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ON N-SEMIGROUPS

By

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A semigroup S is called an N-semigroup, due to Petrich[8], if S is commutative, cancellative, archimedean, nonpotent(without idempotent). The concept of N-semigroups is very important in the theory of commutative semigroups([2], [4], [16]). N-semigroups were employed firstly in 1956, by Hewitt and Zuckerman[4] in a paper on the semilattice decomposition of commutative semigroups. Since then many papers on N-semigroups have appeared. In this paper we shall discuss synthetically major results of these.

1. TAMURA'S REPRESENTATION

In this section we shall discuss a faithful representation for N-semigroups which is fundamental in the study of N-semigroups. The following 1.1, 1.2, 1.3 are due to Tamura[17] and 1.4 is due to Sasaki[14].

Let S be an N-semigroup and let a be a fixed element of S. Then any element x of S is uniquely expressed as $x = a^n p_\alpha$, where $p_\alpha \in S \setminus aS$ and n is a non-negative integer, since $S = \bigcup_{i=0}^{\infty} P_i$, $P_0 = S \setminus aS$, $P_i =$

$a^iS \setminus a^{i+1}S$ and $P_i \neq \square$, $P_i \cap P_j = \square$ for $i \neq j$.

1.1. Group decomposition. We define two relations τ_a and ρ_a on S as follows:

$x \, \tau_a \, y$ if and only if $x = a^n y$ for some integer $n \geq 0$,

$x \, \rho_a \, y$ if and only if $x \, \tau_a \, y$ or $y \, \tau_a \, x$.

Then τ_a is a compatible partial ordering and ρ_a is a congruence generated by τ_a, hence we get the following:

Theorem 1. The factor semigroup $S^*_a = S/\rho_a$ is a commutative group, and is a homomorphic image of S. And each ρ_a-class S_λ of S is an infinite chain with respect to $\tau_a | S_\lambda$ and it contains exactly one $\tau_a | S_\lambda$-maximal element, which is called the prime respecting a.

S^*_a is called the structure group of S with respect to a, and a is called the standard element for S^*_a. We note that the set of all primes respecting a coincides with $P_o = S \setminus aS$. Let p_λ denote the prime respecting a contained in S_λ. If ε denotes the identity of S^*_a, $p_\varepsilon = a$. For p_α, $p_\beta \in P_o$, there exists unique non-negative integer n such that $p_\alpha p_\beta = a^n p_{\alpha\beta}$. We define here $n = I(\alpha, \beta)$, then $I(\alpha, \beta)$ becomes an index function on S^*_a, which shall be called the index function corresponding to S^*_a. By an index function I on a commutative group G we mean a mapping of $G \times G$ into the additive non-negative integers satisfying the following conditions:

(1) $I(\alpha, \beta) = I(\beta, \alpha)$,

(2) $I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma)$,

(3) $I(\varepsilon, \varepsilon) = 1$, ε is the identity of G,

(4) for any $\alpha \in G$, there exists a positive integer m such that $I(\alpha, \alpha^m) > 0$.

1.2. Construction. The following theorem is important.

Theorem 2. Let G be a commutative group and let I be an index function on G. Let S be the product set of the additive non-negative integers and G, and define a binary operation on S by $(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta)$. Then S is an N-semigroup under this operation and $S^*_{(0,\varepsilon)}$, ε is the identity of G, is isomorphic upon G. Also every N-semigroup may be obtained in this manner.

By $S(G: I)$ we denote the N-semigroup constructed from a commutative group G with an index function I in the above mentioned manner.

1.3. Faithful representation. From the foregoing description we see that a mapping $x = a^n p_\alpha \rightarrow (n, \bar{p}_\alpha)$, $\bar{p}_\alpha$ is the ρ_a-class of S containing p_α, gives a faithful representation of S by $S(S^*_a: I)$, which shall play an important role in the study of N-semigroups. In this paper, a representation of an N-semigroup shall mean always one as given in the above.

1.4. Generalized index functions. Consider a mapping I of the product set $G \times G$ of a commutative group G and itself into the additive non-negative integers satisfying the conditions (1), (2) and (3'), (4') below:

(3') $I(\varepsilon, \varepsilon) > 0$,

(4') for any α there exists a positive integer m such that $I(\alpha, \alpha) + \cdots + I(\alpha, \alpha^m) \geqq I(\varepsilon, \varepsilon)$.

