数理解析研究所講究録
第 63 巻 1969 年 85-99

An Application of the PLK-Method for Second-Order Nonlinear Ordinary Differential Equations

By Ken-iti Takahasi

(Tokyo University of Agriculture and Technology)

§1. Introduction. In 1949, M.J.Lighthill [1] proposed a new method for nonlinear differential equations known as PLK-method.. In order to construct a uniform representation of the solution of the initial value problem:

$$(1.1)\quad \left\{x+\sum_{m=1}^{\infty}\varepsilon^m P_m(x,u)\right\}\frac{du}{dx}+q(x)u=r(x)+\sum_{m=1}^{\infty}\varepsilon^m R_m(x,u),\quad u(1)=b$$

over the interval $0 \leqq x \leqq 1$ for every ε ($0 < \varepsilon \leqq \varepsilon_0$, $\varepsilon_0 > 0$ being sufficiently small), he substitutes

$$(1.2)\qquad u=\sum_{m=0}^{\infty}\varepsilon^m u_m(\xi),\qquad x=\xi+\sum_{m=1}^{\infty}\varepsilon^m x_m(\xi)$$

into the equation of Problem (1.1) and determines $u_m(\xi)$ and $x_m(\xi)$ in a suitable manner. In 1966, Y. Sibuya and K. Takahasi [4] proved the uniform convergence of the formal solution (1.2) of the initial value problem: $(x+\varepsilon u)\,du/dx + q(x)u = r(x)$, $u(1) = b$. In 1955, W.A.Wasow mentioned in his paper [3] that Lighthill's method can be justified if $P_m(x,u)$ and $R_m(x,u)$ are polynomials of degree not greater than m with respect to u. In 1967, the author mentioned in his paper [6] that Lighthill's method can be justified for the problem of the form:

$$\left\{x+\varepsilon u+\sum_{m=2}^{s}\varepsilon^m P_m(x,u)\right\}\frac{du}{dx}+q(x)u+r(x)+\sum_{m=1}^{s'}\varepsilon^m R_m(x,u)=0,\quad u(1)=b$$

under some assumptions, even if $P_m(x,u)$ and $R_m(x,u)$ are polynomials of degree greater than m with respect to u . In this paper, our purpose is to show that Lighthill's method can be justified for the initial value problem of the second order differential equation:

$$(1.3)\quad \left\{x+\varepsilon\left(v+p(x)\frac{dv}{dx}\right)\right\}\frac{d^2v}{dx^2}+q(x)\frac{dv}{dx}+s(x)v=r(x),\quad v(1)=b,\ v'(1)=b',$$

where $p(x)$, $q(x)$, $s(x)$ and $r(x)$ are real-valued and analytic functions of x for $0 \leqq x \leqq 1$, and $q_o = q(0)$ is a positive non-integer.

§ 2 Theorems.

In order to consider this problem for $\varepsilon = 0$, we put $\varepsilon = 0$ in Problem (1.3), then we have

$$(2.1)\quad x\frac{d^2v}{dx^2}+q(x)\frac{dv}{dx}+s(x)v = r(x),\quad v(1) = b,\quad v'(1) = b'.$$

Putting $\frac{dv}{dx} = u$, we have

$$(2.2)\quad \begin{cases} x\dfrac{du}{dx} = -q(x)u - s(x)v + r(x), \\ \dfrac{dv}{dx} = u, \quad u(1) = b', \quad v(1) = b. \end{cases}$$

