A trace theorem for Dirichlet forms on fractals

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Dedicated to Professor Shinzo Watanabe on the occasion of his 70th birthday

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Abstract

We consider a trace theorem for self-similar Dirichlet forms on self-similar sets to self-similar subsets. In particular, we characterize the trace of the domains of Dirichlet forms on Sierpinski gaskets and Sierpinski carpets to their boundaries, where the boundaries are represented by triangles and squares that confine the gaskets and the carpets. As an application, we construct diffusion processes on a collection of fractals called fractal fields. These processes behave as an appropriate fractal diffusion within each fractal component of the field.

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1 Introduction

The traces of Sobolev spaces on $\mathbb{R}^n$ to linear subspaces have been studied from various viewpoints as the generalizations of the Sobolev imbedding theorem. Further, the extension of Sobolev, Besov, and Lipschitz spaces from subdomains of $\mathbb{R}^n$ to whole spaces has been extensively studied (see, for example, [1, 26] and references therein). Since the 1980s, the problems for Besov-type spaces have been generalized for more complicated spaces, namely, the so-called Alfors d-regular sets ([16, 29]).

On the other hand, recent developments in the analysis of fractals shed new light on these problems. Diffusion processes and "Laplace" operators are constructed on fractals such as Sierpinski gaskets and Sierpinski carpets. It is observed that the domains of the corresponding Dirichlet forms are Besov-Lipschitz spaces.

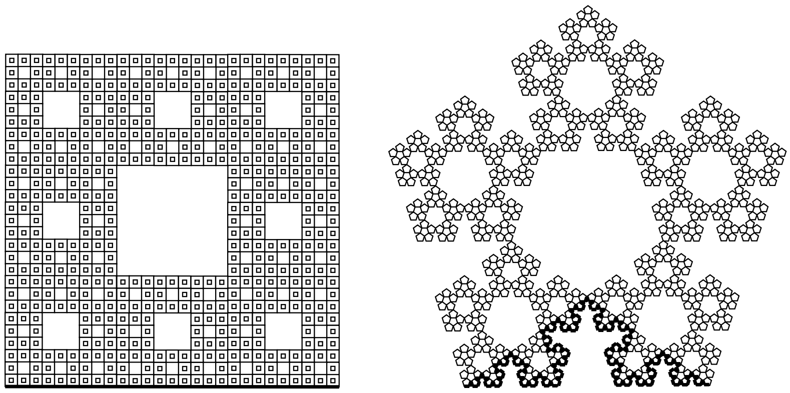

Figure 1: The Sierpinski carpet and the Pentakun

In this paper, we consider the following natural question: Given a Besov-type space on a self-similar fractal K, what is the trace of the space to a self-similar subspace L? Figure 1 shows two examples. The figure on the left is obtained when K is the so-called two-dimensional Sierpinski carpet (see Section 2 and Section 5 3) for the definition) and L is the line on the bottom (indicated by the thick line). The figure on the right is obtained when K is the Pentakun (a self-similar fractal determined by five contraction maps; see Section 5 2) for the definition) and L is a Koch-like curve (indicated by the thick curve). In each case, the domain of the Dirichlet form on K is a Besov-Lipschitz space; however, the trace cannot be obtained by using the general theory given by Jonsson-Wallin ([16]) and Triebel ([29]).

This problem was recently solved by Jonsson ([15]) for one typical case, i.e., when K is a two-dimensional Sierpinski gasket and L is the bottom line. However, his methods strongly depend on the structure of the Sierpinski gasket and its Dirichlet form, and they cannot be applied to the so-called infinitely ramified fractals such as the Sierpinski carpets. Instead, we use the self-similarity of the Dirichlet form and the compactness property of a family of harmonic functions, which can be obtained using the elliptic Harnack inequalities. Our methods can be applied to the Sierpinski carpets (even to the higher dimensional ones), and we can state the trace theorem under some abstract framework. In fact, we would need various assumptions for K and for the Dirichlet form on K, which are stated in Section 2. Unless these conditions are satisfied, nonstandard indices may appear in the trace spaces because of the "complexity" of the space (see Section 5 4) for an example).

In order to prove our trace theorem, we provide a discrete approximation of the Besov-Lipschitz space in Section 3.1. The approximation result is new, and it is regarded as a generalization of the main result in [17]. The restriction theorem is given in Section 3.2; the key estimate (Proposition 3.8) is based on the idea used by one of the authors in [13]. The extension theorem is given in Section 3.3, where a classical construction of the Whitney decomposition and the extension map is modified and generalized to this framework.

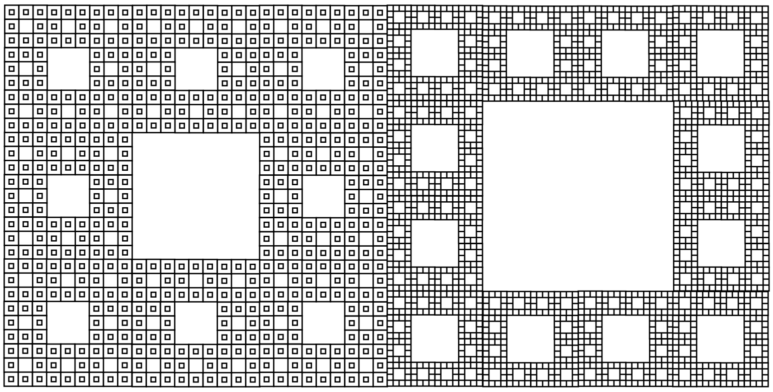

Figure 2: An example of fractal fields

Such a trace theorem has an important application in the penetrating process, which is discussed in Section 6. Let us discuss one concrete example. Two types of Sierpinski carpets are shown in Figure 2 (the carpet on the left is obtained from eight contraction maps with a contraction rate of 1/3; the carpet on the right is obtained from twelve contraction maps with a contraction rate of 1/4). On each carpet, a self-similar diffusion can be constructed; a question arises whether one can construct a diffusion that behaves as appropriate fractal diffusions within each carpet and one that can penetrate each fractal. In order to construct such a diffusion by the superposition of Dirichlet forms on each carpet, the primary problem is whether a sufficient number of functions exist whose restrictions for each carpet are in the domain of each Dirichlet form. To answer this question, it is crucial to obtain the information on the trace space of the domain of the Dirichlet form on each carpet to the line that intersects the two carpets. Indeed, when one of the authors studied this problem regarding fractals in [12, 20], a very strong assumption was required on each fractal because of the lack of information for the trace. Our trace theorem can be applied here, and we can construct the penetrating processes on a much wider class of fractals.

Hereafter, if f and g depend on a variable x ranging in a set A, $f \asymp g$ implies that there exists $C > 0$ such that $C^{-1}f(x) \le g(x) \le C\,f(x)$ for all $x \in A$. We use c, with or without subscripts, to denote strictly positive constants whose values are insignificant.

2 Framework and the main theorem

Let (X, d) be a complete separable metric space. For $\alpha > 1$ and a finite index set W, let $\{F_i\}_{i\in W}$ be a family of α-*similitudes* on X, i.e., $\mathsf{d}(F_i(x), F_i(y)) = \alpha^{-1}\mathsf{d}(x, y)$ for all $x, y \in X$.

Let S be a subset of W, and let N denote the cardinality of S. Since $\{F_i\}_{i\in S}$ is a family of contraction maps, there exists a unique non-void compact set K such that $K = \bigcup_{i\in S} F_i(K)$. We assume that K is connected. Note that the extra indices in W are required in general in order to define a self-similar subset L (such as in the Pentakun example in Section 5 2)). In various important examples such as 1) and 3) in Section 5, we can take $W = S$.

We formulate a relation with the shift space. The one-sided shift space Σ is defined by $\Sigma = W^{\mathbb{N}}$. For $w \in \Sigma$, we denote the i-th element in the sequence by w_i and express $w = w_1 w_2 w_3 \cdots$. When $w \in W^n$, $|w|$ denotes n. For $v \in W^m$ and $w \in W^n$, we define $v\cdot w \in W^{m+n}$ by $v \cdot w = v_1 v_2 \cdots v_m w_1 w_2 \cdots w_n$. For $A \subset W^m$ and $B \subset W^n$, $A \cdot B$ denotes $\{v \cdot w : v \in A,\ w \in B\}$. The set $w \cdot A$ is defined as $\{w\} \cdot A$. By definition, $W^0 = \{\emptyset\}$ and $\emptyset \cdot A = A$.

We assume that there exists a group $\mathfrak{G}$ comprising isometries on K such that the following hold.

- For each $i \in W$, there exist $j = j(i) \in S$ and $\Psi_i \in \mathfrak{G}$ such that $F_i = F_j \circ \Psi_i$.
- For each $(\Psi, \alpha) \in \mathfrak{G} \times S$, there exists $(\hat{\Psi}, \hat{\alpha}) \in \mathfrak{G} \times S$ such that $\Psi \circ F_\alpha = F_{\hat{\alpha}} \circ \hat{\Psi}$.

Note that when $W = S$, we can invariably take $\mathfrak{G}$ as a trivial group comprising one element. We express $F_{w_1\cdots w_n} = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}$ for $w = w_1 w_2 \cdots w_n$. We consider $F_\emptyset$ as an identity map. For $w \in W^n$ and $A \subset W^n$ for some $n \in \mathbb{Z}_+$, we define $K_w = F_w(K)$ and $K_A = \bigcup_{v\in A} K_v$.

Lemma 2.1. *There exist maps* $\Phi\colon \bigcup_{n\in\mathbb{Z}_+} W^n \to \bigcup_{n\in\mathbb{Z}_+} S^n$ *and* $\Psi\colon \bigcup_{n\in\mathbb{Z}_+} W^n \to \mathfrak{G}$ *such that* $F_w = F_{\Phi(w)} \circ \Psi(w)$ *for each* $w \in \bigcup_{n\in\mathbb{Z}_+} W^n$. *In particular,* $K_w = K_{\Phi(w)}$.

Proof. Set $\Phi(\emptyset) = \emptyset$ and $\Psi(\emptyset) =$ the unit element of $\mathfrak{G}$. When $i \in W^1$, it suffices to set $\Phi(i) = j(i)$ and $\Psi(i) = \Psi_i$. Suppose that $\Phi(w)$ is defined for $w \in W^n$. Then, for $w' = w \cdot i$ with $i \in W$, $F_{w'} = F_w \circ F_i = F_{\Phi(w)} \circ \Psi(w) \circ F_{j(i)} \circ \Psi_i$. This is equal to $F_{\Phi(w)} \circ F_{\hat{i}} \circ \hat{\Psi} \circ \Psi_i$ for some $(\hat{\Psi}, \hat{i}) \in \mathfrak{G} \times S$. Therefore, it is sufficient to define $\Phi(w') = \Phi(w) \cdot \hat{i}$ and $\Psi(w') = \hat{\Psi} \circ \Psi_i$. □

We define $\pi\colon \Sigma \to K$ by the relation $\{\pi(w)\} = \bigcap_m K_{w_1\cdots w_m}$ for $w = w_1 w_2 \cdots \in \Sigma$. Further,

$$C_K := \pi^{-1}\left(\bigcup_{i,j\in S, i\neq j} (K_i \cap K_j)\right), \qquad P_K := \bigcup_{n\geq 1} \sigma^n(C_K), \tag{2.1}$$

where $\sigma\colon \Sigma \to \Sigma$ is the left shift map, i.e., $\sigma w = w_2 w_3 \cdots$ if $w = w_1 w_2 w_3 \cdots$.

For $v, w \in W^n$, we write $v \overset{n,K}{\sim} w$ if $K_v \cap K_w \neq \emptyset$. For $w \in W^n$ and $A \subset W^n$, $w \overset{n,K}{\sim} A$ implies that $w \overset{n,K}{\sim} v$ for some $v \in A$. For $A \subset W^n$, define $\mathcal{N}_0(A) = A$ and $\mathcal{N}_k(A) = \{v \in W^n \mid v \overset{n,K}{\sim} \mathcal{N}_{k-1}(A)\}$ for $k \in \mathbb{N}$ inductively. We set $\mathcal{N}_k(w) = \mathcal{N}_k(\{w\})$ for $w \in W^n$.

Let I be a subset of W. We assume that the cardinality N_I of I is less than N. Let L be a unique non-void compact set such that $L = \bigcup_{i\in I} F_i(L)$. Evidently, L is a subset of K due to Lemma 2.1. We denote $F_w(L)$ by L_w for $w \in \bigcup_{n\in\mathbb{Z}_+} I^n$. Let $M \in \mathbb{N}$. For $v,\ w \in I^n$, we write $v \underset{M}{\overset{n,L}{\longleftrightarrow}} w$ if $v \in \mathcal{N}_M(w)$. We fix M such that for each $i,\ j \in I$, there exist $i_1, i_2, \ldots \in I$ satisfying $i \underset{M}{\overset{1,L}{\longleftrightarrow}} i_1 \underset{M}{\overset{1,L}{\longleftrightarrow}} i_2 \underset{M}{\overset{1,L}{\longleftrightarrow}} \cdots \underset{M}{\overset{1,L}{\longleftrightarrow}} j$. Hereafter, we omit M from the notation $\underset{M}{\overset{n,L}{\longleftrightarrow}}$.

Now, we introduce several conditions. In order to avoid burdening the readers, we recommend them to keep in mind the standard two-dimensional Sierpinski carpet (SC(2)) to be K and the bottom line to be L (the left figure in Figure 1) as a typical example. The carpet is constructed as follows. Let (X, d) be $\mathbb{R}^2$ with the Euclidean metric. Let $\alpha = 1/3$, $W = \{1, 2, \dots, 8\}$, and define F_i to be a standard α-similitude on X with center $(0,0)$, $(1/2, 0)$, $(1, 0)$, $(0, 1/2)$, $(1, 1/2)$, $(0, 1)$, $(1/2, 1)$, and $(1, 1)$, for $i = 1, 2, \dots, 8$, respectively. Let $S = W$ and $I = \{1, 2, 3\}$. Then, K is a subset of $[0,1] \times [0,1]$ and L is equal to $[0,1] \times \{0\}$. We can easily verify that P_K is the boundary of $[0,1] \times [0,1]$ in $\mathbb{R}^2$. We can take $M = 1$ and for $v, w \in I^n$, $v \overset{n,L}{\longleftrightarrow} w$ if and only if $L_v \cap L_w \neq \emptyset$. We take a trivial group $\{e\}$ as $\mathfrak{G}$. Then, Φ is the identity map and $\Psi(w) = e$ for all w.

With regard to the general situation, we assume the following:

(A1) $\sup_{n \in \mathbb{Z}_+} \max_{w \in S^n} \#(\mathcal{N}_1(w) \cap S^n) < \infty$ and $C_0 := \sup_{n \in \mathbb{Z}_+} \max_{w \in I^n} \#(\mathcal{N}_M(w) \cap I^n) < \infty$.

(A2) There exist $k_1, k_2 > 0$ such that for $x, y \in L$, $n \in \mathbb{Z}_+$ and $v, w \in I^n$ with $x \in K_v$ and $y \in K_w$, $\mathsf{d}(x, y) < k_1 \alpha^{-n}$ implies $v \overset{n,L}{\leftrightarrow} w$ and $v \overset{n,L}{\leftrightarrow} w$ implies $\mathsf{d}(x, y) < k_2 \alpha^{-n}$.

(A3) There exist $k_1, k_2 > 0$ such that for $x, y \in K$, $n \in \mathbb{Z}_+$ and $v, w \in S^n$ with $x \in K_v$ and $y \in K_w$, $\mathsf{d}(x, y) < k_1 \alpha^{-n}$ implies $v \overset{n,K}{\sim} w$ and $v \overset{n,K}{\sim} w$ implies $\mathsf{d}(x, y) < k_2 \alpha^{-n}$.

In the case of SC(2), the first value in (A1) is 8 and $C_0 = 3$. In (A2), we can take $k_1 = 1$ and $k_2 = 2.01$. In (A3), we can take $k_1 = 1$ and $k_2 = 2\sqrt{2} + 0.01$.

Let $\hat{\mu}$ and $\hat{\nu}$ be the canonical Bernoulli measures on $S^{\mathbb{N}}$ and $I^{\mathbb{N}}$, respectively. In other words, they are infinite product measures of S and I, respectively, with uniformly distributed probability measures. The image measures of $\hat{\mu}$ are denoted by μ in the map $\pi|_{S^{\mathbb{N}}} \colon S^{\mathbb{N}} \to K$. Similarly, the probability measure ν on L is defined. Based on conditions (A1), (A2), and (A3) and [18, Theorem 1.5.7], the Hausdorff dimensions of K and L are equal to $d_f := \log N / \log \alpha$ and $d_I := \log N_I / \log \alpha$, respectively, and μ and ν are equivalent to the Hausdorff measures on K and L, respectively.

Further, we make the following assumptions that are clearly satisfied in the case of SC(2) because both the sets in (A4) are empty sets.

(A4) $\mu(\{x \in K : \#(\pi^{-1}(x) \cap S^{\mathbb{N}}) = \infty\}) = 0$ and $\nu(\{x \in L : \#(\pi^{-1}(x) \cap I^{\mathbb{N}}) = \infty\}) = 0$.

Then, by Theorem 1.4.5 in [18], $\mu(K_w) = N^{-|w|}$ for every $w \in \bigcup_{n \in \mathbb{Z}_+} S^n$ and $\nu(L_w) = N_I^{-|w|}$ for every $w \in \bigcup_{n \in \mathbb{Z}_+} I^n$. Furthermore, $\mu(L) = 0$ holds.