Let G be a commutative group with the above generalized index function I, and let S be the product set of the additive non-negative integers and G. Define a binary operation on S by $(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta)$, then S becomes an N-semigroup. Such S shall be denoted by $S\langle\langle G: I\rangle\rangle$. Especially, in the case $I(\varepsilon, \varepsilon) = 1$, $S\langle\langle G: I\rangle\rangle$ shall be understood as $S(G: I)$. Let $G_{(0,\varepsilon)}$, $I_{(0,\varepsilon)}$ be the structure group of $S\langle\langle G: I\rangle\rangle$ with respect to $(0, \varepsilon)$, ε is the identity of G, and its corresponding index function. Then we can show that $S\langle\langle G: I\rangle\rangle$ is isomorphic upon $S(G_{(0,\varepsilon)}: I_{(0,\varepsilon)})$. Thus we have

Theorem 3. <u>For given any $S\langle\langle G: I\rangle\rangle$ there exist a group-extension G' of a finite cyclic group by G and its corresponding index function I' such that $S\langle\langle G: I\rangle\rangle \cong S(G': I')$.</u>

And we have

Theorem 4. <u>For given any $S(G': I')$, whose G' has a cyclic subgroup of order $c > 1$, there exists $S\langle\langle G: I\rangle\rangle$ such that $S(G': I') \cong S\langle\langle G: I\rangle\rangle$, $I(\varepsilon, \varepsilon) = c$, $G' = G_{(0,\varepsilon)}$, $I' = I_{(0,\varepsilon)}$, where ε is the identity of G, if and only if it holds that $cI'(\overline{(m, \alpha)}, \overline{(n, \beta)}) + J(\overline{(m, \alpha)}, \overline{(n, \beta)}) - m - n = cI'(\overline{(0, \alpha)}, \overline{(0, \beta)}) + J(\overline{(0, \alpha)}, \overline{(0, \beta)})$ for all</u>

$0 \leqq m, n < c$, $\alpha, \beta \in G$, where $\overline{(m, \alpha)}$ is the equivalence class modulo ρ, $S(G: I)/\rho = G'$, containing (m, α) and $J(\overline{(m, \alpha)}, \overline{(n, \beta)})$ is a non-negative integer valued function on $G' \times G'$ such that $\overline{(m, \alpha)}\,\overline{(n, \beta)} = \overline{(J(\overline{(m, \alpha)}, \overline{(n, \beta)}), \alpha\beta)}$, $0 \leqq J(\overline{(m, \alpha)}, \overline{(n, \beta)}) < c$.

2. INDEX FUNCTIONS

It was shown in Theorem 2 that if an index function is defined on a commutative group G, then we can construct an N-semigroup such that its structure group is isomorphic upon G. Given a commutative group G, there always exists an index function, for example $I(\alpha, \beta) = 1$ for all $\alpha, \beta \in G$, but it is not easy to determine all the index functions on G. In this section we will discuss how to determine all the index functions on given a finitely generated commutative group and also discuss the semigroup of all generalized index functions on given a commutative group. The following 2.1 is due to Rigge, Tamura and Sasaki[1] and 2.2 is due to Sasaki[14].

2.1. Determination of all index functions. The results are as follows:

Theorem 5. If G is a cyclic group of order n generated by α, the index function values $I(\alpha, \alpha^k)$, $k = 1, \cdots, n-1$, are independent up to relative size considerations and every other function value is determined from these n-1 values by $I(\alpha^i, \alpha^j) = I(\alpha, \alpha^{i+j-1}) + [j, i-1]_{i-1}$, $i \geqq 2$, where $[m, n]_\ell$ denotes $\sum_{p=0}^{\ell-1}(I(\alpha, \alpha^{m+p}) - I(\alpha, \alpha^{n-p}))$ if

