{[k]} \tag{2.6}$$

with $q_r^{[k]}$ being the k-th iteration of q_r. In particular,

$$p_t^r \geq e^{-Ct/h(r)} \cdot [1 + Ct/h(r) \cdot q_r].$$

(ii) Assume in addition (A1), (A2*) and that M is compact. Then $\mathcal{E}$ and $\mathcal{E}^{(c)}$ are conservative and

$$\begin{aligned}\int_B \int_{M\setminus B} p_t(x,dy)\,\mu(dx) &\geq \int_B \int_{M\setminus B} p_t^{(c)}(x,dy)\,\mu(dx) \\ &\geq \int_B \int_{M\setminus B} p_{\delta^2 t}^{r_n}(x,dy)\,\mu(dx)\end{aligned} \tag{2.7}$$

for each measurable $B \subset M$, each $t > 0$ and each n. Here $p_t, p_t^{(c)}, p_t^{r_n}$ denote the transition semigroups associated with the Dirichlet forms $\mathcal{E}, \mathcal{E}^{(c)}, \mathcal{E}^{r_n}$, resp.

Proof. (i) is a simple calculation and we omit the proof. For (ii), since

$$\mathcal{E} \geq \mathcal{E}^{(c)} \geq \delta \cdot \mathcal{E} \geq \delta^2 \cdot \mathcal{E}^{r_n}$$

on $L^2(M,\mu)$, we have $\exp(tA) \leq \exp(tA^{(c)}) \leq \exp(\delta tA) \leq \exp(\delta^2 tA^{r_n})$ and thus

$$\int_B \int_B p_t(x,dy)\,\mu(dx) \leq \int_B \int_B p_t^{(c)}(x,dy)\,\mu(dx) \leq \int_B \int_B p_{\delta^2 t}^{r_n}(x,dy)\,\mu(dx).$$

According to the compactness of M, $\int_B \int_M p_t^{\bullet}(x,dy)\,\mu(dx) = \mu(B) < \infty$ where $p_t^{\bullet}$ stands for any of the involved transition semigroups. Hence (2.7) follows. $\square$

The inequality (2.7) can be used in combination with the explicit formula (2.6) to deduce lower bounds for $\int_B \int_{M\setminus B} p_t(x,dy)\,\mu(dx)$ and $\int_B \int_{M\setminus B} p_t^{(c)}(x,dy)$ $\mu(dx)$ and to conclude that p_t and $p_t^{(c)}$ are not degenerate.

Remark 2. Let us consider the case where there exists a family of time scales $\{h_\beta(\cdot)\}_{\beta\geq 0}$ such that each h_β satisfies (TD) and for each $0 \leq \beta < \beta'$

$$\limsup_{r\downarrow 0} h_{\beta'}(r)/h_\beta(r) = 0.$$

Given h_β, we denote $\mathcal{E}^r(u)$ in (2.4) as $\mathcal{E}^{\beta,r}(u)$. We also denote $\mathcal{E}^{\beta,*}(u) = \limsup_{n\to\infty} \mathcal{E}^{\beta,r_n}(u)$ and $\mathcal{E}^{\beta,0}(u) = \Gamma\text{-}\lim_{n\to\infty} \mathcal{E}^{\beta,r'_n}(u)$. Note that (A1) for some β implies (A1) for all $\beta' < \beta$ and

$$\mathcal{E}^{\beta',0}(u) = 0 \qquad \text{for all } \; u \in L^2(M,\mu).$$

Indeed, $\mathcal{E}^{\beta,*}(u) < \infty$ implies $\mathcal{E}^{\beta',0}(u) = \mathcal{E}^{\beta',*}(u) = 0$ for all $\beta' < \beta$. Hence, (A1) for β implies $\mathcal{E}^{\beta',0}(u) = 0$ on a dense subset of $L^2(M,\mu)$. By lower semicontinuity this implies $\mathcal{E}^{\beta',0}(u) = 0$ for $u \in L^2(M,\mu)$. Note also that (A1) is always true for $h_\beta(r) = r^2$ since then $\mathcal{C}_0^{\mathrm{Lip}}(M) \subset \mathcal{F}^*$.

There exists at most one β_0 such that simultaneously:

- $\{u \in L^2(M,\mu) : \mathcal{E}^{\beta_0,*}(u) = 0\}$ is not dense in $L^2(M,\mu)$;
- $\{u \in L^2(M,\mu) : \mathcal{E}^{\beta_0,*}(u) < \infty\}$ is dense in $L^2(M,\mu)$.

(Indeed, $\{u \in L^2(M,\mu) : \mathcal{E}^{\beta',*}(u) = 0\} \supset \{u \in L^2(M,\mu) : \mathcal{E}^{\beta'',*}(u) < \infty\}$ for all $\beta' < \beta''$, as discussed above.)

In the case $h_\beta(r) = r^{2\beta}$, the number $d_w := 2\beta_0$ is called *walk dimension.*

3. Graph approximation under volume doubling

In computing concrete examples, it is often useful to work on approximating graphs instead of M itself. In this section, we will introduce a natural sequence of approximating graphs of (M,ρ,μ) when μ satisfies (VD). We further show that approximating forms are comparable to the original jump-type form under some weak assumptions.

We first define *graphs* which approximate M. The following construction is standard, but we write down details for completeness.

Lemma 3.1. *(i) Under (VD), there exists a constant $N_0 \in \mathbb{N}$ such that for each $r > 0$ there exists an open covering $\{B(x_i,r)\}_{i=1}^\infty$ of M with the property that no point in M is contained in more than N_0 of the $B(x_i,r)$, $i \in \mathbb{N}$.*

(ii) For each $x \in M$, the number of balls $B(x_i,r)$ in (i) which intersects with $B(x,2r)$ (say $L_{x,r}$) is bounded by some positive constant L_0 which is independent of x,r.

