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On Finite Controllability of Second-Order Evolution Equations in Hilbert Spaces.

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1. Introduction We consider controllability of a second-order evolution equation in a Hilbert space E ;

$$\frac{d^2u}{dt^2} = Au(t) + Bf(t) \qquad 0 < t \leqq T \tag{1}$$

with the initial condition

$$u(0) = \frac{du}{dt}(0) = 0 \tag{2}$$

where A is a selfadjoint operator in E and B is a bounded linear operator on a Hilbert space F to E. A function f(t) belongs to $C^1([0, T] : F)$ and it is called a control function. A function u(t) is defined on [0, T] and takes values in E. H.O.Fattorini ([3]) studied controllability of a first-order evolution equation in E ;

$$\frac{du}{dt} = Au(t) + Bf(t) \qquad 0 < t \leqq T \tag{3}$$

with the initial condition

$$u(0) = 0 \tag{4}$$

We shall derive the analogous result for controllability of (1), (2).

2. Preliminaries Let E and F be two complex Hilbert spaces and let A be a selfadjoint semibounded above operator with its domain $\mathcal{D}(A)$ in E. We denote the set of all bounded linear operators on a Hilbert space X into a Hilbert space Y by $\mathcal{L}(X, Y)$. Let B be an operator $\in \mathcal{L}(F,E)$.

The norm and the scalar product in E are respectively denoted by $\| \quad \|$ and $(\; , \;)$. A control f(t) is a function belonging to $C^1([0, T] ; F)$ for some positive T. Since A is semibounded above, we find some $\alpha \geq 0$ and $\delta > 0$ such that $((-A+\alpha)u, u) \geq \delta \|u\|^2$ for $u \in \mathcal{D}(A)$. We denote the positive square root of the positive operator $A_\alpha = -A + \alpha$ by $A_\alpha^{\frac{1}{2}}$. $\mathcal{D}(A_\alpha^{\frac{1}{2}})$ becomes a Hilbert space denoted by $H_{\frac{1}{2}}$ with its inner product defined by $(u, v)_{H_{\frac{1}{2}}} = (A_\alpha^{\frac{1}{2}} u, A_\alpha^{\frac{1}{2}} v)$ for $u, v \in \mathcal{D}(A_\alpha^{\frac{1}{2}})$. Putting $u_1 = u$, $u_2 = \frac{du}{dt}$, the second-order evolution equation (1) with the initial condition (2) is reduced formally to the first-order equation

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(t) = \mathfrak{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(t) + \mathfrak{B}f(t) \tag{5}$$

where

$$\mathfrak{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}, \quad \mathfrak{B}f(t) = \begin{pmatrix} 0 \\ Bf(t) \end{pmatrix} \tag{6}$$

with the initial condition

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(0) = 0 \tag{7}$$

We consider the equation (5) in the Hilbert space $\mathfrak{X} = H_{\frac{1}{2}} \times E$. Let $\mathfrak{A}$ be the operator in $H_{\frac{1}{2}} \times E$ with its domain $\mathcal{D}(\mathfrak{A}) = \mathcal{D}(A) \times \mathcal{D}(A_\alpha^{\frac{1}{2}})$ such that $\mathfrak{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ Au_1 \end{pmatrix}$ for $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{D}(\mathfrak{A})$. $\mathfrak{B}$ is the operator $\in \mathcal{L}(F ; \mathfrak{X})$ defined in (6). The operator $\mathfrak{A}$ is the infinitesimal generator of a continuous group in $\mathfrak{X}$ ([8]). We say that an $\mathfrak{X}$-valued function $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(t)$ on $[0, T]$ is a solution of (5) with a given initial value

$\begin{pmatrix} u_{1o} \\ u_{2o} \end{pmatrix} \in \mathfrak{D}(\mathfrak{A})$ if

(i) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(0) = \begin{pmatrix} u_{1o} \\ u_{2o} \end{pmatrix}$

(ii) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(t) \in \mathfrak{D}(\mathfrak{A})$ for $0 < t \leqq T$

(iii) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(t)$ belongs to $C^1([0, T] : \mathfrak{X})$ and satisfies (5) for every $t \in (0, T]$. Since $\mathfrak{A}$ is the infinitesimal generator of a continuous group $e^{t\mathfrak{A}}$ $(-\infty < t < \infty)$, the evolution equation (5) with the initial condition (7) has a unique solution

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(t) = \int_0^t e^{(t-s)\mathfrak{A}}\mathfrak{B}f(s)ds$$

for any $f(t) \in C^1([0, T] : F)$. Let us return to the second-order evolution equation (1) with the initial condition (2). We have a unique solution $u(t)$ of (1), (2) such that

(i) $u(0) = u'(0) = 0$.

(ii) $u(t) \in \mathfrak{D}(A)$, $u'(t) \in \mathfrak{D}(A_\alpha^{\frac{1}{2}})$, $0 < t \leqq T$

(iii) $u(t)$ is twice continuously differentiable in E and satisfies (1) for every $t \in (0, T]$.

For any $T > 0$, we define the attainable set $\mathcal{R}_T$ in $\mathfrak{X}$ by

$$\mathcal{R}_T = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int_0^T e^{(T-s)\mathfrak{A}}\mathfrak{B}f(s)ds \ ; \ f(t) \in C^1([0, T] \ ; \ F) \right\}$$

For given A and B, we say that the evolution equation (1) with the initial condition (2) is completely controllable (completely controllable at time T) if $\overline{\bigcup_{t>0} \mathcal{R}_t} = \mathfrak{X}$ ($\overline{\mathcal{R}_T} = \mathfrak{X}$). For a given operator A in E, the evolution equation (1) with the initial condition (2) is called finitely controllable (finitely controllable at time T) if it is completely controllable (completely controllable at time T) for some finite dimensional linear space F and for some B in $\mathcal{L}(F, E)$ (cf.[3]). For the first-order equation (3) with the initial condition (4) we define the attainable set R_T in E by $R_T=\left\{ u=\int_0^T e^{(T-s)A}Bf(s)ds;\ f(t) \in C^1([0,T] ; F)\right\}$. The definitions of complete controllability (complete controllability at time T) and finite controllability (finite controllability at time T) for (3), (4) are given similarly (cf.[3]). We have $\overline{R_T} = \overline{\bigcup_{t>0} R_t}$ for any finite T 0. In fact, $h \in (R_T)^{\perp}$ (= the orthogonal complement of R_T) is equivalent to that $\left(\int_0^T e^{(T-s)A}Bf(s)ds,\ h\right) = \int_0^T (f(s),\ B^* e^{(T-s)A}h)ds = 0$ for any $f(t) \in C^1([0, T] : F)$, that is, $B^*e^{tA}h=0$ for $0 \leq t \leq T$, which is continued analytically to $0 \leq t < \infty$ since e^{tA} is an analytic semigroup. But $B^*e^{tA}h=0$ for $0 \leq t < \infty$ if and only if $h \in (\overline{\bigcup_{t>0} R_t})^{\perp}$. Thus $(R_T)^{\perp} = (\bigcup_{t>0} R_t)^{\perp}$ and $\overline{R_T} = \overline{\bigcup_{t>0} R_t}$. Consequently complete controllability of (3), (4) at some finite time T is equivalent to complete controllability. On the other hand we have $\overline{\mathcal{R}_T} \subset \overline{\bigcup_{t>0} \mathcal{R}_t}$ but $\overline{\mathcal{R}_T} \supset \overline{\bigcup_{t>0} \mathcal{R}_t}$ does not hold in general since $e^{t\mathfrak{A}}$ is not necessarily an analytic semigroup. We shall ask for a necessary and sufficient condition on A in order that (1), (2) is finitely controllable.