$\ell > 0$, and does 0 if $\ell = 0$. The relative sizes of the independent $I(\alpha, \alpha^k)$, $k = 1, \cdots, n-1$, are determined as follows:

the case $n = 2$, $I(\alpha, \alpha) \geqq 0$,

the case $n = 3$, $I(\alpha, \alpha) \geqq 0$, $I(\alpha, \alpha^2) \geqq \max\{0, I(\alpha, \alpha) - 1\}$,

the case $n = 4$, $I(\alpha, \alpha) \geqq 0$, $I(\alpha, \alpha^2) \geqq 0$, $I(\alpha, \alpha^3) \geqq \max\{0, I(\alpha, \alpha) - I(\alpha, \alpha^2), I(\alpha, \alpha) - 1, I(\alpha, \alpha^2) - 1\}$,

the case $n \geqq 5$, $I(\alpha, \alpha^k) \geqq \overline{m}(k)$, $k = 1, \cdots, n-2$, $I(\alpha, \alpha^{n-1}) \geqq \max\{\overline{m}'(0), \overline{m}'(1), \cdots, \overline{m}'(n-5), \overline{m}(n-1), \max_{1 \leqq i \leqq n-2} \{I(\alpha, \alpha^i) - 1\}\}$,

where $\overline{m}(k) = \max_{0 \leqq i \leqq [\frac{k-1}{2}]} \{[1, k-1]_i\}$ and $\overline{m}'(k) = \max_{1 \leqq i \leqq [\frac{n-k-3}{2}]} \{[k+1, n-2]_i + I(\alpha, \alpha^{i+k+1}) - 1\}$.

Theorem 6. If G is an infinite cyclic group generated by α, the function values $I(\alpha, \alpha^k)$, $k = \pm 1, \cdots$, are independent up to relative size considerations and every other function value is determined from these by $I(\alpha^i, \alpha^j) = I(\alpha, \alpha^{i+j-1}) + [j, i-1]_{i-1}$ if $i \geqq 2$ and $I(\alpha^i, \alpha^j) = I(\alpha, \alpha^i) + [i+1, j-1]_{-i}$ if $i \leqq -1$. And the relative sizes of the $I(\alpha, \alpha^k)$, $k = \pm 1, \cdots$, are given as follows: $I(\alpha, \alpha^k) \geqq \overline{m}(k)$ for $k \geqq 1$, $I(\alpha, \alpha^{-1}) \geqq \overline{n}(-1)$, $\overline{n}'(-k) \geqq I(\alpha, \alpha^{-k}) \geqq \overline{n}(-k)$ for $k \geqq 1$, and for any non-zero integer s there exists a positive integer t_s such that $[st_s, s-1]_s \geqq 0$ if $s \geqq 1$, $[s, st_s - 1]_{-s} \geqq 0$ if $s \leqq -1$, where $\overline{n}(-k) = \max_{0 \leqq i} \{[1, i-k]_i\}$, $\overline{n}'(-k) = \min_{0 \leqq i \leqq [\frac{k-2}{2}]} \{I(\alpha, \alpha^{-1}) + 1 + [-i-1, i-k]_i\}$.

Theorem 7. Suppose that the direct product $G = A \times B$ of two commutative groups A, B and that index function values I_A for A and I_B for B are already given. Then the set $I_{A,B}$ of function values $I((\alpha, \varepsilon'), (\varepsilon, \beta))$, $\alpha \in A \setminus \{\varepsilon\}$, $\beta \in B \setminus \{\varepsilon'\}$, where ε, ε' are the identities of A and B, are independent up to relative size considerations and every other value $I((\alpha_1, \beta_1), (\alpha_2, \beta_2))$ is determined from I_A, I_B and $I_{A,B}$ by $I((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = I((\alpha_1, \varepsilon'), (\alpha_2, \varepsilon')) + I((\varepsilon, \beta_1), (\varepsilon, \beta_2)) + I((\alpha_1\alpha_2, \varepsilon'), (\varepsilon, \beta_1\beta_2)) - I((\alpha_1, \varepsilon'), (\varepsilon, \beta_1)) - I((\alpha_2, \varepsilon'), (\varepsilon, \beta_2))$.

The complete solution for the relative sizes of the independent elements in the above Theorem 7 is not yet obtained.