Proof. We first prove (i). Since M is a locally compact separable metric space, there is an increasing sequence of compact sets $\{K_n\}_{n\ge1}$ such that $\cup_{n\ge1} K_n = M$. Now, take $x_1^1 \in K_1$ and choose $x_2^1, x_3^1, \cdots \in K_1$ by letting x_{i+1}^1 be any point in $K_1 \setminus \cup_{j=1}^i B(x_j^1,r)$. We do this until we can no longer proceed. Since K_1 is compact, there is a finite subset $\{x_i\}_{i=1}^{l_1} \subset \{x_i^1\}_i$ such that $K_1 \subset \cup_{i=1}^{l_1} B(x_i,r)$. We next choose $x_1^2, x_2^2, \cdots \in K_2$ by letting x_{i+1}^2 be any point in $K_2 \setminus (\cup_{i=1}^{l_1} B(x_i,r) \cup \cup_{j=1}^i B(x_j^2,r))$. Again we do this until we can no longer proceed. By doing this procedure iteratively, we obtain a desired open covering of M. Note that the x_i must be at least r distance apart, so that the balls $\{B(x_i,r/2)\}_i$ are disjoint. Now suppose y is in N of the balls $B(x_i,r)$, $i \in \mathbb{N}$ (N may be infinite at this stage). Using (2.5), there exists $N_0 = C_3 \cdot 10^{\eta_1}$ such that for each of these we have $V(y,2r)/V(x_i,r/2) \le N_0$. Since $B(y,2r)$ contains N disjoint balls $B(x_i,r/2)$,

$$V(y,2r) \ge \sum_{i:y\in B(x_i,r)} V(x_i,r/2) \ge N N_0^{-1} V(y,2r),$$

which implies $N \le N_0$, independent of y and r.

We next prove (ii). If $A_{x,r}$ is a set of such $\{x_i\}_i$, then

$$N_0 V(x,4r) \geq \sum_{x_i \in A_{x,r}} V(x_i,r) \geq L_{x,r} \cdot \min_{x_i \in A_{x,r}} V(x_i,r),$$

since each point in M is covered by at most N_0 of the balls $B(x_i,r)$ and $B(x_i,r) \subset B(x,4r)$. Using (2.5) as before, we have $V(x,4r)/V(x_i,r) \leq c_1$, so that $N_0 c_1 \geq L_{x,r}$. We can thus take $L_0 = N_0 c_1$. $\square$

Let $V_r = \{x_i\}_i$. We say that x and y are connected by *bonds* (which we denote $\{x,y\} \in B_r$) if $B(x,r) \cap B(y,r) \neq \emptyset$. In this way, we can define a graph (V_r, B_r). The definition of (V_r, B_r) depends on the choice of the open covering of M; — in the following, for each $r > 0$, we choose one open covering with the above mentioned property and fix the graph (V_r, B_r). For each sequence (r_m) which converges to zero, the set $\cup_m V_{r_m}$ is dense in M. Note that (V_r, B_r) has bounded degree, i.e. $\sup_{x \in V_r} \sharp\{y \in V_r : \{x,y\} \in B_r\} < \infty$.

For each $r > 0$, the mean value operator $\mu_r : L^1(M,\mu) \to l(V_r)$ is given by

$$\mu_r f(s) = \fint_{B(s,r)} f(y)\mu(dy) \qquad \forall s \in V_r, \forall f \in L^1(M,\mu).$$

Here $l(V_r)$ denotes the set of all maps $f : V_r \to \mathbb{R}$. We define the *discrete Dirichlet form* on the graph (V_r, B_r) by

$$E^r(f) = \frac{1}{h(r)} \sum_{\{x,y\} \in B_r} (f(x) - f(y))^2 \cdot V(x,r) \qquad \forall f \in l(V_r). \tag{3.1}$$

We will show that this form is comparable with the approximating Dirichlet form $\mathcal{E}^r$ on the original metric space (M,ρ) under the following *Poincaré inequality.*

(PI(h)) There exists $c_{3.1} > 0$ such that for each $f \in L^2(M,\mu)$ and each $r > 0$,

$$\frac{1}{h(r)} \sum_{s \in V_r} \int_{B(s,r)} |f(x) - \mu_r f(s)|^2 \mu(dx) \leq c_{3.1} \underline{E}^0(f),$$

where $\underline{E}^0(f) = \Gamma\text{-}\liminf_{n \to \infty} E^{r_n}(\mu_{r_n} f)$.

Lemma 3.2. *Assume (VD) and (PI(h)). Then there exists $c_{3.2} > 0$ such that*

$$\mathcal{E}^r(f) \leq c_{3.2}\{\underline{E}^0(f) + E^r(\mu_r f)\} \qquad \forall f \in L^2(M,\mu),\ \forall r > 0.$$

Proof. For $i \geq 1$, define $B_r^{(i)} = \{\{s,t\} : s,t \in V_r$, there exists $\{u_j\}_{j=0}^L \subset V_r$ $(L \leq i)$ such that $s = u_0, t = u_L$ and $\{u_j, u_{j+1}\} \in B_r$ for $0 \leq j \leq L-1\}$. Note that if $x \in B(s,r)$, $s \in V_r$ and $\rho(x,y) \leq r$, then $y \in B(t,r)$ for some $t \in V_r$ where either $t = s$ or $\{t,s\} \in B_r^{(L_0)}$. (Recall that L_0 is the number of balls $B(x_i,r)$ which intersects with $B(x,2r)$.) Using this fact and (VD), we get

$$\mathcal{E}^r(f) \leq \frac{c_1}{h(r)} \sum_{s \in V_r} \int_{B(s,r)} \frac{\mu(dx)}{V(s,r)} \sum_{\substack{t \in B_r : t = s \\ \text{or } \{s,t\} \in B_r^{(L_0)}}} \int_{B(t,r)} |f(x) - f(y)|^2 \mu(dy). \tag{3.2}$$

Since the degree of (V_r, B_r) is uniformly bounded with respect to r, by a simple computation using triangle inequalities, we have