In order to construct a solution of Problem (2.2), first of all, let us consider a solution of the equation of Problem (2.2) of the form:

$$(2.3)\qquad u = \varphi(x) = \sum_{m=0}^{\infty} a_m x^m,\quad v = \psi(x) = \sum_{m=0}^{\infty} b_m x^m$$

in the neighborhood of $x = 0$ Inserting the series (2.3) into the equation of Problem (2.2), we have

$$(2.4)\quad \begin{cases} (m+q_o)a_m = F_m(a_o, \ldots, a_{m-1}, b_o, \ldots, b_m), \\ (m+1)b_{m+1} = a_m, \quad (m = 0,1,2,\ldots\ldots), \end{cases}$$

where F_m is a polynomial of $a_o, \ldots, a_{m-1}, b_o, \ldots, b_m$. Since we assumed that $q_o > 0$, we can construct a solution of the equation of Problem (2.2) which is analytic and real-valued in the neighborhood of $x = 0$. Since (2.2) is linear, $u = \varphi(x)$, $v = \psi(x)$ is analytic on $0 \leqq x \leqq 1$. It is evident that $\varphi(x)$, $\psi(x)$ are real-valued on $0 \leqq x \leqq 1$. Thus the solution of (2.2) is given by

$$(2.5)\quad \begin{cases} u = u_o(x) = C_1\varphi_1(x) + C_2 x^{-q_o}\varphi_2(x) + \varphi(x), \\ v = v_o(x) = C_1\psi_1(x) + C_2 x^{-q_o}\psi_2(x) + \psi(x), \end{cases}$$

where $(\varphi_1(x), \psi_1(x))$, $(x^{-q_o}\varphi_2(x), x^{-q_o}\psi_2(x))$ is a fundamental system of solutions of the following system of homogeneous equations:

$$x\frac{du}{dx} = -q(x)u - s(x)v, \quad \frac{dv}{dx} = u$$

and C_1 and C_2 are constants such that $u_o(1) = b'$, $v_o(1) = b$.

Our main theorems are stated as follows:

THEOREM 1. Assume that

(I) $p(x)$, $q(x)$, $s(x)$ and $r(x)$ are real-valued and analytic functions of x for $0 \leqq x \leqq 1$;

(II) $q_o = q(0)$ is a positive non-integer.

Then there exist functions $x(\xi,\zeta)$, $u(\xi,\zeta)$ and $v(\xi,\zeta)$ defined by the power series

$$(2.6)\quad \begin{cases} x(\xi,\zeta) = \xi + \sum_{m=1}^{\infty} x_m(\xi)\zeta^m, \\ u(\xi,\zeta) = \sum_{m=0}^{\infty} u_m(\xi)\zeta^m, \\ v(\xi,\zeta) = \sum_{m=0}^{\infty} v_m(\xi)\zeta^m \end{cases}$$

such that (a) the power series $x(\xi,\zeta)$, $u(\xi,\zeta)$ and $v(\xi,\zeta)$ are uniformly convergent for $0 < \xi \leqq 1$, $|\zeta| \leqq \delta$, ($\delta > 0$ being sufficiently small) *with coefficients* $x_m(\xi)$, $u_m(\xi)$ *and* $v_m(\xi)$ *functions real-valued and analytic for* $0 < \xi \leq 1$; (b) $x(1,\zeta)=1$, $u(1,\zeta)=b'$, $v(1,\zeta)=b$; (c) $x = x(\xi, \varepsilon\xi^{-q_o})$, $u=u(\xi, \varepsilon\xi^{-q_o})$, $v=v(\xi, \varepsilon\xi^{-q_o})$ *is a parametric representation of the solution* $u(x)$, $v(x)$ *of the following Problem*:

$$(2.7)\quad \begin{cases} \{x+\varepsilon(v+p(x)u)\}\dfrac{du}{dx} + q(x)u + s(x)v = r(x), \\ \dfrac{dv}{dx} = u, \quad u(1) = b', \quad v(1) = b. \end{cases}$$

THEOREM 2. *Assume the same assumptions* (I) *and* (II) *as in Theorem 1. Then there exist positive constants* δ *and* ε_o *such that*

