# [D] Full-harmonic Structures on a Green Space

## (Abstract)

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## Chap. I. Full-harmonic structures and associated kernels.

### § 1. Full-harmonic structure.

Let $\Omega$ be a non-compact space of type $\mathcal{E}$ in the sense of Brelot-Choquet [1] or an open Riemann surface. For an open set $G \subset \Omega$, let $H(G)$ be the linear space of all harmonic functions on $G$. Let $\mathcal{A}$ be the family of all domains $D$ in $\Omega$ such that $D$ is not relatively compact and its relative boundary $\partial D$ is compact. Let $\mathcal{G}$ be the family of all open sets $G$ in $\Omega$ such that $\partial G$ is compact. Then every component $D$ of $G \in \mathcal{G}$ is either relatively compact or $D \in \mathcal{A}$.

Suppose there is given a family $\mathcal{H} = \left\{ \widetilde{H}(D) \right\}_{D \in \mathcal{A}}$ satisfying the following three conditions:

(i)   Each $\widetilde{H}(D)$ is a linear subspace of $H(D)$;

(ii)   If $D$, $D' \in \mathcal{A}$, $D' \subset D$ and $u \in \widetilde{H}(D)$, then $u|_{D'} \in \widetilde{H}(D')$;

(iii)   If $u \in H(D)$ and if there exists a compact set $K$ in $\Omega$ such that $u|_{D'} \in \widetilde{H}(D')$ for every component $D'$ of $D - K$, then $u \in \widetilde{H}(D)$.

A domain $D \in \mathcal{A}$ is called $\mathcal{H}$-regular if, for any non-negative continuous function $f$ on $\partial D$, there exists a unique non-negative function $u_f$ on $\overline{D}$ such that $u_f|_D \in \widetilde{H}(D)$ and $u_f|_{\partial D} = f$. An open set $G \in \mathcal{G}$ is called $\mathcal{H}$-regular if each relatively compact component of $G$ is regular with respect to the Dirichlet problem and each component $D$ of $G$ such that $D \in \mathcal{A}$ is $\mathcal{H}$-regular.

$\mathcal{H}$ is called a <u>full-harmonic structure</u> if it satisfies the following fourth condition (cf. [6]):

(iv)  For any compact set $K_0$ in $\Omega$, there exists another compact set $K$ in $\Omega$ such that $K^i$(the interior of $K$)$\supset K_0$ and $\Omega - K$ is $\mathcal{H}$-regular.

In this note, we shall assume the following condition (iv)' instead of (iv):

(iv)'  If $D \in \mathcal{O}$ has smooth boundary, then it is $\mathcal{H}$-regular.

Let $D$ be an $\mathcal{H}$-regular domain.  For $f \in C(\partial D)$, we define $u_f = u_{f+} - u_{f-}$.  Then $u_f \in \widetilde{H}(D)$ and the mapping $f \to u_f$ is linear. For each $x \in D$, the mapping $f \to u_f(x)$ is positive linear, so that there exists a measure $\mu_x^D \equiv \mu_{\mathcal{H},x}^D$ on $\partial D$ such that $u_f(x) = \int f d\mu_x^D$ for all $f \in C(\partial D)$.  If $G \in \mathcal{G}$ is $\mathcal{H}$-regular, then we define $\mu_x^G \equiv \mu_{\mathcal{H},x}^G$ as follows:  Let $D$ be the component containing $x$.  If $D \in \mathcal{O}$, then define $\mu_x^G = \mu_x^D$.  If $D$ is relatively compact, then let $\mu_x^G$ be the usual harmonic measure of $D$ with respect to $x$.

§ 2.  Full-superharmonic functions.

Let $\mathcal{H}$ be a full-harmonic structure and let $D \in \mathcal{O}$.  A superharmonic function $s$ on $D$ is called $\mathcal{H}$-full-superharmonic if there exists a sequence $\{K_n\}$ of compact sets such that $\partial D \subset K_1^i$, $K_n \subset K_{n+1}^i$, $\bigcup_{n=1}^{\infty} K_n = \Omega$, each $\Omega - K_n$ is $\mathcal{H}$-regular and

$$s(x) \geq \int s d\mu_{\mathcal{H},x}^{D-K_n}$$

for all $n$ and for all $x \in D - K_n$.