Theorem 8. Let $G = A_1 \times \cdots \times A_n$ be the direct product of n commutative groups $A_1, \cdots, A_n$. Suppose that I-values I_i for A_i, $i = 1, \cdots, n$, are already given, and consider the sets I'_j of function values: $I'_j((\alpha_1, \cdots, \alpha_{j-1}, \varepsilon_j, \cdots, \varepsilon_n), (\varepsilon_1, \cdots, \varepsilon_{j-1}, \alpha_j, \varepsilon_{j+1}, \cdots, \varepsilon_n))$, $j = 2, \cdots, n$, where $(\alpha_1, \cdots, \alpha_{j-1}, \varepsilon_j, \cdots, \varepsilon_n) \neq (\varepsilon_1, \cdots, \varepsilon_n)$, $(\varepsilon_1, \cdots, \varepsilon_{j-1}, \alpha_j, \varepsilon_{j+1}, \cdots, \varepsilon_n) \neq (\varepsilon_1, \cdots, \varepsilon_n)$ and ε_k is the identity of A_k. Then the union of I'_j, $j = 2, \cdots, n$, is a set of I-values independent up to relative size considerations and every other value is determined from $I_1, \cdots, I_n, I'_2, \cdots, I'_n$.

Since every finitely generated commutative group is the direct product of a finite number of cyclic groups, the results obtained above can be applied to any finitely generated commutative group.

All the index functions on groups of order $\leqq 4$ are given as the following tables:

Group of order 1.

	ε
ε	1

Group of order 2.

	ε	α
ε	1	1
α	1	i

$i \geqq 0.$

Group of order 3.

	ε	α	α^2
ε	1	1	1
α	1	i	j
α^2	1	j	j+1-i

$i \geqq 0,$
$j \geqq \max\{0,\ i-1\}.$

Group of order 4.

cyclic group,

	ε	α	α^2	α^3
ε	1	1	1	1
α	1	i	j	k
α^2	1	j	j+k-i	1+k-i
α^3	1	k	1+k-i	1+k-j

$i \geqq 0,$
$j \geqq 0,$
$k \geqq \max\{0,\ i-j,\ i-1,\ j-1\}.$

Klein's group.

	ε	α	β	$\alpha\beta$
ε	1	1	1	1
α	1	i	j	i+1-j
β	1	j	k	k+1-j
$\alpha\beta$	1	i+1-j	k+1-j	k+i+1-2j

$j \geqq 0,$
$i \geqq \max\{0,\ j-1\},$
$k \geqq \max\{0,\ j-1,\ 2j-i-1\}.$

2.2. <u>Semigroups of generalized index functions.</u> Let $\mathfrak{J}(G)$ be the set of all generalized index functions satisfying (1), (2), (3'), (4') in 1.4 on given a commutative group G. $\mathfrak{J}(G)$ is not empty. Define I_1+I_2 and aI_1 for $I_1,\ I_2 \in \mathfrak{J}(G)$ and non-negative integer a as follows:

$$(I_1+I_2)(\alpha,\beta) = I_1(\alpha,\beta) + I_2(\alpha,\beta), \quad (aI_1)(\alpha,\beta) = a(I_1(\alpha,\beta)).$$

Then $I_1 + I_2 \in \mathfrak{J}(G)$, $aI_1 \in \mathfrak{J}(G)$ and

Theorem 9. $\mathfrak{J}(G)$ forms a commutative, cancellative, nonpotent semigroup under $(+)$ and satisfies $a(I_1 + I_2) = aI_1 + aI_2$, $(a + b)I_1 = aI_1 + bI_1$, $(ab)I_1 = a(bI_1)$.

Let I_1, I_2 be any elements of $\mathfrak{J}(G)$. Consider the direct product of $S(\!(G: I_1)\!)$ and $S(\!(G: I_2)\!)$ and define a relation σ on it as follows:

$((m, \alpha), (n, \beta))\ \sigma\ ((m', \alpha'), (n', \beta'))$ if and only if $m + n = m' + n'$, $\alpha = \alpha'$, $\beta = \beta'$.

Then we have that $S(\!(G: I_1 + I_2)\!) \cong (S(\!(G: I_1)\!) \times S(\!(G: I_2)\!))/\sigma$. And it is easily seen that $S(\!(G: I_1)\!)$ is isomorphic upon a subsemigroup of $S(\!(G: aI_1)\!)$ for any positive integer a. Thus we have

Theorem 10. $S(\!(G: I_1 + I_2)\!)$, $I_1, I_2 \in \mathfrak{J}(G)$, can be embedded isomorphically onto a homomorphic image of the direct product $S(\!(G: I_1)\!) \times S(\!(G: I_2)\!)$, and $S(\!(G: I_1)\!)$ be embedded into $S(\!(G: aI_1)\!)$ for any positive integer a.