(i) *if* $C_2p(0)\varphi_2(0) > 0$, (a) *the function*

$$(2.8)\quad F \equiv x(\xi, \varepsilon\xi^{-q_o})+\varepsilon\{v(\xi, \varepsilon\xi^{-q_o})+p(x(\xi,\varepsilon\xi^{-q_o}))u(\xi,\varepsilon\xi^{-q_o})\}$$

never vanishes in the interval

$$(2.9)\quad (\varepsilon/\delta)^{1/q_o} \leq \xi \leq 1$$

for every ε *in the interval*

$$(2.10)\quad 0 < \varepsilon \leq \varepsilon_o;$$

(b) *the equation* $x(\xi, \varepsilon\xi^{-q_o}) = 0$ *has a unique solution* $\xi = \hat{\xi}(\varepsilon)$ *in the interval* (2.9) *for every* ε *in the interval* (2.10) *and the function* $\hat{\xi}(\varepsilon)$ *behaves asymptotically as*

$$(2.11)\quad \hat{\xi}(\varepsilon) = \varepsilon^{1/(q_o+1)}\left\{\left(\frac{C_2p(0)\varphi_2(0)}{q_o+1}\right)^{1/(q_o+1)} + o(1)\right\}, \ (\varepsilon\to +0),$$

and, consequently, the value of the solution $u(x)$, $v(x)$ *at* $x = 0$ *is* *given by* $u(\hat{\xi}(\varepsilon))$, $v(\hat{\xi}(\varepsilon))$ *for* $0 < \varepsilon \leqq \varepsilon_o$; (c) *the solution* $u(x)$, $v(x)$ *defined by*

(2.12) $\quad x=x(\xi,\ \varepsilon\xi^{-q_o}),\ u=u(\xi,\ \varepsilon\xi^{-q_o}),\ v=v(\xi,\ \varepsilon\xi^{-q_o})$

does not have any singular point on the interval $0 \leqq x \leqq 1$ *for every* ε $(0 < \varepsilon \leqq \varepsilon_o)$;

(ii) *if* $C_2 p(0)\varphi_2(0) < 0$, *the function* F *has a zero* $\xi = \tilde{\xi}(\varepsilon)$ *in the interval* (2.9) *for every* ε *in the interval* (2.10), *hence the solution* $u(x)$, $v(x)$ *defined by* (2.12) *has a singular point with respect to* x *on the interval* $0 \leqq x \leqq 1$.

§ 3. *Construction of a Formal Solution of Equation* (2.7). Putting

$$(3.1)\qquad \begin{cases} x = \xi + \eta, \\ u = \xi^{-q_o}\,(\bar{u}_o(\xi) + \bar{u}), \\ v = \xi^{-q_o}\,(\bar{v}_o(\xi) + \bar{v}\,) \end{cases}$$

in (2.7), where $\bar{u}_o(\xi) = \xi^{q_o} u_o(\xi)$, $\bar{v}_o(\xi) = \xi^{q_o} v_o(\xi)$, we have

$$(3.2)\qquad \begin{cases} \left[\xi+\eta+\zeta\left\{\bar{v}_o+\bar{v}+p(\xi+\eta)(\bar{u}_o+\bar{u})\right\}\right]\left[-q(\xi)\bar{u}_o-s(\xi)\bar{v}_o+\xi^{q_o}r(\xi)-q_o\bar{u}+\xi\dfrac{d\bar{u}}{d\xi}\right] \\ \quad +\xi\left[q(\xi+\eta)(\bar{u}_o+\bar{u})+s(\xi+\eta)(\bar{v}_o+\bar{v})-\xi^{q_o}r(\xi+\eta)\right]\left(1+\dfrac{d\eta}{d\xi}\right)=0, \\ \xi\dfrac{d\bar{v}}{d\xi} = \zeta\bar{u}+q_o\bar{v}+\zeta(\bar{u}_o+\bar{u})\dfrac{d\eta}{d\xi}, \quad (\zeta = \varepsilon\xi^{-q_o}). \end{cases}$$