If E is a separable Hilbert space, E has an ordered representation relative to the selfadjoint operator A ([2]). That is, there exist a positive measure μ defined and finite on bounded Borel set in $(-\infty, \infty)$ vanishing outside $\sigma(A)$, a decreasing sequence of Borel sets e_n, n=1, 2, ... in $(-\infty, \infty)$ with $\sigma(A) = e_1$ and a unitary operator U on E into $X = \sum_{n=1}^{\infty} L^2(e_n : \mu)$ such that we have

$\mathcal{D}(UAU^{-1})=\{f(\lambda)=(f_1(\lambda),\cdots,f_n(\lambda),\cdots)\in\sum_{n=1}^{\infty}L^2(e_n,\mu);\lambda f(\lambda)\in\sum_{n=1}^{\infty}L^2(e_n,\mu)\}$ and that

$(UAU^{-1}f)_n(\lambda)=\lambda f_n(\lambda)$ for $f(\lambda)\in\mathcal{D}(UAU^{-1})$.

If $\mu(e_n)>0$ for $n\leq m$ and $\mu(e_{m+1})=0$, we say that A has multiplicity $m(A)=m$. If $\mu(e_n)>0$ for any n, we say that A has infinite multiplicity $m(A)=\infty$.

3 Finite Controllability of Second-Order Evolution Equations

H.O.Fattorini proved the following theorem on finite controllability of the first-order evolution equations.

THEOREM 1 (Fattorini [3]) Let A be a selfadjoint semibounded above operator in a separable Hilbert space E. Then in order that the first-order evolution equation (3) with the initial condition (4) is finitely controllable it is necessary and sufficient that A has finite multiplicity. Moreover if A has finite multiplicity m, we can choose an m-dimensional linear space F and an operator $B\in\mathcal{L}(F, E)$ which makes (3), (4) completely controllable and (3), (4), is not completely controllable for any F with its dimension $<$ m.

REMARK 1 In [3], Fattorini remarked that the result of Theorem 1 can be extended further to a normal operator with a connected resolvent. Let us consider finite controllability of the second-order evolution equation (1) in its matricial first-order form (5). The operator $e^{t\mathfrak{A}}$ is normal but it does not necessarily have a connected resolvent and that the operator has a special form given in (6). Therefore we cannot apply Theorem 1 directly.

<u>REMARK 2</u> In Theorem 1, finite controllability is equivalent to finite controllability at any finite time. For the second-order evolution equations, finite controllability does not always imply finite controllability at some finite time. We shall give in Theorem 2 a result analogous to Theorem 1. When (1), (2) is finitely controllable, using the result of Theorem 1 we can construct a finite demensional linear space F and $B \in \mathcal{L}(F, E)$ which makes (1), (2) completely controllable <u>at any finite time</u>.

<u>THEOREM 2</u> Let A be a selfadjoint semibounded above operator in a separable Hilbert space E. Then in order that the second-order evolution equation (1) with the initial condition (2) is finitely controllable it is necessary and sufficient that A has finite multiplicity. Moreover if A has finite multiplicity m, we can choose an m-dimensional linear space F and an operator $B \in \mathcal{L}(F, E)$ which makes (1), (2) completely controllable at any finite time and (1), (2) is not completely controllable for any F with its dimension $<$ m.

<u>LEMMA 1</u> Let A be the infinitesimal generator of a continuous semigroup in a Banach space X. If $g \in \mathcal{D}(A^{\infty}) = \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$ and $\sum_{n=0}^{\infty} \frac{|t|^n \| A^n g \|}{n!}$ is convergent uniformly on an interval $I=[0, t_0]$ with $0 < t_0 < \infty$, then $\sum_{n=1}^{\infty} \frac{t^n A^n g}{n\,!}$ converges to $e^{tA} g$ uniformly on I.

PROOF Let $J_n = (I - \frac{A}{n})^{-1}$. If A is the infinitesimal generator of an equicontinuous semigroup then $\sup_{\substack{n=1,2,\dots \\ m=1,2,\dots}} \| J_n^{\ m} \| = M < \infty$ and the result follows since $e^{tAJ_k} g$ converges to $\sum_{n=0}^{\infty} \frac{t^n A^n g}{n!}$ uniformly on I. If $\sup_{\substack{n=1,2,\dots \\ m=1,2,\dots}} \| J_n^{\ m} \| = \infty$ then $\sup_{\substack{n=1,2,\dots \\ m=1,2,\dots}} \| (I - \frac{A-\beta}{n})^{-m} \|$ is finite for some constant β ([9]) and that $e^{t(A-\beta)J_n} g$ converges to $e^{t(A-\beta)} g$ uniformly on I.

Since $e^{tAJ_n} g = e^{t\beta J_n} e^{t(A-\beta)J_n} g$ and that $e^{t\beta J_n}$ converges to $e^{t\beta} I$ uniformly on I in the uniform operator topology, we have the desired result.

LEMMA 2 If $g_1, g_2 \in \mathcal{D}(A^\infty)$ and that both $\sum_{n=0}^{\infty} \frac{t^{2n} \| A^n A_\alpha g_1 \|}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{t^{2n} \| A^n A_\alpha^{\frac{1}{2}} g_2 \|}{(2n)!}$ converge uniformly on a finite interval $I = [0, t_0]$, then

$$e^{t\mathcal{A}} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^{2n} A^n g_1}{(2n)!} + \sum_{n=0}^{\infty} \frac{t^{2n+1} A^n g_2}{(2n+1)!} \\ \sum_{n=0}^{\infty} \frac{t^{2n+1} A^{n+1} g_1}{(2n+1)!} + \sum \frac{t^{2n} A^n g_2}{(2n)!} \end{pmatrix} \text{ in } \mathfrak{X} \text{ uniformly on I.}$$

PROOF Since $g_i \in \mathcal{D}(A^\infty)$ $(i=1, 2)$, it is clear that $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{D}(\mathcal{A}^\infty)$.