Suppose that we are given a strong local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$. $\mathcal{F}$ is equipped with a norm $\|f\|_{\mathcal{F}} = (\mathcal{E}(f) + \|f\|^2_{L^2(\mu)})^{1/2}$. Hereafter, for each quadratic form $E(\cdot, \cdot)$, we abbreviate $E(f, f)$ as $E(f)$. We assume the following:

(A5) (Self-similarity) For each $f \in \mathcal{F}$ and $i \in S$, $F_i^* f \in \mathcal{F}$ where $F_i^* f = f \circ F_i$. Further, there exists $\rho > 0$ such that

$$\mathcal{E}(f) = \rho \sum_{i \in S} \mathcal{E}(F_i^* f), \quad f \in \mathcal{F}.$$

(A6) For every $\Psi \in \mathfrak{G}$, $\Psi^* \mathcal{F} = \mathcal{F}$, that is, $\{f \circ \Psi : f \in \mathcal{F}\} = \mathcal{F}$. Further, $\mathcal{E}(\Psi^* f) = \mathcal{E}(f)$ for all $f \in \mathcal{F}$.

(A7) Let $d_w = (\log \rho N)/(\log \alpha)$. Then, $d_w > d_f - d_I$.

(B1) The space $\mathcal{F}$ is compactly imbedded in $L^2(K, \mu)$, and $\mathcal{E}(f) = 0$ if and only if f is a constant function.

In the case of SC(2), there exists a Dirichlet form on $L^2(K, \mu)$ satisfying (A5)–(A7) and (B1). (See Section 5 3) for details).

For each subset A of W^m for some $m \in \mathbb{Z}_+$, let $\mathcal{F}_A$ be a function space on K_A such that $\{f|_{K_A} : f \in \mathcal{F}\} \subset \mathcal{F}_A \subset \{f \in L^2(K_A) : F_w^* f \in \mathcal{F} \text{ for all } w \in A\}$. The space $\mathcal{F}_A$ will be specified later for some class of Dirichlet forms in Section 4. For $f, g \in \mathcal{F}_A$, we define

$$\mathcal{E}_A(f, g) = \rho^m \sum_{w \in A} \mathcal{E}(F_w^* f, F_w^* g). \tag{2.2}$$

We assume that $\mathcal{F}_A = \mathcal{F}_{A \cdot S^n}$ for all $n \in \mathbb{N}$ and $(\mathcal{E}_A, \mathcal{F}_A)$ is a closed form on $L^2(K_A, \mu|_{K_A})$. Hereafter, we always consider $\mathcal{F}_A$ as a normed space with norm $\|f\|_{\mathcal{F}_A} = (\mathcal{E}_A(f) + \|f\|^2_{L^2(K_A)})^{1/2}$. By (A5), $\mathcal{E}_A(f) = \mathcal{E}_{A \cdot S^n}(f)$ holds for any $f \in \mathcal{F}_A$, and $\mathcal{E}_{\Phi(A)}(f) = \mathcal{E}_A(f)$ if $\#\Phi(A) = \#A$ by (A6). When $A = \{w\}$, we use the notation $\mathcal{E}_w$ instead of $\mathcal{E}_{\{w\}}$. Functions in $\mathcal{F}$ can be naturally considered as elements in $\mathcal{F}_A$ due to the restriction of the domain. For notational convenience, we often simply write f in the place of $f|_{K_A}$ when $f \in \mathcal{F}$ is regarded as an element of $\mathcal{F}_A$.

Definition 2.2. Let A be a nonempty subset of W^m for some $m \in \mathbb{Z}_+$. We say that A is $\mathcal{E}_A$-connected if, for $f \in \mathcal{F}_A$, $\mathcal{E}_A(f) = 0$ implies that f is constant on K_A.

Definition 2.3. Let $A \subset W^m$ and $B \subset W^n$ for some m and n. We say that A and B are of the same type if there exist a homeomorphism $F \colon K_A \to K_B$ and a bijection $\chi \colon A \to B$ such that $F \circ F_u = F_{\chi(u)}$ for all $u \in A$ and $F^*(\mathcal{F}_B) = \mathcal{F}_A$.

We assume the following:

(B2) There exists $\hat{I} \subset W$ such that the following hold:

(1) $\hat{I} \supset I$ and $\#\hat{I} < N$.

(2) For each $w \in I^n$, $\mathcal{N}_M(w) \cap \hat{I}^n$ is an $\mathcal{E}_{\mathcal{N}_M(w) \cap \hat{I}^n}$-connected set.

(3) There exist finite elements $u_1, \dots, u_k \in \bigcup_{n \in \mathbb{Z}_+} I^n$ such that for any $w \in \bigcup_{n \in \mathbb{Z}_+} I^n$, there exists $j \in \{1, \dots, k\}$ such that $\mathcal{N}_M(w) \cap \hat{I}^{|w|}$ and $\mathcal{N}_M(u_j) \cap \hat{I}^{|u_j|}$ are of the same type; moreover, $F(L_{\mathcal{N}_M(w) \cap \hat{I}^{|w|}}) = L_{\mathcal{N}_M(u_j) \cap \hat{I}^{|u_j|}}$, where F is provided in Definition 2.3.

(4) $C_1 := \sup_{n \in \mathbb{Z}_+} \max_{w \in \hat{I}^n} \#(\mathcal{N}_M(w) \cap I^n) < \infty$ and $C_2 := \sup_{n \in \mathbb{Z}_+} \max_{w \in S^n} \#\{v \in \hat{I}^n : \Phi(v) = w\} < \infty$.

Let us check (B2) in the case of SC(2). We define $\mathcal{F}_A$ as in (4.3). Then, we can take $\hat{I} = I$, $k = 3$, and $\{u_1, u_2, u_3\} = \{1, 2, 3\}$. In (B2) (4), $C_1 = 3$ and $C_2 = 1$.

For an open set $U \subset K$, we define the capacity of U by

$$\mathrm{Cap}(U) = \inf\{\|u\|_{\mathcal{F}}^2 : u \in \mathcal{F}, u \geq 1 \ \mu\text{-a.e. on } U\}.$$

The capacity of any set $D \subset K$ is defined as the infimum of the capacity of open sets that contain D. We denote a quasi-continuous modification of $f \in \mathcal{F}$ by $\tilde{f}$. We assume the following:

(A8) There exists some $c > 0$ such that $\nu(D) \leq c\,\mathrm{Cap}(D)$ for every compact set $D \subset K$.

It is known that (A8) is equivalent to the following (see, for example, [7, Theorem 3.1]):

(A8)' The measure ν charges no set of zero capacity and $f \mapsto \tilde{f}|_L$ is a continuous map from $\mathcal{F}$ to $L^2(L, \nu)$.

We will provide sufficient conditions for (A8) in Section 4.

For each $n \in \mathbb{Z}_+$, define $Q_n \colon L^1(L, \nu) \to \mathbb{R}^{I^n}$ as

$$Q_n f(w) = \fint_{L_w} f(y) d\nu(y), \quad w \in I^n,$$

where, in general, $\fint_A \cdots d\lambda(y) := \lambda(A)^{-1} \int_A \cdots d\lambda(y)$ denotes the normalized integral on A. Then, one can easily check

$$N_I^{-1} \sum_{j \in I} Q_{m+1} f(w \cdot j) = Q_m f(w), \quad w \in I^m. \tag{2.3}$$

Let $m \in \mathbb{N}$, $A \subset S^m$, and $J \subset I^m$. We define $\mathcal{F}(J, A) = \{f \in \mathcal{F} : f = 0 \text{ on } K_{S^m \setminus A},\ Q_m(\tilde{f}|_L) = 0 \text{ on } J\}$. Further, we define a closed subspace $\mathcal{H}(J, A)$ of $\mathcal{F}$ by

$$\mathcal{H}(J, A) = \{h \in \mathcal{F} : \mathcal{E}(h, f) = 0 \ \text{ for all } f \in \mathcal{F}(J, A)\}.$$

When J is an empty set, we omit it from the notation. We assume the following:

(B3) There exist some $l_0, m_0 \in \mathbb{Z}_+$, $C > 0$, a proper subset $D'(w)$ of $S^{|w|}$ with $w \in D'(w)$ for each $w \in \bigcup_{n \in \mathbb{Z}_+} \Phi(\hat{I}^{n+m_0})$, a finite subset $\Xi \subset \bigcup_{n \in \mathbb{Z}_+} \Phi(\hat{I}^{n+m_0})$, and subsets $D^\sharp(v)$ of $D'(v)$ with $v \in D^\sharp(v)$ for each $v \in \Xi$ such that the following hold:

  (1) For each $n \in \mathbb{Z}_+$ and $w \in \Phi(\hat{I}^{n+m_0})$,

    (a) $w \in D'(w)$ and $D'(w) \subset \mathcal{N}_{l_0}(w) \cap (\Phi(\hat{I}^n) \cdot S^{m_0})$,

    (b) there exists $v \in \Xi$ such that

$$F_w^*(\{h \in \mathcal{H}(I^{|w|}, D'(w)) : \int_{K_{D'(w)}} h\, d\mu = 0,\ \rho^{-|w|} \mathcal{E}_{D'(w)}(h) \leq 1\}) \subset F_v^*(\{h \in \mathcal{H}(I^{|v|}, D^\sharp(v)) : \|h\|_{\mathcal{F}_{D^\sharp(v)}} \leq C\}).$$

  (2) For each $v \in \Xi$, the operator $F_v^* \colon \mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}} \to \mathcal{F}$ is a compact operator, where $\mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}}$ is regarded as a subspace of $\mathcal{F}_{D^\sharp(v)}$.

We set $D(w) = D'(\Phi(w))$ for $w \in \bigcup_{n\in\mathbb{Z}_+} \hat{I}^{n+m_0}$. For a sufficient condition regarding (B3), see Section 4.

In the case of SC(2), let $l_0 = 12$ and $m_0 = 3$. For $k \in \mathbb{N}$, $m \geq 3$, and $w \in I^m$, we define $\Lambda_k(w)$ as the intersection of K and a cube in $\mathbb{R}^2$ whose center is identical to the center of K_w and whose length is $(2k+1)3^{-m}$. Further, we also define $D'(w) \subset S^m$ such that $K_{D'(w)} = \Lambda_6(w)$. We take $\Xi = I^3$ and $D^\sharp(v) = \Lambda_3(v)$ for $v \in \Xi$. For $w \in I^{n+3}$, we take $v \in \Xi$ such that there exists a similitude from $K_{D'(w)}$ to $K_{D'(v)}$ and the image of K_w is K_v. Then, (B3) (1) (a) is clearly satisfied. (B3) (1) (b) with a sufficiently large C, and (B3) (2) can be verified by Propositions 4.6 and 4.7. See also Section 5 3).

The following assumption (B4) will be used in the restriction theorem.

(B4) For $f \in \mathcal{F}$, if $\mathcal{E}_{S^m \setminus \Phi(\hat{I}^m)}(f) = 0$ for every $m \in \mathbb{Z}_+$, then f is a constant function.

Next, we introduce Besov spaces.

Definition 2.4. For $1 \leq p < \infty,\ 1 \leq q \leq \infty,\ \beta \geq 0$ and $m \in \mathbb{Z}_+$, we set

$$a_m(\beta, f) := \gamma^{m\beta} \left(\gamma^{md_f} \iint_{\{(x,y)\in K\times K:\mathsf{d}(x,y)<c\gamma^{-m}\}} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \right)^{1/p}$$

for $f \in L^p(K, \mu)$, where $1 < \gamma < \infty,\ 0 < c < \infty$. Further, we define a Besov space $\Lambda^\beta_{p,q}(K)$ as a set of all $f \in L^p(K, \mu)$ such that $\bar{a}(\beta, f) := \{a_m(\beta, f)\}_{m=0}^\infty \in l^q$. $\Lambda^\beta_{p,q}(K)$ is a Banach space with the norm $\|f\|_{\Lambda^\beta_{p,q}(K)} := \|f\|_{L^p(K)} + \|\bar{a}(\beta, f)\|_{l^q}$. Let $\hat{\Lambda}^\beta_{p,q}(K)$ denote the closure of $\Lambda^\beta_{p,q}(K) \cap C(K)$ in $\Lambda^\beta_{p,q}(K)$. $\Lambda^\beta_{p,q}(L)$ and $\hat{\Lambda}^\beta_{p,q}(L)$ are defined in a similar manner by replacing (K, μ) by (L, ν).

This definition is valid for general Alfors regular compact sets K with a normalized Hausdorff measure μ. We use the notation $\Lambda^\beta_{p,q}(K)$, as used in [11]. $\Lambda^\beta_{p,q}(K)$ was denoted as Lip $(\beta, p, q)(K)$ in [14, 20] and $\Lambda^{p,q}_\beta(K)$ in [28]. Note that different selections of $c > 0$ and $\gamma > 1$ provide the same space $\Lambda^\beta_{p,q}(K)$ with equivalent norms. Hereafter, we will take $\gamma = \alpha$.

Now, we state our main theorems. Let $\beta = d_w/2 - (d_f - d_I)/2$.

Theorem 2.5. *Suppose that* (A1)–(A8) *and* (B1)–(B4) *hold. Then, for every* $f \in \mathcal{F}$, $\tilde{f}|_L$ *belongs to* $\hat{\Lambda}^\beta_{2,2}(L)$. *Moreover, there exists* $c > 0$ *such that* $\|\tilde{f}|_L\|_{\Lambda^\beta_{2,2}(L)} \leq c\|f\|_\mathcal{F}$ *for every* $f \in \mathcal{F}$.

Theorem 2.6. *Suppose that* (A1)–(A8) *and* (C1)–(C2) *hold. (Conditions* (C1) *and* (C2) *will be defined in Section 3.3). Then, there exists a bounded linear map* ξ *from* $\hat{\Lambda}^\beta_{2,2}(L)$ *to* $\mathcal{F}$ *such that* $\xi(\Lambda^\beta_{2,2}(L) \cap C(L)) \subset \mathcal{F} \cap C(K)$ *and* $\widetilde{\xi f}|_L = f$ *ν-a.e. for all* $f \in \hat{\Lambda}^\beta_{2,2}(L)$.

Hereafter, we often express $\mathcal{F}|_L = \hat{\Lambda}^\beta_{2,2}(L)$ to denote the assertions of both the abovementioned theorems.

Remark 2.7. In the following two cases, we can prove $\hat{\Lambda}^\beta_{2,2}(L) = \Lambda^\beta_{2,2}(L)$.
1) $L \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $\beta < 1$. In this case, according to [16], the following trace theorem holds: $B^{2,2}_{\beta+(n-d_I)/2}(\mathbb{R}^n)|_L = \Lambda^\beta_{2,2}(L)$, where $B^{2,2}_\gamma(\mathbb{R}^n)$ is a classical Besov space with a

smoothness order γ. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $B_\gamma^{2,2}(\mathbb{R}^n)$ for $\gamma > 0$, it follows that the functions in $C_0^\infty(\mathbb{R}^n)$ restricted to L are dense in $\Lambda_{2,2}^\beta(L)$.
2) $\beta > d_I/2$. In this case, according to [11] Theorem 8.1, the following holds: $\Lambda_{2,\infty}^\beta(L) \subset \mathcal{C}^{\beta - d_I/2}(L)$, where $\mathcal{C}^\lambda(L)$ is a Hölder space defined as follows: $u \in \mathcal{C}^\lambda(L)$ if

$$\|u\|_{\mathcal{C}^\lambda(L)} := \|u\|_{L^\infty(L)} + \nu - \operatorname*{ess\,sup}_{x,y\in L,\ x\neq y} \frac{|u(x)-u(y)|}{\mathsf{d}(x,y)^\lambda} < \infty. \tag{2.4}$$

Since $\Lambda_{2,2}^\beta(L) \subset \Lambda_{2,\infty}^\beta(L)$, we observe that any element in $\Lambda_{2,2}^\beta(L)$ is continuous in this case.

Remark 2.8. Since ν is smooth with respect to $(\mathcal{E},\mathcal{F})$, we can consider the time-changed Markov process with respect to the positive continuous additive functional associated with ν via the Revuz correspondence. According to the general theory of Dirichlet forms, this has an associated regular Dirichlet form $(\check{\mathcal{E}},\check{\mathcal{F}})$ on $L^2(L,\nu)$ with $\check{\mathcal{F}} = \{f \in L^2(L,\nu) : f = \tilde{u}\ \nu\text{-a.e. on } L \text{ for some } u \in \mathcal{F}_e\}$, where $\mathcal{F}_e$ is the family of μ-measurable functions u on K such that $|u| < \infty$ μ-a.e., and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}_{n\in\mathbb{N}}$ of functions in $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ μ-a.e. As observed in the following proposition, $\mathcal{F}_e = \mathcal{F}$ in our framework. Therefore, our main theorems determine the function space $\check{\mathcal{F}}$.

Proposition 2.9. *Under the condition* (B1), $\mathcal{F}_e = \mathcal{F}$.

Proof. By (B1), there exists some $c > 0$ such that

$$\left\| f - \int_K f\,d\mu \right\|_{L^2(K,\mu)}^2 \le c\mathcal{E}(f), \quad f \in \mathcal{F}. \tag{2.5}$$

Let $u \in \mathcal{F}_e$. We consider $\{u_n\}_{n\in\mathbb{N}}$ from $\mathcal{F}$ as in the definition of $\mathcal{F}_e$ in Remark 2.8. Further, we define $g_n = u_n - \int_K u_n\,d\mu$ for each n. Then, $\{g_n\}_{n\in\mathbb{N}}$ is an $\mathcal{E}$-Cauchy sequence. Since $\int_K g_n\,d\mu = 0$, (2.5) implies that $\{g_n\}$ is an $L^2(K,\mu)$-Cauchy sequence. Therefore, g_n converges to some g in $\mathcal{F}$. By considering a subsequence, we may assume that $g_n \to g$ μ-a.e. Thus, $\int_K u_n\,d\mu\,(= u_n - g_n)$ converges to some $C \in \mathbb{R}$. In particular, $\int_K u_n\,d\mu$ converges to C in $\mathcal{F}$ as a sequence of constant functions. Therefore, u_n converges to $g + C$ in $\mathcal{F}$. This implies that $u = g + C$ belongs to $\mathcal{F}$. □

3 Proof of main theorems

3.1 Discrete approximation

In this section, we assume (A1)–(A8). For $n \in \mathbb{Z}_+$, we define a bilinear form on I^n as

$$E_{(n)}(g,g) = \sum_{v,w\in I^n,\ v \overset{n,L}{\leftrightarrow} w} (g(v) - g(w))^2 \quad \text{for} \quad g \in \mathbb{R}^{I^n}.$$

Then, we obtain the following discrete characterization of $\Lambda_{2,q}^\beta(L)$ (for related results, see [17]).