(3.3)
$$[\text{LHS of (3.2)}] \le \frac{c_2}{h(r)} \sum_{s\in V_r} \sum_{\substack{t\in B_r : t=s \\ \text{or } \{s,t\}\in B_r}} \int_{B(s,r)} \frac{\mu(dx)}{V(s,r)} \int_{B(t,r)} |f(x)-f(y)|^2 \mu(dy).$$

Computing each term in the sum of the right hand side, we deduce

$$\begin{aligned}
&\int_{B(s,r)} \frac{\mu(dx)}{V(s,r)} \int_{B(t,r)} |f(x)-f(y)|^2 \mu(dy) \\
&= \frac{V(t,r)}{V(s,r)} \int_{B(s,r)} f(x)^2 \mu(dx) + \int_{B(t,r)} f(y)^2 \mu(dy) - 2V(t,r)\mu_r f(s)\mu_r f(t) \\
&= \frac{V(t,r)}{V(s,r)} \int_{B(s,r)} |f(x)-\mu_r f(s)|^2 \mu(dx) + \int_{B(t,r)} |f(y)-\mu_r f(t)|^2 \mu(dy) \\
&\quad +V(t,r)(\mu_r f(s) - \mu_r f(t))^2. \qquad (3.4)
\end{aligned}$$

Now summing up the last term of (3.4) and using (VD), we see that the right hand side of (3.3) is greater than or equal to

$$c_3\{\frac{1}{h(r)} \sum_{s\in V_r} \int_{B(s,r)} |f(x)-\mu_r f(s)|^2 \mu(dx) + E^r(\mu_r f)\}.$$

Using (PI(h)), we obtain the result. □

The opposite inequality is easier.

Lemma 3.3. *Assume (VD) and (TD). Then there exists $c_{3.3} > 0$ such that*

$$E^r(\mu_r f) \le c_{3.3} \mathcal{E}^r(f) \qquad \forall f \in L^2(M,\mu),\ \forall r > 0.$$

Proof. First, by similar computations as (3.2) and (3.3), using (VD), (TD) and the fact that $\{B(s,r)\}_{s\in V_r}$ covers each point of M at most finite number of times, we have $\mathcal{E}^r(f) \ge c_1 \mathcal{E}^{4r}(f)$ for some $c_1 > 0$. Thus, it is enough to prove $E^r(\mu_r f) \le c_2 \mathcal{E}^{4r}(f)$.

Note that if $x \in B(s,r)$ and $\{s,t\} \in B_r$, then $B(t,r) \subset \{y \in M : \rho(x,y) < 4r\}$. Since $\{B(s,r)\}_{s\in V_r}$ covers each point of M at most finite number of times, using (VD), we have

$$\mathcal{E}^{4r}(f) \ge \frac{c_3}{h(4r)} \sum_{\{s,t\}\in B_r} \int_{B(s,r)} \frac{\mu(dx)}{V(s,r)} \int_{B(t,r)} |f(x)-f(y)|^2 \mu(dy). \qquad (3.5)$$

Now by the same computation as (3.4) and using (TD), we obtain that the right hand side of (3.5) is greater than or equal to $c_4 E^r(\mu_r f)$ for some $c_4 > 0$. □

As in the case of $\mathcal{E}^r$, for a sequence $(r_n)_n$ of positive numbers decreasing to 0, put

$$E^*(u) := \limsup_{n\to\infty} E^{r_n}(\mu_{r_n} u), \qquad F^* := \{u \in \mathcal{C}_0(M) :\ E^*(u) < \infty\}.$$

Assumptions 3.

(B1) F^* is dense in $\mathcal{C}_0(M)$.

(B2) There exists $\delta > 0$ such that for all $u \in L^2(M, \mu)$ and all $(u_n)_n \subset L^2(M, \mu)$ that converges to u in L^2,

$$\liminf_{n\to\infty} E^{r_n}(\mu_{r_n} u_n) \geq \delta \cdot E^*(u).$$

(B2*) There exists $\delta > 0$ such that for all $u \in L^2(M, \mu)$ and all $(u_n)_n \subset L^2(M, \mu)$ that converges to u in L^2,

$$\liminf_{n\to\infty} E^{r_n}(\mu_{r_n} u_n) \geq \delta \cdot \sup_n E^{r_n}(\mu_{r_n} u).$$

Our main result in this section is that the Assumptions 3 together with (PI(h)) for the discrete Dirichlet forms on the induced graphs (V_r, B_r) imply the Assumptions 1 for the Dirichlet forms on the abstract metric space M.

Proposition 3.1. *Asume (VD), (TD) and (PI(h)). Then (B1) $\Rightarrow$ (A1), (B2) $\Rightarrow$ (A2), and (B2*) $\Rightarrow$ (A2*).*

Proof. The first implication is clear from Lemma 3.2 and the fact that $\underline{E}^0(u) \leq E^*(u)$. For the second one, note that by (B2) and Lemma 3.2, we have $\mathcal{E}^*(u) \leq c_1\Gamma\text{-}\liminf E^{r_n}(\mu_{r_n} u)$. By this and Lemma 3.3, we have $\mathcal{E}^*(u) \leq c_2\Gamma\text{-}\liminf \mathcal{E}^{r_n}(u)$ so that (A2) holds. The last inequality can be proved similarly. □

4. Domains of Dirichlet forms with two-sided heat kernel estimates

In the following, we will prove that if a heat kernel of a regular Dirichlet form satisfies suitable two-sided estimates, then the domain of the form is the Lipschitz space. We assume that the heat kernel satisfies the following estimate for all $0 < t \leq 1,\ x, y \in M$,

$$\text{(4.1)} \quad \frac{1}{V(x, h^{-1}(t))}\Phi_1(\frac{h(\rho(x,y))}{t}) \leq p_t(x,y) \leq \frac{1}{V(x, h^{-1}(t))}\Phi_2(\frac{h(\rho(x,y))}{t}),$$

where h^{-1} is a inverse function of h and Φ_1, Φ_2 are monotone decreasing positive functions on $[0, \infty)$. (We will assume that Φ_2 decay sufficiently fast at $+\infty$ in Theorem 4.1). Note that we only assume short time heat kernel estimates.