As $\mathcal{A}^{2n+1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} A^n g_2 \\ A^{n+1} g_1 \end{pmatrix}$ and that $\mathcal{A}^{2n} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} A^n g_1 \\ A^n g_2 \end{pmatrix}$ for $n = 0, 1, 2, \dots$, we have formally

$$\sum_{n=0}^{\infty} \frac{t^n \mathcal{A}^n}{n!} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{t^{2n} \mathcal{A}^{2n}}{(2n)!} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{t^{2n+1} \mathcal{A}^{2n+1}}{(2n+1)!} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^{2n} A^n g_1}{(2n)!} + \sum_{n=0}^{\infty} \frac{t^{2n+1} A^n g_2}{(2n+1)!} \\ \sum_{n=0}^{\infty} \frac{t^{2n+1} A^{n+1} g_2}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{t^{2n} A^n g_2}{(2n)!} \end{pmatrix}.$$

The validity of the above equality is assured by Lemma 1 since

$$\left\| \sum_{n=0}^{\infty} \frac{t^{2n} A^n g_1}{(2n)!} + \sum_{n=0}^{\infty} \frac{t^{2n+1} A^n g_2}{(2n+1)!} \right\|_{H_{\frac{1}{2}}} \leq \| A_\alpha^{-\frac{1}{2}} \| \sum_{n=0}^{\infty} \frac{t^{2n} \| A^n A_\alpha g_1 \|}{(2n)!} + t_0 \sum_{n=0}^{\infty} \frac{t^{2n} \| A^n A_\alpha^{\frac{1}{2}} g_2 \|}{(2n)!} < \infty$$

and

$$\left\| \sum_{n=0}^{\infty} \frac{t^{2n+1} A^{n+1} g_1}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{t^{2n} A^n g_2}{(2n)!} \right\| \leq t_0 \| A A_\alpha^{-1} \| \sum_{n=0}^{\infty} \frac{t^{2n} \| A^n A_\alpha g_1 \|}{(2n)!} + \| A_\alpha^{-\frac{1}{2}} \| \sum \frac{t^{2n} \| A^n A_\alpha^{\frac{1}{2}} g_2 \|}{(2n)!} < \infty$$

uniformly on I.

LEMMA 3 If $g \in \mathcal{D}(A^\infty)$ and that $\| A^n g \|_{H_{\frac{1}{2}}} \leq c R^n n!$, $n = 0, 1, 2, \dots$, for some $c > 0$ and $R > 0$, then $e^{t\mathcal{A}} \begin{pmatrix} 0 \\ g \end{pmatrix}$ is holomorphic in $(-\infty, \infty)$ and it has a representation $e^{t\mathfrak{a}}\begin{pmatrix}0\\ g\end{pmatrix} = \begin{pmatrix}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}A^n g\\ \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}A^n g\end{pmatrix}$ uniformly on any finite interval in $(-\infty,\infty)$.

PROOF Let $g_1=0$, $g_2=g$, then $\begin{pmatrix}g_1\\ g_2\end{pmatrix}$ satisfies the assumption of Lemma 2 because

$$\sum_{n=0}^{\infty}\frac{\tau^{2n}\|A^n A_\alpha^{\frac{1}{2}} g_2\|}{(2n)!} \leq c\sum_{n=0}^{\infty}\frac{(\tau^2 R)^n}{n!} < \infty$$

LEMMA 4 For $\varepsilon > 0$, we have $\|A^n e^{\varepsilon A}\| \leq \frac{n!}{\varepsilon^n}$ for sufficiently large n.

PROOF Since $\|A^n e^{\varepsilon A}u\|^2 = \int_{-\infty}^{\mu}\lambda^{2n} e^{2\varepsilon\lambda}\, d\|E(\lambda)u\|^2 \leq \sup_{-\infty<\lambda\leq\mu}\lambda^{2n}e^{2\varepsilon\lambda}\|u\|^2$ we have $\|A^n e^{\varepsilon A}\| \leq \sup_{-\infty<\lambda<\mu}|\lambda|^n e^{\varepsilon\lambda}$. For $\mu<0$, $\|A^n e^{\varepsilon A}\| \leq \sup_{-\infty<\lambda\leq 0}(-\lambda)^n e^{\varepsilon\lambda}$

$= \sup_{\lambda\geq 0}(\lambda^n/e^{\varepsilon\lambda}) \leq \sup_{\lambda\geq 0}(\lambda^n/\frac{(\varepsilon\lambda)^n}{n!}) \leq n!\,\varepsilon^{-n}$ $(n=1, 2, \dots)$.

For $\mu \geq 0$. $\|A^n e^{\varepsilon A}\| \leq \max(\sup_{-\infty<\lambda\leq 0}(-\lambda)^n e^{\varepsilon\lambda}, \sup_{0\leq\lambda\leq\mu}\lambda^n e^{\varepsilon\lambda}) \leq \max(n!\,\varepsilon^{-n}, \mu^n e^{\varepsilon\mu})$

If $n > 2(\varepsilon\mu e^{\varepsilon\mu})^2$, the right hand side becomes $n!\,\varepsilon^{-n}$ because

$n!\,\varepsilon^{-n} \geq (\frac{n}{2})^{\frac{n}{2}}\varepsilon^{-n} = (\varepsilon\mu e^{\varepsilon\mu})^n\varepsilon^{-n} = \mu^n e^{n\varepsilon\mu} \geq \mu^n e^{\varepsilon\mu}$.

PROOF of THEOREM 2 (Sufficiency) Let A has finite multiplicity m, Then by Theorem 1 the evolution equation (3) with the initial condition (4) is completely controllable at any finite time T for some m-dimensional linear space F and for some $B \in \mathcal{L}(F, E)$. In the following we show that $B_\varepsilon = e^{\varepsilon A}A_\alpha^{-\frac{1}{2}}B \in \mathcal{L}(F; E)$ makes (1), (2) completely controllable at any time. For any s with $0\leq s\leq T$, $g_\varepsilon(s) = e^{\varepsilon A}A_\alpha^{-\frac{1}{2}}Bf(s)$ satisfies the assumption of Lemma 3. In fact, in view of Lemma 4, $\|A^n g_\varepsilon\|_{H_{\frac{1}{2}}} = \|A^n e^{\varepsilon A}A_\alpha^{-\frac{1}{2}}Bf\| \leq \|A^n e^{\varepsilon A}\|\|A_\alpha^{-\frac{1}{2}}B\| \leq \|A_\alpha^{-\frac{1}{2}}B\|^{n!/\varepsilon^n}$. By Lemma 3, $e^{t\mathfrak{a}}\begin{pmatrix}0\\ g_\varepsilon\end{pmatrix} = \begin{pmatrix}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}A^n g_\varepsilon\\ \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}A^n g_\varepsilon\end{pmatrix}$ uniformly on any finite interval in $(-\infty,\infty)$.