It can be seen that if $s$ is $\mathcal{H}$-full-superharmonic on $D$, then for any compact set $K$ such that $K^i \supset \partial D$ and $\Omega - K$ is $\mathcal{H}$-regular,

$$s(x) \geq \int s d\mu_{\mathcal{F},x}^{D-K} \text{ for all } x \in D - K.$$

For $D \in \mathcal{O}$, $u \in \tilde{H}(D)$ if and only if $u$ and $-u$ are both $\mathcal{F}$-full-superharmonic. Thus, $u \in \tilde{H}(D)$ is said to be $\mathcal{F}$-full-harmonic on $D$.

Full-superharmonic functions have many properties analogous to those of superharmonic functions (cf. [6]). For example, we have:

Minimum Principle: Let $D \in \mathcal{O}$ be such that $\bar{D} \neq \Omega$. If $u$ is a full-superharmonic function on $D$ and if $\varliminf_{x \to \xi, x \in D} u(x) \geq 0$ for all $\xi \in \partial D$, then $u \geq 0$.

§ 3. Full-superharmonic functions of potential type.

Let a full-harmonic structure $\mathcal{F}$ be given. We take a domain $\Omega_0 \in \mathcal{O}$ such that $\bar{\Omega}_0 \neq \Omega$ and fix it in the sequel.

It is shown that any non-negative full-superharmonic function on $\Omega_0$ has the greatest full-harmonic minorant (cf. [6]).

Definition. A non-negative $\mathcal{F}$-full-superharmonic function on $\Omega_0$ is called of potential type if its greatest $\mathcal{F}$-full-harmonic minorant on $\Omega_0$ is zero. We denote by $\mathcal{P} = \mathcal{P}(\Omega_0; \mathcal{F})$ the set of all $\mathcal{F}$-full-superharmonic functions of potential type.

Any non-negative $\mathcal{F}$-full-superharmonic function $u$ on $\Omega_0$ is decomposed into $u = v + h$ with $v \in \mathcal{P}$ and $h$ $\mathcal{F}$-full-harmonic.

Let $\mathcal{P}_b = \mathcal{P}_b(\Omega_0; \mathcal{F})$ be the set $\mathcal{P}(\Omega_0; \mathcal{F}) \cap H(\Omega_0)$.

§ 4. $\mathcal{F}$-Green kernel.

The domain $\Omega_0$ has the Green function $G_x(y) \equiv G_x^{\Omega_0}(y)$ (cf. [1]). For a full-harmonic structure $\mathcal{F}$, an $\mathcal{F}$-Green kernel is a non-negative extended real valued function $M_x(y) \equiv M_{\mathcal{F},x}(y) \equiv M_{\mathcal{F},x}^{\Omega_0}(y)$ on $\Omega_0 \times \Omega_0$ such that

(i) for each $x \in \Omega_0$, $M_x - G_x$ is harmonic on $\Omega_0$;

(ii) for each $x \in \Omega_0$, $M_x \in \wp(\Omega_0; \mathcal{F})$ and $M_x$ is $\mathcal{F}$-full-harmonic on $\Omega_0 - \{x\}$.

Theorem. The $\mathcal{F}$-Green kernel $M_x \equiv M_{\mathcal{F},x}^{\Omega_0}$ always exists and is unique. It is continuous on $\Omega_0 \times \Omega_0$. Furthermore, $M_x = G_x + U_x$ with $U_x \in \wp_b(\Omega_0; \mathcal{F})$.

Sketch of the proof: For each $x \in \Omega_0$, let $\mathcal{U}_x = \{u \in \wp; G_x + u \in \wp\}$.

(i) $\mathcal{U}_x$ is non-empty. To show it, choose a closed sphere $K_0$ such that $K_0 \cap \overline{\Omega}_0 = \emptyset$ and let $h(y) = \int d\mu_{\mathcal{F},y}^{\Omega-K_0}$ and $h_0$ be the Dirichlet solution on $\Omega - K_0$ with boundary values 1 on $\partial K_0$, 0 on the ideal boundary. $h - h_0$ is $\mathcal{F}$-full-superharmonic and non-negative on $\Omega_0$. Let $h - h_0 = v + h_1$, where $v \in \wp(\Omega_0; \mathcal{F})$ and $h_1$ is $\mathcal{F}$-full-harmonic on $\Omega_0$. Choose a compact set $K$ with smooth boundary such that $K^i \supset \partial \Omega_0 \cup \{x\}$ and let
$$\lambda = \sup_{y \in \partial K \cap \Omega_0} [G_x(y)/h_0(y)].$$ Then $0 < \lambda < +\infty$. We can show that

$G_x + \lambda v \in \wp$, so that $\lambda v \in \mathcal{U}_x$.