Moreover, we impose the following "fast time growth" condition:

(FTG) There exists $\eta_3 > 0$ and $C_1 > 0$ such that $h(T)/h(t) \geq C_1(T/t)^{\eta_3}$ for all $0 < t < T$.

Our main theorem in this section is the following.

Theorem 4.1. *Let (M, ρ, μ) be a metric measure space. Assume that (VD) holds for μ and that a time scale h is given which satisfies (TD) and (FTG). Assume further that there is a local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on*

$L^2(M,\mu)$ with the following properties: there exists a symmetric transition density $p_t(x,y)$ for $(\mathcal{E},\mathcal{F})$ with respect to μ which satisfies (4.1). Here we assume that the function $\varphi(s) := s^{(\eta_1+\eta_2)/\eta_3}\Phi_2(s)$ is monotone decreasing in $[s_0,\infty)$ for some $s_0 > 0$ and

$$\int_1^\infty \Phi_2(s)s^{\frac{\eta_1+\eta_2}{\eta_3}-1}ds < \infty. \tag{4.2}$$

Then, for all $\alpha > 1$ there exist $c_{4.1}(\alpha), c_{4.2} > 0$ such that the following holds.

$$c_{4.1}(\alpha)\mathcal{E}(f) \le \limsup_{m\to\infty} \mathcal{E}^{\alpha^{-m}}(f) \le \sup_{r>0} \mathcal{E}^r(f) \le c_{4.2}\mathcal{E}(f) \qquad \textit{for} \;\; f \in \mathcal{F}. \tag{4.3}$$

Here $\mathcal{E}^r$ is the approximating Dirichlet form as defined in (2.4). In particular,

$$\mathcal{F} = Lip_\mu(h,2,\infty)(M).$$

Remark 3. By (4.3), we see that (A1) holds. Thus, by Theorem 2.1 i), $\mathcal{E}^0 := \Gamma\text{-}\lim_{n\to\infty}\mathcal{E}^{r'_n}$ exists. If in addition

$$\mathcal{E}^0(f) \ge c_{4.3}\mathcal{E}(f) \qquad \text{for} \;\; f \in \mathcal{F} \tag{4.4}$$

or

$$\underline{E}^0(f) := \Gamma\text{-}\liminf{}_{n\to\infty} E^{r_n}(\mu_{r_n}f) \ge c_{4.3}\mathcal{E}(f) \qquad \text{for} \;\; f \in \mathcal{F} \tag{4.5}$$

holds, then by (4.3) and Lemma 3.3, we see that (A2*) (thus (A2)) holds. We note that this theorem is an extension of [7] Theorem 4.2 and [18] Theorem 1.

Proof. We first prove $\sup_{r>0}\mathcal{E}^r(f) \le c_1\mathcal{E}(f)$ which in turn immediately will imply $\mathcal{F} \subset$ Lip. For $t \le 1$ and $f \in L^2(M,\mu)$, let $\mathcal{E}_t(f) := \frac{1}{t}(f - P_t f, f)_{L^2}$, where P_t is the semigroup corresponding to $(\mathcal{E},\mathcal{F})$. Then,

$$\begin{aligned}
\mathcal{E}_t(f) &= \frac{1}{2t}\int\int_{M\times M}(f(x)-f(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\ge \frac{1}{2t}\int\int_{\rho(x,y)\le h^{-1}(t)}(f(x)-f(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\ge \frac{\Phi_1(1)}{2t}\int\int_{\rho(x,y)\le h^{-1}(t)}\frac{(f(x)-f(y))^2}{V(x,h^{-1}(t))}\mu(dx)\mu(dy), \qquad (4.6)
\end{aligned}$$

where we use the lower bound of (4.1) in the last inequality. Taking $t = h(r)$ for $r > 0$, we see that the RHS of (4.6) is equal to $\frac{1}{c_1}\mathcal{E}^r(f)$ for some $c_1 > 0$. It is well known that $\mathcal{E}_t(f) \nearrow \mathcal{E}(f)$ as $t \downarrow 0$ ([6], Lemma 1.3.4). Thus the claim follows.

We next prove $c_2\mathcal{E}(f) \le \sup_{m\in\mathbb{N}}\mathcal{E}^{\alpha^{-m}}(f)$ which then will imply $\mathcal{F} \supset$ Lip. For each $t \le 1$ and $g \in$ Lip,

$$\begin{aligned}
\mathcal{E}_t(g) &= \frac{1}{2t}\int\int_{M\times M}(g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&= \frac{1}{2t}\int\int_{\substack{x,y\in M\\ \rho(x,y)>1}}(g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&+ \frac{1}{2t}\int\int_{\substack{x,y\in M\\ \rho(x,y)\le 1}}(g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy) =: A(t)+B(t).
\end{aligned}$$

We first estimate $A(t)$. (Note that this part is empty if $diam(M) \le 1$.) Since

$$A(t) = \frac{1}{2t}\sum_{m=0}^{\infty}\int\int_{\alpha^m<\rho(x,y)\le\alpha^{m+1}} (g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy),$$

for $\alpha > 1$, using the fact $(a+b)^2 \le 2(a^2+b^2)$ and the symmetry, we have

$$A(t) \le \frac{2}{t}\sum_{m=0}^{\infty}\int_M g(x)^2\mu(dx)\int_{\alpha^m<\rho(x,y)\le\alpha^{m+1}} p_t(x,y)\mu(dy).$$