Lemma 3.1. *Let $\beta > 0$ and $q \in [1,\infty]$. Then, there exists $c_1 > 0$ such that for each $f \in L^2(L,\nu)$,*

$$
\begin{aligned}
& c_1 \left\| \left\{ \alpha^{n\beta} \left(\alpha^{nd_I} \iint_{\{(x,y)\in L\times L : \mathsf{d}(x,y)<k_1\alpha^{-n}\}} |f(x)-f(y)|^2 \, d\nu(x)\, d\nu(y) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^q} \\
\le & \left\| \left\{ \alpha^{n\beta} \left(\alpha^{-nd_I} E_{(n)}(Q_n f) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^q} \\
\le & \left\| \left\{ \alpha^{n\beta} \left(\alpha^{nd_I} \iint_{\{(x,y)\in L\times L : \mathsf{d}(x,y)<k_2\alpha^{-n}\}} |f(x)-f(y)|^2 \, d\nu(x)\, d\nu(y) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^q} . \qquad (3.1)
\end{aligned}
$$

Here, k_1 and k_2 are provided in (A2).

Proof. Due to the selection of M, there exists some $c_2 > 0$ such that

$$
\sum_{i\in I} \left(g(i) - N_I^{-1} \sum_{j\in I} g(j) \right)^2 \le c_2 E_{(1)}(g), \quad g \in \mathbb{R}^I .
$$

For $f \in L^2(L,\nu)$ and $n \in \mathbb{Z}_+$, we have

$$
\begin{aligned}
& \iint_{\{(x,y)\in L\times L : \mathsf{d}(x,y)<k_1\alpha^{-n}\}} |f(x)-f(y)|^2 \, d\nu(x)\, d\nu(y) \\
\le & \sum_{(v,w)\in I^n\times I^n,\, v \overset{n,L}{\leftrightarrow} w} \iint_{L_v\times L_w} |f(x)-f(y)|^2 \, d\nu(x)\, d\nu(y) \qquad \text{(by (A2))} \\
\le & \sum_{(v,w)\in I^n\times I^n,\, v \overset{n,L}{\leftrightarrow} w} \iint_{L_v\times L_w} 3\{ |f(x)-Q_n f(v)|^2 + |Q_n f(v) - Q_n f(w)|^2 \\
& \quad + |Q_n f(w) - f(y)|^2 \} \, d\nu(x)\, d\nu(y) \\
\le & \; 6C_0 N_I^{-n} \sum_{v\in I^n} \int_{L_v} (f(x) - Q_n f(v))^2 \, d\nu(x) + 3N_I^{-2n} E_{(n)}(Q_n f),
\end{aligned}
$$

where C_0 is similar to that in (A1). With regard to the first term, we have

$$
\begin{aligned}
& \sum_{v\in I^n} \int_{L_v} (f(x) - Q_n f(v))^2 \, d\nu(x) \\
= & \int_L f(x)^2 \, d\nu(x) - N_I^{-n} \sum_{v\in I^n} Q_n f(v)^2 \\
= & \sum_{m=n}^{\infty} \left(N_I^{-(m+1)} \sum_{v\in I^{m+1}} Q_{m+1} f(v)^2 - N_I^{-m} \sum_{w\in I^m} Q_m f(w)^2 \right) \\
= & \sum_{m=n}^{\infty} N_I^{-(m+1)} \sum_{w\in I^m} \sum_{i\in I} \left(Q_{m+1} f(w\cdot i) - N_I^{-1} \sum_{j\in I} Q_{m+1} f(w\cdot j) \right)^2
\end{aligned}
$$

$$\leq \ c_2 \sum_{m=n}^{\infty} N_I^{-(m+1)} \sum_{w \in I^m} E_{(1)}(Q_{m+1} f(w \cdot *))$$
$$\leq \ c_2 \sum_{m=n}^{\infty} N_I^{-(m+1)} E_{(m+1)}(Q_{m+1} f),$$

where the martingale convergence theorem was used in the second equality and (2.3) was used in the third equality. Note that $\alpha^{d_I} = N_I$. Suppose that $q \in [1, \infty)$. Then,

$$\sum_{n=0}^{\infty} \alpha^{n(\beta + d_I/2)q} \left(\iint_{\{(x,y) \in L \times L : \mathsf{d}(x,y) < k_1 \alpha^{-n}\}} |f(x) - f(y)|^2 \, d\nu(x) \, d\nu(y) \right)^{q/2}$$
$$\leq \ \sum_{n=0}^{\infty} \alpha^{n(\beta + d_I/2)q} \left(6 c_2 C_0 N_I^{-n} \sum_{m=n}^{\infty} N_I^{-(m+1)} E_{(m+1)}(Q_{m+1} f) + 3 N_I^{-2n} E_{(n)}(Q_n f) \right)^{q/2}$$
$$\leq \ c_3 \sum_{n=0}^{\infty} \alpha^{n\beta q} \left(\sum_{m=n}^{\infty} \alpha^{-m d_I} E_{(m)}(Q_m f) \right)^{q/2}$$
$$\leq \ c_4 \sum_{m=0}^{\infty} \alpha^{m(\beta - d_I/2)q} E_{(m)}(Q_m f)^{q/2}$$
$$= \ c_4 \left\| \left\{ \alpha^{n\beta} \left(\alpha^{-n d_I} E_{(n)}(Q_n f) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^q}^q ,$$

where in the third inequality, we used (A7) and the following inequality for $\gamma > 0$:

$$\sum_{i=0}^{\infty} 2^{\gamma i} \left(\sum_{j \in \Lambda_i} a_j \right)^p \leq c \sum_{j=0}^{\infty} 2^{\gamma j} a_j^p \quad \text{for } \gamma \neq 0, \ p > 0, \ a_j \geq 0, \tag{3.2}$$

where $\Lambda_i = \{i, i+1, \ldots\}$ when $\gamma > 0$ and $\Lambda_i = \{0, 1, \ldots, i\}$ when $\gamma < 0$. When $0 < p \leq 1$, this is obvious since $(x+y)^p \leq x^p + y^p$ for x, $y \geq 0$. When $p > 1$, this is proved by the application of Hölder's inequality (see, for example, [22]).

When $q = \infty$, letting $\gamma = \left\| \left\{ \alpha^{n\beta} \left(\alpha^{-n d_I} E_{(n)}(Q_n f) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^\infty}$, for every $n \in \mathbb{Z}_+$, we have

$$\left| \alpha^{n(\beta + d_I/2)} \left(\iint_{\{(x,y) \in L \times L : \mathsf{d}(x,y) < k_1 \alpha^{-n}\}} |f(x) - f(y)|^2 \, d\nu(x) \, d\nu(y) \right)^{1/2} \right|^2$$
$$\leq \ c_5 \alpha^{n(2\beta + d_I)} \left(N_I^{-n} \sum_{m=n}^{\infty} N_I^{-(m+1)} E_{(m+1)}(Q_{m+1} f) + N_I^{-2n} E_{(n)}(Q_n f) \right)$$
$$\leq \ c_5 \alpha^{2n\beta} \sum_{m=n}^{\infty} \alpha^{-2m\beta} \gamma^2$$
$$= \ \frac{c_5}{1 - \alpha^{-2\beta}} \gamma^2 .$$

Thus, the first inequality in (3.1) is proved.

Next, we have

$$
\begin{aligned}
E_{(n)}(Q_n f) &= \sum_{(v,w)\in I^n\times I^n,\, v\overset{n,L}{\leftrightarrow} w} \left| N_I^{2n} \iint_{L_v\times L_w} \{f(x)-f(y)\}\, d\nu(x)\, d\nu(y)\right|^2 \\
&\le \sum_{(v,w)\in I^n\times I^n,\, v\overset{n,L}{\leftrightarrow} w} N_I^{2n} \iint_{L_v\times L_w} |f(x)-f(y)|^2\, d\nu(x)\, d\nu(y) \\
&\le \alpha^{2nd_I} \iint_{\{(x,y)\in L\times L:\mathsf{d}(x,y)<k_2\alpha^{-n}\}} |f(x)-f(y)|^2\, d\nu(x)\, d\nu(y),
\end{aligned}
$$

which deduces the second inequality of (3.1). □

Remark 3.2. Recently, M. Bodin ([8]) provided a discrete characterization of $\Lambda_{p,q}^{\beta}(K)$ for the Alfors d-regular set K that has a regular triangular system with some property (property (B) in the thesis).

3.2 Proof of the restriction theorem

In this section, we assume (A1)–(A8) and (B1)–(B4) and prove Theorem 2.5. The following lemma is immediately proved by equation (2.2).

Lemma 3.3. *Let $A\subset W^m$, $B\subset W^n$, $f\in\mathcal{F}_A$, and $g\in\mathcal{F}_B$. Suppose that there exists a bijection ι from A to B and $F_v^* f = F_{\iota(v)}^* g$ for every $v\in A$. Then, $\rho^{-m}\mathcal{E}_A(f)=\rho^{-n}\mathcal{E}_B(g)$.*

Let $n\in\mathbb{Z}_+$ and $w\in I^n$. Let $A=\mathcal{N}_M(w)\cap\hat{I}^n$. We define $\mathcal{G}_w=\{f\in\mathcal{F}_A : Q_n(\tilde{f}|_{L_A}) = 0 \text{ on } \mathcal{N}_M(w)\cap I^n\}$ and $\mathcal{K}_w=\{h\in\mathcal{F}_A : \mathcal{E}_A(h,f)=0 \text{ for all } f\in\mathcal{G}_w\}$. Hereafter, we use notations $Q_n(\tilde{f}|_{L_A})$ (on A) and $\mathcal{E}_A(f)$ for $f\in\mathcal{F}_A$ in the obvious sense.

Lemma 3.4. (1) *There exists some $c>0$ such that $\|f\|_{L^2(K_A)}^2\le c\mathcal{E}_A(f)$ for all $f\in\mathcal{G}_w$.*

(2) *For each $g\in\mathcal{F}_A$, there exists $h_g\in\mathcal{K}_w$ such that $Q_n(\tilde{h}_g|_{L_A})=Q_n(\tilde{g}|_{L_A})$ on $\mathcal{N}_M(w)\cap I^n$ and $\mathcal{E}_A(h_g)\le\mathcal{E}_A(g)$.*

Proof. (1) Suppose that the claim does not hold. Then, there exists a sequence $\{f_k\}_{k\in\mathbb{N}}\subset\mathcal{G}_w$ such that $\|f_k\|_{L^2(K_A)}=1$ and $\lim_{k\to\infty}\mathcal{E}_A(f_k)=0$. We may assume that f_k converges weakly to some f in $\mathcal{F}_A$ and $F_w^* f_k$ converges to $F_w^* f$ weakly in $\mathcal{F}$ for every $w\in A$. By (B1), $F_w^* f_k$ converges to $F_w^* f$ in $L^2(K)$ for each $w\in A$. Thus, f_k converges to f in $L^2(K_A)$. Further, we have $\mathcal{E}_A(f)\le\liminf_{k\to\infty}\mathcal{E}_A(f_k)=0$. Therefore, $\mathcal{E}_A(f)=0$. Based on (B2)(2), f is constant on K_A. Since f belongs to $\mathcal{G}_w$ by (A8)', we conclude that $f=0$ on K_A, which contradicts the fact that $\|f\|_{L^2(K_A)}=\lim_{k\to\infty}\|f_k\|_{L^2(K_A)}=1$.

(2) Let $\mathcal{F}_g=\{f\in\mathcal{F}_A : Q_n(\tilde{f}|_{L_A})=Q_n(\tilde{g}|_{L_A}) \text{ on } \mathcal{N}_M(w)\cap I^n\}$. We consider a sequence $\{h_k\}_{k\in\mathbb{N}}\subset\mathcal{F}_g$ such that $\mathcal{E}_A(h_k)$ converges to the infimum of $\{\mathcal{E}_A(f) : f\in\mathcal{F}_g\}$. Since

$$
\|h_k\|_{L^2(K_A)}\le\|h_k-g\|_{L^2(K_A)}+\|g\|_{L^2(K_A)}\le c^{1/2}\mathcal{E}_A(h_k-g)^{1/2}+\|g\|_{L^2(K_A)}, \tag{3.3}
$$

we have $\sup_k\|h_k\|_{L^2(K_A)}<\infty$. There exists a weak limit $h\in\mathcal{F}_A$ of a subsequence of $\{h_k\}_{k\in\mathbb{N}}$ in $\mathcal{F}_A$. Then, $h\in\mathcal{F}_g$ and h attains the infimum of $\{\mathcal{E}_A(f) : f\in\mathcal{F}_g\}$. Dividing both sides of the inequality $\mathcal{E}_A(h+\epsilon f)-\mathcal{E}_A(h)\ge 0$ by ϵ for $f\in\mathcal{G}_w$ and letting $\epsilon\to 0$, we obtain $h\in\mathcal{K}_w$. □

Lemma 3.5. *There exists some $c_1 > 0$ such that*

$$c_1\rho^{-n}\mathcal{E}_{\Phi(\hat{I}^n)}(f) \geq E_{(n)}(Q_n(\tilde{f}|_L)) \quad \textit{for all } f \in \mathcal{F} \textit{ and } n \in \mathbb{Z}_+. \tag{3.4}$$

Proof. First, we prove that $\mathcal{K}_w$ is a finite-dimensional vector space. For each $i \in \mathcal{N}_M(w) \cap I^n$, take a function $g_i \in \mathcal{F}$ such that $Q_n(\tilde{g}_i|_L)(j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$ for all $j \in \mathcal{N}_M(w) \cap I^n$. The existence of such functions is established by the regularity of the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Further, we define a linear map $\Theta\colon \mathcal{K}_w \to \mathbb{R}^{\mathcal{N}_M(w)\cap I^n}$ by $\Theta(f) = \{\mathcal{E}_A(f, g_i)\}_{i\in\mathcal{N}_M(w)\cap I^n}$. Suppose f belongs to the kernel of Θ. Then, $\mathcal{E}_A(f, g) = 0$ for every $g \in \mathcal{F}_A$, which implies that f is constant on K_A by (B2) (2). Therefore, $\mathcal{K}_w$ is finite dimensional.

Since $\mathcal{E}_A(h) = 0$ implies $\sum_{v\in A}(Q_n(\tilde{h}|_{L_A})(v) - Q_n(\tilde{h}|_{L_A})(w))^2 = 0$ for $h \in \mathcal{K}_w$, there exists $c_2 > 0$ such that $\sum_{v\in A}(Q_n(\tilde{h}|_{L_A})(v) - Q_n(\tilde{h}|_{L_A})(w))^2 \leq c_2\rho^{-n}\mathcal{E}_A(h)$ for every $h \in \mathcal{K}_w$. By (B2) (3) and Lemma 3.3, we can independently take c_2 with respect to $w \in \bigcup_{n\in\mathbb{Z}_+} I^n$. Therefore, for any $f \in \mathcal{F}$ and $n \in \mathbb{Z}_+$, by considering $h_f \in \mathcal{K}_w$ as in Lemma 3.4 (2),

$$\begin{aligned}
\sum_{v\in\mathcal{N}_M(w)\cap I^n} (Q_n(\tilde{f}|_L)(v) - Q_n(\tilde{f}|_L)(w))^2 &= \sum_{v\in\mathcal{N}_M(w)\cap I^n} (Q_n(\tilde{h}_f|_{L_A})(v) - Q_n(\tilde{h}_f|_{L_A})(w))^2 \\
&\leq c_2\rho^{-n}\mathcal{E}_A(h_f) \\
&\leq c_2\rho^{-n}\mathcal{E}_A(f).
\end{aligned}$$

This implies that

$$\begin{aligned}
E_{(n)}(Q_n(\tilde{f}|_L)) &= \sum_{w\in I^n}\sum_{v\in\mathcal{N}_M(w)\cap I^n} (Q_n(\tilde{f}|_L)(v) - Q_n(\tilde{f}|_L)(w))^2 \\
&\leq c_2\rho^{-n}\sum_{w\in I^n}\mathcal{E}_{\mathcal{N}_M(w)\cap\hat{I}^n}(f) \\
&\leq c_2C_1\rho^{-n}\mathcal{E}_{\hat{I}^n}(f) \leq c_2C_1C_2\rho^{-n}\mathcal{E}_{\Phi(\hat{I}^n)}(f),
\end{aligned}$$

where C_1 and C_2 are provided in (B2) (4). □

Recall that the finite sets Ξ and $D^\sharp(v)$ for $v \in \Xi$ were introduced in (B3).