(ii) Let $U_x = \inf \mathcal{U}_x$. It is easy to see that $U_x \in \wp_b$ and $G_x + U_x \in \wp$. By Theorem 3 of [6], we also see that $G_x + U_x$ is full-harmonic on $\Omega_0 - \{x\}$. Hence $M_x = G_x + U_x$ satisfies (i) and (ii).

(iii) The continuity follows from the fact that, given $\varepsilon > 0$, $U_{x'} + \varepsilon v \in \mathcal{U}_x$ if $x'$ is sufficiently close to $x$.

§ 5. Integral representation of $\wp$-functions.

Let $u \in \wp(\Omega_0; \mathcal{F})$. There exists a unique positive measure $\mu$ on $\Omega_0$ such that

$$u = h + \int_{\Omega_0} M^{\Omega_0}_{\mathcal{F},x} d\mu(x)$$

with $h \in \mathcal{P}_b(\Omega_0; \mathcal{F})$ (cf. [5]).

In order to consider the integral representation of $h \in \mathcal{P}_b(\Omega_0; \mathcal{F})$, we construct an ideal boundary associated with $\mathcal{F}$: Fix $x_0 \in \Omega_0$ and let

$$K^{\Omega_0}_{\mathcal{F},x}(y) = \begin{cases} \dfrac{M^{\Omega_0}_{\mathcal{F},x}(y)}{M^{\Omega_0}_{\mathcal{F},x}(x_0)} & \text{if } x \neq y \text{ or } x \neq x_0 \\ \\ 1 & \text{if } y = x = x_0. \end{cases}$$

For each $y \in \Omega_0$, $x \to K_x(y) \equiv K^{\Omega_0}_{\mathcal{F},x}(y)$ is continuous on $\Omega_0$. We can define ideal boundary points with respect to $K_x$ just as the Martin boundary points. The ideal boundary thus obtained is denoted by $\Delta_{\mathcal{F}}(\Omega_0)$. The corresponding metric defines a topology on $\Omega_0 \cup \Delta_{\mathcal{F}}(\Omega_0)$ and $\Delta_{\mathcal{F}}(\Omega_0)$ is compact with respect to this topology. It is shown that $\Delta_{\mathcal{F}}(\Omega_0)$ does not depend on $x_0 \in \Omega_0$. For each $\xi \in \Delta_{\mathcal{F}}(\Omega_0)$ let $K_\xi(y) = \lim\limits_{x \to \xi, x \in \Omega_0} K_x(y) \, (y \in \Omega_0)$.

On $\Delta_{\mathcal{F}}(\Omega_0)$, there appear $\mathcal{F}$-minimal points and $\mathcal{F}$-non-minimal points. The set of $\mathcal{F}$-minimal points on $\Delta_{\mathcal{F}}(\Omega_0)$ is a $G_\delta$-set and is denoted by $\Delta^1_{\mathcal{F}}(\Omega_0)$. As in the case of the Martin boundary, we obtain the following integral representation theorem (cf. [2], [5] and [6]):

**Theorem.** Given $u \in \mathcal{P}_b(\Omega_0; \mathcal{F})$, there exists a unique positive measure $\mu$ on $\Delta_{\mathcal{F}}(\Omega_0)$ such that $\mu(\Delta_{\mathcal{F}} - \Delta^1_{\mathcal{F}}) = 0$ and

$$u = \int_{\Delta_{\mathcal{F}}} K_\xi \, d\mu(\xi).$$

Thus, for any $u \in \mathcal{P}(\Omega_0; \mathcal{F})$, there exists a unique positive measure $\nu$ on $\Omega_0 \cup \Delta_{\mathcal{F}}(\Omega_0)$ such that $\nu(\Delta_{\mathcal{F}} - \Delta_{\mathcal{F}}^1) = 0$ and

$$ u = \int_{\Omega_0 \cup \Delta_{\mathcal{F}}} K_x d\nu(x) . $$

### § 6. Ordering among full-harmonic structures.

Let $\mathcal{F}_1$, and $\mathcal{F}_2$ be two full-harmonic structures. If, for any $D \in \mathcal{O}$ with smooth boundary and for any $x \in D$, $\mu_{\mathcal{F}_1,x}^D \leq \mu_{\mathcal{F}_2,x}^D$

(i.e., $\int f d\mu_{\mathcal{F}_1,x}^D \leq \int f d\mu_{\mathcal{F}_2,x}^D$ for all $f \in C(\partial D)$ such that $f \geq 0$),

then we write $\mathcal{F}_1 \leq \mathcal{F}_2$. This is an order relation in the class of full-harmonic structures on $\Omega$.