Set $H_m = \{y \in M : \alpha^m < \rho(x,y) \le \alpha^{m+1}\}$. By (4.1), we have

$$\int_{H_m} p_t(x,y)\mu(dy) \le \int_{H_m}\frac{1}{V(x,h^{-1}(t))}\Phi_2(\frac{h(\alpha^m)}{t})\mu(dy)$$
$$\le \frac{\mu(H_m)}{V(x,h^{-1}(t))}\Phi_2(\frac{c_3\alpha^{m\eta_3}}{t}) \le \frac{V(x,\alpha^{m+1})}{V(x,h^{-1}(t))}\Phi_2(\frac{c_3\alpha^{m\eta_3}}{t}),$$

where we use (FTG) and the fact $0 < h(1) < \infty$ in the second inequality. Using (2.5), we have $V(x,\alpha^{m+1})/V(x,h^{-1}(t)) \le c_4(\alpha^m/h^{-1}(t))^{\eta_1}$. Note that by (FTG), if t' is small we have $h(1)/h(t') \ge c_5/t'^{\eta_3}$. Taking $t = h(t')$, we have $1/h^{-1}(t) \le c_6/t^{1/\eta_3}$. Combining these facts, we have

$$A(t) \le \frac{c_7}{t}\|g\|_{L^2}^2\sum_{m=0}^{\infty}(\frac{\alpha^{m\eta_3}}{t})^{\eta_1/\eta_3}\Phi_2(\frac{c_3\alpha^{m\eta_3}}{t}) =: \frac{c_8}{t}\|g\|_{L^2}^2\sum_{m=0}^{\infty}F(t,m),$$

for small $t \le 1$. for small $t > 0$. By a simple calculation, we have $F(t,m) = \varphi(c_3\alpha^{m\eta_3}/t)\cdot(\alpha^{m\eta_3}/t)^{-\eta_2/\eta_3} \le \varphi(c_3/t)t^{\eta_2/\eta_3}/\alpha^{m\eta_2}$. By the assumption that φ is monotone decreasing and (4.2), we have $\lim_{s\to\infty}\varphi(s) = 0$. Thus, noting that $\eta_2 \ge \eta_3$, we obtain

$$A(t) \le c'\|g\|_{L^2}^2\varphi(c_3/t)t^{\eta_2/\eta_3-1} \tag{4.7}$$

for small t and thus $A(t) \xrightarrow{t\to 0} 0$.

Next we estimate $B(t)$. By (4.1) again, we have

$$\begin{aligned}
&B(t)\\
&\le \frac{1}{2t}\sum_{m=1}^{\infty}\int\mu(dx)\int_{H_{-m}}\frac{1}{V(x,h^{-1}(t))}\Phi_2(\frac{h(\alpha^{-m})}{t})(g(x)-g(y))^2\mu(dy)\\
&\le \frac{c_9}{t}\sum_{m=1}^{\infty}\int\mu(dx)\frac{V(x,\alpha^{1-m})}{V(x,h^{-1}(t))}\fint_{B(x,\alpha^{1-m})}\Phi_2(\frac{h(\alpha^{-m})}{t})(g(x)-g(y))^2\mu(dy)\\
&\le c_9(\sup_{m\in\mathbb{N}}\mathcal{E}^{\alpha^{-m}}(g))\sum_{m=1}^{\infty}\{c_{10}(\frac{\alpha^{1-m}}{h^{-1}(t)})^{\eta_1}\vee 1\}\frac{h(\alpha^{1-m})}{t}\Phi_2(\frac{h(\alpha^{-m})}{t}) =: I_1,
\end{aligned}$$

where we use (2.5) in the last inequality. We now compute the sum in I_1. Note that by (2.5) and (FTG), we have $c_{11}(z'/z)^{\eta_3} \le h(z')/h(z) \le c_{12}(z'/z)^{\eta_2}$

for $z' \geq z$ and $h(z')/h(z) \leq c_{13}(z'/z)^{\eta_3}$ for $z' < z$. Let $t' = h^{-1}(t)$ and take $m_0 = m_0(t')$ so that $\alpha^{-m_0-1} < t' \leq \alpha^{-m_0}$. Then,

$$
\begin{aligned}
& I_1 \\
&\leq c_{14}(\sup_{m\in\mathbb{N}} \mathcal{E}^{\alpha^{-m}}(g))\{\sum_{m=1}^{m_0\vee 0}(\frac{\alpha^{-m}}{t'})^{\eta_1+\eta_2}\Phi_2(c_{15}(\frac{\alpha^{-m}}{t'})^{\eta_3}) + \sum_{m=(m_0+1)\vee 1}^{\infty}(\frac{\alpha^{-m}}{t'})^{\eta_3}\} \\
&\leq c_{16}(\sup_{m\in\mathbb{N}} \mathcal{E}^{\alpha^{-m}}(g))\{\int_{c_{17}}^{\infty}\Phi_2(s)s^{\frac{\eta_1+\eta_2}{\eta_3}-1}ds + \sum_{m=(m_0+1)\vee 1}^{\infty}(\frac{\alpha^{-m}}{t'})^{\eta_3}\} \\
&\leq c_{18}\cdot \sup_{m\in\mathbb{N}} \mathcal{E}^{\alpha^{-m}}(g),
\end{aligned}
$$

where we use (4.2) in the last inequality. Combining this with (4.7), we obtain

$$
\mathcal{E}_t(g) \leq c'\|g\|_{L^2}^2\varphi(c_3/t)t^{\eta_2/\eta_3-1} + c_{18}\cdot\sup_{m\in\mathbb{N}}\mathcal{E}^{\alpha^{-m}}(g)
$$

for small $t \leq 1$. Thus, $\mathcal{E}(g) = \lim_{t\to 0}\mathcal{E}_t(g) \leq c_{18}\cdot\sup_{m\in\mathbb{N}}\mathcal{E}^{\alpha^{-m}}(g)$.

Now replacing $A(t)$ and $B(t)$ in the previous argument by

$$
A_k(t) = \frac{1}{2t}\int\int_{\rho(x,y)>\alpha^{-k}}(g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy)
$$

and

$$
B_k(t) = \frac{1}{2t}\int\int_{\rho(x,y)\leq\alpha^{-k}}(g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy)
$$

yields $\mathcal{E}(g) \leq c_{18}\cdot\sup_{m\geq k}\mathcal{E}^{\alpha^{-m}}(g)$ for each $k\in\mathbb{N}$ and thus

$$
\mathcal{E}(g) \leq c_{18}\cdot\limsup_{m\to\infty}\mathcal{E}^{\alpha^{-m}}(g).
$$

□

5. Examples

In this section, we present some examples where we can obtain non-trivial processes by the knowledge of the last section. We begin with the definition of a self-similar space: see [1], [14] for more details and examples.