Lemma 3.6. *For each $v \in \Xi$, $F_v^*\colon \mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}} \to \mathcal{F}$ is a compact operator. Here, $\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}}$ is regarded as a subspace of $\mathcal{F}_{K_{D^\sharp(v)}}$.*

Proof. We define $I(v) = \{w \in I^{|v|} : L_w \not\subset K_{S^{|v|}\setminus D^\sharp(v)}\}$. Note that $\mathcal{H}(I^{|v|}, D^\sharp(v)) = \mathcal{H}(I(v), D^\sharp(v))$. For each $i \in I(v)$, we consider a function g_i in $\mathcal{F}(D^\sharp(v))$ such that $Q_{|v|}(\tilde{g}_i|_L)(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$ for all $j \in I(v)$. Further, we define a linear map $\Theta\colon \mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}} \to \mathbb{R}^{I(v)}$ by $\Theta(f) = \{\mathcal{E}(f, g_i)\}_{i\in I(v)}$. Then, the kernel of Θ is equal to $\mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}}$. The homomorphism theorem implies that $\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}} \big/ \mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}}$ is isomorphic to $\Theta(\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}})$ as a vector space. Therefore, there exists a finite-dimensional vector space Z of $\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}}$ such that $\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}}$ is a direct sum of $\mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}}$ and Z. Condition (B3) (2) concludes this assertion. □

Lemma 3.7. *Let $m \in \mathbb{N}$; A, a proper subset of S^m; and J, a subset of I^m. For $g \in \mathcal{F}$, there exists a unique function g' in $\mathcal{H}(J, A)$ such that $g' = g$ on $K_{S^m \setminus A}$ and $Q_m(\widetilde{g'}|_L) = Q_m(\tilde{g}|_L)$ on J. Moreover, there exists $c > 0$ such that*

$$\|g'\|_{\mathcal{F}_A} \le c\|g\|_{\mathcal{F}_A}, \quad \mathcal{E}(g') \le \mathcal{E}(g) \tag{3.5}$$

for all $g \in \mathcal{F}$. Further, if $g \ge 0$ μ-a.e., then $g' \ge 0$ μ-a.e.

Proof. First, we prove that there exists some $c' > 0$ such that $\|f\|^2_{L^2(K_A)} \le c'\mathcal{E}_A(f)$ for every $f \in \mathcal{F}(A)$. Suppose this does not hold. Then, there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}(A)$ such that $\|f_n\|_{L^2(K_A)} = 1$ for every n and $\mathcal{E}_A(f_n)$ converges to 0 as $n \to \infty$. We may assume that f_n converges weakly to some f in $\mathcal{F}$. Then, f_n converges to $f \in \mathcal{F}$ in $L^2(K)$ by (B1), and $\mathcal{E}(f) \le \liminf_{n\to\infty} \mathcal{E}(f_n) = 0$. Therefore, $\mathcal{E}(f) = 0$ and f is constant on K. Since $f \in \mathcal{F}(A)$ and $A \ne S^m$, f is identically 0, which is contradictory to the fact that $\|f\|_{L^2(K)} = 1$.

Now, given that $g \in \mathcal{F}$, let $\mathcal{F}_g = \{f \in \mathcal{F} : f = g \text{ on } K_{S^m \setminus A} \text{ and } Q_m(\tilde{f}|_L) = Q_m(\tilde{g}|_L) \text{ on } J\}$. Then, in a similar manner to the proof of Lemma 3.4 (2), there exists $h \in \mathcal{F}_g$ attaining the infimum of $\{\mathcal{E}(f) : f \in \mathcal{F}_g\}$ and $h \in \mathcal{H}(J, A)$. Such functions exist uniquely; indeed, if both h and h' attain the abovementioned infimum, we have

$$\mathcal{E}\left(\frac{h - h'}{2}\right) = \frac{1}{2}(\mathcal{E}(h) + \mathcal{E}(h')) - \mathcal{E}\left(\frac{h + h'}{2}\right) \le 0,$$

which implies that $h - h'$ is a constant. Since $h - h' = 0$ on $K_{S^m \setminus A}$, we conclude that $h = h'$. On the other hand, it can be observed that g' should attain the abovementioned infimum. Therefore, g' is uniquely determined. By an inequality similar to (3.3), we conclude (3.5). The last assertion follows from the characterization of g' and the Markov property of the Dirichlet form. □

The following is the primary proposition. Condition (B4) will be used (only) here.

Proposition 3.8. *There exist $0 < c_0 < 1$ and $b_0 \in \mathbb{N}$ such that the following holds for all $n \in \mathbb{Z}_+$ and $h \in \mathcal{H}(I^n, \Phi(\hat{I}^n))$:*

$$\mathcal{E}_{\Phi(\hat{I}^{n+b_0})}(h) \le c_0 \mathcal{E}_{\Phi(\hat{I}^n)}(h).$$

Moreover, for all $i \ge j \ge 1$, $b = 0, 1, \ldots, b_0 - 1$, and $h \in \mathcal{H}(I^{b_0 j}, \Phi(\hat{I}^{b_0 j}))$,

$$\mathcal{E}_{\Phi(\hat{I}^{b_0 i + b})}(h) \le c_0^{i-j} \mathcal{E}_{\Phi(\hat{I}^{b_0 j + b})}(h).$$

Proof. It is sufficient to prove the first claim. Recall l_0 and m_0 in condition (B3). By (B3), $C := \sup_{n \in \mathbb{Z}_+} \max_{w \in \hat{I}^{n+m_0}} \#D(w)$ is finite. Let $n \in \mathbb{Z}_+$ and $w \in \hat{I}^{n+m_0}$. We define

$$\begin{aligned} \mathcal{C}_w &= \{F_w^* f : f \in \mathcal{H}(I^{n+m_0}, D(w)),\ \int_{K_{D(w)}} f\,d\mu = 0,\ \rho^{-(n+m_0)}\mathcal{E}_{D(w)}(f) \le 1\}, \\ \mathcal{C} &= \text{the closure of } \bigcup_{w \in \bigcup_{n \in \mathbb{Z}_+} \hat{I}^{n+m_0}} \mathcal{C}_w \text{ in } \mathcal{F}. \end{aligned}$$

Then, $\mathcal{C}$ is a compact subset in $\mathcal{F}$ by Lemma 3.6 and (B3). Let $\delta = 1/(4C^2)$; we define $\mathcal{C}(\delta) = \{f \in \mathcal{C} : \mathcal{E}(f) \geq \delta\}$. Since (B4) holds, for each $f \in \mathcal{C}(\delta)$, there exist $m(f) \in \mathbb{N}$ and $a(f) \in (0,1)$ such that $\mathcal{E}_{\Phi(\hat{I}^m)}(f) < a(f)\mathcal{E}(f)$ for all $m \geq m(f)$. By continuity, $\mathcal{E}_{\Phi(\hat{I}^m)}(g) < a(f)\mathcal{E}(g)$ for all $m \geq m(f)$ for any g in some neighborhood of f in $\mathcal{F}$. Since $\mathcal{C}(\delta)$ is compact in $\mathcal{F}$, there exist $m_1 \in \mathbb{N}$ and $a_1 \in (0,1)$ such that $\mathcal{E}_{\Phi(\hat{I}^{m_1})}(f) < a_1\mathcal{E}(f)$ for every $f \in \mathcal{C}(\delta)$. In particular,

$$\mathcal{E}(f) \leq a_2\mathcal{E}_{S^{m_1}\setminus\Phi(\hat{I}^{m_1})}(f), \quad f \in \mathcal{C}(\delta) \tag{3.6}$$

with $a_2 = (1-a_1)^{-1} > 1$.

Now, consider h as in the claim of the proposition. We construct an oriented graph such that the set of vertices is $\Phi(\hat{I}^{n+m_0})$ and a set of oriented edges is $E = \{(v,w) \in \Phi(\hat{I}^{n+m_0})\times\Phi(\hat{I}^{n+m_0}) : v \in D'(w),\ \mathcal{E}_w(h) > 0 \text{ and } \mathcal{E}_w(h) \geq 2C\mathcal{E}_v(h)\}$. This graph does not allow any loops. Let Y be the set of all elements w in $\Phi(\hat{I}^{n+m_0})$ such that $\mathcal{E}_w(h) > 0$ and w is not a source of any edges. For $w \in Y$, we define $N_0(w) = \{w\}$, $N_k(w) = \{v \in \Phi(\hat{I}^{n+m_0}) \setminus \bigcup_{l=0}^{k-1} N_l(w) : (v,u) \in E \text{ for some } u \in N_{k-1}(w)\}$ for $k \in \mathbb{N}$ inductively, and $N(w) = \bigcup_{k\geq 0} N_k(w)$. It is evident that $\#N_k(w) \leq C^k$ and $\mathcal{E}_v(h) \leq (2C)^{-k}\mathcal{E}_w(h)$ for all $k \geq 0$ and $v \in N_k(w)$. Then, for each $w \in Y$,

$$\mathcal{E}_{N(w)}(h) = \sum_{k=0}^{\infty}\sum_{v\in N_k(w)}\mathcal{E}_v(h) \leq \sum_{k=0}^{\infty} C^k(2C)^{-k}\mathcal{E}_w(h) = 2\mathcal{E}_w(h). \tag{3.7}$$

Suppose $w \in Y$ and $\mathcal{E}_w(h) \geq \delta\mathcal{E}_{D'(w)}(h)$. Then, since

$$F_w^*\left(\left(h - \fint_{K_{D'(w)}} h\,d\mu\right) \times \rho^{(n+m_0)/2}\mathcal{E}_{D'(w)}(h)^{-1/2}\right) \in \mathcal{C}(\delta),$$

(3.6) implies that $\mathcal{E}(F_w^*h) \leq a_2\mathcal{E}_{S^{m_1}\setminus\Phi(\hat{I}^{m_1})}(F_w^*h)$, namely,

$$\mathcal{E}_w(h) \leq a_2\mathcal{E}_{w\cdot(S^{m_1}\setminus\Phi(\hat{I}^{m_1}))}(h).$$

Next, suppose $w \in Y$ and $\mathcal{E}_w(h) < \delta\mathcal{E}_{D'(w)}(h)$. Since w is not a source of any edges, $\mathcal{E}_v(h) < 2C\mathcal{E}_w(h)$ for every $v \in D'(w)\cap\Phi(\hat{I}^{n+m_0})$. Then,

$$\mathcal{E}_{D'(w)\cap\Phi(\hat{I}^{n+m_0})}(h) < C\cdot 2C\mathcal{E}_w(h) < 2C^2\delta\mathcal{E}_{D'(w)}(h) = \frac{1}{2}\mathcal{E}_{D'(w)}(h),$$

which implies that $\mathcal{E}_{D'(w)\cap\Phi(\hat{I}^{n+m_0})}(h) < \mathcal{E}_{D'(w)\cap((\Phi(\hat{I}^n)\cdot S^{m_0})\setminus\Phi(\hat{I}^{n+m_0}))}(h)$ by (B3) (1) (a). In particular,

$$\mathcal{E}_w(h) < \mathcal{E}_{D'(w)\cap((\Phi(\hat{I}^n)\cdot S^{m_0})\setminus\Phi(\hat{I}^{n+m_0}))}(h).$$

Therefore, in any case, for $w \in Y$, we have

$$\begin{aligned}\mathcal{E}_w(h) &\leq a_2\mathcal{E}_{w\cdot(S^{m_1}\setminus\Phi(\hat{I}^{m_1}))\cup((D'(w)\cap((\Phi(\hat{I}^n)\cdot S^{m_0})\setminus\Phi(\hat{I}^{n+m_0})))\cdot S^{m_1})}(h)\\ &\leq a_2\mathcal{E}_{(D'(w)\cdot S^{m_1})\cap((\Phi(\hat{I}^n)\cdot S^{b_0})\setminus\Phi(\hat{I}^{n+b_0}))}(h),\end{aligned} \tag{3.8}$$

where $b_0 = m_0 + m_1$; further, it should be noted that $\Phi(\hat{I}^{n+b_0}) \subset \Phi(\hat{I}^{n+m_0})\cdot S^{m_1}$. Then, we have

$$\mathcal{E}_{\Phi(\hat{I}^{n+b_0})}(h) \leq \mathcal{E}_{\Phi(\hat{I}^{n+m_0})}(h)$$

$$
\begin{aligned}
&\leq \sum_{w\in Y}\mathcal{E}_{N(w)}(h)\\
&\leq 2a_2\sum_{w\in Y}\mathcal{E}_{(D'(w)\cdot S^{m_1})\cap((\Phi(\hat{I}^n)\cdot S^{b_0})\setminus\Phi(\hat{I}^{n+b_0}))}(h)\\
&\leq 2a_2C_3\mathcal{E}_{(\Phi(\hat{I}^n)\cdot S^{b_0})\setminus\Phi(\hat{I}^{n+b_0})}(h).
\end{aligned}
$$

Here, we used (3.7) and (3.8) in the third inequality and $C_3 := \sup_{n\in\mathbb{Z}_+}\max_{v\in S^{n+m_0}}\#(\mathcal{N}_{l_0}(v)\cap S^{n+m_0})$ is finite by (A1). Hence, the claim of the proposition holds when $c_0 = 2a_2C_3/(1+2a_2C_3)$. □

Proof of Theorem 2.5. We set $b\in\{0,1,\dots,b_0-1\}$. For each $f\in\mathcal{F}$, we can take $g_m\in\mathcal{H}(I^{b_0m+b},\Phi(\hat{I}^{b_0m+b}))$ such that $g_m=f$ on $K_{S^{b_0m+b}\setminus\Phi(\hat{I}^{b_0m+b})}$ and $Q_{b_0m+b}(\tilde{g}_m|_L)=Q_{b_0m+b}(\tilde{f}|_L)$ by Lemma 3.7. By using the relations $\|g_m\|_{\mathcal{F}}\leq c\|f\|_{\mathcal{F}}$, $\mathcal{E}(g_m)\leq\mathcal{E}(f)$ (by Lemma 3.7), and $g_m\to f$ μ-a.e., we will prove $g_m\to f$ in $\mathcal{F}$ as $m\to\infty$. Here, note that the constant c is taken independently of m, which derives from the fact that c depends only on c' in the proof of Lemma 3.7. We first obtain that g_m converges weakly to f in $\mathcal{F}$ and $\limsup_{m\to\infty}\mathcal{E}(g_m-f)=\limsup_{m\to\infty}\mathcal{E}(g_m)-\mathcal{E}(f)\leq 0$. Therefore, $\mathcal{E}(g_m-f)\to 0$ as $m\to\infty$. By (B1), $g_m-f-\int_K(g_m-f)\,d\mu$ converges to 0 in $L^2(K)$. Since $\|g_m-f\|_{L^2(K)}\leq c\|f\|_{\mathcal{F}}+\|f\|_{L^2(K)}$, we have $\int_K(g_m-f)\,d\mu\to 0$ as $m\to\infty$, which implies that $\|g_m-f\|_{L^2(K)}\to 0$ as $m\to\infty$. Thus, $g_m\to f$ in $\mathcal{F}$ as $m\to\infty$.

Let $f_m=g_m-g_{m-1}$, where we set $g_{-1}\equiv 0$. Then, $f=\sum_{m=0}^\infty f_m$. Since $f_i\in\mathcal{F}(I^{b_0j+b},\Phi(\hat{I}^{b_0j+b}))$ for $i>j$ and $f_j\in\mathcal{H}(I^{b_0j+b},\Phi(\hat{I}^{B_0j+b}))$, we have $\mathcal{E}(f_i,f_j)=0$ for $i\neq j$; therefore,

$$
\mathcal{E}(f)=\sum_{m=0}^\infty\mathcal{E}(f_m). \tag{3.9}
$$

Now, for each $f\in\mathcal{F}$,

$$
\begin{aligned}
(E_{(b_0i+b)}(Q_{b_0i+b}(\tilde{f}|_L)))^{1/2} &= (E_{(b_0i+b)}(Q_{b_0i+b}(\tilde{g}_i|_L)))^{1/2}=\left(E_{(b_0i+b)}\left(\sum_{j=0}^i Q_{b_0i+b}(\tilde{f}_j|_L)\right)\right)^{1/2}\\
&\leq \sum_{j=0}^i(E_{(b_0i+b)}(Q_{b_0i+b}(\tilde{f}_j|_L)))^{1/2}\leq\sum_{j=0}^i(c_1\rho^{-b_0i-b}\mathcal{E}_{\Phi(\hat{I}^{b_0i+b})}(f_j))^{1/2}\\
&\leq \sum_{j=0}^i(c_1\rho^{-b_0i-b}c_0^{i-j}\mathcal{E}_{\Phi(\hat{I}^{b_0j+b})}(f_j))^{1/2}\\
&\leq \sum_{j=0}^i(c_1\rho^{-b_0i-b}c_0^{i-j}\mathcal{E}(f_j))^{1/2},
\end{aligned} \tag{3.10}
$$

where we apply Minkowski's inequality to the first inequality, (3.4) to the second inequality, and Proposition 3.8 to the third inequality.