In order to satisfy the equation (3.2), it is sufficient to determine η, $\bar{u}$, $\bar{v}$ such that

$$(3.3)\qquad \begin{cases} \xi\dfrac{d\eta}{d\xi} = \eta+\zeta\left\{\bar{v}_o+\bar{v}+p(\xi+\eta)(\bar{u}_o+\bar{u})\right\}, \\ \xi\dfrac{d\bar{u}}{d\xi} = (q_o-q(\xi+\eta))\bar{u}-s(\xi+\eta)\bar{v}-(q(\xi+\eta)-q(\xi))\bar{u}_o \end{cases}$$

$$-(s(\xi+\eta)-s(\xi))\bar{v}_o+\ \xi^{q_o}(r(\xi+\eta)-r(\xi)),$$

$$\xi\frac{d\bar{v}}{d\xi}=\xi\bar{u}+q_o\bar{v}+(\bar{u}_o+\bar{u})\left[\eta+\zeta\{\bar{v}_o+\bar{v}+p(\xi+\eta)(\bar{u}_o+\bar{u})\}\right].$$

Putting

$$p(\xi+\eta)=\sum_{m=0}^{\infty}p_m(\xi)\eta^m,\quad q(\xi+\eta)=\sum_{m=0}^{\infty}q_m(\xi)\eta^m,$$

$$s(\xi+\eta)=\sum_{m=0}^{\infty}s_m(\xi)\eta^m,\quad r(\xi+\eta)=\sum_{m=0}^{\infty}r_m(\xi)\eta^m$$

in (3.3), we have

$$(3.4)\quad\begin{cases}\xi\dfrac{d\eta}{d\xi}=\eta+g_1(\xi,\eta,\bar{u},\bar{v},\zeta),\\ \xi\dfrac{d\bar{u}}{d\xi}=(q_o-q_o(\xi))\bar{u}-s_o(\xi)\bar{v}+g_2(\xi,\eta,\bar{u},\bar{v},\zeta),\\ \xi\dfrac{d\bar{v}}{d\xi}=\xi\bar{u}+q_o\bar{v}+g_3(\xi,\eta,\bar{u},\bar{v},\zeta),\end{cases}$$

where

$$g_1=\zeta\left\{\bar{v}_o+\bar{v}+(\bar{u}_o+\bar{u})\sum_{m=0}^{\infty}p_m(\xi)\eta^m\right\},$$

$$g_2=-\sum_{m=1}^{\infty}\{(\bar{u}_o+\bar{u})q_m(\xi)+(\bar{v}_o+\bar{v})s_m(\xi)-\xi^{q_o}r_m(\xi)\}\eta^m,$$

$$g_3=(\bar{u}_o+u)\left[\eta+\zeta\left\{\bar{v}_o+\bar{v}+(\bar{u}_o+\bar{u})\sum_{m=0}^{\infty}p_m(\xi)\eta^m\right\}\right].$$

We want to construct a formal solution of Equation (3.4) in the form:

$$(3.5)\quad\begin{cases}\eta=\sum_{m=1}^{\infty}\bar{x}_m(\xi)\zeta^m,\ (\bar{x}_m(1)=0,\ m=1,2,\dots\dots),\\ u=\sum_{m=1}^{\infty}\bar{u}_m(\xi)\zeta^m,\ (\bar{u}_m(1)=0,\ m=1,2,\dots\dots),\\ v=\sum_{m=1}^{\infty}\bar{v}_m(\xi)\zeta^m,\ (\bar{v}_m(1)=0,\ m=1,2,\dots\dots).\end{cases}$$

