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ON THE CLOSURE OF TRANSLATIONS IN $L^p(R_k)$

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0. Introduction

We shall discuss some aspect of "closure of translations theorem". (Cf. References listed at the end of this paper.)

Throughout this paper, we denote a conjugate exponent of p ($1<p<\infty$) by q, that is, $1<p<\infty$, $1<q<\infty$, $1/p + 1/q =1$. Let us define a sub-class W_k^q of $L^q(R_k)$ by

$$W_k^q = \left\{ \varphi(x) \in L^q(R_k) \cap L^\infty(R_k) \ ; \|\varphi\|_W = \sup_{\xi} \left\{ \int_{R_k} (1+|x|^\beta)^q |\varphi(x-\xi)|^q dx \right\} < \infty \right\}$$

where $\beta = (k-1)/p$. Note that if $k \geqq 2$ and $1<p<2$ then $W_k^q \subseteqq L^2(R_k)$. For $\sigma > 0$, let us denote $\varphi(x)\exp(-\sigma|x|)$ by $\varphi_\sigma(x)$ and its Fourier transform (in $L^1(R_k)$) by $\hat{\varphi}_\sigma(t)$. We denote the zeros of the Fourier transform $\hat{f}(t)$ of $f(x) \in L^1(R_k)$ by $Z(\hat{f})$.

§1. Key Theorems.

<u>Theorem 1.</u> Suppose $f(x) \in L^p(R_k) \cap L^1(R_k)$, $\varphi(x) \in W_k^q$, and $f * \varphi = 0$. Then we have $\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t) = 0$ on the complement of $Z(\hat{f})$. Especially, the above limit exists uniformly on any closed interval contained in the complement of $Z(\hat{f})$.

The theorem for the case k=1 was proved by H.Pollard [1], and there is no essential difference between the proof for the case k=1 and for the case $k \geqq 2$. We shall repeat the argument due to Pollard for the sake of completeness.

Let us put

$$U(\sigma, t, y) = \int_{R_k} \varphi(x)\exp(-\sigma|x+y|)\exp(it\cdot x)\, dx .$$

Take any closed interval I contained in the complement of $Z(\hat{f})$. Then we have a real number $a > 0$ such that

$$\inf_{t \in I} |\hat{f}(t)| > 2 \int_{|y| \geq a} |f(y)|\, dy .$$

The assumption $f * \varphi = 0$ implies

$$(1.1) \qquad \int_{R_k} f(y-\xi)\exp(it\cdot(y-\xi))\, U(\sigma, t, y)\, dy = 0.$$

Make the difference between $U(\sigma, t, \xi)\hat{f}(t)$ and (1.1). Then we have

$$U(\sigma, t, \xi)\hat{f}(t) - 0$$

$$= \int_{R_k} f(y-\xi)\exp(it\cdot(y-\xi)) \left[U(\sigma, t, \xi) - U(\sigma, t, y)\right] dy$$

$$= \int_{|y-\xi| \leq a} + \int_{|y-\xi| > a} = J_1 + J_2 \text{ , say.}$$

An elementary calculation shows that

$$|U(\sigma, t, \xi) - U(\sigma, t, y)|$$

$$\leq \{ M \|\varphi\|_W \sigma^{1-1/p} |y - \xi| \}$$

$$(1.2) \qquad = M' \sigma^{1-1/p} |y - \xi| ,$$

where we have denoted constants by M's. By (1.2), we easily see that

$$|J_1| \leq M' \sigma^{1-1/p} \int_{|y| \leq a} |y| |f(y)| dy$$

and we have

$$|J_2| \leqq 2 \sup_{\xi} |U(\sigma, t, \xi)| \int_{|y|>a} |f(y)|\, dy,$$

where we have fixed $b > 0$. Now we are ready to conclude the following inequality;

$$\sup_{\xi} |U(\sigma, t, \xi)| \left\{ |\hat{f}(t)| - 2 \int_{|y|>a} |f(y)|\, dy \right\}$$

$$\leqq M'' \sigma^{1-1/p} \quad ,$$

which shows the conclusion of Theorem 1.

<u>Theorem 2.</u> Suppose $f(x) \in L^p(R_k) \cap L^1(R_k)$, $\varphi(x) \in L^q(R_k) \cap L^2(R_k)$ and $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma(t) = 0$ on the complement of $Z(\hat{f})$. Then we have $f * \varphi = 0$.