By applying (3.10) and noting that $\alpha^{d_w-d_f}=\rho$, we have

$$
\sum_{i=0}^\infty\alpha^{(d_w-d_f)(b_0i+b)}E_{(b_0i+b)}(Q_{b_0i+b}(\tilde{f}|_L)) \leq \sum_{i=0}^\infty\rho^{b_0i+b}\left(\sum_{j=0}^i(c_1\rho^{-b_0i-b}c_0^{i-j}\mathcal{E}(f_j))^{1/2}\right)^2
$$

$$
\begin{aligned}
&= c_1 \sum_{i=0}^{\infty} c_0^i \left(\sum_{j=0}^{i} (c_0^{-j} \mathcal{E}(f_j))^{1/2} \right)^2 \\
&\le c_2 \sum_{j=0}^{\infty} c_0^j c_0^{-j} \mathcal{E}(f_j) = c_2 \sum_{j=0}^{\infty} \mathcal{E}(f_j) = c_2 \mathcal{E}(f).
\end{aligned}
$$

Here, we use (3.2) in the second inequality and (3.9) in the last equality. Thus, we have

$$
\sum_{n=0}^{\infty} \alpha^{(d_w - d_f)n} E_{(n)}(Q_n(\tilde{f}|_L)) \le b_0 c_2 \mathcal{E}(f).
$$

By combining this with Lemma 3.1 and (A8)', we have $\|\tilde{f}|_L\|_{\Lambda^{\beta}_{2,2}(L)} \le c_3 \|f\|_{\mathcal{F}}$; therefore, $\mathcal{F}|_L \subset \Lambda^{\beta}_{2,2}(L)$ and $(\mathcal{F} \cap C(K))|_L \subset \Lambda^{\beta}_{2,2}(L) \cap C(L)$. The claim of Theorem 2.5 follows from a simple limiting procedure by the fact that $\mathcal{F} \cap C(K)$ is dense in $\mathcal{F}$ due to the regularity of $(\mathcal{E}, \mathcal{F})$. □

Remark 3.9. Even if (B4) does not hold, the relation $\mathcal{F}|_L \subset \hat{\Lambda}^{\beta}_{2,\infty}(L)$ holds. Indeed, for each $f \in \mathcal{F}$ and $n \in \mathbb{Z}_+$, by Lemma 3.5, we have

$$
c_1 \mathcal{E}(f) \ge c_1 \mathcal{E}_{\Phi(\hat{I}_n)}(f) \ge \rho^n E_{(n)}(Q_n(\tilde{f}|_L)) = \alpha^{(2\beta - d_I)n} E_{(n)}(Q_n(\tilde{f}|_L)).
$$

Hence,

$$
\left\| \left\{ \alpha^{n\beta} \left(\alpha^{-n d_I} E_{(n)}(Q_n(\tilde{f}|_L)) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^\infty} \le c_1^{1/2} \mathcal{E}(f)^{1/2};
$$

therefore, the same argument as the abovementioned one yields the result.

3.3 Proof of the extension theorem

In this section, we assume (A1)–(A8) and (C1)–(C2), and prove Theorem 2.6. Conditions (C1) and (C2) are defined as follows:

In order to construct an extension map ξ, we first define a Whitney-type decomposition and an associated partition of unity. Let $\Omega^{(n)} = \bigcup_{m=0}^{n} I^m$ for $n \in \mathbb{Z}_+$. For $w \in I^0 = \{\emptyset\}$, we set $A_w = W \setminus \mathcal{N}_2(I)$ and $B_w = W \setminus \mathcal{N}_1(I)$. For $w \in I^n$ with $n \in \mathbb{N}$, we set $A_w = (\mathcal{N}_2(w) \cdot W) \setminus \mathcal{N}_2(I^{n+1})$, $\hat{A}_w = \mathcal{N}_2(w) \cdot W$, $B_w = (\mathcal{N}_3(w) \cdot W) \setminus \mathcal{N}_1(I^{n+1})$, and $\hat{B}_w = \mathcal{N}_3(w) \cdot W$. Evidently, $K_{A_w} \subset K_{B_w}$, $K_{\hat{A}_w} \subset K_{\hat{B}_w}$, $K_{A_w} \cap K_{W^{|w|+1} \setminus B_w} = \emptyset$, and $K_{\hat{A}_w} \cap K_{W^{|w|+1} \setminus \hat{B}_w} = \emptyset$.

By (A3), the following holds for $w, w' \in \bigcup_{n \in \mathbb{Z}_+} I^n$:

$$
c_1 \alpha^{-|w|} \le \mathsf{d}(L, K_{B_w}) \le c_2 \alpha^{-|w|} \text{ if } B_w \ne \emptyset, \tag{3.11}
$$

$$
\text{there exists } l > 0 \text{ such that if } |w'| \ge |w| + l, \text{ then } K_{B_w} \cap K_{\hat{B}_{w'}} = \emptyset. \tag{3.12}
$$

For $n \in \mathbb{N}$ and $w \in \Omega^{(n)}$, we set

$$
A_w^{(n)} = \begin{cases} A_w & \text{if } |w| < n \\ \hat{A}_w & \text{if } |w| = n \end{cases}, \quad B_w^{(n)} = \begin{cases} B_w & \text{if } |w| < n \\ \hat{B}_w & \text{if } |w| = n \end{cases},
$$

and $R_w^{(n)} = \{w' \in \Omega^{(n)} : K_{B_w^{(n)}} \cap K_{B_{w'}^{(n)}} \ne \emptyset\}$. We assume the following:

(C1) There exists a finite subset Γ of $\bigcup_{n\in\mathbb{N}}(\{n\}\times\Omega^{(n)})$ such that for any $n\in\mathbb{N}$ and $w\in\Omega^{(n)}$, there exist $(m,v)\in\Gamma$, a bijection $\iota\colon R_w^{(n)}\to R_v^{(m)}$, and a homeomorphism $F\colon K_{\bigcup_{u\in R_w^{(n)}}B_u^{(n)}}\to K_{\bigcup_{u\in R_v^{(m)}}B_u^{(m)}}$ such that for every $u\in R_w^{(n)}$, $A_u^{(n)}$ and $A_{\iota(u)}^{(m)}$, and $B_u^{(n)}$ and $B_{\iota(u)}^{(m)}$ are of the same type for the homeomorphism F.

For each $(m,v)\in\Gamma$, we take a function $\bar{\varphi}_v^{(m)}\in\mathcal{F}\cap C(K)$ such that $0\le\bar{\varphi}_v^{(m)}\le 1$, $\bar{\varphi}_v^{(m)}(x)=1$ on $K_{A_v^{(m)}}$, and $\bar{\varphi}_v^{(m)}(x)=0$ on $K_{W^{|v|+1}\setminus B_v^{(m)}}$. Such a function exists since $(\mathcal{E},\mathcal{F})$ is regular. For $n\in\mathbb{N}$ and $w\in\Omega^{(n)}$, we define $\varphi_w^{(n)}(x)=\begin{cases}\bar{\varphi}_v^{(m)}(F(x)) & \text{if } x\in B_w^{(n)}\\ 0 & \text{otherwise}\end{cases}$, where m, v, and F are given in (C1). We assume

(C2) $\varphi_w^{(n)}\in\mathcal{F}\cap C(K)$ for every $n\in\mathbb{N}$ and $w\in\Omega^{(n)}$.

In the case of SC(2), it is sufficient to take that $\Gamma=\bigcup_{n=1}^{3}(\{n\}\times\Omega^{(n)})$ to ensure (C1), and (C2) clearly holds.

For $n\in\mathbb{N}$ and $w\in\Omega^{(n)}$, we define

$$\psi_w^{(n)}(x)=\frac{\varphi_w^{(n)}(x)}{\sum_{w'\in\Omega^{(n)}}\varphi_{w'}^{(n)}(x)},\quad x\in K.$$

This is well-defined since the sum in the denominator is not less than 1. $\psi_w^{(n)}$ is continuous and takes values between 0 and 1. Since $\varphi_w^{(n)}\in\mathcal{F}$ and vanishes outside $K_{B_w^{(n)}}$, so does $\psi_w^{(n)}$. For each $f\in\Lambda_{2,2}^{\beta}(L)\cap C(L)$, we define

$$\xi^{(n)}f(x)=\sum_{w\in\Omega^{(n)}}\psi_w^{(n)}(x)Q_{|w|}f(w)=\sum_{w\in\Omega^{(n)}}\psi_w^{(n)}(x)⨍_{L_w}f(s)d\nu(s).$$

$\xi^{(n)}$ is a linear map from $\Lambda_{2,2}^{\beta}(L)\cap C(L)$ to $\mathcal{F}\cap C(K)$. For $x\in K\setminus L$, $\xi^{(n)}f(x)$ is independent of n if n is sufficiently large because of (3.12). Therefore, for $f\in\Lambda_{2,2}^{\beta}(L)\cap C(L)$,

$$\xi f(x):=\begin{cases}\lim_{n\to\infty}\xi^{(n)}f(x), & x\in K\setminus L\\ f(x), & x\in L\end{cases}\tag{3.13}$$

is well-defined and $\xi^{(n)}f$ converges to ξf μ-a.e.

Proof of Theorem 2.6. First, we prove that ξf is continuous on K. Since ξf is continuous on $K\setminus L$ by the construction, it is enough to show that for each $x_0\in L$,

$$\lim_{\substack{x\to x_0\\ x\in K\setminus L}}\xi f(x)=f(x_0).\tag{3.14}$$

Since f is uniformly continuous on L, if we set $\omega_a(f)=\sup\{|f(s)-f(t)|: s,t\in L,\ \mathsf{d}(s,t)\le a\}$ for $a>0$, then $\lim_{a\to 0}\omega_a(f)=0$. Let $x_0\in L$, $x\in K\setminus L$, and $\delta=\mathsf{d}(x,x_0)$. Suppose that $w\in\bigcup_{n\in\mathbb{N}}I^n$ satisfies $x\in K_{B_w}$. Then, $c_1\alpha^{-|w|}\le\mathsf{d}(L,K_{B_w})\le\mathsf{d}(x_0,x)=\delta$ by (3.11). Next, we consider $y \in L_w$ and select $z \in K_{B_w}$ that satisfies $\mathsf{d}(y,z) = \mathsf{d}(y, K_{B_w}) \le c_2\alpha^{-|w|}$. Then, since $\operatorname{diam}(K_{B_w}) \asymp \alpha^{-|w|}$, we have

$$\mathsf{d}(y,x_0) \le \mathsf{d}(y,z) + \mathsf{d}(z,x) + \mathsf{d}(x,x_0) \le c_2\alpha^{-|w|} + c_3\alpha^{-|w|} + \delta \le c_4\delta.$$

Therefore, $⨍_{L_w} |f(y) - f(x_0)|\, d\nu(y) \le \omega_{c_4\delta}(f)$. Now, n is taken to be sufficiently large such that $x \notin \bigcup_{w\in I^n} K_{\hat{B}_w}$. Then, $\xi^{(n)} f(x) = \xi f(x)$ and

$$\begin{aligned} |\xi f(x) - f(x_0)| &= \left| \sum_{w\in\Omega^{(n)}} \psi_w^{(n)}(x) ⨍_{L_w} (f(y) - f(x_0))\, d\nu(y) \right| \\ &\le \sum_{w\in\Omega^{(n-1)},\ x\in K_{B_w}} \psi_w^{(n)}(x) ⨍_{L_w} |f(y) - f(x_0)|\, d\nu(y) \\ &\le \omega_{c_4\delta}(f). \end{aligned}$$

Thus, (3.14) is proved.

Next, we prove that $\{\xi^{(n)} f\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{F}$. It should be noted that $\int_K \psi_w^{(n)}(x)\, d\mu(x) \le c_5\alpha^{-d_f|w|}$ for all $n \in \mathbb{N}$ and $w \in \Omega^{(n)}$ for some $c_5 > 0$; therefore, we have

$$\begin{aligned} \|\xi^{(n)} f\|^2_{L^2(K,\mu)} &= \int_K \left(\sum_{w\in\Omega^{(n)}} \psi_w^{(n)}(x) ⨍_{L_w} f(s)\, d\nu(s) \right)^2 d\mu(x) \\ &\le \int_K \left(\sum_{w\in\Omega^{(n)}} \psi_w^{(n)}(x) ⨍_{L_w} f(s)^2\, d\nu(s) \right) d\mu(x) \\ &\le \sum_{w\in\Omega^{(n)}} c_5\alpha^{-d_f|w|}\alpha^{d_I|w|} \int_{L_w} f(s)^2 d\nu(s) \\ &= c_5 \sum_{k=0}^{n} \alpha^{(d_I - d_f)k} \|f\|^2_{L^2(L,\nu)} \\ &\le \frac{c_5}{1-\alpha^{d_I-d_f}} \|f\|^2_{L^2(L,\nu)}. \end{aligned} \tag{3.15}$$

For $n \in \mathbb{N}$, $w \in \Omega^{(n)}$ with $m = |w|$, let $\bar{R}_w^{(n)} = \bigcup_{v\in R_w^{(n)}} v \cdot I^{m+l-|v|} \subset I^{m+l}$, where l is given by (3.12). For $g \in L^2(L,\nu)$, we define

$$E_w^{(n)}(g) = \sum_{\substack{u,v\in\bar{R}_w^{(n)} \\ u \overset{m+l,L}{\longleftrightarrow} v}} (Q_{m+l}g(u) - Q_{m+l}g(v))^2, \quad \bar{E}_w^{(n)}(g) = \mathcal{E}_{\Phi(B_w^{(n)})}\left(\sum_{v\in R_w^{(n)}} Q_{|v|}g(v)\psi_v^{(n)} \right).$$

Both $E_w^{(n)}(g)$ and $\bar{E}_w^{(n)}(g)$ are determined only from the values $\{Q_{m+l}g(u)\}_{u\in\bar{R}_w^{(n)}}$. If $E_w^{(n)}(g) = 0$, then $Q_{m+l}g$ is constant on $\bar{R}_w^{(n)}$, which implies that $\bar{E}_w^{(n)}(g) = 0$. Therefore, there exists $c_w^{(n)} > 0$ such that $\bar{E}_w^{(n)}(g) \le c_w^{(n)} E_w^{(n)}(g)$ for every $g \in \mathcal{F}$. Due to (C1) and Lemma 3.3, there exists some $c_6 > 0$ such that $\bar{E}_w^{(n)}(g) \le c_6 \rho^{|w|} E_w^{(n)}(g)$ for all $n \in \mathbb{N}$, $w \in \Omega^{(n)}$, and $g \in \mathcal{F}$. Further, there exists $c_7 > 0$ independent of m such that $\sum_{w \in I^m} E_w^{(n)}(g) \le c_7 E_{(m+l)}(Q_{m+l} g)$ for all n and $g \in L^2(L, \nu)$. Then, we have

$$\begin{aligned} \mathcal{E}(\xi^{(n)} f) &\le \sum_{w \in \Omega^{(n)}} \mathcal{E}_{\Phi(B_w^{(n)})}(\xi^{(n)} f) = \sum_{w \in \Omega^{(n)}} \bar{E}_w^{(n)}(f) \\ &\le c_6 \sum_{m=0}^{n} \sum_{w \in I^m} \rho^m E_w^{(n)}(f) \le c_6 c_7 \sum_{m=0}^{n} \rho^m E_{(m+l)}(Q_{m+l} f) \\ &\le c_8 \sum_{m=0}^{\infty} \rho^m E_{(m)}(Q_m f). \end{aligned}$$

Since $\alpha^{2\beta - d_I} = \alpha^{d_w - d_f} = \rho$, we obtain $\mathcal{E}(\xi^{(n)} f) \le c_8 \|f\|^2_{\Lambda^\beta_{2,2}(L)}$ by Lemma 3.1.

By combining this with (3.15), $\{\xi^{(n)} f\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{F}$; therefore, we conclude that $\xi f \in \mathcal{F}$ and $\|\xi f\|_{\mathcal{F}} \le c_9 \|f\|_{\Lambda^\beta_{2,2}(L)}$ for some $c_9 > 0$.

Next, we take any $\Lambda^\beta_{2,2}(L)$-Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset \Lambda^\beta_{2,2}(L) \cap C(L)$ and let $f \in \Lambda^\beta_{2,2}(L)$ be the limit point. By the abovementioned result, $\{\xi f_n\}_{n \in \mathbb{N}} \subset \mathcal{F} \cap C(K)$ is a $\mathcal{E}_1$-Cauchy sequence. Let $g \in \mathcal{F}$ be the limit point. Since $\xi f_n|_L = f_n$ and a subsequence ξf_{n_k} converges to $\tilde{g}$ q.e., $\tilde{g}|_L = f$ ν-a.e. Thus, ξ can extend to a continuous map from $\hat{\Lambda}^\beta_{2,2}(L)$ to $\mathcal{F}$ such that $\widetilde{\xi f}|_L = f$ ν-a.e. for $f \in \hat{\Lambda}^\beta_{2,2}(L)$. □

Remark 3.10. Let $\{L_i\}_{i=1}^m$ be a finite number of self-similar subsets of K, where each L_i is constructed by the same number of contraction maps and satisfies (A2), the second identity of (A4), (A7), and (A8) in Section 2. Let $L = \bigcup_{i=1}^m L_i$. With suitable changes for A_w, B_w, etc., we can consider conditions (C1)*–(C2)* that correspond to (C1)–(C2). We define $\Lambda^\beta_{2,2}(L)$ as in Definition 2.4. Then, under such conditions, Theorem 2.6 is still valid, i.e., there is a linear map ξ from $\Lambda^\beta_{2,2}(L)$ to $\mathcal{F}$ such that $\xi\big(\Lambda^\beta_{2,2}(L) \cap C(L)\big) \subset \mathcal{F} \cap C(K)$, $\widetilde{\xi f}|_L = f$, and

$$\|\xi f\|_{\mathcal{F}} \le c_1 \sum_{i=1}^m \|f|_{L_i}\|_{\Lambda^\beta_{2,2}(L_i)}, \quad f \in \Lambda^\beta_{2,2}(L).$$

4 Complementary results

In this section, we provide sufficient conditions with regard to (A8) and (B3) and discuss a suitable selection of $\mathcal{F}_A$ for $A \subset W^m$. First, we define fractional diffusions in the sense of [2] Definition 3.5.

Definition 4.1. Let (X, d) be a complete metric space, where d exhibits the midpoint property; for each $x, y \in X$, there exists $z \in X$ such that $\mathsf{d}(x,y) = \mathsf{d}(x,z)/2 = \mathsf{d}(z,y)/2$. For simplicity, we assume $\operatorname{diam} X = 1$. Let μ be a Borel measure on X such that there exists $d_f > 0$ with $\mu(B(x,r)) \asymp r^{d_f}$ for all $0 < r \le 1$. A Markov process $\{Y_t\}_{t \ge 0}$ is a fractional diffusion on X if 1) Y is a μ-symmetric conservative Feller diffusion and

2) Y has a symmetric jointly continuous transition density $p_t(x,y)$ $(t>0, x,y\in X)$, which satisfies the Chapman-Kolmogorov equations and has the following estimate:

$$\begin{aligned} & c_1 t^{-d_f/d_w}\exp(-c_2(\mathsf{d}(x,y)^{d_w}t^{-1})^{1/(d_w-1)}) \le p_t(x,y) \\ \le\ & c_3 t^{-d_f/d_w}\exp(-c_4(\mathsf{d}(x,y)^{d_w}t^{-1})^{1/(d_w-1)}) \qquad \text{for all } 0<t<1,\ x,y\in X, \end{aligned}$$

with some constant $d_w \ge 2$.