3. STRUCTURE GROUPS

In this section we shall consider the relations between the structure groups of an N-semigroup S with respect to a and b of S and their corresponding index functions. All results in this section are due to Sasaki[12].

Let $S = S(G: I)$ be an N-semigroup. Let $G_{(m,\alpha)}$ and $G_{(n,\beta)}$ be the structure groups of S with respect to (m, α) and $(n, \beta) \in S$ respectively, and let $\rho_{(m,\alpha)}$, $\rho_{(n,\beta)}$ be congruences on S such that $G_{(m,\alpha)} = S/\rho_{(m,\alpha)}$, $G_{(n,\beta)} = S/\rho_{(n,\beta)}$ respectively. For simplicity we shall denote by $\overline{(\ell, \xi)}$ and $\overline{\overline{(\ell, \xi)}}$ the equivalence classes of S modulo $\rho_{(m,\alpha)}$ and $\rho_{(n,\beta)}$ containing $(\ell, \xi) \in S$ respectively. Then the following holds:

Theorem 11. <u>$G_{(m,\alpha)}/[\overline{(n, \beta)}]$ is isomorphic upon $G_{(n,\beta)}/[\overline{\overline{(m, \alpha)}}]$, where $[\eta]$ means the cyclic group generated by η.</u>

From the above theorem we easily see that $G_{(m,\varepsilon)}$ is the group-extension of $[\overline{(0, \varepsilon)}]$ by G.

For $(m, \alpha) \in S(G: I)$, $\xi \in G$ and for a non-negative integer p, we put $\Pi_p((m, \alpha), \xi) = mp + \rho_1^{p-1}(\alpha) + I(\alpha^p, \alpha^{-p}\xi)$, where $\rho_r^s(\alpha)$ means $\sum_{i=r}^{s} I(\alpha, \alpha^i)$ if $s-r \geqq 0$, does 0 if $s-r = -1$, and $-\sum_{i=s+1}^{r-1} I(\alpha, \alpha^i)$ if $s-r \leqq -2$. Then the following three conditions are equivalent:

a) $(0, \alpha^{-p}\xi) \in S \setminus (m, \alpha)S$;

b) $m + I(\alpha, \alpha^{-p-1}\xi) > 0$,

c) $\Pi_{p+1}((m, \alpha), \xi) - \Pi_p((m, \alpha), \xi) > 0$.

Therefore we can state how to construct $G_{(n,\beta)}$ from given $G_{(m,\alpha)}$:

(i) Select all primes of type $(0, \xi)$ respecting (m, α),

(ii) for each $(0, \xi)$ in (i), take $(0, \xi), (0, \alpha\xi), \cdots, (0, \alpha^p\xi)$, where p is a positive integer such that $0 = mp + \rho_1^{p-1}(\alpha) + I(\alpha^p, \xi) < m(p+1) + \rho_1^p(\alpha) + I(\alpha^{p+1}, \xi)$,

(iii) and put $\overline{\overline{(0, \alpha^i\xi)}}, \overline{\overline{(1, \alpha^i\xi)}}, \cdots, \overline{\overline{(n-1+I(\beta, \beta^{-1}\alpha^i\xi), \alpha^i\xi)}}$ for all $i (0 \leqq i \leqq p)$ such that $n + I(\beta, \beta^{-1}\alpha^i\xi) > 0$.

Then all members in (iii) form just the structure group $G_{(n,\beta)}$.

Let $(r, \xi), (s, \eta)$ be any primes of $S(G: I)$ respecting (m, α). Then the index function $I_{(m,\alpha)}$ corresponding to $G_{(m,\alpha)}$ is given by $I_{(m,\alpha)}(\overline{(r, \xi)}, \overline{(s, \eta)}) = p$, where p is a non-negative integer such that $\prod_p((m, \alpha), \xi\eta) \leqq r + s + I(\xi, \eta) < \prod_{p+1}((m, \alpha), \xi\eta)$. Let $I_{(m,\alpha)}$ and $I_{(n,\beta)}$ be the index functions corresponding to $G_{(m,\alpha)}$ and $G_{(n,\beta)}$ respectively. Then we get the following:

Theorem 12. Let (r, ξ) and (s, η) be primes of S respecting (m, α). If $\prod_p((m, \alpha), \xi\eta) \leqq r + s + I(\xi, \eta) < \prod_{p+1}((m, \alpha), \xi\eta)$ and $\prod_q((n, \beta), \xi\eta) \leqq r + s + I(\xi, \eta) < \prod_{q+1}((n, \beta), \xi\eta)$, then $I_{(m,\alpha)}(\overline{(r, \xi)}, \overline{(s, \eta)}) = p$ and $I_{(n,\beta)}(\overline{\overline{(r, \xi)}}, \overline{\overline{(s, \eta)}}) = q - k - h$, where k and h are non-negative integers such that $\prod_k((n, \beta), \xi) \leqq r < \prod_{k+1}((n, \beta), \xi)$, $\prod_h((n, \beta), \eta) \leqq s < \prod_{h+1}((n, \beta), \eta)$.

4. THE ISOMORPHISM PROBLEM

The problem of distinct representations for isomorphic N-semigroups was proposed by Tamura[17], and was discussed by Sasaki[11], [12]. Lately Higgins[5] gave an isomorphism theorem, for finitely generated N-semigroups, which depends on the canonical representation. The following theorem is due to Sasaki [11], [12].

Theorem 13. Let $S(G: I)$, $S(G': I')$ be N-semigroups. $S(G: I)$ is isomorphic upon $S(G': I')$ if and only if there exist cyclic subgroups $[\omega]$ of G and $[\omega']$ of G' such that $G/[\omega] \cong G'/[\omega']$(under ψ) and there exist representative systems $\Gamma = \{\xi_\alpha\}$ of the cosets of $[\omega]$ in G and $\Gamma' = \{\xi'_\alpha\}$ of the cosets of $[\omega']$ in G' satisfying

(1) for any ξ_α, $\xi_\beta \in \Gamma$, if $\xi_\alpha \xi_\beta = \omega^{\ell}\xi_\gamma$, $\xi_\gamma \in \Gamma$, then $\xi_\alpha\tau \cdot \xi_\beta\tau = \omega'^{\ell'} \cdot \xi_\gamma\tau$ and $I'(\xi_\alpha\tau, \xi_\beta\tau) = n'\ell' + \ell + \rho'^{\,0}_{-\ell'}(\omega') - I'(\omega'^{-\ell'}, \xi_\alpha\tau \cdot \xi_\beta\tau)$, $\ell' = -n\ell + I(\xi_\alpha, \xi_\beta) - \rho^{0}_{-\ell}(\omega) + I(\omega^{-\ell}, \xi_\alpha\xi_\beta)$,

(2) for any integer s and $\xi_\alpha \in \Gamma$, $\xi'_\beta \in \Gamma'$

(i) $s + n't(s) + \rho'^{\,t(s)-1}_{1}(\omega') + I'(\omega'^{t(s)}, \xi_\alpha\tau) \geqq 0$, $t(s) = -ns - \rho^{s-1}_{1}(\omega) - I(\omega^{s}, \xi_\alpha)$ and

(ii) $s + nt'(s) + \rho^{t'(s)-1}_{1}(\omega) + I(\omega^{t'(s)}, \xi'_\beta\tau^{-1}) \geqq 0$, $t'(s) = -n's - \rho'^{\,s-1}_{1}(\omega') - I'(\omega'^{s}, \xi'_\beta)$, where τ is a mapping of Γ onto Γ' such that $\tau: \xi_\alpha \longrightarrow \xi'_\alpha$ if $([\omega]\xi_\alpha)\psi = [\omega']\xi'_\alpha$ and n, n' are non-negative integers such that $h = n'h' + \rho'^{h'}_{1}(\omega')$ and $h' = nh + \rho^{h}_{1}(\omega)$ for the orders h and h' of ω and ω', where h = 0 means the order of ω is infinite and h' does so. $\rho'^{s}_{r}(\alpha')$ is the symbol obtained by replacing I, α with I', α' in $\rho^{s}_{r}(\alpha)$.