For any $h=\begin{pmatrix}h_1\\ h_2\end{pmatrix}$ in $(\mathcal{R}_T)^+$, we have

$$\left(\int_0^T e^{(T-s)\mathfrak{a}}\begin{pmatrix}0\\ g_\varepsilon(s)\end{pmatrix}ds,\ h\right)_{\mathcal{X}} = \int_0^T\left(\begin{pmatrix}\sum_{n=0}^{\infty}\frac{(T-s)^{2n+1}}{(2n+1)!}A^n e^{\varepsilon A}A_\alpha^{-\frac{1}{2}}Bf(s)\\ \sum_{n=0}^{\infty}\frac{(T-s)^{2n}}{(2n)!}A^n e^{\varepsilon A}A_\alpha^{-\frac{1}{2}}Bf(s)\end{pmatrix}, h\right)_{\mathcal{X}} ds = 0$$

for any $f(t) \in C^1([0, T] ; F)$, that is,

$$\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (A^n e^{\varepsilon A} B)^* A_\alpha^{\frac{1}{2}} h_1 + \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} (A^n e^{\varepsilon A} A_\alpha^{-\frac{1}{2}} B)^* h_2 = 0 \tag{8}$$

for $0 \leq t \leq T$. By Lemma 4, $\| (A^n e^{\varepsilon A} B)^* \| = \| A^n e^{\varepsilon A} B \| \leq \| B \| \varepsilon^{-n} n!$ and $\| A^n e^{\varepsilon A} A_\alpha^{-\frac{1}{2}} B \| \leq \| A_\alpha^{-\frac{1}{2}} B \| \varepsilon^{-n} n!$ for sufficiently large n. Thus the left hand side of (8) is holomorphic in $(-\infty, \infty)$ and that $(A^n e^{\varepsilon A} B)^* A_\alpha^{\frac{1}{2}} h_1 = (A^n e^{\varepsilon A} A_\alpha^{-\frac{1}{2}} B)^* h_2 = 0$. The estimate $\| t^n A^n e^{\varepsilon A} \| \leq n! (\frac{t}{\varepsilon})^n$ together with Lemma 1 implies that

$$e^{tA} e^{\varepsilon A} u = \sum_{n=0}^{\infty} \frac{t^n A^n e^{\varepsilon A}}{n!} u \qquad \text{for } u \in E$$

uniformly on $[0, \frac{\varepsilon}{2}]$. Therefore we have $((e^{(t+\varepsilon)A} B)^* A_\alpha^{\frac{1}{2}} h_1 , u)$
$= (A_\alpha^{\frac{1}{2}} h_1 , e^{tA} e^{\varepsilon A} Bu) = (A_\alpha^{\frac{1}{2}} h_1 , \sum_{n=0}^{\infty} \frac{t^n A^n e^{\varepsilon A} B}{n!} u) = ((\sum_{n=0}^{\infty} \frac{t^n A^n e^{\varepsilon A} B}{n!})^* A_\alpha^{\frac{1}{2}} h_1 , u)$
$= 0$ and $((e^{(t+\varepsilon)A} A_\alpha^{-\frac{1}{2}} B)^* h_2 , u) = (h_2 , e^{tA} e^{\varepsilon A} A_\alpha^{-\frac{1}{2}} Bu) = (h_2 , \sum_{n=0}^{\infty} \frac{t^n A^n e^{\varepsilon A} A_\alpha^{-\frac{1}{2}} B}{n!} u)$
$= ((\sum_{n=0}^{\infty} \frac{t^n A^n e^{\varepsilon A} A_\alpha^{-\frac{1}{2}} B}{n!})^* h_2, u) = 0$ for $t \in [0, \frac{\varepsilon}{2}]$ and $u \in E$. Thus
$(e^{(t+\varepsilon)A} B)^* A_\alpha^{\frac{1}{2}} h_1 = (e^{(t+\varepsilon)A} A_\alpha^{-\frac{1}{2}} B)^* h_2 = 0$ for $t \in [0, \frac{\varepsilon}{2}]$. By analytic continuation

$$(e^{tA} B)^* h_1 = (e^{tA} A_\alpha^{-\frac{1}{2}} B)^* h_2 = 0 \text{ for any } t \geq 0.$$

For any $f(t)$ in $C^1([0, T] ; F)$, we have

$$(\int_0^T e^{(T-s)A} Bf(s)ds, h_1) = \int_0^T (f(s), (e^{(T-s)A} B)^* h_1) \, ds = 0$$

and

$$\begin{aligned} (\int_0^T e^{(T-s)A} Bf(s)ds, A_\alpha^{-\frac{1}{2}} h_2) &= \int_0^T (f(s), B^* e^{(T-s)A} A_\alpha^{-\frac{1}{2}} h_2) ds \\ &= \int_0^T (f(s), (e^{(T-s)A} A_\alpha^{-\frac{1}{2}} B)^* h_2) ds \\ &= 0 . \end{aligned}$$

Therefore h_1 and $A_\alpha^{-\frac{1}{2}} h_2$ belong to $(R_T)^\perp = \{ 0 \}$. Thus $h_1 = h_2 = 0$ and sufficiency is proved.

(Necessity) Let (1), (2) be finitely controllable. Then there exists a finite dimensional linear space $F = C^n$ and $B \in \mathcal{L}(F, E)$ which makes (1), (2) completely controllable. Let $(u_1, \ldots, u_n)$, $u_i \in F$ $i=1, \ldots, n$ be an orthonormal basis for F, then $Bf(t) = \sum_{i=1}^{2} g_i f_i(t)$ where $g_i = Bu_i$ and $f_i(t)=(f(t), u_i)$. Let us prove that $m(A) \leqq n$. Suppose that $m(A) \geqq n+1$. Putting $Ug_i(\lambda)=(g_{i1}(\lambda), g_{i2}(\lambda), \ldots, g_{ij}(\lambda), \ldots)$, $1 \leqq i \leqq n$, where $g_{ij}(\lambda) \in L^2(e_j, \mu)$ with $\sum_{j=1}^{\infty} \int_{e_j} |g_{ij}(\lambda)|^2 \mu(d\lambda) < \infty$, $i=1, \ldots, n$, We have for any $h \in E$,

$$(E(e)g_i, h) = (UE(e)g_i, Uh)_X = (\chi(e)Ug_i(\lambda), Uh(\lambda))_X$$
$$= \sum_{j=1}^{\infty} \int_{e_1} \chi(e_j) g_{ij}(\lambda) h_j(\lambda) \mu(d\lambda) .$$