Let $I = \{1, 2, \cdots, N\}$. The one-sided shift space Σ is defined by $\Sigma = I^{\mathbb{N}}$. For $w \in \Sigma$, we denote the i-th element in the sequence by w_i and write $w = w_1w_2w_3\cdots$.

Definition 5.1. 1) Let M be a compact metrizable space and for each $s \in I$, $F_s : M \to M$ be a continuous injection. Then, $\mathcal{L} = (M, I, \{F_s\}_{s\in I})$ is said to be a self-similar structure on M if there exists a continuous surjection $\pi : \Sigma \to M$ such that $\pi\circ\widetilde{\sigma}_s = F_s\circ\pi$ for every $s \in I$, where $\widetilde{\sigma}_s : \Sigma\to\Sigma$ is

defined by $\widetilde{\sigma}_s w = sw$ for $s \in I$.
2) Let $\mathcal{L} = (M, I, \{F_s\}_{s\in I})$ be a self-similar structure on M. Then, $C(\mathcal{L})$ (the critical set of $\mathcal{L}$) and $P(\mathcal{L})$ (the post critical set of $\mathcal{L}$) are defined by

$$C(\mathcal{L}) = \pi^{-1}(\bigcup_{s,t\in I, s\neq t} (F_s(M) \cap F_t(M))), \qquad P(\mathcal{L}) = \bigcup_{n\geq 1} \sigma^n(C(\mathcal{L})),$$

where $\sigma : \Sigma \to \Sigma$ is the left shift map, i.e. $\sigma w = w_2 w_3 \cdots$ if $w = w_1 w_2 \cdots$.

Set $V_0 = \pi(P(\mathcal{L}))$; we call V_0 the boundary of M. A Bernoulli (probability) measure on M is a measure μ on M such that $\mu(F_i(M)) = \mu_i > 0$, where $\sum_{i=1}^N \mu_i = 1$.

In the following, we will demonstrate two classes of connected self-similar sets $(M, I, \{F_s\}_{s\in I})$, which have non-trivial processes under the framework of the last section.

5.1. P.c.f. self-similar sets

We call the self-similar set $(M, I, \{F_s\}_{s\in I})$ a post critically finite (p.c.f. for short) self-similar sets if the post critical set $P(\mathcal{L})$ is a finite set. This condition implies that M is finitely ramified.

These sets were introduced by Kigami ([13]). It is shown that, provided a 'regular harmonic structure' exists, (which roughly means there exists a non-degenerate fixed point for a non-linear renormalization map), then a local regular local Dirichlet form exists. There are many fractals which satisfy this assumption (typical example is the Sierpinski gasket). We will make this assumption for p.c.f. self-similar sets and introduce the results we need concerning the properties of their Dirichlet forms.

The resistance $R(p,q)$ between points $p \neq q \in M$ can be defined using the Dirichlet form $(\mathcal{E}, \mathcal{F})$, by

$$R(p,q) = \left(\inf\{\mathcal{E}(f,f) : f(p) = 0, f(q) = 1\}\right)^{-1},$$

where we set $\inf \emptyset = \infty$. We set $R(p,p) = 0$ for $p \in M$. Then, the function $R(\cdot,\cdot)$ determines a metric, which we call the (effective) resistance metric, on M.

Theorem 5.1. ([14]) *1) There exists a local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M,\mu)$ which has the following property:*

$$|f(p) - f(q)|^2 \leq R(p,q)\mathcal{E}(f,f) \quad \text{for} \quad f \in \mathcal{F}, \quad p, q \in M, \tag{5.1}$$

$$\mathcal{E}(f,g) = \sum_{i=1}^N \rho_i \mathcal{E}(f \circ F_i, g \circ F_i) \quad \text{for} \quad f, g \in \mathcal{F}, \tag{5.2}$$

where $\rho_i > 1$ $(i \in I)$. Especially, all the elements in $\mathcal{F}$ are continuous functions. Further, if we set $\mathcal{E}_{(\beta)}(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \beta(\cdot,\cdot)_{L^2(M,\mu)}$ for $\beta > 0$, then, $\mathcal{E}_{(\beta)}$ admits a positive symmetric continuous reproducing kernel.
2) Let $\mathcal{L}_\mu$ be the self-adjoint operator on $L^2(M,\mu)$ associated with the Dirichlet

form $(\mathcal{E},\mathcal{F})$, and define $n^\mu(x) = \#\{\lambda : \lambda$ is an eigenvalue of $-\mathcal{L}_\mu \le x.\}$. Let $d^e_s(\mu) > 0$ be the unique positive number satisfying $\sum_{i=1}^N (\mu_i/\rho_i)^{d^e_s(\mu)/2} = 1$, then

$$0 < \liminf_{x\to\infty} n^\mu(x)/x^{d^e_s(\mu)/2} \le \limsup_{x\to\infty} n^\mu(x)/x^{d^e_s(\mu)/2} < \infty.$$

Further, let ν be the Bernoulli measure satisfying

$$\nu_i = \rho_i^{-S} \qquad \text{for all } \ i \in I, \tag{5.3}$$

where S is the unique constant which satisfies $\sum_{i=1}^N \rho_i^{-S} = 1$. Then the maximum of $d^e_s(\mu)/2$ over Bernoulli measures on M is attained only at ν, and the maximum value is $S/(S+1)$.

For this special case, (i.e. $\mu = \nu$) detailed estimates on the heat kernel $p_t(x,y)$ are obtained in [8]. In the following, we will explain a version of the result of [8].