Proposition 4.2. (A8) *holds for the following three cases:*
1) *There exists* $c>0$ *such that* $\|f\|_{L^\infty(K)} \le c\|f\|_{\mathcal{F}}$ *for all* $f\in\mathcal{F}$.
2) *The diffusion process corresponding to* $(\mathcal{E},\mathcal{F})$ *is the fractional diffusion and* (A7) *holds.*
3) $K\subset\mathbb{R}^n$, (A7) *holds, and* $\mathcal{F}=\Lambda^{d_w/2}_{2,\infty}(K)$.

Proof. Suppose that 1) holds. Then, for any nonempty set D of K, $\mathrm{Cap}(D)\ge c^{-2}$. Therefore, $\nu(D)\le 1\le c^2\,\mathrm{Cap}(D)$.

When 2) holds, the proof is similar to Lemma 2.5 of [5]; however, we provide it for completeness. Let $g_1(\cdot,\cdot)$ be the 1-order Green density given by

$$E^x\left[\int_0^\infty e^{-t}f(X_t)dt\right] = \int_K g_1(x,y)f(y)d\mu$$

for any Borel measurable function f, where $\{X_t\}$ is the diffusion corresponding to $(\mathcal{E},\mathcal{F})$. Then, since $\{X_t\}$ is a fractional diffusion, we have

$$g_1(x,y) \asymp \begin{cases} c_1\mathsf{d}(x,y)^{d_w-d_f} & \text{if } d_f>d_w, \\ -c_2\log\mathsf{d}(x,y)+c_3 & \text{if } d_f=d_w, \\ c_4 & \text{if } d_f<d_w. \end{cases} \tag{4.1}$$

See [2] Proposition 3.28 for the proof. If $d_f<d_w$, then the points have a strictly positive capacity, and the result is immediate. We prove the result for $d_f>d_w$: the proof for $d_f=d_w$ is similar. It is well known that for each compact set $M\subset K$,

$$\mathrm{Cap}(M)=\sup\left\{m(M):\ \begin{array}{l} m \text{ is a positive Radon measure, } \operatorname{supp} m\subset M, \\ G_1m(x)\equiv\int_M g_1(x,y)m(dy)\le 1 \text{ for every } x\in K \end{array}\right\}. \tag{4.2}$$

Using the abovementioned estimates of $g_1(\cdot,\cdot)$,

$$\begin{aligned} \int_M g_1(x,y)\nu(dy) &\le \int_K g_1(x,y)\nu(dy) \le \sum_{n=0}^\infty \int_{\alpha^{-n-1}\le\mathsf{d}(x,y)<\alpha^{-n}} g_1(x,y)\nu(dy) \\ &\le c_5\sum_n \alpha^{n(d_f-d_w)}\nu(\alpha^{-n-1}\le\mathsf{d}(x,y)<\alpha^{-n}) \\ &\le c_6\sum_n \alpha^{n(d_f-d_w-d_I)} \equiv c_7<\infty \end{aligned}$$

because of the assumption $d_f-d_I<d_w$. Thus, by setting $\nu_M(\cdot)\equiv\nu(\cdot\cap M)$, we have $G_1\nu_M\le c_7$. Using (4.2), $\mathrm{Cap}(M)\ge\nu(M)/c_7$ for each compact set M.

For 3), we will use the results by Jonsson-Wallin in [16] and Triebel in [29]. We denote the Lipschitz spaces and the Besov spaces in the sense of Jonsson-Wallin by $\mathrm{Lip}_{JW}(\alpha, p, q, K)$ and $B^{p,q}_{\alpha,JW}(K)$ (see page 122–123 in [16] for definition). Note that $\mathrm{Lip}_{JW}(\alpha, p, q, K) \subset B^{p,q}_{\alpha,JW}(K)$ and they are equal when $\alpha \notin \mathbb{N}$ (page 125 in [16]). For each $f \in \Lambda^{d_w/2}_{2,\infty}(K)$, $(f, 0, \ldots, 0) \in \mathrm{Lip}_{JW}(d_w/2, 2, \infty, K)$. Thus, by using the extension theorem in page 155 of [16], we have

$$\Lambda^{d_w/2}_{2,\infty}(K) \subset \mathrm{Lip}_{JW}(d_w/2, 2, \infty, K) \subset B^{2,2}_{d_w/2,JW}(K) \subset \Lambda^{2,\infty}_{\gamma}(\mathbb{R}^n)|_K,$$

where $\gamma = (d_w + n - d_f)/2$ and $\Lambda^{p,q}_{\gamma}(\mathbb{R}^n)$ is a classical Besov space on $\mathbb{R}^n$. Now, since $d_w - d_f > -d_I$ (due to (A7)), $\Lambda^{2,\infty}_{\gamma}(\mathbb{R}^n) \subset \Lambda^{2,1}_{(n-d_I)/2}(\mathbb{R}^n)$. Finally, by Corollary 18.12 (i) in [29], we have $\mathrm{tr}_L\, \Lambda^{2,1}_{(n-d_I)/2}(\mathbb{R}^n) = L^2(L, \nu)$. (Note that this trace in the sense of Triebel is simply a restriction and there is no corresponding extension.) By combining these facts, we have $\mathcal{F}|_L \subset L^2(L, \nu)$, which implies $\|\tilde{f}|_L\|_{L^2(L,\nu)} \le c_9 \|f\|_{\mathcal{F}}$ for all $f \in \mathcal{F}$. Therefore, (A8)' holds. □

Now, we consider one concrete selection of $\mathcal{F}_A$ for $A \subset W^m$ and show that such a selection is suitable for Dirichlet forms whose corresponding processes are fractional diffusions. By (A5), (A6), and the self-similarity of μ, for any $w \in \bigcup_{n\in\mathbb{Z}_+} W^n$, there exists $c > 0$ such that $\mathrm{Cap}(D) \le c\,\mathrm{Cap}(F_w(D))$ for any $D \subset K$. We assume the converse as follows:

(A*) For any $w \in \bigcup_{n\in\mathbb{Z}_+} W^n$, there exists $c > 0$ such that $\mathrm{Cap}(F_w(D)) \le c\,\mathrm{Cap}(D)$ for any $D \subset K$.

For a subset A of W^m for some $m \in \mathbb{N}$, we say that a collection $\{f_w\}_{w\in A}$ of functions in $\mathcal{F}$ is compatible if $\tilde{f}_v(F_v^{-1}(x)) = \tilde{f}_w(F_w^{-1}(x))$ q.e. on $K_v \cap K_w$ for every $v, w \in A$. Note that this is well-defined by (A*). We define

$$\mathcal{F}_A = \{f \in L^2(K_A, \mu|_{K_A}) : F_w^* f \in \mathcal{F} \text{ for all } w \in A \text{ and } \{F_w^* f\}_{w\in A} \text{ is compatible}\}. \qquad (4.3)$$

If we equip A with a graph structure such that $v \in A$ and $w \in A$ are connected if $\mathrm{Cap}(K_v \cap K_w) > 0$, then A is $\mathcal{E}$-connected when A is a connected graph. This is verified by (B1).

Lemma 4.3. *For $A \subset W^m$ with $m \in \mathbb{Z}_+$, $(\mathcal{E}_A, \mathcal{F}_A)$ is a strong local Dirichlet form on $L^2(K_A, \mu|_{K_A})$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_A$ be a Cauchy sequence in $\mathcal{F}_A$. Let g be the limit in $L^2(K_A)$. Let $w \in A$. Since $\{F_w^* f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{F}$, $F_w^* f_n \to F_w^* g$ in $\mathcal{F}$. Further, it is easily deduced that $\{F_w^* g\}_{w\in A}$ is compatible. Therefore, $g \in \mathcal{F}_A$ and $f_n \to g$ in $\mathcal{F}_A$. This implies that $(\mathcal{E}_A, \mathcal{F}_A)$ is a closed form on $L^2(K_A)$. The Markov property and strong locality are inherited from those of $(\mathcal{E}, \mathcal{F})$ via relation (2.2). □

Corollary 4.4. *Assume case* 1) *or* 2) *in Proposition 4.2. Then,* (A*) *holds.*

Proof. In case 1), non-empty sets have uniform positive capacities, which implies (A*). In case 2), (A*) is easily obtained from (4.1) and (4.2). □

Hereafter, in this section, we will discuss the sufficient conditions for (B3).

Lemma 4.5. *Let $A \subset W_m$, $m \in \mathbb{Z}_+$. Then, $\mathcal{F}_A$ is compactly imbedded in $L^2(K_A, \mu|_{K_A})$. Suppose that A is $\mathcal{E}_A$-connected. Then, when we set $\mathcal{A} = \{f \in \mathcal{F}_A : \fint_{K_A} f\, d\mu = 0,\ \mathcal{E}_A(f) \le C\}$ for a constant $C > 0$, $\mathcal{A}$ is bounded in $\mathcal{F}_A$.*

Proof. Let $\mathcal{B}$ be a bounded subset of $\mathcal{F}_A$. For each $v \in A$, $\{F_v^* f : f \in \mathcal{B}\}$ is bounded in $\mathcal{F}$. By (B1), we can take a sequence $\{f_n\}_{n\in\mathbb{N}}$ from $\mathcal{B}$ such that $F_v^* f_n$ converges in $L^2(K)$. Therefore, we can take a sequence from $\mathcal{B}$ converging in $L^2(K_A)$. This implies the first assertion.

By combining this with the $\mathcal{E}_A$-connectedness of K_A, there exists $c > 0$ such that $\|f - \fint_{K_A} f\, d\mu\|^2_{L^2(K_A)} \le c\mathcal{E}_A(f)$ for every $f \in \mathcal{F}_A$. The latter assertion immediately follows from this. □

Now, we give a sufficient condition for (B3) (2).

Proposition 4.6. *The following condition* (EHI1) *implies* (B3) (2).

(EHI1) *For any $v \in \Xi$, there exist some $c_1 > 0$ and subsets $D''(v)$ and $D'''(v)$ of $D^\sharp(v)$ such that $D'''(v) \subset D''(v) \subset D^\sharp(v)$, $K_{D''(v)} \cap K_{S^{|v|}\setminus D^\sharp(v)} = \emptyset$, $K_v \cap K_{S^{|v|}\setminus D'''(v)} = \emptyset$, and $\operatorname{ess\,sup}_{x\in K_{D'''(v)}} h(x) \le c_1 \operatorname{ess\,inf}_{x\in K_{D'''(v)}} h(x)$ for every $h \in \mathcal{H}(D^\sharp(v))$ with $h \ge 0$ μ-a.e.*

Proof. First, we apply Lemma 3.7 to $g \in \mathcal{F}$ with $A = D^\sharp(v)$ and $J = \emptyset$ and denote g' by Hg. We follow the proof of Theorem 2.2 in [13]. For $h \in \mathcal{H}(D^\sharp(v))$ with $h \ge 0$ μ-a.e., by (EHI1), we have

$$\operatorname*{ess\,sup}_{x\in K_{D'''(v)}} h(x) \le c_1 \operatorname*{ess\,inf}_{x\in K_{D'''(v)}} h(x) \le c_2\|h\|_{L^2(K_{D'''(v)})}.$$

For $h \in \mathcal{H}(D^\sharp(v))$, let $h_+(x) = \max\{h(x), 0\}$ and $h_-(x) = \max\{-h(x), 0\}$. Since $h = Hh = Hh_+ - Hh_-$ and $Hh_\pm \ge 0$ μ-a.e., we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{x\in K_{D'''(v)}} |h(x)| &\le \operatorname*{ess\,sup}_{x\in K_{D'''(v)}} Hh_+(x) + \operatorname*{ess\,sup}_{x\in K_{D'''(v)}} Hh_-(x) \\
&\le c_2(\|Hh_+\|_{L^2(K_{D'''(v)})} + \|Hh_-\|_{L^2(K_{D'''(v)})}) \\
&\le c_3(\|h_+\|_{\mathcal{F}_{D''(v)}} + \|h_-\|_{\mathcal{F}_{D''(v)}}) \\
&\le 2c_3\|h\|_{\mathcal{F}_{D''(v)}}. \qquad (4.4)
\end{aligned}$$

In order to prove (B3) (2), it suffices to prove the following:

($*$) If a sequence $\{h_l\}$ in $\mathcal{H}(D^\sharp(v))$ converges weakly to 0 in $\mathcal{F}_{D^\sharp(v)}$, then there exists a subsequence $\{h_{l(k)}\}$ such that $F_v^* h_{l(k)}$ converges strongly to 0 in $\mathcal{F}$.

Indeed, suppose ($*$) holds. Let $\{f_m\}$ be a sequence in $\mathcal{H}(D^\sharp(v))$ that is bounded in $\mathcal{F}_{D^\sharp(v)}$. We can take a subsequence $\{f_{m(l)}\}$ and $f \in \mathcal{F}_{D^\sharp(v)}$ such that $f_{m(l)}$ converges weakly to f in $\mathcal{F}_{D^\sharp(v)}$. We take $g_l \in \mathcal{H}(D^\sharp(v))$ such that $g_l \to f$ in $\mathcal{F}_{D^\sharp(v)}$. By applying ($*$) to $h_l := f_{m(l)} - g_l$, we can take a sequence $\{l(k)\}$ diverging to ∞ such that $F_v^* f_{m(l(k))} \to F_v^* f$ in $\mathcal{F}$. This implies (B3) (2).

In order to prove ($*$), recall the notion of energy measure. For $f \in \mathcal{F} \cap L^\infty(K)$, the energy measure $\mu_{\langle f\rangle}$ is a unique positive Radon measure on K such that the following identity holds for every $g \in \mathcal{F} \cap C(K)$:

$$\int_K g\, d\mu_{\langle f\rangle} = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g).$$

Now, by (4.4), $C := \operatorname{ess\,sup}_{x\in K_{D'''(v)}} |h_l(x)|$ is bounded in l. We define $\hat{h}_l = ((-C)\vee h_l)\wedge C$. Since $\mathcal{F}_{D^\sharp(v)}$ is compactly imbedded in $L^2(K_{D^\sharp(v)})$ by Lemma 4.5, $\{h_l\}$ converges to 0 in $L^2(K_{D^\sharp(v)})$. We consider a subsequence $\{h_{l'}\}$ converging to 0 μ-a.e. on $K_{D^\sharp(v)}$. Since $(\mathcal{E},\mathcal{F})$ is regular, we can take $\varphi \in \mathcal{F}\cap C(K)$ such that $0\le\varphi\le 1$ on K, $\varphi = 1$ on K_w, and $\varphi = 0$ outside $K_{D'''(v)}$. We have

$$0 = 2\mathcal{E}(h_{l'}, \hat{h}_{l'}\varphi) = 2\mathcal{E}(\hat{h}_{l'}, \hat{h}_{l'}\varphi) = \mathcal{E}(\hat{h}_{l'}^2, \varphi) + \int_K \varphi\, d\mu_{\langle \hat{h}_{l'}\rangle}$$

because $\hat{h}_{l'}\varphi$ vanishes outside $K_{D'''(v)}$. Note that $\mathcal{E}(\hat{h}_{l'}^2) \le 4C^2\mathcal{E}(h_{l'})$, which is bounded in l'. A suitable subsequence $\hat{h}_{l''}$ can be considered such that $\{\hat{h}_{l''}^2\}$ converges weakly to some g in $\mathcal{F}$. Since $g = 0$ on $K_{D^\sharp(v)}$, $\mathcal{E}(\hat{h}_{l''}^2,\varphi) \to \mathcal{E}(g,\varphi) = 0$ as $l''\to\infty$. On the other hand,

$$\begin{aligned}\int_K \varphi\, d\mu_{\langle \hat{h}_{l''}\rangle} &= \sum_{z\in S^{|v|}} \rho^{|v|}\int_K F_z^*\varphi\, d\mu_{\langle F_z^*\hat{h}_{l''}\rangle}\\ &\ge \rho^{|v|}\int_K F_v^*\varphi\, d\mu_{\langle F_v^*\hat{h}_{l''}\rangle}\\ &= \rho^{|v|}\mu_{\langle F_v^*\hat{h}_{l''}\rangle}(K) = 2\rho^{|v|}\mathcal{E}(F_v^*\hat{h}_{l''}) = 2\rho^{|v|}\mathcal{E}(F_v^* h_{l''}).\end{aligned}$$

By combining these estimates, we obtain $\overline{\lim}_{l''\to\infty}\mathcal{E}(F_v^* h_{l''}) \le 0$. Therefore, $F_v^* h_{l''}$ converges to 0 in $\mathcal{F}$. This proves $(*)$. □

Next, we provide the sufficient conditions for (B3) (1) (b).

Proposition 4.7. *The following conditions imply* (B3) (1) (b)*.*

(1) *$\mathcal{F} = \Lambda^\beta_{2,\infty}(K)$ for some $\beta > 0$.*

(2) *For each $v\in\Xi$, $D'(v)$ is $\mathcal{E}_{D'(v)}$-connected.*

(3) *For each $w \in \bigcup_{n\in\mathbb{Z}_+}\hat{I}^{n+m_0}$, there exist subsets $D^\sharp(w)$, $D^{(1)}(w)$, and $D^{(2)}(w)$ of $D'(w)$ such that $D^\sharp(w)\subset D^{(1)}(w)\subset D^{(2)}(w)\subset D'(w)$ and the following hold:*

   (a) *There exists $v\in\Xi$ such that both $D'(w)$ and $D'(v)$, and $D^\sharp(w)$ and $D^\sharp(v)$, are of the same type by the same map F.*

   (b) $K_{D^\sharp(w)}\cap K_{S^{|w|}\setminus D^{(1)}(w)} = K_{D^{(2)}(w)}\cap K_{S^{|w|}\setminus D'(w)} = \emptyset$.