Theorem. If $\mathcal{F}_1 \leq \mathcal{F}_2$ then $m_{\mathcal{F}_1,x}^{\Omega_0} \leq m_{\mathcal{F}_2,x}^{\Omega_0}$ for all $x \in \Omega_0$.

Let $G \in \mathcal{O}$ be regular with respect to the Dirichlet problem. For $f \in C(\partial G)$, the Dirichlet solution with boundary values $f$ on $\partial G$, $0$ on the ideal boundary is denoted by $h_{0,f}^G$. Let

$$ \tilde{H}_0(D) = \left\{ u \in H(D); \text{ there exists a compact set } K \text{ such that } \atop K^i \supset \partial D \text{ and } \Omega - K \text{ is regular and } h_{0,u}^{D-K} = u \text{ on } D - K \right\}. $$

Then, $\mathcal{F}_0 = \left\{ \tilde{H}_0(D) \right\}_{D \in \mathcal{O}}$ is a full-harmonic structure and the $\mathcal{F}_0$-Green kernel on $\Omega_0$ is the Green function $G_x^{\Omega_0}(y)$. $\tilde{\mathcal{F}}_0$ is the smallest full-harmonic structure. The corresponding ideal boundary $\Delta_{\tilde{\mathcal{F}}_0}(\Omega - K_0)$ is the Martin boundary.

## Chap. II. Boundary value problems and examples of full-harmonic structures.

Let $\Omega$ be a Green space or a hyperbolic Riemann surface, and

let $K$ be a compact set in $\Omega$ with smooth boundary $\partial K (\neq \emptyset)$. Let $C^1(\partial K)$ be the set of functions which are $C^1$ in a neighborhood of $\partial K$. We consider a compactification $\Omega^*$ of $\Omega$ and we shall treat the problem to find a harmonic function $u$ on $\Omega - K$ satisfying a preassigned boundary condition on $\Delta = \Omega^* - \Omega$ and $u = f$ on $\partial K$.

## § 1. Dirichlet problem. (cf. [2] and [4])

For an extended real valued function $g$ on $\Delta$, we consider the classes

$$\overline{\mathscr{S}}_g = \left\{ s; \text{ superharmonic, bounded below on } \Omega, \varliminf_{x \to \xi, x \in \Omega} s(x) \geq g(\xi) \text{ for any } \xi \in \Delta \right\} \cup \{\infty\}$$

$$\underline{\mathscr{S}}_g = \left\{ -s; \ s \in \overline{\mathscr{S}}_{-g} \right\},$$

and let $\overline{H}_g(x) = \inf \left\{ s(x); \ s \in \overline{\mathscr{S}}_g \right\}$ and $\underline{H}_g(x) = \sup \left\{ s(x); \ s \in \underline{\mathscr{S}}_g \right\}$. If $\overline{H}_g = \underline{H}_g$ and is finite, then we say that $g$ is resolutive and we write $H_g = \overline{H}_g = \underline{H}_g$. $H_g$ is a harmonic function on $\Omega$. If any $g \in C(\Delta)$ is resolutive, then $\Omega^*$ is called a resolutive compactification.

<u>Assumption 1.</u> $\Omega^*$ is a resolutive compactification.

The harmonic measure $\mu_x (x \in \Omega)$ is a positive Radon measure on $\Delta$ such that $H_g(x) = \int g d\mu_x$ for any $g \in C(\Delta)$.

Let $g$ be a resolutive function on $\Delta$ and let $f \in C^1(\partial K)$. Then

$$g_1 = \begin{cases} g \text{ on } \Delta \\ \\ f \text{ on } \partial K \end{cases}$$

is resolutive with respect to the compactification $\Delta \cup (\Omega - K) \cup \partial K$ of $\Omega - K$. The solution will be denoted by $H_{g,f}^{\Delta,\partial K}$. We shall write

$$u_{0,f}^K \equiv H_{0,f}^{\Delta,\partial K} \quad \text{and} \quad v_g^K \equiv H_{g,0}^{\Delta,\partial K}.$$

## § 2. HD-space.

For an open set G and two harmonic functions u and v on G, let $\langle u, v \rangle_G$ be the mixed Dirichlet integral over G and let $\|u\|_G^2 = \langle u, u \rangle_G$. Let HD(G) be the set of all harmonic functions u on G such that $\|u\|_G < +\infty$.