We find solutions $h_j(\lambda) \in L^2(e_j, \ )$ $j = 1, \ldots, n+1$ of the equation

$$\sum_{j=1}^{n+1} g_{ij}(\lambda) h_j(\lambda) = 0 \qquad \mu\text{-a.e. in } e_{n+1}, \ 1 \leqq i \leqq n \tag{9}$$

such that $(h_1(\lambda), \ldots, h_{n+1}(\lambda), 0, \ldots,)$ is non-null in X. In fact, let $e^{(k)} = e_{n+1} \cap (-k, -k)$ for $k=0, 1, 2, \ldots$, then $e_{n+1} = \bigcup_{k=0}^{\infty} e^{(k)}$. Since $\mu(e_{n+1}) > 0$ and $e^{(k)}$ is a bounded Borel set, we have $0 < \mu(e^{(k_0)}) < \infty$ for some k_0. If $g_{ij}(\lambda)=0$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n+1$, μ-a.e. in $e^{(k_0)}$ then non-null μ-measurable functions $h_j(\lambda) = \chi_{e^{(k_0)}}(\lambda)$, $1 \leqq j \leqq n+1$, satisfy (9) and belong to $L^2(e_{n+1}, \mu)$ since $\int_{e_{n+1}} |h_j(\lambda)|^2 \mu(d\lambda) = \mu(e^{(k_0)}) < \infty$. Otherwise we find a Borel set $e_0 \subset e^{(k_0)}$ with $\mu(e_0) > 0$ such that rank $\{ g_{ij}(\lambda), 1 \leqq i \leqq n, 1 \leqq j \leqq n+1 \} = n_0 \geqq 1$ for μ-almost every λ in e_0. Let $M_{\substack{i_1 \cdots i_{n_0} \\ j_1 \cdots j_{n_0}}} = \{ \lambda \in e_0 ; \det(g_{ij}(\lambda), i=i_1, \ldots, i_{n_0}, j = j_1, \ldots, j_{n_0}) \neq 0 \}$ where $i_1 \ldots i_{n_0}$ and $j_1 \ldots j_{n_0}$ are respectively a distinct combination from 12 ... n and 12 ... n+1. Then $M_{i_1\cdots i_{n_0}}^{j_1\cdots j_{n_0}}$ are μ-measurable and $e_0=\bigcup M_{i_1\cdots i_{n_0}}^{j_1\cdots j_{n_0}}$ where $i_1\ldots i_n$ and $j_1\ldots j_n$ are taken from all such combinations. As $\mu(e_0)>0$, an $M_0=M_{i_1\cdots i_{n_0}}^{j_1\cdots j_{n_0}}$ has a positive μ-measure for some $i_1\ldots i_{n_0}$ and $j_1\ldots j_{n_0}$. Changing if necessary rows and columns of the matrix $\{g_{ij}(\lambda)\ ;\ i=i_1,\ldots,i_{n_0},\ j=j_1,\ldots,j_{n_0}\}$, we may assume that $(i_1,\ldots,i_{n_0})=(j_1,\ldots,j_{n_0})=(1,2,\ldots,n_0)$. Take $h_j(\lambda)=\chi_{M_0}(\lambda)$ for $n_0+1\leq j\leq n+1$ and define the remaining $h_j(\lambda)$, $1\leq j\leq n_0$ by

$$\sum_{j=1}^{n_0} g_{ij}(\lambda)\,h_j(\lambda)=-\sum_{j=n_0+1}^{n+1} g_{ij}(\lambda)\ \text{for}\ \lambda\in M_0,\quad h_j(\lambda)=0\ \text{for}\ \lambda\in e_{n+1}-M_0$$

Replacing each $h_j(\lambda)$ by $h_j(\lambda)/\sum_{j=1}^{n+1}|h_j(\lambda)|$, we have non-null bounded μ-mesurable functions satisfying (9). As M_0 is a bounded Borel set, $h_j(\lambda)$ $(1\leq j\leq n+1)$ is in $L^2(e_{n+1},\mu)$ since $h_j(\lambda)$ vanishes on $e_{n+1}-M_0$ and bounded on M_0. In the complement of e_{n+1} we define $h_j(\lambda)=0$.

If we put $h=U^{-1}(h_1(\lambda),\ \ldots\ ,h_{n+1}(\lambda),0,\ \ldots\ ,)$, we have $(E(e)g,h)=0$ for any Borel set e in $\mathfrak{S}(A)$.

For $g_{iN}=\int_{-N}^{\mu} dE(\lambda)g_i$, we have by Lemma 3

$$\left(e^{t\alpha}\begin{pmatrix}0\\ g_{iN}\end{pmatrix},\begin{pmatrix}0\\ h\end{pmatrix}\right)_{\mathfrak{X}}=\sum_{n=0}^{\infty}\int_{-N}^{\mu}\frac{t^{2n}}{(2n)!}\lambda^n d(E(\lambda)g_i,h)=0$$

Letting $N\to\infty$, we have $\left(e^{t\alpha}\begin{pmatrix}0\\ g_i\end{pmatrix},\begin{pmatrix}0\\ h\end{pmatrix}\right)_{\mathfrak{X}}=0$. Hence $\left(\int_0^T e^{(t-s)}\begin{pmatrix}0\\ g_if_i(s)\end{pmatrix}ds,\begin{pmatrix}0\\ h\end{pmatrix}\right)=0$ for any $T>0$ and for any $f_i\in C^1([0,T]\ ;\ C)$ which implies that $\begin{pmatrix}0\\ h\end{pmatrix}\in(\mathcal{R}_T)^{\perp}$ for any $T>0$. Thus $(\bigcup_{t>0}\mathcal{R}_T)^{\perp}\neq\{0\}$ and $\overline{\bigcup_{t>0}\mathcal{R}_t}\neq\mathfrak{X}$ contrary to the assumption.

Proof of sufficiency and necessity given above also shows the validity of the second statement in Theorem 2.

4. Applications

Example 1 We consider the initial-boundary value problem for one-dimensional wave equation ;

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} + q(x)u(x, t) = g(x)f(t) ,$$
$$0 < t \leq T, \quad 0 < x \leq \ell < \infty \tag{10}$$

with the boundary conditions

$$a_0 u(0, t) + a_1 u_x(0, t) = b_0 u(0, t) + b_1 u_x(0, t) = 0$$
$$0 \leq t \leq T \tag{11}$$

and with the initial condition

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0 , \qquad x \in (0, \ell) . \tag{12}$$

where $q(x) \in C[0, \ell]$, $g(x) \in L^2[0, \ell]$ and a_i, b_i are real constants such that $a_0{}^2 + a_1{}^2 \neq 0$, $b_0{}^1 + b_1{}^2 \neq 0$.