(EHI2) *There exists $c>0$ such that $\operatorname{ess\,sup}_{x\in K_{D^{(1)}(w)}} h(x) \le c\operatorname{ess\,inf}_{x\in K_{D^{(1)}(w)}} h(x)$ for $h\in\mathcal{H}(D^{(2)}(w))$ with $h\ge 0$ μ-a.e.*

Proof. Let g be a function in $\mathcal{F}$ such that $\int_{K_{D'(w)}} g\, d\mu = 0$ and $\rho^{|w|}\mathcal{E}_{D'(w)}(g)\le 1$. Let $f(x) = g(F^{-1}(x))$, $x\in K_{D'(v)}$. Then, $f\in\mathcal{F}_{D'(v)}$, $\int_{K_{D'(v)}} f\, d\mu = 0$, and $\rho^{|v|}\mathcal{E}_{D'(v)}(f)\le 1$. By Lemma 4.5, $\|f\|_{\mathcal{F}(D'(v))}\le C$, where C is a constant independent of w. Moreover, suppose that $g\in\mathcal{H}(I^{|w|}, D'(w))$. Apply Lemma 3.7 to g with $A = D^{(2)}(w)$ and $J=\emptyset$ and denote g' by g_1. Let $g_2 = g - g_1$. By (EHI2) and the same argument in the first part of the proof of Proposition 4.6, g_1 is bounded on $D^{(1)}(w)$. We consider a function $\psi\in\mathcal{F}$ such that $0\le\psi\le 1$,

$\psi = 0$ on $K_{S^{|w|}\setminus D^{(1)}(w)}$, and $\psi = 1$ on $K_{D^\sharp(w)}$. Then, $g_1\psi \in \mathcal{F}$. Since both $g_1\psi$ and g_2 vanish on $K_{S^{|w|}\setminus D^{(2)}(w)}$ when we set $f'(x) = \begin{cases} (g_1\psi + g_2)(F^{-1}(x)), & x \in K_{D'(v)} \\ 0, & x \in K \setminus K_{D'(v)} \end{cases}$, f' belongs to $\mathcal{F}$ by using the fact that $\mathcal{F} = \Lambda^{\beta}_{2,\infty}(K)$. Since $f' = f$ on $K_{D^\sharp(v)}$, we have $f' \in \mathcal{H}(I^{|v|}, D^\sharp(v))$ and $\|f'\|_{\mathcal{F}_v} = \|f\|_{\mathcal{F}_v} \le C$. These conclude the assertion. □

5 Examples

In this section, we choose $\mathcal{F}_A$ as in (4.3) for $A \subset W^m$.

1) Sierpinski gaskets: Let $\{a_0, a_1, \ldots, a_n\} \subset \mathbb{R}^n$ be the vertices of an n-dimensional simplex. Let $W = S = \{0, 1, \ldots, n\}$ and let $F_i(x) = (x - a_i)/2 + a_i$ for $x \in \mathbb{R}^n$ and $i = 0, 1, \ldots, n$. Then, the unique non-void compact set K, which satisfies $K = \bigcup_{i=0}^{n} F_i(K)$, is an n-dimensional Sierpinski gasket. The map Φ in Lemma 2.1 is an identity map. It is well known (see [2, 6, 18]) that there is a self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$, where the corresponding diffusion is fractional diffusion. In particular, $\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K) \subset C(K)$, where $d_w = (\log(n+3))/(\log 2)$. Note that $d_w - d_f > 0$ in this case. Let L be the $(n-1)$-dimensional gasket determined by $\{F_i\}_{i=0}^{n-1}$. In other words, $I = \{0, 1, \ldots, n-1\}$. Let $\hat{I} = I$, $M = 1$. It is evident that $\mathcal{F}_A = \mathcal{F}|_{K_A}$ for each $A \subset W^m$. Then, (A1)–(A7), (B2), and (C1)–(C2) are easy to verify with $\rho = (n+3)/(n+1)$. (A8) holds by Proposition 4.2 and (B1) holds by [18] Lemma 3.4.5. For (B3), we define $l_0 = 0$, $m_0 = 0$, $D'(w) = \{w\}$ for $w \in \bigcup_{n\in\mathbb{Z}_+} I^n$, $\Xi = \{\emptyset\}$, and $D^\sharp(\emptyset) = \{\emptyset\}$. It is easy to verify (B3) (1) by using Lemma 3.3 and Lemma 4.5. Since $\mathcal{H}_{D'(w)}(D'(w))$ is a finite-dimensional space, (B3) (2) is clearly true. We will prove (B4). Let $f \in \mathcal{F}$ and $\mathcal{E}_{S^m\setminus I^m}(f) = 0$ for some $m \in \mathbb{Z}_+$. Then, for each $w \in S^m \setminus I^m$, $\mathcal{E}(F_w^* f) = \rho^{-m}\mathcal{E}_w(f) = 0$. Therefore, f is constant on K_w for each $w \in S^m \setminus I^m$. We consider an unoriented graph with a vertex set $V = S^m \setminus I^m$ and an edge set $\{(v, w) \in V \times V : \mathrm{Cap}(K_v \cap K_w) > 0\}$. Then, V is a connected set. Note that (v, w) is an edge if and only if $K_v \cap K_w \neq \emptyset$. Therefore, f should be constant on $K_{S^m\setminus I^m}$ and thus constant on $K \setminus L$. This concludes that (B4) holds. Therefore, by Theorem 2.5, Theorem 2.6, and Remark 2.7, we have

$$\mathcal{F}|_L = \Lambda^{\beta}_{2,2}(L), \qquad \text{where} \quad \beta = \frac{d_w}{2} - \frac{\log(1 + 1/n)}{2\log 2}.$$

When $n = 2$, this characterization was obtained in [15].

2) Pentakun: Let $a_k = e^{2k\sqrt{-1}\pi/5 + \sqrt{-1}\pi/2} \in \mathbb{C}$, $k = 0, 1, 2, 3, 4$. Let $W = \{0, 1, 2, 3, 4, 5, 6\}$, $S = \{0, 1, 2, 3, 4\}$, $I = \hat{I} = \{2, 3, 5, 6\}$, and $M = 1$. Let $\mathfrak{G} = \{G_k\}_{k=0}^{4}$ with $G_k \colon \mathbb{C} \to \mathbb{C}$ defined by $G_k(z) = e^{2k\sqrt{-1}\pi/5} z$. For $i = 0, 1, 2, 3, 4$, we define a contraction map $F_i \colon \mathbb{C} \to \mathbb{C}$ by $F_i(z) = \alpha^{-1}(z - a_i) + a_i$, where $\alpha = (3 + \sqrt{5})/2$. We also define $F_5 = F_2 \circ G_1$ and $F_6 = F_3 \circ G_4$. Then, the resulting nested fractal K is called a Pentakun and the subset L is a Koch curve (see Figure 1). The Hausdorff dimensions of K and L are $(\log 5)/(\log \alpha)$ and $(\log 4)/(\log \alpha)$, respectively. There exists a canonical Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$, where the corresponding diffusion is fractional diffusion (see [2, 19, 23]); therefore, $\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K)$. It is known (see [19]) that $d_w = (\log \frac{\sqrt{161}+9}{2})/(\log \alpha)$ and we can check all the assumptions in a

similar manner to the case of the Sierpinski gasket. Note that C_2 given in (B2) (4) is equal to 4. Thus, by Theorem 2.5, Theorem 2.6, and Remark 2.7,

$$\mathcal{F}|_L = \Lambda^{\beta}_{2,2}(L), \qquad \text{where} \quad \beta = \frac{d_w}{2} - \frac{\log 5 - \log 4}{2\log\alpha}.$$

For the Pentakun K, let $I' = \{2,3\}$. Then, the corresponding self-similar subset L' is a Cantor set with Hausdorff dimension $(\log 2)/(\log\alpha)$. In this case, we should set $\hat{I} = \{2,3,5,6\}$; therefore, $I \neq \hat{I}$. Further, we can check all the assumptions in a similar manner; therefore, by Theorem 2.5, Theorem 2.6, and Remark 2.7,

$$\mathcal{F}|_{L'} = \Lambda^{\beta'}_{2,2}(L'), \qquad \text{where} \quad \beta' = \frac{d_w}{2} - \frac{\log 5 - \log 2}{2\log\alpha}.$$

In general, if K is a nested fractal satisfying (A3), then there exists a canonical Dirichlet form on $L^2(K,\mu)$, where the corresponding diffusion is fractional diffusion (see [2, 19, 23]). Let L be a self-similar subset of K given in a manner similar to that in the first part of Section 2 and satisfying (A2). In most cases, all the assumptions except (B4) can be checked in a similar manner to the case of the Sierpinski gasket; therefore, we can use Theorem 2.5 and Theorem 2.6 to characterize the trace space if (B4) holds. However, there are cases where (B4) does not hold (see 4)).

3) Sierpinski carpets: We consider general Sierpinski carpets. Let $H_0 = [0,1]^n$, $n \geq 2$, and let $l \in \mathbb{N}$, $l \geq 2$ be fixed. We set $\mathcal{Q} = \{\Pi_{i=1}^n [(k_i - 1)/l, k_i/l] : 1 \leq k_i \leq l,\ k_i \in \mathbb{N}\ (1 \leq i \leq n)\}$, let $N \leq l^n$, and $W = S = \{1,\dots,N\}$. Let F_i, $i \in S$ be orientation preserving affine maps of H_0 onto some element of $\mathcal{Q}$. We assume that the sets $F_i(H_0)$ are distinct. We set $H_1 = \bigcup_{i\in I} F_i(H_0)$. Then, the unique non-void compact set K, which satisfies $K = \bigcup_{i=1}^N F_i(K)$, is called the generalized Sierpinski carpet if the following holds:

(SC1) (Symmetry) H_1 is preserved by all the isometries of the unit cube H_0.

(SC2) (Connected) H_1 is connected.

(SC3) (Non-diagonality) Let B be a cube in H_0, which is the union of 2^n distinct elements of $\mathcal{Q}$. (Therefore, B has a side length of $2l^{-1}$.) Then, if $\mathrm{Int}(H_1 \cap B)$ is non-empty, it is connected.

(SC4) (Borders included) H_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_n = 0\}$.

Here, (see [3]) (SC1) and (SC2) are essential, while (SC3) and (SC4) are included for technical convenience. The Sierpinski carpets are infinitely ramified: the critical set C_K in (2.1) is an infinite set, and K cannot be disconnected by removing a finite number of points.

It is known (see [3, 4, 21]) that there is a self-similar Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$, where the corresponding diffusion is fractional diffusion. In particular, $\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K)$, where $d_w = (\log \rho N)/(\log l)$ and ρ is given in (A5). Let $\mathfrak{G} = \{\text{the identity map}\}$ and $L = ([0,1]^{n-1} \times \{0\}) \cap K$ (cf. Figure 1). Let $I = \{i \in S : F_i(K) \cap L \neq \emptyset\}$, $N_I = \#I$, and assume

$$\rho N_I > 1. \tag{5.1}$$

For simplicity, we assume that the $(n-1)$-dimensional Sierpinski carpet L also satisfies the conditions corresponding to (SC1)–(SC4). Then, (A1)–(A6) and (C1)–(C2) are easy to check with $M = 1$. (A7) holds by (5.1) because $\beta = d_w/2 - (d_f - d_I)/2 = (\log \rho N_I)/(2 \log l)$. (A8) holds by Proposition 4.2. It is known that the corresponding self-adjoint operator has compact resolvents (see [3, 4, 21]); therefore, (B1) holds. Letting $\hat{I} = I$, we can check (B2). For $w \in I^m$, $m \in \mathbb{Z}_+$, let $x_0(w) \in [0,1]^n$ be the center of K_w and $\Lambda_k(w)$ be the intersection of K and a cube in $\mathbb{R}^n$ with center $x_0(w)$ and length $(2k+1)l^{-m}$ for $k \in \mathbb{N}$. In order to ensure (B3), we assume for the moment that there exists some $k \geq 6$ such that $\Lambda_k(w)$ is connected for all $w \in \bigcup_{m \in \mathbb{Z}_+} I^m$. Let $l_0 = (2k+1)n$ and consider $m_0 \in \mathbb{N}$ such that $l^{m_0} \geq 2k+1$. For each $w \in I^{m+m_0}$, $m \in \mathbb{Z}_+$, we consider $D'''(w) \subset D''(w) \subset D^\sharp(w) \subset D^{(1)}(w) \subset D^{(2)}(w) \subset D'(w)$ such that $K_{D'''(w)} = \Lambda_1(w)$, $K_{D''(w)} = \Lambda_2(w)$, $K_{D^\sharp(w)} = \Lambda_3(w)$, $K_{D^{(1)}(w)} = \Lambda_4(w)$, $K_{D^{(2)}(w)} = \Lambda_5(w)$, and $K_{D'(w)} = \Lambda_k(w)$. With the use of Proposition 4.6 and Proposition 4.7, (B3) can be checked. Here, the Harnack inequalities (EHI1) and (EHI2) are ensured by [3, 4, 21]. To be more precise, let $\hat{K} = \bigcup_{x \in \{-1,0\}^n} (K + x)$, which is a subset of $[-1,1]^n$. Then, one can construct a Dirichlet form on $\hat{K}$ whose corresponding diffusion is fractal diffusion in the same manner as in [3, 4, 21]. Indeed, $\hat{K}$ has sufficient symmetry to employ the coupling arguments in [3]. In this manner, the Harnack inequalities (EHI1) and (EHI2) are ensured. If for each k, there exists $w \in \bigcup_{m \in \mathbb{Z}_+} I^m$ such that $\Lambda_k(w)$ is not connected, we consider the connected component of $\Lambda_k(w)$ including K_w instead of $\Lambda_k(w)$ and discuss in a similar manner as mentioned above. By the covering argument, we can check (B3). (B4) is confirmed by an argument similar to the case of Sierpinski gaskets. Thus, by Theorem 2.5 and Theorem 2.6, we have

$$\mathcal{F}|_L = \hat{\Lambda}^\beta_{2,2}(L), \qquad \text{where} \quad \beta = \frac{d_w}{2} - \frac{1}{2}\left(\frac{\log N}{\log l} - \dim_H L\right).$$

Note that when $\partial[0,1]^n \subset K$, $0 < \beta < 1$; therefore, (5.1) holds and $\mathcal{F}|_L = \Lambda^\beta_{2,2}(L)$ by Remark 2.7. Indeed, let $K_2 = [0,1]^n$ and K_1 be a generalized Sierpinski carpet in $\mathbb{R}^n$ with $\partial[0,1]^n \subset K_1$, which is determined by $\{F_i\}_i$ where $F_i([0,1]^n) \cap \partial[0,1]^n \neq \emptyset$ for all i. Evidently, $K_1 \subset K \subset K_2$. For each K_i, one can construct the self-similar Dirichlet form. Let ρ_i be the scaling factor given in (A5). By the shorting and cutting laws for electrical networks (see [9]), $\rho_2 \leq \rho \leq \rho_1$. Then, $\rho_2 = l^{2-n}$ and

$$\frac{2}{l^{n-1}} + \frac{l-2}{l^{n-1} - (l-2)^{n-1}} \leq \rho_1 \leq \frac{l}{l^{n-1} - (l-2)^{n-1}} \tag{5.2}$$

by (5.9) in [3]. Since $L = [0,1]^{n-1} \times \{0\}$ and $N_I = l^{n-1}$ in this case, we have $\rho N_I \geq \rho_2 N_I = l \geq 2$; therefore, (5.1) holds and $\beta > 0$. Using (5.2),

$$\rho N_I \leq \rho_1 N_I \leq \frac{l^n}{l^{n-1} - (l-2)^{n-1}} < l^2,$$

where the last inequality is a simple computation. Thus, $\beta < 1$.

4) The Vicsek set: Let $a_1 = (0,0), a_2 = (1,0), a_3 = (1,1), a_4 = (0,1)$ and $a_5 = (1/2, 1/2)$ be points in $\mathbb{R}^2$. We define $F_i(x) = (x - a_i)/3 + a_i$ for $x \in \mathbb{R}^2$ and $i = 1, \ldots, 5$. The unique non-void compact set K, which satisfies $K = \bigcup_{i=1}^5 F_i(K)$, is the Vicsek set. Similar to the case of 1), there is a self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ with $\rho = 3$, where the corresponding

diffusion is fractional diffusion. In particular, $\mathcal{F} = \Lambda_{2,\infty}^{d_w/2}(K)$, where $d_w = (\log 15)/(\log 3)$. Let L be the line segment from $(0,0)$ to $(1,1)$. Then, (B4) does not hold. In this case, the trace of Brownian motion on the Vicsek set is the Brownian motion on the line segment. Indeed, one can easily check conditions (H_1)–(H_3) in Section 8 of [6] on the one-dimensional Sierpinski gasket, which is a line. Therefore, by [6] Theorem 8.1, it can be observed that the trace of Brownian motion on the Vicsek set is a constant time change of the Brownian motion on the line. Thus,

$$\mathcal{F}|_L = \Lambda_{2,\infty}^1(L),$$

which is greater than $\Lambda_{2,2}^1(L)$. This shows that (B4) is necessary for Theorem 2.5.