We denote by $R_D(\Delta)$ the set of all resolutive functions g on $\Delta$ such that $H_g \in HD(\Omega)$. We know that if $g \in R_D(\Delta)$ and $f \in C^1(\partial K)$, then $H_{g,f}^{\Delta,\partial K} \in HD(\Omega - K)$; in particular, $u_{0,f}^K$, $v_g^K \in HD(\Omega - K)$.

Doob's lemma (cf. [3] and [4]). There exists $M > 0$, depending only on K and x, such that, for any $g \in R_D(\Delta)$,

$$\int g^2 d\mu_x \leq M \|v_g^K\|^2.$$

Corollary. $R_D(\Delta) \subseteq L^2(\mu)$.

## § 3. Normal derivative on an ideal boundary.

Let $x_0 \in \Omega$ be fixed and let $\mu \equiv \mu_{x_0}$. Let A be a $\mu$-measurable set on $\Delta$. Given $u \in HD(\Omega - K)$ and a $\mu$-measurable function $\varphi$ on A, we say that u has a normal derivative $\varphi$ on A, or $\varphi$ is a normal derivative of u on A, if, for any $g \in R_D(\Delta)$ such that $g = 0$ $\mu$-a.e. on $\Delta - A$, $\varphi g$ is $\mu$-summable on A and

$$\langle u, v_g^K \rangle_{\Omega - K} = - \int_A \varphi g \, d\mu.$$

$\varphi$ may not be uniquely (even $\mu$-a.e.) determined by u. But, under the following assumption, if u, v $\in HD(\Omega)$ have the same normal

derivative on $\Delta$, then $u = v + \text{const}$.

Assumption 2. For any $u \in HD(\Omega)$, there exists $u^* \in R_D(\Delta)$ such that $u = H_{u^*}$.

Such $u^*$ is uniquely determined $\mu$-a.e. We hereafter assume Assumptions 1 and 2. The Wiener, Royden, Kuramochi and Martin compactifications are examples.

We can show that, for any $u \in HD(\Omega - K)$ such that $u = f \in C^1(\partial K)$ on $\partial K$, there exists $u^* \in R_D(\Delta)$ (uniquely determined $\mu$-a.e.) such that $u = H_{u^*,u}^{\Delta,\partial K}$ on $\Omega - K$.

§ 4. Boundary value problems.

Let $A_0$ be a $\mu$-measurable subset of $\Delta$ and let $\beta(\xi)$ be a non-negative $\mu$-measurable function on $\Delta - A_0$, finite $\mu$-a.e. We consider the following boundary condition $(A_0, \beta)$:

$$\begin{cases} u^* = 0 & \text{on } A_0 \quad \mu\text{-a.e.} \\ \varphi = \beta u^* & \text{on } \Delta - A_0 \quad \mu\text{-a.e. for a normal derivative } \varphi \\ & \text{of } u \text{ on } \Delta - A_0. \end{cases}$$

If we write

$$\alpha = \begin{cases} 0 & \text{on } A_0 \\ \dfrac{1}{1+\beta} & \text{on } \Delta - A_0, \end{cases}$$

then $\alpha$ is a $\mu$-measurable function on $\Delta$, $0 \leqslant \alpha \leqslant 1$ and the condition $(A_0, \beta)$ is equivalent to

($\alpha$): $(1 - \alpha)u^* = \alpha\varphi$ $\mu$-a.e. for a normal derivative $\varphi$ of $u$ on $\Delta - A_0$.

Given $f \in C^1(\partial K)$, if there exists $u \in HD(\Omega - K)$ such that $u = f$ on $\partial K$ and $u$ satisfies the condition ($\alpha$) or $(A_0, \beta)$, then $u$ will be denoted by $u_{\alpha,f}^K$. If $\alpha \equiv 0$, then our problem reduces to the Dirichlet problem, so that $u_{0,f}^K$ coincides with the one

defined in § 1.

Theorem 1. $u_{\alpha,f}^{K}$ is unique if it exists. If $f \geq 0$ and $\alpha_1 \leq \alpha_2$, then $u_{\alpha_1,f}^{K} \leq u_{\alpha_2,f}^{K}$; in particular, for any $\alpha$,

$$0 \leq u_{0,f}^{K} \leq u_{\alpha,f}^{K} \leq u_{1,f}^{K}.$$

§ 5. Existence theorem.

Theorem 2. If $\beta$ is bounded, then $u_{\alpha,f}^{K}$ exists for any $K$ and $f$.