Theorem 5.2. *There exists a jointly continuous symmetric transition density $p_t(x,y)$ for $(\mathcal{E},\mathcal{F})$ with respect to ν which satisfies the following,*

$$\begin{aligned} c_{5.1} t^{-\frac{S}{S+1}} \exp\left(-c_{5.2}\left(\frac{R(x,y)^{S+1}}{t}\right)^{\gamma_1}\right) &\le p_t(x,y) \\ &\le c_{5.3} t^{-\frac{S}{S+1}} \exp\left(-c_{5.4}\left(\frac{R(x,y)^{S+1}}{t}\right)^{\gamma_2}\right) \end{aligned} \tag{5.4}$$

for all $0 < t \le 1,\ x,y \in M$, where $0 < \gamma_2 \le \gamma_1$ and $c_{5.1},\cdots,c_{5.4}$ are positive constants which depend only on M.

Proof. This can be obtained by a simple modification of [8] Corollary 1.2, i.e.,

$$p_t(x,y) \le c_1 t^{-\frac{S}{S+1}} \exp\left(-c_2\left(\frac{R(x,y)^{S+1}}{t}\right)^{d^c_{k(m,t)}(x,y)/(S+1-d^c_{k(m,t)}(x,y))}\right),$$

for all $x,y \in M$ with $e^{-m-1} \le R(x,y) < e^{-m}$ and all $0 < t \le 1$, with the same lower bound (with different constants c_3 and c_4). Here

$$d^c_k(x,y) := \frac{1}{k}\log N_{m+k}(x,y), \tag{5.5}$$

$N_m(x,y)$ is the shortest path counting function for the resistance metric at level m (see [8] for detailed definition), and

$$k = k(m,t) := \inf\{j : N_{m+j}(x,y)e^{-(S+1)(m+j)} < t\}. \tag{5.6}$$

It is enough to estimate $d^c_k(x,y)$ uniformly from above and below when $R(x,y)^{S+1}/t$ is (thus k is) large. Now, using self-similarity, Lemma 3.3, Lemma 3.4 and (3.11) of [8], we have

$$c_5 \exp(k) \le N_{m+k}(x,y) \le c_6 \exp\left(\frac{(S+1)k}{2}\right).$$

(From this we see that $S \geq 1$.) Substituting this to (5.5), we have

$$1 - \frac{c_7}{k} \leq d_k^c(x,y) \leq \frac{S+1}{2} + \frac{c_8}{k},$$

and the result holds. □

We remark that we cannot take $\gamma_1 = \gamma_2$ in general, as shown in [8] Section 6.

We now show that the domain $\mathcal{F}$ of the Dirichlet form is the Lipschitz space. First, note that $\mathcal{F}$ is embedded in the space of continuous functions on M and it is independent of the choice of μ. We thus take $\mu = \nu$. Then, by (5.4), we can apply Theorem 4.1 with $\rho(\cdot,\cdot) = R(\cdot,\cdot)$, $\beta_0 = d_w/2 = (S+1)/2$, $h(t) = t^{S+1}$ and $c_1 r^S \leq V(x,r) \leq c_2 r^S$. We thus have the following.

Proposition 5.1.

$$\mathcal{F} = Lip_\nu(\frac{S+1}{2}, 2, \infty)(M).$$

Further, the Lipschitz norm is comparable to $\mathcal{E}$ in the sense of (4.3).

Note that we can also prove this proposition by the similar way as [15] Theorem 4.3, but as we have detailed estimates (5.4) of the heat kernel, the proof in Theorem 4.1 is much easier. Using (5.4) again, we can prove that for all $\beta > (S+1)/2$, $\mathrm{Lip}_\nu(\beta, 2, \infty)(M)$ consists only of constant functions (cf. [7] Theorem 4.2 and [18] Proposition 2).

By this proposition, (A1) holds. In this case, $\mathcal{E}(f)$ is a monotone increasing limit of quadratic forms on approximating graphs. Further, elements of the domains of the forms are continuous functions. We thus obtain $E^0(f) := \Gamma\text{-}\lim_{n\to\infty} E^{r_n}(f) = \mathcal{E}(f)$ (cf. Section 3 in [8]), in particular (4.5) holds. Therefore (A2*) (thus (A2)) holds in this case.

5.2. Sierpinski carpets and their Dirichlet forms

Let $H_0 = [0,1]^d$, and let $l \in \mathbb{N}$, $l \geq 2$ be fixed. Set $\mathcal{Q} = \{\Pi_{i=1}^d [(k_i - 1)/l, k_i/l] : 1 \leq k_i \leq l,\ k_i \in \mathbb{N}\ (1 \leq i \leq d)\}$, let $l \leq N \leq l^d$ and let F_i, $1 \leq i \leq N$ be orientation preserving affine maps of H_0 onto some element of $\mathcal{Q}$. (We assume that $F_1(x) = l^{-1}x$ for $x \in H_0$ and that the sets $F_i(H_0)$ are distinct.) Set $H_1 = \cup_{i\in I} F_i(H_0)$. Then, there exists a unique non-empty compact set $M \subset H_0$ such that $M = \cup_{i\in I} F_i(M)$ and $(M, I, \{F_s\}_{s\in I})$ is a self-similar structure. M is called a Sierpinski carpet if the following holds:

(SC1) (Symmetry) H_1 is preserved by all the isometries of the unit cube H_0.

(SC2) (Connected) H_1 is connected.

(SC3) (Non-diagonality) Let B be a cube in H_0 which is the union of 2^d distinct elements of $\mathcal{Q}$. (So B has side length $2l^{-1}$.) Then if $\mathrm{Int}(H_1 \cap B)$ is non-empty, it is connected.

(SC4) (Borders included) H_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_d = 0\}$.

Here (see [3]) (SC1) and (SC2) are essential, while (SC3) and (SC4) are included for technical convenience. The main difference from p.c.f. self-similar sets is that Sierpinski carpets are infinitely ramified: the critical set $C(\mathcal{L})$ in Definition 5.1 is infinite, and M cannot be disconnected by removing a finite number of points. In fact, for the classical Sierpinski carpet in $\mathbb{R}^d$ with $l = 3$ and $N = 3^d - 1$ we have $V_0 = \partial[0,1]^d$. Write $D = \log N/\log l$ for the Hausdorff dimension of M with respect to the Euclidean metric. Let

$$H_n := \bigcup_{i_1,\cdots,i_n\in I} F_{i_1}\circ\cdots\circ F_{i_n}(H_0), \quad M_{pre} := \bigcup_{n\geq 0} l^n H_n \;\text{ and }\; \tilde{M} := \bigcup_{n\geq 0} l^n M.$$

M_{pre} is called a pre-carpet and $\tilde{M}$ is called an unbounded carpet (see [3]).