Suppose that S is a finitely generated N-semigroup. Then all the structure groups of S are finite(Chrislock[3]), so there is an element $a \in S$ such that the structure group G_a of S with respect to a is of minimal order. Such a is called a normal standard. Let G_a and I_a be the structure group of S with respect to a normal standard element a of S and its corresponding index function. Then the representation $S(G_a : I_a)$ for S is called a canonical, due to Higgins[5]. We should note that an N-semigroup S may have many normal standard elements, so the canonical representation for S is not unique in general. Higgins[5] gave the following theorem:

Theorem 14. Let S and S' be finitely generated N-semigroups and let S(G: I) and S(G': I') be canonical representations for S and S' respectively. Then S is isomorphic upon S' if and only if G" with I" equivalent to G' with I' can be obtained from G with I by changing normal standard elements.

We say that G" with I" is equivalent to G' with I' if G" is isomorphic upon G' under some mapping φ such that $I''(\xi, \eta) = I'(\xi\varphi, \eta\varphi)$ for $\xi, \eta \in G''$.

The theorem corresponding to Theorem 14 in general case is given as follows(Sasaki[13]):

Theorem 15. Let S and S' be N-semigroups and let $S(G: I)$ and $S(G': I')$ ~~be~~ representations for S and S' respectively. Then S is isomorphic upon S' if and only if there is an element $(0,\alpha) \in S$ such that $G_{(0,\alpha)}$ with $I_{(0,\alpha)}$ is equivalent to G' with I', where $G_{(0,\alpha)}$, $I_{(0,\alpha)}$ are the structure group of S with respect to $(0,\alpha)$ and its corresponding index function.

5. FINITELY GENERATED N-SEMIGROUPS

The following classification for N-semigroups is due to Petrich[8]:

- N-semigroups
  - power joined
    - finitely generated
      - 1 generator
      - 2 generators
      - more than 2 generators
    - not finitely generated
      - with indecomposable element(indec. e.)
      - without indec. e.
  - not-power-joined
    - with indec. e.
    - without indec. e.

An element a is said to be indecomposable if $a \neq bc$ for all b, c.

Chrislock[3] showed the following in a more general form:

An N-semigroup S is a finitely generated if and only if all the structure groups of S are finite, S is a power joined, i.e. for any a, b there exist positive integers m, n such that $a^m = b^n$, if and only if all the structure groups of S are periodic.

Petrich[8] has given a representation for N-semigroups with two generators by the set of ordered pairs of non-negative integers and Higgins[5] has given a new and somewhat distinct representation for finitely generated N-semigroups. In this section we shall treat of Higgins' representation for finitely generated N-semigroups, that is, of the embedding of finitely generated N-semigroups in the direct product of a finite commutative group and the additive positive integers. Excepting theorems 16 and 17, all the results in this section are due to Higgins[5].

5.1. Homomorphisms. We shall consider the homomorphisms of finitely generated N-semigroups into the additive positive integers and onto groups. The following two theorems are required:

Theorem 16(Tamura[18]). Let Ξ be a set of implications. Let τ be the class of all semigroups satisfying all implications in Ξ. Then every semigroup has a greatest homomorphic image of type τ.

Theorem 17(Levin and Tamura[6]). Any power joined and power cancellative N-semigroup can be embedded in the additive positive rational numbers.

Let S be a finitely generated N-semigroup. Then, by Theorems 16 and 17, S has a greatest power joined, power cancellative homomorphic image M which is isomorphic upon a subsemigroup of the additive positive integers. Since any homomorphism between subsemigroups of the additive positive integers is an isomorphism, any positive integer homomorphic image of S is isomorphic upon M. Thus we have

Theorem 18. Let S be a finitely generated N-semigroup. Then the homomorphic image of S into the additive positive integers is uniquely determined.

Let G be any group homomorphic image of a finitely generated N-semigroup S. Take some element a of the pre-image of the identity of G, then G is a homomorphic image of the structure group of S with respect to a. Therefore we get

Theorem 19. Let S be a finitely generated N-semigroup. Let G be any group homomorphic image of S. Then G is a homomorphic image of some structure group of S.

5.2. J-functions. Let G be a finite commutative group. Consider a mapping J of G into the positive integers satisfying the following conditions:

(1) $J(e) = |G|$, where e is the identity of G,

(2) for all $a, b \in G$, it holds $J(a) + J(b) - J(ab) = k|G|$, where k is a non-negative integer,

(3) for every $a \in G$, there is a positive integer n such that

$J(a) + J(a^n) - J(a^{n+1}) = h|G|$, and h is a positive integer. Then we can show that there is one and only one J-function which can be defined on any structure group of a finitely generated N-semigroup S and that J(a) is equal to the number of elements of S which are prime to a, hence the J-function may be considered as a function defined on S. And we can also show that the J-function on a finitely generated N-semigroup S is a homomorphism of S into the additive positive integers. Thus the foregoing homomorphic image M of S in 5.1 may be constructed from the J-function.