Denote these series by $S(\xi,\zeta)$, $\bar{S}(\xi,\zeta)$, $\bar{\bar{S}}(\xi,\zeta)$ and let

$$g_1(\xi,s,\bar{s},\bar{\bar{s}},\zeta)=\sum_{m=1}^{\infty} H_m(\bar{x}_1,\dots,\bar{x}_{m-1},\bar{u}_1,\dots,\bar{u}_{m-1},\bar{v}_{m-1})\zeta^m,$$

$$g_2(\xi,s,\bar{s},\bar{\bar{s}},\zeta)=\sum_{m=1}^{\infty} \bar{H}_m(\bar{x}_1,\dots,\bar{x}_m,\bar{u}_1,\dots,\bar{u}_{m-1},\bar{v}_1,\dots,\bar{v}_{m-1})\zeta^m,$$

$$g_3(\xi,s,\bar{s},\bar{\bar{s}},\zeta)=\sum_{m=1}^{\infty} \bar{\bar{H}}_m(\bar{x}_1,\dots,\bar{x}_m,\bar{u}_1,\dots,\bar{u}_{m-1},\bar{v}_1,\dots,\bar{v}_{m-1})\zeta^m ,$$

where the quantities H_m, $\bar{H}_m$, $\bar{\bar{H}}_m$ are polynomials of their arguments with coefficients analytic in a suitable sector S_o in the complex ξ - plane such that the vertex of S_o is $\xi = 0$, S_o contains the line segment $0 < \xi \leqq 1$ in its interior. In particular

$$H_1 = \bar{v}_o(\xi) + \bar{u}_o(\xi)p_o(\xi),$$

$$\bar{H}_1 = - \bar{x}_1(\xi)\left\{\bar{u}_o(\xi)q_1(\xi)+\bar{v}_o(\xi)s_1(\xi)- \xi^{q_o} r_1(\xi)\right\} ,$$

$$\bar{\bar{H}}_1 = \bar{u}_o(\xi)\left\{x_1 + \bar{v}_o(\xi) + \bar{u}_o(\xi)p_o(\xi)\right\} .$$

Hence we determine $\bar{x}_m$, $\bar{u}_m$ and $\bar{v}_m$ by

$$(3.6)\qquad \begin{cases} \xi\dfrac{d\bar{x}_m}{d\xi}=(mq_o+1)\bar{x}_m+H_m(\bar{x}_1,\dots,\bar{v}_{m-1}), \ \bar{x}_m(1)=0, \\ \xi\dfrac{d\bar{u}_m}{d\xi}=\left\{(m+1)q_o-q_o(\xi)\right\}\bar{u}_m-s_o(\xi)\bar{v}_m+\bar{H}_m(\bar{x}_1,\dots,\bar{v}_{m-1}), \ \bar{u}_m(1)=0, \\ \xi\dfrac{d\bar{v}_m}{d\xi}=\xi\bar{u}_m+(m+1)q_o\bar{v}_m+\bar{\bar{H}}_m(\bar{x}_1,\dots,\bar{v}_{m-1}), \ \bar{v}_m(1)=0, (m=1,2,\dots). \end{cases}$$

The solution of Problem $(3.6)_1 (m=1)$ is given by

$$(3.7)\qquad \bar{x}_1 = \xi^{q_o+1}\int_1^{\xi} t^{-q_o-2}H_1(t)dt,$$

where $H_1(t)= \bar{v}_o(t) + \bar{u}_o(t)p_o(t)$, and the integral may be taken along any path from $t = 1$ to $t = \xi$ within the sector S_o. Noticing that $(\xi^{2q_o}\varphi_1(\xi), \ \xi^{2q_o}\psi_1(\xi))$, $(\xi^{q_o}\varphi_2(\xi), \ \xi^{q_o}\psi_2(\xi))$ is a fundamental system of solutions of the following system:

$$\begin{cases} \xi \dfrac{d\bar{u}_1}{d\xi} = (2q_o - q_o(\xi))\bar{u}_1 - s_o(\xi)\bar{v}_1, \\ \xi \dfrac{d\bar{v}_1}{d\xi} = \xi \bar{u}_1 + 2q_o\bar{v}_1, \end{cases}$$

we get the following solution of Problem $(3.6)_{2,3}(m=1)$:

$$(3.8) \quad \begin{cases} \bar{u}_1 = \xi^{2q_o} \varphi_1(\xi) \displaystyle\int_1^{\xi} \frac{t^{-2q_o-1}W_1(t)}{W(t)} dt \\ \qquad + \xi^{q_o} \varphi_2(\xi) \displaystyle\int_1^{\xi} \frac{t^{-q_o-1}\hat{W}_1(t)}{W(t)} dt, \\ \bar{v}_1 = \xi^{2q_o} \psi_1(\xi) \displaystyle\int_1^{\xi} \frac{t^{-2q_o-1}W_1(t)}{W(t)} dt \\ \qquad + \xi^{q_o} \psi_2(\xi) \displaystyle\int_1^{\xi} \frac{t^{-q_o-1}\hat{W}_1(t)}{W(t)} dt, \end{cases}$$

where

$$W(t) = \begin{vmatrix} \varphi_1(t) & \varphi_2(t) \\ \psi_1(t) & \psi_2(t) \end{vmatrix}, \quad W_1(t) = \begin{vmatrix} \bar{H}_1(t) & \varphi_2(t) \\ \bar{\bar{H}}_1(t) & \psi_2(t) \end{vmatrix}$$

$$\hat{W}_1(t) = \begin{vmatrix} \varphi_1(t) & \bar{H}_1(t) \\ \psi_1(t) & \bar{\bar{H}}_1(t) \end{vmatrix} \quad ,$$

$$\bar{H}_1(t) = -\bar{x}_1(t)\left\{\bar{u}_o(t)q_1(t) + \bar{v}_o(t)s_1(t) - t^{q_o}r_1(t)\right\},$$

$$\bar{\bar{H}}_1(t) = \bar{u}_o(t)\left\{\bar{x}_1(t) + \bar{v}_o(t) + \bar{u}_o(t)p_o(t)\right\}.$$

Assuming that we can get the solutions of Problem (3.6)$(m=1,\ldots,k-1)$, the solution of $(3.6)_1(m=k)$ is given by

$$(3.9) \qquad \bar{x}_k = \xi^{kq_o+1} \int_1^{\xi} t^{-kq_o-2} H_k(t)dt,$$

where

$$H_k(t) = H_k(\bar{x}_1(t),\ldots,\bar{x}_{k-1}(t),\bar{u}_1(t),\ldots,\bar{u}_{k-1},\bar{v}_{k-1}(t)),$$

and then the solutions of Problem $(3.6)_{2,3}(m=k)$ are given by

$$(3.10)\quad \begin{cases} \bar{u}_k = \xi^{(k+1)q_o} \quad \varphi_1(\xi)\displaystyle\int_1^{\xi} \frac{t^{-(k+1)q_o-1}W_k(t)}{W(t)}\,dt \\ \qquad + \xi^{kq_o} \quad \varphi_2(\xi)\displaystyle\int_1^{\xi} \frac{t^{-kq_o-1}\hat{W}_k(t)}{W(t)}\,dt, \\ \bar{v}_k = \xi^{(k+1)q_o} \quad \psi_1(\xi)\displaystyle\int_1^{\xi} \frac{t^{-(k+1)q_o-1}W_k(t)}{W(t)}\,dt \\ \qquad + \xi^{kq_o} \quad \psi_2(\xi)\displaystyle\int_1^{\xi} \frac{t^{-kq_o-1}\hat{W}_k(t)}{W(t)}\,dt, \end{cases}$$

where

$$W_k(t) = \begin{vmatrix} \bar{H}_k(t) & \varphi_2(t) \\ \bar{\bar{H}}_k(t) & \psi_2(t) \end{vmatrix}, \quad \hat{W}_k(t) = \begin{vmatrix} \varphi_1(t) & \bar{H}_k(t) \\ \psi_1(t) & \bar{\bar{H}}_k(t) \end{vmatrix},$$

$$\bar{H}_k(t) = \bar{H}_k(\bar{x}_1(t),\ldots,\bar{x}_k(t),\bar{u}_1(t),\ldots,\bar{u}_{k-1}(t),\bar{v}_1(t),\ldots,\bar{v}_{k-1}(t)),$$

$$\bar{\bar{H}}_k(t) = \bar{\bar{H}}_k(\bar{x}_1(t),\ldots,\bar{x}_k(t),\bar{u}_1(t),\ldots,\bar{u}_{k-1}(t),\bar{v}_1(t),\ldots,\bar{v}_{k-1}(t)).$$

Thus, by mathematical induction, we can get the solutions of Problem (3.6).