For the proof of Theorem 2, we use a tecnique developed by A.Beurling [2] . Put $\psi = f * \varphi$, then $\psi(x) \in L^2(R_k)$. By the Parseval relation and the Schwarz inequality, we have

$$I(\sigma) = \int_{R_k} |\hat{f}(t)\, \hat{\varphi}_\sigma(t) - \hat{\psi}_\sigma(t)|^2\, dt$$

$$= M \int_{R_k} dx \left| \int_{R_k} f(y)\, \varphi(x-y) \left\{ \exp(-\sigma|x-y|) - \exp(-\sigma|x|) \right\} dy \right|^2$$

$$= M \|f\|_1 \int_{R_k} |f(y)|\, dy \int_{R_k} |\varphi(x-y)|^2 \left| \exp(-\sigma|x-y|) - \exp(-\sigma|x|) \right|^2 dx \; .$$

Hence, by the Lebesgue theorem, $\lim_{\sigma \to 0+} I(\sigma) = 0$.
Since $\lim_{\sigma \to 0+} \| \psi - \psi_\sigma \|_2 = 0$, we get $\lim_{\sigma \to 0+} \| \hat{\psi} - \hat{f}\, \hat{\varphi}_\sigma \|_2 = 0$, that is,

$\hat{\psi}(t) = \hat{f}(t)\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t) = 0$. So we have $\psi = f * \varphi = 0$.

Theorem 3. We suppose $\varphi(x) \in W_k^q \cap L^2(R_k)$ or $\varphi(x) \in L^q(R_1) \cap L^\infty(R_1)$. Let F be a closed sub-set of R_k . If $\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t) = 0$ on the complement of F, then the above limit exists uniformly on any closed interval contained in the complement of F.

Under the assumption" $\varphi \in L^q(R_1) \cap L^\infty(R_1)$", Theorem 3 was proved in[3]. We give a proof under the assumption "$\varphi \in W_k^q \cap L^2(R_k)$, $(k \geq 1)$ ". Take any closed interval I which is contained in the complement of F. Find a function $f(x) \in L^p(R_k) \cap L^1(R_k)$ such that $I \subset \mathrm{supp}(\hat{f}) \subset CF$. By the asumption, we have $\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t) = 0$ on the complement of $Z(\hat{f})$. Since $\varphi(x) \in L^2(R_k)$, applying Theorem 2, we have $f * \varphi = 0$. Now we can apeal to Theorem 1, and we have $\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t) = 0$ uniformly on $t \in I$.

§ 2. Closure problems (1).

We shall introduce several notions for the discussion of "closure of translations problem".

A linear subfamily W of L^q may introduce such the weakest topology into $L^p(R_k)$ that it makes only elements of W continuous linear functionals on $L^p(R_k)$. We call such a topology mentioned above by "W-topology". We denote the closure of the linear manifold spanned by the translates of $f(x) \in L^p(R_k)$ by $T[f;W]$, where the closure is considered under W-topology.

A closed sub-set F of R_k is said to be a $(U;W)$-set if the relations

$$\lim_{\sigma \to 0+} \hat{\varphi}_{\sigma}(t) = 0 \text{ on } t \in CF, \quad \varphi \in W \subset L^q(R_k)$$

imply that $\varphi(x) = 0$, a.e..

Using the notions introduced above, we can interprete Theorems 1 and 2 in the following forms.

<u>Theorem 4.</u> Let $f(x) \in L^p(R_k) \cap L^1(R_k)$. If $Z(\hat{f})$ is a $(U;W_k^q)$-set, then $T[f;W_k^q] = L^p(R_k)$.

<u>Theorem 5.</u> Let $f(x) \in L^p(R_k) \cap L^1(R_k)$. If $T[f;\ L^q(R_k) \cap L^2(R_k)] = L^p(R_k)$, then $Z(\hat{f})$ is a $(U;L^q(R_k) \cap L^2(R_k))$-set.

<u>Theorem 6.</u> Let $f(x) \in L^p(R_k) \cap L^1(R_k)$. Then

$$T[f;\ W_k^q \cap L^2(R_k)] = L^p(R_k)$$

if and only if $Z(\hat{f})$ is a $(U;\ W_k^q \cap L^2(R_k))$-set.

§3. Closure problems (2).