6 Application: Brownian motion penetrating fractals

In [12], one of the authors constructed Brownian motions on *fractal fields*—a collection of fractals with (in general) different Hausdorff dimensions (see also [20]). They are diffusion processes that behave as appropriate fractal diffusions within each fractal component of the field and they penetrate each fractal. In [12], a restrictive assumption (Assumption 2.2 in [12]) was required to construct such processes because the corresponding function spaces were not known. Our result in this paper can be applied here, and we can construct such penetrating diffusions without the restrictive assumption.

Let A_0 be a countable set and let $\{K_i\}_{i\in A_0} \subset \mathbb{R}^n$ be a family of self-similar sets along with strong local, regular, and self-similar Dirichlet forms $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ on $L^2(K_i, \mu_i)$, where K_i and μ_i lie within the framework of Section 2. We also regard μ_i as a measure on $\mathbb{R}^n$ by letting $\mu_i(\mathbb{R}^n \setminus K_i) = 0$. We set $G = \bigcup_{i\in A_0} K_i$.

Let A_1 be another countable set and let $\{D_j\}_{j\in A_1} \subset \mathbb{R}^n$ be a family of disjoint domains in $\mathbb{R}^n \setminus G$. We denote the closure of D_j in $\mathbb{R}^n$ by K_j and the Lebesgue measure restricted on K_j by μ_j. Further, $\tilde{G} = G \cup \left(\bigcup_{j\in A_1} K_j\right)$. $\tilde{G}$ is called a *fractal field* generated by $\{K_i\}_{i\in A_0}$ and $\{D_j\}_{j\in A_1}$. (When G is connected as in the introduction, we also refer to G as a *fractal field* or a *fractal tiling*.)

We denote the disjoint union of A_0 and A_1 by A. For $i, j \in A$ with $i \neq j$, let $\Gamma_{ij} = K_i \cap K_j$. We define $\Gamma = \bigcup_{i,j\in A, i\neq j} \Gamma_{ij}$. For $x \in \Gamma$, let $J_x := \{i \in A : x \in K_i\}$ and define $N_x := \bigcup_{i,j\in J_x, i\neq j} \Gamma_{ij}$. Throughout this section, we impose the following assumption:

Assumption A (1) For each compact set $C \subset \mathbb{R}^n$, $\#\{i \in A : C \cap K_i \neq \emptyset\} < \infty$.
(2) For each $i \in A_1$, $K_i \setminus D_i$ is a null set with respect to the Lebesgue measure on $\mathbb{R}^n$.

For each $i \in A_1$, we define $\mathcal{D}(\mathcal{E}_{K_i}) = \{u \in C_0(K_i) : u|_{D_i} \in W^{1,2}(D_i)\}$ and

$$\mathcal{E}_{K_i}(u, v) = \frac{1}{2}\int_{D_i} (\nabla u(x), \nabla v(x))_{\mathbb{R}^n}\, dx, \quad \text{for } u, v \in \mathcal{D}(\mathcal{E}_{K_i}).$$

Then, $(\mathcal{E}_{K_i}, \mathcal{D}(\mathcal{E}_{K_i}))$ is closable on $L^2(K_i, \mu_i)$. Its closure is denoted by $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$. It is evident that $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is a strong local regular Dirichlet form.

For $x \in \Gamma$ and $i \in J_x$, we define $\beta_{x,i} = d_w(K_i)/2 - (d_f(K_i) - d_f(N_x \cap K_i))/2$. Here, $d_w(K_i)$ is defined in (A7) for $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ if $i \in A_0$; it is equal to 2 if $i \in A_1$. Further, $d_f(K_i)$ and $d_f(N_x \cap K_i)$ are the Hausdorff dimensions of K_i and $N_x \cap K_i$, respectively.

Further, we assume the following throughout this section.

Assumption B (1) For $i \in A_0$, $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is a strong local regular Dirichlet form on $L^2(K_i, \mu_i)$, which satisfies (A1), (A3), the first identity of (A4), (A5), and (A6) in Section 2.
(2) For each $x \in \Gamma$ and $i \in J_x \cap A_0$, $N_x \cap K_i$ is a finite number of the union of compact self-similar sets $\{L_j\}$ that are constructed using the same number of contraction maps and each of which satisfies (A2), the second identity of (A4), (A7), and (A8) in Section 2. Further, (C1)*–(C2)* in Remark 3.10 hold when $K = K_i$ and $L = N_x \cap K_i$.
(3) For each $x \in \Gamma$ and $i \in J_x \cap A_1$, $N_x \cap K_i$ is a closed Alfors $d_{x,i}$-regular set with some $d_{x,i}$.
(4) For every $x \in \Gamma$, $\beta_{x,i} > 0$ for all $i \in J_x$, and the set $\Lambda_x := \{f \in C_0(N_x) : f|_{N_x \cap K_i} \in \Lambda_{2,2}^{\beta_{x,i}}(N_x \cap K_i)$ for all $i \in J_x\}$ is dense in $C_0(N_x)$.

Now, we make several remarks. When $i \in A_1$, we have $d_f(K_i) = n$ and $d_f(K_i \cap N_x) = d_{x,i}$. The set Λ_x is closed under the operation of the normal contraction; $(0 \vee f) \wedge 1 \in \Lambda_x$ for $f \in \Lambda_x$. If N_x is an Alfors regular set and $\beta_{x,i} \in (0, 1)$ for all $i \in J_x$, then $\Lambda_{2,2}^{\max_{i \in J_x} \beta_{x,i}}(N_x) \cap C_0(N_x)$ (a subset of Λ_x) is dense in $C_0(N_x)$ by Chapter V, Proposition 1 in [16], and Theorem 3 in [27]. The condition $\beta_{x,i} \in (0, 1)$ holds, for example, if $i \in N_x \cap A_1$ and $d_{x,i} \in (n-2, n)$, because then $\beta_{x,i} = 1 - (n - d_{x,i})/2 \in (0, 1)$.

Define a measure $\tilde{\mu}$ on $\tilde{G}$ by $\tilde{\mu} = \sum_{i \in A} \mu_i$. Now, we define a bilinear form $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ on $L^2(\tilde{G}, \tilde{\mu})$ as follows:

$$\begin{aligned} \tilde{\mathcal{E}}(u, v) &= \sum_{i \in A} \mathcal{E}_{K_i}(u|_{K_i}, v|_{K_i}) \quad \text{for } u, v \in \mathcal{D}(\tilde{\mathcal{E}}), \\ \mathcal{D}(\tilde{\mathcal{E}}) &= \{u \in C_0(\tilde{G}) : u|_{K_i} \in \mathcal{F}_{K_i} \text{ for all } i \in A \text{ and } \tilde{\mathcal{E}}(u, u) < \infty\}. \end{aligned}$$

Then, the following is easy to check.

Lemma 6.1. (1) *$(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ is closable in $L^2(\tilde{G}, \tilde{\mu})$.*

(2) *$\mathcal{D}(\tilde{\mathcal{E}})$ is an algebra.*

(3) *For $i \in A$, $x \in K_i$, and for $U(x)$—a neighborhood of x in K_i—there exists $f \in \mathcal{F}_{K_i} \cap C_0(K_i)$ such that $f(x) > 0$ and* supp $f \subset U(x) \cap K_i$, *where* supp f *denotes the support of f.*

Now, let $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ be the closure of $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$. We then have the following:

Theorem 6.2. *$(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is a strong local regular Dirichlet form on $L^2(\tilde{G}, \tilde{\mu})$.*

Note that the strong local property of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ can be easily deduced from those of the original forms on $\{K_i\}_{i \in A}$. Therefore, it is sufficient to prove the regularity of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. To prove the regularity, primarily the following should be proved:

Proposition 6.3. (1) *For each $x \neq y \in \tilde{G}$, there exists $g \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $g(x) \neq g(y)$.*

(2) *For any compact set L in $\tilde{G}$, there exists $f \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $f = 1$ on L.*

Once this proposition is proved, it is easy to prove the regularity of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ (see [12]); therefore, we only prove the proposition.

Proof of Proposition 6.3. Let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius r. When x or y is in the compliment of Γ, then (1) is clear by Lemma 6.1 (3); therefore, we consider the case $x, y \in \Gamma$. By Assumption A (1), $\#J_x < \infty$. Since each K_j is closed, by Assumption A (1), there exists $r_x > 0$ such that $B(x, r_x) \cap K_j \neq \emptyset$ if and only if $j \in J_x$, and $y \notin B(x, r_x)$. Since Λ_x is dense in $C_0(N_x)$ by Assumption B (4), there exists $u \in \Lambda_x$ such that $u|_{B(x,r_x/2)} = 1$ and $u|_{B(x,3r_x/4)^c} = 0$.

Now, by Assumption B (1), (2), and the extension theorem (Remark 3.10), for each $i \in J_x \cap A_0$, there exists $\hat{u}_i \in \mathcal{F}_{K_i} \cap C(K_i)$ such that $\hat{u}_i|_{N_x \cap K_i} = u$. For each $i \in J_x \cap A_1$, since $N_x \cap K_i$ is a closed Alfors $d_{x,i}$-regular set, we have

$$W^{1,2}(\mathbb{R}^n)|_{N_x \cap K_i} = \Lambda_{2,2}^{1-(n-d_{x,i})/2}(N_x \cap K_i) \tag{6.1}$$

(see [16]). By carefully tracing the proof of the extension theorem in (6.1), we see that there exists $\hat{u}_i \in W^{1,2}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ such that $\hat{u}_i|_{N_x \cap K_i} = u$ (see, for example, pages 77–78 in [20]). For both the cases, since $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is regular, by multiplying a function in $\mathcal{F}_{K_i} \cap C_0(K_i)$ that is 1 in $B(x, 3r_x/4)$ and 0 outside $B(x, r_x)$, we may assume $\operatorname{supp} \hat{u}_i \subset B(x, r_x)$. We define $g \in C_0(\tilde{G})$ as $g|_{K_i} = \hat{u}_i$ for $i \in J_x$ and $g|_{K_i} \equiv 0$ otherwise. Then, $g \in \mathcal{D}(\tilde{\mathcal{E}})$, $g(x) = 1$ and $g(y) = 0$. We thus obtain the desired function.

The proof of (2) is quite similar; therefore, we omit it (see Proposition 2.6 (2) in [12]). □

The 1-capacity associated with $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ are denoted by Cap_{K_i} and $\mathrm{Cap}_{\tilde{G}}$, respectively. By definition, it is evident that $u|_{K_i} \in \mathcal{F}_{K_i}$ for any $i \in A$ and $u \in \tilde{\mathcal{F}}$. Further, $\mathrm{Cap}_{K_i}(H) \leq \mathrm{Cap}_{\tilde{G}}(H)$ for any $i \in A$ and $H \subset K_i$. For $i \in A$, let $\mathcal{F}_{K_i'} = \{f \in \mathcal{F}_{K_i} : \tilde{f} = 0 \text{ q.e. on } K_i \cap \Gamma\}$ and $\tilde{\mathcal{F}}_i = \{f \in \tilde{\mathcal{F}} : \tilde{f} = 0 \text{ q.e. on } \bigcup_{j \in A \setminus \{i\}} K_j\}$, where $\tilde{f}$ is a (corresponding) quasi-continuous modification of f.

We will denote the diffusion process corresponding to $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ by $(\{\tilde{X}_t\}_{t \geq 0}, \{\tilde{P}_x\}_{x \in \tilde{G}})$. The following proposition shows that $\{\tilde{X}_t\}$ behaves on K_i in the same manner as the diffusion process associated with $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ until the process reaches Γ.

Proposition 6.4. *$(\mathcal{E}_{K_i}, \mathcal{F}_{K_i'})$ and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}_i)$ yield the same Dirichlet forms on $L^2(K_i, \mu_i|_{K_i \setminus \Gamma})$ by identifying the measure space $(\tilde{G}, \mu_i|_{K_i \setminus \Gamma})$ with $(K_i, \mu_i|_{K_i \setminus \Gamma})$. In particular, the corresponding parts of the processes on $K_i \setminus \Gamma$ are the same.*

Proof. It is evident that $f \in \tilde{\mathcal{F}}_i$ satisfies $f|_{K_i} \in \mathcal{F}_{K_i'}$; therefore, we prove the converse. Let $f \in \mathcal{F}_{K_i'}$. By Theorem 4.4.3 of [10], we can take an approximation sequence of f from $\mathcal{F}_{K_i'} \cap C_0(K_i \setminus \Gamma)$. Therefore, the 0-extension of f outside K_i is an element of $\tilde{\mathcal{F}}_i$. □

For each distinct $i, j \in A$, we denote $K_i \sim K_j$ if $\mathrm{Cap}_{K_l}(\Gamma_{ij}) > 0$ for $l = i$ and j. We now assume the following in addition to Assumptions A and B.

Assumption C (1) For each $i \in A_0$, $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is irreducible.
(2) For each distinct $i, j \in A$, there exist $k \in \mathbb{N}$ and a sequence $i_0, i_1, \ldots, i_k \in A$ such that $K_{i_0} = K_i$, $K_{i_k} = K_j$ and $K_{i_l} \sim K_{i_{l+1}}$ for $l = 0, 1, \ldots, k-1$.
(3) For each distinct $i, j \in A$ with $K_i \sim K_j$, there exists a positive Radon measure ν_{ij} on Γ_{ij} such that $\nu_{ij}(\Gamma_{ij}) > 0$ and ν_{ij} is smooth with respect to both $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ and $(\mathcal{E}_{K_j}, \mathcal{F}_{K_j})$.

Note that when $i \in A_1$, $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is irreducible since D_i is connected. (See, for example, Theorem 4.5 in [24] for the proof.)

For each nearly Borel set $B \subset \mathbb{R}^n$, we define $\sigma_B = \inf\{t > 0 : \tilde{X}_t \in B\}$. The next proposition shows that $\tilde{X}_t$ penetrates into each K_i.

Proposition 6.5. *The following holds for any nearly Borel set B with $\mathrm{Cap}_{\tilde{G}}(B) > 0$:*

$$\tilde{P}^x(\sigma_B < \infty) > 0 \text{ for } (\tilde{\mathcal{E}}, \tilde{\mathcal{F}})\text{-quasi every } x \in \tilde{G}. \tag{6.2}$$

In particular, if B is a subset of a certain K_i with $\mathrm{Cap}_{K_i}(B) > 0$, then (6.2) *holds.*

Proof. Due to Theorem 4.6.6 in [10], it is sufficient to prove that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is irreducible. First, we recall the following fact. Let $(\mathcal{E}, \mathcal{F})$ be a local Dirichlet form. (Here, the locality represents the condition that $\mathcal{E}(f, g) = 0$ if $fg = 0$ a.e. All Dirichlet forms appearing in this paper are local in this sense; see [25].) Let Y be a measurable subset of the state space and $\mathcal{C}$ be a dense set in $\mathcal{F}$. Then, Y is an invariant set if and only if $1_Y \cdot u \in \mathcal{F}$ for any $u \in \mathcal{C}$. This is verified by Theorem 1.6.1 in [10] and a usual approximation argument.

Now, let M be an invariant set for $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. We set $i \in A$ and take $u \in \mathcal{F}_{K_i} \cap C_0(K_i)$. We can take $v \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $v = 1$ on $\operatorname{supp} u$ by Proposition 6.3 (2). Then, $1_M \cdot v \in \tilde{\mathcal{F}}$, which implies that $(1_M \cdot v)|_{K_i} \in \mathcal{F}_{K_i}$. Therefore, $u \cdot (1_M \cdot v)|_{K_i} = u \cdot 1_{M \cap K_i}$ also belongs to $\mathcal{F}_{K_i}$. Since $\mathcal{F}_{K_i} \cap C_0(K_i)$ is dense in $\mathcal{F}_{K_i}$, we conclude that $M \cap K_i$ is an invariant set for $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$. By the irreducibility of $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$, either $\mu_i(M \cap K_i) = 0$ or $\mu_i(K_i \setminus M) = 0$ holds.

By this argument, there exists a subset A' of A such that $M = \bigcup_{i \in A'} K_i$ $\tilde{\mu}$-a.e. We assume that M is a nontrivial invariance set. Then, $A' \neq \emptyset$, $A' \neq A$, and there exist $i \in A'$ and $j \in A \setminus A'$ such that $K_i \sim K_j$ by Assumption C (2). We consider a compact set $H \subset \Gamma_{ij}$ such that $\nu_{ij}(H) > 0$ and a relatively compact open set H' including H. Further, $v \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $v = 1$ on H' and let $u = 1_M \cdot v \in \tilde{\mathcal{F}}$. We denote the quasi-continuous modification of u w.r.t. $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ by $\tilde{u}$. Then, $\tilde{u}|_{K_l}$ is also quasi-continuous w.r.t. $(\mathcal{E}_{K_l}, \mathcal{F}_{K_l})$ for $l = i, j$. Since $\tilde{u} = 1$ μ-a.e. on $H' \cap K_i$, we have $\tilde{u} = 1$ $\mathcal{E}_{K_i}$-q.e. on $H \subset H' \cap K_i$. By Assumption C (3), $\tilde{u} = 1$ ν_{ij}-a.e. on H. On the other hand, since $\tilde{u} = 0$ μ-a.e. on $H' \cap K_j$, we have $\tilde{u} = 0$ $\mathcal{E}_{K_j}$-q.e. on H. Therefore, $\tilde{u} = 0$ ν_{ij}-a.e. on H. This is a contradiction, which reveals that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is irreducible. □

The fractal field in Figure 2 satisfies Assumptions A, B, and C; therefore, a penetrating diffusion exists on the field.

In [12], detailed properties of $\tilde{X}_t$ such as heat kernel bounds and large deviation estimates are established under strong assumptions such as Assumption 2.2 in [12]. By using the results given in this section, the assumption can be relaxed and the same results can be obtained by the proof given in [12] when each Dirichlet form exhibits the resistance form in the sense of [18].

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