§ 4. Estimates of the Coefficients $x_m(\xi)$, $u_m(\xi)$ and $v_m(\xi)$. In this section we shall prove the following lemma:

LEMMA 1. There exists a positive constant M such that

$$\sup_{\xi\in S_0} |\bar{x}_m(\xi)| \leq M \sup_{\xi\in S_0} |H_m(\xi)| ,$$

$$\sup_{\xi\in S_0} |\bar{u}_m(\xi)| , \sup_{\xi\in S_0} |\bar{v}_m(\xi)| \leq M\left\{\sup_{\xi\in S_0} |\bar{H}_m(\xi)| + \sup_{\xi\in S_0} |\bar{\bar{H}}_m(\xi)|\right\} .$$

PROOF. We shall prove the first inequality, since the others can be proved in the same manner. Since the quantities $\varphi_1(\xi)$, $\varphi_2(\xi)$, $\psi_1(\xi)$, $\psi_2(\xi)$ and $|W(\xi)|^{-1}$ are all bounded on the closure of S_o, there exists a constant C such that

$$|\bar{x}_m(\xi)| \leqq C|\xi|^{mq_o+1} \sup_{t\in S_o}|H_m(t)|\left\{\left|\int_{C_1} t^{-mq_o-2}dt\right| + \left|\int_{C_2} t^{-mq_o-2}\,dt\right|\right\},$$

where the paths of integration C_1 and C_2 are respectively the line segment $\overline{1,|\xi|}$ and the circular arc $\widehat{|\xi|,\xi}$. On the other hand, we can prove the following inequality in the same way as in [4] :

$$\left|\int_{C_1} t^{-mq_o-2}\,dt\right| + \left|\int_{C_2} t^{-mq_o-2}\,dt\right| \leqq C'\,|\xi|^{-mq_o-1} \quad \text{in } S_o,$$

where C' is a sufficiently large constant. This proves our lemma.

§ 5. Construction of Majorant. Since $p(\xi+\eta)$, $q(\xi+\eta)$, $s(\xi+\eta)$ and $r(\xi+\eta)$ are analytic for $|\eta| \leqq \delta_o$ ($\delta_o > 0$ is sufficiently small) and $\xi \in S_o$, there exists a constant K such that

$$|\bar{u}_o(\xi)|\ ,\ |\bar{v}_o(\xi)| \leqq K,$$

$$|p_m(\xi)|,\ |q_m(\xi)|,\ |s_m(\xi)|,\ |r_m(\xi)| = K/\delta_o^{\,m}\ .$$

Let us consider

$$(5.1)\qquad \begin{cases} \eta = MG_1(\eta,\bar{u},\zeta), \\ \bar{u} = \bar{v} = M\{G_2(\eta,\bar{u},\zeta)+G_3(\eta,\bar{u},\zeta)\}\ , \end{cases}$$

where

$$G_1 = \zeta(K+\bar{u})\left(1+\frac{K\delta_o}{\delta_o-\eta}\right)\ ,\quad G_2 = \frac{K\eta(2\bar{u}+2K+1)}{\delta_o-\eta}\ ,$$

$$G_3 = (K+\bar{u})\left\{\eta + \zeta(K+\bar{u})\left(1+\frac{K\delta_o}{\delta_o-\eta}\right)\right\}$$