Let A be a differential operator $\frac{\partial^2}{\partial x^2} - q(x)$ with its domain $\mathcal{D}(A) = \left\{ u(x) \in \mathcal{E}^2_{L^2}(0, \ell) \right.$; u(x) satisfies the boundary conditions (10) in $\left. E = L^2(0, \ell) \right\}$. Then A is a selfadjoint semibounded above operator in E and A has a sequence of simple eigenvalues $\{\lambda_n\}$ n=0, 1, 2, ... strictly decreasing and diverging at $-\infty$. Multiplicity of A is 1. Let $\{\varphi_n\}$ n=0, 1, 2,... be eigenfunctions corresponding to eigenvalues $\{\lambda_n\}$, n=0, 1, 2, ... which forms a complete orthonormal basis in $L^2(0, \ell)$.

The following asymptotic properties hold, that is, for $\omega_n = \sqrt{-\lambda_n}$ (n=0, 1, 2,..)

$$\liminf_{n \to \infty} (\omega_{n+1} - \omega_n) = \frac{1}{D} , \quad \lim_{n \to \infty} \frac{\omega_n}{n} = D \tag{13}$$

where D is a positive constant. (cf. f.g., [7]) LEMMA 5 The evolution equation ;

$$\frac{\partial u(x,t)}{\partial t} = Au(x,t) + g(x)f(t), \; u(x,0) = 0 , \qquad (14)$$

$$0 < t \leq T, \; 0 < x < \ell$$

in $L^2(0,\ell)$ is completely controllable if and only if $(g, \varphi_n) \neq 0$ for $n=0, 1, 2, \ldots$

PROOF Let $h \in (R_T)^{\perp}$, then we have

$$\Big(\sum_{n=0}^{\infty}\int_0^T e^{\lambda_n(T-s)} f(s)(g,\varphi_n)_{L^2(0,\ell)}\varphi_n ds, \; h\Big)_{L^2(0,\ell)} = 0$$

for $f(t) \in C^1([0, T] \; ; \mathbb{C})$, that is,

$$\sum_{n=0}^{\infty} e^{\lambda_n t} (g, \varphi_n)(\varphi_n, h) = 0 \qquad (15)$$

for $t \in [0, T]$. By analytic continuation, (15) holds for $t \in [0,\infty)$.

For any $\lambda \neq \lambda_n$ $(n=0, 1, 2 \ldots)$ with $\mathrm{Re}\lambda > \mu$,

$$0 = \sum_{n=0}^{\infty}\int_0^{\infty} e^{(\lambda_n-\lambda)t} g_n \overline{h}_n dt = \sum_{n=0}^{\infty} \frac{g_n \overline{h}_n}{\lambda_n - \lambda} \qquad (16)$$

where $g_n = (g, \varphi_n)$, $h_n = (h, \varphi_n)$. By analyticity we have

$$\sum_{n=0}^{\infty} \frac{g_n \overline{h}_n}{\lambda_n - \lambda} = 0 \quad \text{for } \lambda \neq \lambda_n \quad (n=0, 1, 2 \; \ldots)$$

Let $\Gamma_n = \{ z \in \mathbb{C} \; ; \; |z - \lambda_n| = \varepsilon_n \}$ where ε_n is a positive number such that $\lambda_n \notin \Gamma_m$ for $m \neq n$. Then we have $g_n \overline{h}_n = \frac{1}{2\pi i}\int_{\Gamma_n} \sum_{m=0}^{\infty} \frac{g_m \overline{h_m}}{z - \lambda_m} dz = 0$ for $n=0, 1, 2, \ldots$. Thus $(R_T)^{\perp} = \{0\}$ is equivalent to that $g_n \neq 0$ for $n=0, 1, 2, \ldots$.

PROPOSITION 1 Let $g(x) = \sum_{n=0}^{\infty} g_n \varphi_n$, where

$$g_n \neq 0 \text{ and } |g_n| \leq M e^{\varepsilon \lambda_n} \text{ for some } M > 0 \text{ and } \varepsilon > 0 \; (n=0, 1, 2, \ldots) \qquad (17)$$

Then the initial-boundary value problem for wave equation (10). (11), (12) is completely controllable at any time $T > 0$.

PROOF Consider controllability of the second-order evolution equation ;

$$\frac{\partial^2 u(t)}{\partial t^2} = Au(t) + gf(t), \quad 0 < t \leq T, \; u(0)=u'(0)=0 \tag{18}$$

in $L^2(0,\ell)$. If we put $g_{n,\varepsilon} = (\lambda_n+1)^{\frac{1}{2}} e^{\frac{\varepsilon\lambda_n}{2}} g_n$, we see that $\sum_{n=0}^{\infty} |g_{n,\varepsilon}|^2 < \infty$ and $g_{n,\varepsilon} \neq 0$ by (17). It follows from Lemma 5 that the first-order evolution equation (14) is completely controllable at time T if g(x) in (14) is replaced by $g_\varepsilon(x) = \sum_{n=0}^{\infty} g_{n,\varepsilon}\varphi_n$. Thus $g(x) = e^{\frac{\varepsilon}{2}A} A_1^{-\frac{1}{2}} g_\varepsilon(x)$ makes (18) completely controllable (see proof of Theorem 2).

REMARK 3 If q(x) is nonnegative, $\omega_n = \sqrt{-\lambda_n} \geq 0$ for n=0, 1, 2, ... and (17) is weaken to

$$g_h \neq 0, \quad |g_n| \leq M e^{-\varepsilon\omega_n} \qquad (n=0,\ 1,\ 2,\ \ldots\ )$$

PROOF Firstly we show that $e^{t\mathfrak{A}}\binom{0}{g}$ is holomorphic in $(-\infty,\infty)$.