We write ν for the Bernoulli measure with weights $\nu_i = 1/N$: ν is a multiple of the Hausdorff measure on M. In [2], [16], [3], [9] a non-degenerate Dirichlet form $\mathcal{E}'$ on $L^2(M,\nu)$ is constructed on these spaces, with the property that $\mathcal{E}'$ is invariant under local isometries of M – and in particular $\mathcal{E}'$ is the same on each k-complex. The uniqueness of $\mathcal{E}'$ is an open problem – see [3]. If $\mathcal{E}'$ were unique then (5.2) would follow immediately. However, without requiring uniqueness, in [16] (see also Remark 5.11 in [3]) a compactness argument is used to construct a Dirichlet form $\mathcal{E}$ with the same invariances as $\mathcal{E}'$ and in addition satisfying (5.2) in the case when, for a constant ρ_M depending on M,

$$\rho_i = \rho_M, \quad \text{for all} \;\; 1\leq i\leq N.$$

Let $t_M = N\rho_M$ and let $\widehat{X} = (\widehat{X}_t, t\geq 0)$ be the diffusion associated with $\mathcal{E}$ and $L^2(M,\nu)$. We define $d_w = \log t_M/\log l$, the walk dimension of M, and $d_s = 2\log N/\log t_M$, the spectral dimension of M. In [4] (together with the results in [2], [3]), it is proved that $\widehat{X}$ enjoys the following transition density estimates.

$$\begin{aligned}(5.7)\qquad & c_{5.5}t^{-\frac{d_s}{2}}\exp\big(-c_{5.6}(\frac{\|x-y\|^{d_w}}{t})^{\frac{1}{d_w-1}}\big)\leq p_t(x,y)\\ &\leq c_{5.7}t^{-\frac{d_s}{2}}\exp\big(-c_{5.8}(\frac{\|x-y\|^{d_w}}{t})^{\frac{1}{d_w-1}}\big)\end{aligned}$$

for all $0<t\leq 1,\; x,y\in M$, where $c_{5.5},\cdots,c_{5.8}$ are positive constants which depend only on M and $\|\cdot-\cdot\|$ is the Euclidean metric. Thus, we can apply Theorem 4.1 with $\rho(\cdot,\cdot) = \|\cdot-\cdot\|$, $\beta_0 = d_w/2 =$, $h(t) = t^{d_w}$ and $c_1 r^{\log N/\log l} \leq V(x,r)\leq c_2 r^{\log N/\log l}$ and we have the following (cf. [15], [18]).

Proposition 5.2.

$$\mathcal{F} = Lip_\nu(\frac{d_w}{2}, 2, \infty)(M).$$

Further, the Lipschitz norm is comparable to $\mathcal{E}$ in the sense of (4.3).

By this proposition, (A1) holds in this case. Further, (4.5) holds by Proposition 5.2 and Theorem 5.4 in [16] (in general, Lemma 4.1 and Lemma 4.3 in [9]). Therefore (A2*) (thus (A2)) holds in this case.

For the pre-carpet M_{pre}, one can construct a reflected Brownian motion on M_{pre} since its interior is a Lipschitz domain. Let $(\mathcal{E}_{pre}, \mathcal{F}_{pre})$ be the corresponding Dirichlet form. Then the transition density with respect to the Lebesgue measure m satisfies

$$c_{5.9} t^{-\frac{d}{2}} \exp(-c_{5.10} \frac{\|x-y\|^2}{t}) \le p_t^{pre}(x,y) \le c_{5.11} t^{-\frac{d}{2}} \exp(-c_{5.12} \frac{\|x-y\|^2}{t})$$

for all $0 < t \le 1,\ x, y \in M_{pre}$, and it satisfies (5.7) for $t \ge 1 \vee \|x-y\|,\ x, y \in M_{pre}$ (see Theorem 1.4 in [3]). Thus, by Theorem 4.1, we have

$$\mathcal{F}_{pre} = \mathrm{Lip}_m(1, 2, \infty)(M_{pre}).$$

For the unbounded carpet $\tilde{M}$, let $\tilde{\nu}$ be the normalized Hausdorff measure. Then, one can construct a local regular Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\tilde{M}, \tilde{\nu})$ which satisfies

$$\tilde{\mathcal{E}}(f, g) = \rho_M \tilde{\mathcal{E}}(f \circ F_1, g \circ F_1) \quad \text{for} \quad f, g \in \tilde{\mathcal{F}}.$$

The corresponding diffusion enjoys the transition density estimate (5.7) for all $t > 0,\ x, y \in \tilde{M}$ (cf. [4, 2, 3]). Thus, by Theorem 4.1, we have

$$\tilde{\mathcal{F}} = \mathrm{Lip}_{\tilde{\nu}}(\frac{d_w}{2}, 2, \infty)(\tilde{M}). \tag{5.8}$$

In [15], $\tilde{\mathcal{F}}$ is characterized as $\{u \in L^2(\tilde{M}, \tilde{\nu}) : \sup_{n \in \mathbb{Z}} \mathcal{E}^{\alpha^{-n}}(u) < \infty\}$. By (4.3), we see that this space coincides with the right hand side of (5.8) and $\sup_{n \in \mathbb{Z}}$ can be replaced by $\limsup_{n \to \infty}$.

Acknowledgements. This research was initiated while the first named author was visiting Universität Bonn with the support of the SFB 611. The work was completed when the second named author was visiting Kyoto University with the support by Grant-in-Aid for Scientific Research (A)(1) 14204010. Both of us would like to thank the host institutes for hospitality. We also thank Laurent Saloff-Coste for valuable comments to the first version of this paper.

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