According to R.E.Edwards [4] , we shall introduce a notion of thin-set. A closed sub-set F of R_k is said to be $(p;W)$-thin if the relations

$$\mathrm{supp}(\hat{\varphi}) \subset F, \quad \varphi \in W \subset L^q(R_k) \cap L^{\infty}(R_k)$$

imply that $\varphi(x) = 0$, a.e. , where $\mathrm{supp}(\hat{\varphi})$ means the support of the generalized Fourier transform (i.e. a pseudo measure) $\hat{\varphi}$ of $\varphi \in L^{\infty}(R_k)$.

<u>Theorem 7.</u> A closed sub-set $F \subseteq R_k$ is $(p; W_k^q \wedge L^2(R_k))$-thin, if and only if F is a $(U; W_k^q \wedge L^2(R_k))$-set.

For the case k=1, we can exclude the word " $L^2(R_k)$ " from the above statement (cf. Edwards [4]), that is, the notion $(U; L^q(R_1) \wedge L^{\infty}(R_1))$ is equivalent to the notion $(p; L^q(R_1) \wedge L^{\infty}(R_1))$.

For the proof of Theorem 7, suppose F is $(p; W_k^q \wedge L^2(R_k))$. In order to show that F is $(U; W_k^q \wedge L^2(R_k))$, it is enough to show that the relations

$$\lim_{\sigma \to 0+} \hat{\varphi}_\sigma(t) = 0 \text{ on } t \in CF, \ \varphi \in W_k^q \wedge L^2(R_k)$$

imply $\mathrm{supp}(\hat{\varphi}) \subseteq F$. For this purpos, take any closed interval $I \subset CF$. Consider any function $\psi \in \mathcal{S}$ (the Schwartz space) such that $\mathrm{supp}(\psi) \subseteq I$. Since $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma = \hat{\varphi}$ in $\mathcal{S}'$ (the temperate distribution space, the dual space of $\mathcal{S}$), we have

$$\langle \hat{\varphi}, \psi \rangle = \lim_{\sigma \to 0+} \int_{R_k} \hat{\varphi}_\sigma(t)\psi(t)\,dt = \lim_{\sigma \to 0+} \int_I \hat{\varphi}_\sigma(t)\psi(t)\,dt .$$

By Theorem 3, $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma(t) = 0$, uniformly on $t \in I$. Therefore, we have $\langle \hat{\varphi}, \psi \rangle = 0$, which shows that $\mathrm{supp}(\hat{\varphi}) \subseteq F$.

Conversely, we suppose that F is $(U; W_k^q \wedge L^2(R_k))$. In order to show that F is $(U; W_k^q \wedge L^2(R_k))$, we have to show that the relations

$$\mathrm{supp}(\hat{\varphi}) \subseteq F, \qquad \varphi \in W_k^q \wedge L^2(R_k)$$

imply $\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t) = 0$ on $t \in CF$.

Let $Sp(\varphi)$ be the spectrum of $\varphi(x) \in L^\infty(R_k)$, that is,

$$Sp(\varphi) = \bigcap_{g\in J} Z(\hat{g}), \quad J = \left\{ g \in L^1(R_k) \; ; \; g * \varphi = 0 \right\} .$$

Then $Sp(\varphi) = Supp(\hat{\varphi})$ (cf. J.P.Kahane [5] and Edwards[4]).*)

Since $supp(\hat{\varphi}) \subseteq F$, we have $CF \subseteq C\,Sp(\varphi)$. Take any point $t_o \in CF \subseteq C\,Sp(\varphi)$, then, by the definition of $Sp(\varphi)$, there exists $f(x) \in L^1(R_k)$ such that $f * \varphi = 0$ but $\hat{f}(t_o) \neq 0$.

We may suppose that $f(x) \in L^p(R_k) \cap L^1(R_k)$. Now apply Theorem 1, then we have $\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t_o) = 0$, which completes the proof.

In the second part of the above argument, we did not use L^2-property of φ. In fact, we have proved

Theorem 8. If F is $(U; W_k^q)$, then F is $(p; W_k^q)$.

§4. Uniqueness theorem for the Poisson summability of trigonometric integrals.

We may interprete Theorem 7 in the following form:

Theorem 9. Let $\varphi(x) \in W_k^q \cap L^2(R_k)$ and F be $(p; W_k^q \cap L^2(R_k))$. If

$$\lim_{\sigma\to 0+} \int_{R_k} \varphi(x)\exp(it\cdot x)\exp(-\sigma|x|)\,dx = 0$$

on $t \in CF$, then $\varphi(x) = 0$, a.e. . (For the case $k=1$, we can exclude the word "$L^2(R_k)$" from the above assumptions.)