In fact, $\| A^n A_1 g\|^2 = \| \sum_{k=0}^{\infty} (\lambda_k^n (-\lambda_k+1)^{\frac{1}{2}} g_k \varphi_k \|^2 = \| \sum_{k=0}^{\infty} \lambda_k^{2n} (-\lambda_k+1) g_k^2 \|^2$

$$\leq M^2 \sum_{K=0}^{\infty} \omega_K^{4n} (\omega_K^2 + 1) e^{-2\varepsilon\omega_K}$$

For any δ with $0 < \delta < \varepsilon$, we have

$$\omega_K^{4n} (\omega_K^2+1)\, e^{-2\varepsilon\omega_K} = (\omega_K^{2n} / e^{\delta\omega_K})^2 (\omega_K^2+1)\, e^{2(\delta-\varepsilon)\omega_K}$$

$$\leq (\omega_K^{2n} / (2n!)^{-1} (\delta\omega_K)^{2n})^2 (\omega_K^2+1) e^{2(\delta-\varepsilon)\omega_K} \leq (2n!)^2 \delta^{-4n} (\omega_K^2+1) e^{2(\delta-\varepsilon)\omega_K}$$

Putting $C(\varepsilon,\delta) = (\sum_{K=0}^{\infty} (\omega_K^2+1) e^{2(\delta-\varepsilon)\omega_K})^{\frac{1}{2}}$, we have an estimate

$$\| A^n A_1^{\frac{1}{2}} g \| \leq C(\varepsilon,\delta) M\, (2n)!\, \delta^{-2n}$$

Thus $\displaystyle\sum_{n=0}^{\infty} \frac{t^{2n} \| A^n A_1^{\frac{1}{2}} g\|}{(2n)!} \leq C(\varepsilon,\delta) M \sum_{n=0}^{\infty} \left|\frac{t}{\delta}\right|^{2n}$.

It follows from Lemma 2 applied to $g_1 = 0$, $g_2 = g$ that $e^{t\mathfrak{A}}\binom{0}{g}$ is holomorphic in $(-\delta,\delta)$. But since $e^{t\mathfrak{A}}\binom{0}{g} = e^{t_0\mathfrak{A}} e^{(t-t_0)\mathfrak{A}}\binom{0}{g}$ for any $t_0 \in (-\infty,\infty)$, $e^{t\mathfrak{A}}\binom{0}{g}$ is holomorphic in $\{ t : |t-t_0| < \delta \}$.

Hence it is holomorphic in $(-\infty, \infty)$. As in proof of Theorem 2 if $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in (\mathcal{R}_T)^\perp$ then $\left(e^{t\mathcal{Q}}\begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right) = 0$ for $0 \leq t \leq T$, which is continued analytically to $t \in (0, \infty)$. Thus $(\mathcal{R}_T)^\perp$ and $\overline{\mathcal{R}_T}$ do not depend on T. By Lemma 6 given below $\overline{\mathcal{R}_T} = \mathfrak{X}$ if $T > 2\pi D$ and therefore $\overline{\mathcal{R}_T} = \mathfrak{X}$ for any $T > 0$.

<u>LEMMA 6</u> For any $g = \sum_{n=0}^{\infty} g_n \varphi_n$ with $g_n \neq 0$, we have

$$\overline{\mathcal{R}_T} = \mathfrak{X} \quad \text{if } T > 2\pi D .$$

<u>REMARK 4</u> If $0 < T < 2\pi D$, $\overline{\mathcal{R}_T} = \mathfrak{X}$ does not hold in general unless some more strong condition is imposed on g.

<u>DEFINITION</u> A subset $\mathcal{M}$ of $L^2[0, T]$ is said to be linearly independent if every $f \in \mathcal{M}$ does not belong to the smallest closed subspace spanned by $\mathcal{M} - \{f\}$

<u>PROOF of LEMMA 6</u> As a sequence $g^N = \sum_{n=0}^{N} g_n \varphi_n$, $N \geq 0$ satisfies the assumption of Lemma 3 we have

$$e^{t\mathcal{Q}}\begin{pmatrix} 0 \\ g^N \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\left(\sum_{k=0}^{N} g_k \lambda_k^n \varphi_k\right) \\ \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\left(\sum_{k=0}^{N} g_k \lambda_k^n \varphi_k\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{N} \frac{\sin\sqrt{-\lambda_k}\,t}{\sqrt{-\lambda_k}} g_k \varphi_k \\ \sum_{k=0}^{N} \cos\sqrt{-\lambda_k}\,t\, g_k \varphi_k \end{pmatrix}$$

Letting $N \to \infty$

$$e^{t\mathcal{Q}}\begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\sin\sqrt{-\lambda_k}\,t}{\sqrt{-\lambda_k}} g_k \varphi_k \\ \sum_{k=0}^{\infty} \cos\sqrt{-\lambda_k}\,t\, g_k \varphi_k \end{pmatrix}$$

uniformly on any finite interval in $(-\infty, \infty)$. Therefore h belongs to $(\mathcal{R}_T)^\perp$ if and only if

$$\sum_{k=0}^{\infty} \frac{\sin\sqrt{-\lambda_k}\,t}{\sqrt{-\lambda_k}} g_k h_k^1 + \sum_{k=0}^{\infty} \cos(\sqrt{-\lambda_k}\,t) g_k h_k^2 = 0 \tag{19}$$

uniformly on $[0, T]$ where $h_k^1 = (h_1, \varphi_k)_{H^{\frac{1}{2}}}$ and $h_k^2 = (h_2, \varphi_k)_E$.

Let $\mathcal{M} = \{\cos\sqrt{-\lambda_k}\,t, \sin\sqrt{-\lambda_k}\,t \mid k=0, 1, 2, \ldots\}$ if $\lambda_0 \neq 0$ and $\mathcal{M} = \{1, t, \cos\sqrt{-\lambda_k}\,t, \sin\sqrt{-\lambda_k}\,t \mid k = 1, 2, \ldots\}$ if $\lambda_0 = 0$. As $T > 2\pi D$, the estimate (13) implies that $\mathcal{M}$ is linearly independent in $L^2[0, T]$ (cf, f.g., [4] and [6]). Noting that (19) holds in $L^2[0, T]$-topology, we have that

$$g_k h_k^1 = g_k h_k^2 = 0 \qquad \text{for } k = 0, 1, 2, \dots$$

i.e.,

$$h_k^1 = h_k^2 = 0 \qquad \text{for } k = 0, 1, 2, \dots$$

Hence $(\mathcal{R}_T)^{\perp} = \{0\}$ and $\overline{\mathcal{R}_T} = \mathcal{X}$ if $T > 2\pi D$.