Sketch of the proof (cf. [3] and [4]): The case $\alpha \equiv 0$ $\mu$-a.e. (i.e., $A_0 = \Delta$ ) is already known. Thus, we assume $\mu( \Delta - A_0) > 0$.

Case I. $\beta \equiv 0$. Consider the family

$$\overline{\Phi}_f = \left\{ u \in HD(\Omega - K); \; u = f \text{ on } \partial K, \; u^* = 0 \;\; \mu\text{-a.e. on } A_0 \right\}.$$

$\overline{\Phi}_f$ is non-empty, convex and complete with respect to the norm $\| \; \|_{\Omega-K}$. Hence there exists $u_0 \in \overline{\Phi}_f$ such that $\|u_0\|_{\Omega-K} = \min\limits_{u \in \overline{\Phi}_f} \|u\|_{\Omega-K}$ and we see that $0$ is a normal derivative of $u_0$ on $\Delta - A_0$.

Case II. $\beta \not\equiv 0$ on $\Delta - A_0$ and $\beta$ is bounded. Let $A_1 = \left\{ \xi \in \Delta - A_0; \; \beta(\xi) = 0 \right\}$ and $A' = \left\{ \xi \in \Delta - A_0; \; \beta(\xi) > 0 \right\}$. We may assume $\mu(A') > 0$. Using the Doob's lemma, we can prove

Lemma. Given $\varphi \in L^2(\mu; \; \Delta - A_0)$, there exists a unique $u \in \overline{\Phi}_0$ such that $\varphi$ is a normal derivative of $u$ on $\Delta - A_0$.

Next, consider the space $\mathcal{L} = \left\{ \varphi \in L^2(\mu; \; A'); \; \int_{A'} \frac{\varphi^2}{\beta} d\mu < \infty \right\}$. $\mathcal{L}$ is a Hilbert space with respect to $(\varphi, \psi)_\beta = \int_{A'} \frac{\varphi \psi}{\beta} d\mu$. By the above lemma, for each $\varphi \in \mathcal{L}$, there exists a unique $u_\varphi \in \overline{\Phi}_0$ such that $u$ has a normal derivative $\varphi$ on $A'$, $0$ on $A_1$. The

mapping $\Gamma : \varphi \to \beta(u^*_\varphi|_{A'})$ is a symmetric, negative definite operator on $\mathcal{L}$, so that $\Gamma - I$ is invertible. Let $v \in \mathfrak{C}_f$ be such that $0$ is a normal derivative of $v$ on $A_1 \cup A'$ and $v = f$ on $\partial K$, determined in Case I. Then $\beta(v^*|_{A'}) \in \mathcal{L}$ and there exists $\psi \in \mathcal{L}$ such that $\beta u^*_\psi - \psi = -\beta v^*$ on $A'$. Then we see that $u = v + u_\psi$ is our solution.

§ 6. $\alpha$-full-harmonic structure $\mathcal{H}_\alpha$.

Let $\alpha$ be a function as above for which $u^K_{\alpha,f}$ always exists. For $D \in \mathcal{O}$ , we define
$$\widetilde{H}_\alpha(D) = \left\{ u \in H(D); \text{ there exists a compact set } K \text{ with smooth boundary such that } K^i \supset \partial D \text{ and } u = u^{\partial K \cap D}_{\alpha,u} \text{ on } D - K. \right\}.$$

Then, $\mathcal{H}_\alpha = \left\{ \widetilde{H}_\alpha(D) \right\}_{D \in \mathcal{O}}$ is a full-harmonic structure. By Theorem 1, if $\alpha_1 \le \alpha_2$, then $\mathcal{H}_{\alpha_1} \le \mathcal{H}_{\alpha_2}$.

Let $K_0$ be a closed sphere in $\Omega$. Let us denote the $\mathcal{H}_\alpha$-Green kernel on $\Omega - K_0$ by $M_{\alpha,x}(y)$. Remark that $M_{0,x} = G_x$ and $M_{1,x}$ is the N-Green function $N_x$ of Kuramochi. Thus, if $\alpha_1 \le \alpha_2$, then

$$0 \le G_x \le M_{\alpha_1,x} \le M_{\alpha_2,x} \le N_x.$$

The ideal boundary $\Delta_{\mathcal{H}_1}(\Omega - K_0)$ is the Kuramochi boundary.

**Theorem 3.** For any $\alpha$, $M_{\alpha,x}(y) = M_{\alpha,y}(x)$ $(\forall x, y \in \Omega - K_0)$.

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