R.E.Edwards [4] gave several examples of p-thin sets. For example, any discrete set is $(p;\, L^q(R_k) \cap C_o(R_k))$-thin,

*) C.S.Herz [7] reduced the closure problem to the spectral synthesis problem.

where $C_o(R_k)$ denotes the space of continuous functions on R_k which tends to zero at infinity. Combined this fact and Theorem 9, we have the following uniqueness theorem:

Theorem 10. A discrete set is a set of uniqueness for the Poisson summability of trigonometric integrals of $\varphi \in W_k^q \cap C_o(R_k) \cap L^2(R_k)$ or $\varphi \in L^q(R_1) \cap C_o(R_1)$.

From the fact that any discrete set is $(p;L^q(R_k) \cap C_o(R_k))$-thin, Edwards concluded that if $f \in L^p(R_k) \cap L^1(R_k)$ and if $Z(\hat{f})$ is discrete then $T[f;L^q] = L^p(R_k)$. This was proved also by I.E. Segal [6] .

§5. Simple proof of uniquness theorem for the Poisson summability of trigonometric integrals.

The proof of Theorem 7 suggests us a simple proof of uniqueness theorem for the Poisson summability of trigonometric integrals: The following simple result is a key for the problem.

Theorem 11. Suppose $\varphi \in L^\infty(R_k)$. If $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma(t)=0$ for all $t \in R_k$, and if the above limit exists uniformly on any finite closed interval in R_k , then $\varphi = 0$, a.e. .

The proof of Theorem 11 is nothing but the repeat of the proof of Theorem 7. In fact, we have $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma = \hat{\varphi}$, distributionally. The assumption of uniformity implies that $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma = 0$. This means that $\hat{\varphi} = 0$, that is, $\varphi = 0$, a.e. .

As a consequence of Theorem 11, when we want to conclude " $\varphi = 0$, a.e. " from " $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma(t) = 0$, everywhere " , it is enough to show that the above limit exists uniformly on any closed interval in R_k . Therefore, combine Theorem 3 and Theorem 11, we have

Theorem 12. Suppose $\varphi \in W_k^q \cap L^2(R_k)$, $k \geqq 1$. If $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma(t) = 0$, everywhere on R_k , then $\varphi = 0$, a.e. .

We have to remark that we do not drop the assumption " L^2 " from the above theorem even for the case k = 1. The situation is as follows: For the proof of Theorem 3 under the assumption " $\varphi \in L^q(R_1) \cap L^\infty(R_1)$ " , we need the uniqueness theorem.(Cf. Proof of Theorem B in [1] and Proof of Lemma 2 in [3]) . However, when we prove Theorem 3 under the assumption " $\varphi \in W_k^q \cap L^2(R_k)$, $k \geqq 1$ " , we do not need the uniqueness theorem.

When the case k = 1, we can generalize Theorem 12 in the following way:

Theorem 13. Suppose $\varphi \in L^q(R_1) \cap L^\infty(R_1)$, and

(5.1) $$\int_I |\hat{\varphi}_\sigma(t)|^2 \, dt \leqq C(I) < \infty, \text{ for } \sigma > 0 \text{ and for any finite interval } I \text{ in } R_1 ,$$

where C(I) is constant depending only on I.
If $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma(t) = 0$, everywhere, then $\varphi = 0$, a.e. .

For the proof of Theorem 13, we have to show that the limit $\lim_{\sigma \to 0+} \hat{\varphi}_\sigma(t) = 0$ exists uniformly on I. In order to establish the above, we need a theorem corresponding to Theorem 3, and hence we want to have a result of type of Theorem 2. For this purpose, just repeat the argument in [2] , then we have

<u>Theorem 14.</u> Suppose $|x|^{1/2}f(x) \in L^1(R_1)$, $\varphi \in L^\infty(R_1)$ and (5.1). If $\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t) = 0$ on the complement of $Z(\hat{f})$, then $f * \varphi = 0$.

From Theorem 14 and Theorem 1 of the case k = 1, we have the following:

<u>Theorem 15.</u> Suppose $\varphi \in L^q(R_1) \cap L^\infty(R_1)$ and (5.1). Let F be a closed subset of R_1 . If $\lim_{\sigma\to 0+} \hat{\varphi}_\sigma(t) = 0$ on the complement of F, then the above limit exists uniformly on any closed interval contained in the complement of F.

From the above setting, we can conclude Theorem 13.

Remark that (5.1) holds if $\varphi \in L^2(R_1)$. This follows from the Parseval relation.

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