<u>EXAMPLE 2</u> Let A be a differential operator $\frac{\partial^2}{\partial x^2}$ in $L^2(-\infty,\infty)$ with its domain $\mathcal{D}(A) = \mathcal{E}^2_{L^2}(-\infty,\infty)$. As for finite controllability of (1), (2) we have a following

<u>PROPOSITION 2</u> The initial value problem for one-dimensional wave equation :

$$\frac{\partial^2 u}{\partial t^2} = Au + \sum_{i=1}^{2} g_{i\varepsilon}(x) f_i(t) \qquad 0 < t \leq T, \ -\infty < x < \infty$$

$$u(x, 0) = u_t(x, 0) = 0 \tag{20}$$

is completely controllable at any time T if

$$g_{i\varepsilon}(x) = \mathcal{F}^{-1}\left(e^{-\varepsilon|\lambda|^2}(1+|\lambda|^2)^{-\frac{1}{2}}\right)\mathcal{F} g_i , \qquad i = 1, 2.$$

where

$$\mathcal{F} g(s) = \hat{g}(s) = (2\pi)^{-\frac{1}{2}} \underset{N\to\infty}{\text{l. i.m.}} \int_{-N}^{N} e^{is} g(x)dx \quad \text{for } g \in L^2(-\infty,\infty) \text{ and}$$

$g_1(x)$ is a non-null function in $L^2(-\infty,\infty)$ with compact support and $g_2(x) = g_1(x-h)$, $h \neq 0$.

<u>LEMMA 7 (Fattorini [3])</u> The operator A has multiplicity 2.

<u>PROOF</u> It is clear that $e_1 = \sigma(A) = (-\infty, 0]$. We set $e_2 = e_1$ and $\mu = d\lambda / 2|\lambda|^{\frac{1}{2}}$ which is a measure on e_i, $i=1, 2$.

Let U be an operator on $L^2(-\infty,\infty)$ onto $X = \sum_{i=1}^{2} L^2(e_i, \mu)$ defined by $Uu(\lambda) = (\hat{u}(|\lambda|^{\frac{1}{2}}), \hat{u}(-|\lambda|^{\frac{1}{2}}))$. Then U is a unitary operator because
$\|u\|^2_{L^2(-\infty,\infty)} = \|\hat{u}\|^2_{L^2(-\infty,\infty)} = \int_0^\infty |\hat{u}(x)|^2 dx + \int_{-\infty}^0 |\hat{u}(x)|^2 dx =$
$\int_0^\infty |\hat{u}(|\lambda|^{\frac{1}{2}})|^2 d\lambda/2|\lambda|^{\frac{1}{2}} + \int_{-\infty}^0 |\hat{u}(-|\lambda|^{\frac{1}{2}})|^2 d\lambda/2|\lambda|^{\frac{1}{2}} = \|Uu\|^2_X$. Let $f \in \sum_{i=1}^{2} L^2(e_i, \mu)$, then $U\,AU^{-1}f(\lambda) = (\widehat{AU^{-1}f}(|\lambda|^{\frac{1}{2}}), \widehat{AU^{-1}f}(-|\lambda|^{\frac{1}{2}})) =$
$(\lambda\,\widehat{U^{-1}f}(|\lambda|^{\frac{1}{2}}), \widehat{U^{-1}f}((-|\lambda|^{\frac{1}{2}})) = \lambda f(\lambda)$.

<u>LEMMA 8</u> (Fattorini [3]) The first-order evolution equation in $L^2(-\infty,\infty)$

$$\frac{\partial u}{\partial t} = Au + \sum_{i=1}^{2} g_i(x)f_i(t) \tag{21}$$

with the initial condition

$$u(0) = 0$$

is completely controllable where g_i are given in Proposition 2.

<u>PROOF</u> If $h \in (R_T)^\perp$, then

$$\sum_{i=1}^{2}\int_0^T \int_{-\infty}^0 e^{\lambda(t-s)} d(E(\lambda)g_i, h)f_i(s)ds = 0 \quad \text{for } f_i \in C[0, T],$$

that is, $\int_{-\infty}^0 e^{\lambda t} d(E(\lambda)g_i, h) = 0$ for $0 \leq t \leq T$.

For any μ with $\mathrm{Re}\,\mu > 0$, $0 = \int_{-\infty}^0 e^{(\lambda-\mu)t} d(E(\lambda)g_i, h) = \int_{-\infty}^0 \frac{1}{\lambda-\mu} d(E(\lambda)g_i, h)$.

By analytic continuation, $0 = \int_{-\infty}^0 \frac{1}{\lambda-\mu} d(E(\lambda)g_i, h)$ for any complex number $\mu \notin (-\infty, 0]$. Therefore (cf, f.g.,[2]), $(E(a, b)g_i, h) = \frac{1}{2\pi i} \times$
$\lim_{\delta\to+0}\lim_{\varepsilon\to+0}\int_{a+\delta}^{b-\delta}((R(\mu-\varepsilon i, A) - R(\mu+\varepsilon i, A))g_i, h)\,d\mu = \frac{1}{2\pi i} \times$
$\lim_{\delta\to+0}\lim_{\varepsilon\to+0}\int_{a+\delta}^{b-\delta} d\mu \int_{-\infty}^0 ((\mu-\varepsilon i-\lambda)^{-1} - (\mu+\varepsilon i-\lambda)^{-1}) d(E(\lambda)g_i, h) = 0$
for $-\infty < a < b < \infty$. Thus we have $0 = (E(e)g_i, h) = (UE(e)g_i, Uh) =$
$\int_e (\hat{g}_i(\sqrt{-\lambda})\hat{h}(\sqrt{-\lambda}) + g_i(-\sqrt{-\lambda})h(-\sqrt{-\lambda}))\frac{d\lambda}{2\sqrt{-\lambda}} = 0$ for every Borel set e in $(-\infty, 0)$. Hence

$$\begin{cases} \hat{g}_1(\sqrt{-\lambda})\,\hat{h}(\sqrt{-\lambda}) + \hat{g}_1(-\sqrt{-\lambda})\,h(-\sqrt{-\lambda}) = 0 \\ \hat{g}_2(\sqrt{-\lambda})\,\hat{h}(\sqrt{-\lambda}) + \hat{g}_2(-\sqrt{-\lambda})\,h(-\sqrt{-\lambda}) = 0 \end{cases} \tag{22}$$

for μ-almost every λ in $(-\infty,0)$.

Since $g_2(x) = g_1(x - h)$, we have $g_2(\sqrt{\lambda}) = e^{i\sqrt{\lambda}h} g(\sqrt{\lambda})$, $\hat{g}_1(\sqrt{-\lambda})g_2(-\sqrt{-\lambda})$ $-\hat{g}_1(-\sqrt{-\lambda})g_2(\sqrt{-\lambda}) = -2i\,\sin(\sqrt{-\lambda}h)\hat{g}_1(\sqrt{\lambda})\hat{g}_1(-\sqrt{\lambda}) \neq 0$ for almost every λ .
It follows from (22) that $\hat{h}(\lambda) = 0$ and $(R_T)^{\perp} = \{0\}$.

<u>PROOF of PROPOSITION 2.</u> The assertion is proved by Theorem 2 and Lemma 8 because $g_{i\varepsilon} = e^{\varepsilon A} A_1^{-\frac{1}{2}} g_i$, $i=1,2$.

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