5.3. Subdirect products. A semigroup S is a subdirect product of the direct product $R \times T$ of two semigroups R and T if S is a subsemigroup of $R \times T$ and the projections of S into R and T exactly coincide with R and T respectively. It is easily seen that S is isomorphic upon a subdirect product of $R \times T$ if and only if there exist homomorphisms H and K of S onto R and T, and the intersection of the pre-images of $r \in R$ and $t \in T$ in S contains at most one element of S. Let S be a finitely generated N-semigroup, and let Q be a homomorphism of S onto a finite commutative group G, which may be chosen as the structure group of some representation for S by Theorem 19. As a homomorphism of S into the additive positive integers we may use the J-function. Let M be the homomorphic image of S under J. Then we can show that the intersection of the pre-images of $r \in G$

and $t \in M$ in S, as given by Q and J, contains at most one element of S. Therefore

Theorem 20. A finitely generated N-semigroup is isomorphic upon a subdirect product of the direct product of a finite commutative group and a subsemigroup of the additive positive integers. The converse also holds.

The following is also obtained.

Theorem 21. A finitely generated N-semigroup S is isomorphic upon the direct product of a structure group G and a subsemigroup of the additive positive integers if and only if, for some representation for S in terms of G and its corresponding index function I, every element of the form $(0, \alpha)$ is a normal standard element.

6. POWER JOINED N-SEMIGROUPS

In this section we shall show that in the case of power joined N-semigroups we can obtain the theorems corresponding to Higgins' theorems 18 ~ 21. The results in this section are due to Tamura and Sasaki[15].

6.1. Homomorphisms. Let R' be a subsemigroup of the additive positive rational numbers R. Then R' is the union of ascending chain of finitely generated subsemigroups of R:

$$R' = \bigcup_{n=1}^{\infty} R'_n, \quad R'_1 \subseteq R'_2 \subseteq \cdots \quad , R'_n = R' \cap [1/n!].$$

Using Theorems 16 and 17, therefore, we get

Theorem 22. Let S be a power joined N-semigroup. Then the homomorphic image of S into the additive positive rational numbers is uniquely determined.

And we have easily

Theorem 23. Let S be a power joined N-semigroup. Let G be any group homomorphic image of S. Then G is a homomorphic image of some structure group of S.

6.2. $\bar{J}$-functions. Let G be a periodic commutative group. Define a mapping $\bar{J}$ of G into the additive positive rational numbers as follows:

(1) $\bar{J}(e) = 1$, e is the identity of G,

(2) for all a, $b \in G$, $\bar{J}(a) + \bar{J}(b) - \bar{J}(ab)$ is a non-negative integer.

Then we can prove that there is one and only one $\bar{J}$-function which can be defined on any structure group of a power joined N-semigroup S. Let G and its corresponding index function I be a representation for S: $S = S(G: I)$. Then we may say the $\bar{J}$-function on G is given as $\bar{J}(\alpha) = \rho_1^s(\alpha)/s$, where s is the order of α. For $(m, \alpha) \in S$ we define $\bar{\mathfrak{J}}((m, \alpha)) = m + \bar{J}(\alpha)$. Then the function $\bar{\mathfrak{J}}$ on S becomes a homomorphism of S into the additive positive rational numbers, hence the foregoing homomorphic image of S in Theorem 22 may be constructed from the $\bar{\mathfrak{J}}$-function

6.3. Subdirect products. Using the same argument with the case of finitely generated, we obtain the following theorems:

Theorem 24. A power joined N-semigroup is isomorphic upon a subdirect product of the direct product of a periodic commutative group and a subsemigroup of the additive positive rational numbers. The converse also holds.

Theorem 25. A power joined N-semigroup is isomorphic upon the direct product of a structure group G and a subsemigroup M of the additive positive rational numbers if and only if M is a subsemigroup of the additive positive integers and for some representation for S in terms of G and its corresponding index function I, $\mathfrak{Z}$-values of all elements of the form $(0, \alpha)$ are equal.

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