and M is the constant in Lemma 1. Notice that G_1, G_2 and G_3 are

respectively majorants of g_1, g_2 and g_3 uniformly for ξ in S_o.
Equation (5.1) has a solution

$$(5.2)\quad \begin{cases} \eta = S(\zeta) = \sum_{m=1}^{\infty} S_m \zeta^m, \\ \bar{u} = \bar{v} = \bar{S}(\zeta) = \sum_{m=1}^{\infty} \bar{S}_m \zeta^m, \end{cases}$$

such that $S_m, \bar{S}_m \geqq 0$ and these series converge for $|\zeta| \leqq \delta$, if $\delta > 0$ is sufficiently small. Let

$$G_1(S(\zeta), \bar{S}(\zeta), \zeta) = \sum_{m=1}^{\infty} K_m \zeta^m,$$

$$G_2(S(\zeta), \bar{S}(\zeta), \zeta) + G_3(S(\zeta), \bar{S}(\zeta), \zeta) = \sum_{m=1}^{\infty} \bar{K}_m \zeta^m.$$

It is easily seen that $S_m = MK_m$, $\bar{S}_m = M\bar{K}_m$, in particular

$$S_1 = MK_1 = MK(1+K),$$

$$\bar{S}_1 = M\bar{K}_1 = MK^2(1+K)\{M(1+2K) + \delta_o(1+M)\} / \delta_o .$$

Next we shall prove

$$(5.3)\quad \sup_{\xi \in S_o} |\bar{x}_m(\xi)| \leqq S_m, \quad \sup_{\xi \in S_o} |\bar{u}_m(\xi)|, \quad \sup_{\xi \in S_o} |\bar{v}_m(\xi)| \leqq \bar{S}_m, (m=1,2, \ldots).$$

In order to prove (6.3), first of all let us consider the case m=1.
We have

$$\sup_{\xi \in S_o} |\bar{x}_1(\xi)| \leqq M \sup_{\xi \in S_o} |\bar{H}_1(\xi)| \leqq M \sup_{\xi \in S_o} |\bar{v}_o(\xi) + \bar{u}_o(\xi) p_o(\xi)|$$
$$\leqq MK(1+K) = S_1,$$

$$\sup_{\xi \in S_o} |\bar{u}_1(\xi)| = \sup_{\xi \in S_o} |\bar{v}_1(\xi)| \leqq M \left\{ \sup_{\xi \in S_o} |\bar{H}_1(\xi)| + \sup_{\xi \in S_o} |\bar{\bar{H}}_1(\xi)| \right\}$$
$$= M \left[\sup_{\xi \in S_o} \left| \bar{x}_1(\xi) \{ \bar{u}_o(\xi) q_1(\xi) + \bar{v}_o(\xi) s_1(\xi) - \xi^q o r_1(\xi) \} \right| \right.$$

$$+ \sup_{\xi\in S_o} \left|\bar{u}_o(\xi)\left\{\bar{x}_1(\xi)+\bar{v}_o(\xi)+\bar{u}_o(\xi)p_o(\xi)\right\}\right| \Big]$$

$$\leqq M\{MK^2(1+K)(1+2K)/\delta_o + K^2(1+K)(1+M)\}$$

$$= MK^2(1+K)\ \{M(1+2K)+\delta_o(1+M)\}\ /\ \delta_o\ .$$

Assuming that

$$\sup_{\xi\in S_o}|\bar{x}_n(\xi)| \leqq S_n,\quad \sup_{\xi\in S_o}|\bar{u}_n(\xi)| \leqq \bar{S}_n,\quad \sup_{\xi\in S_o}|\bar{v}_n(\xi)| = \bar{S}_n,\quad (n=1,\dots,m-1),$$

we have

$$\sup_{\xi\in S_o}|H_m(\xi)| \leqq K_m,\quad \sup_{\xi\in S_o}|\bar{H}_m(\xi)| + \sup_{\xi\in S_o}|\bar{\bar{H}}_m(\xi)| \leqq \bar{K}_m\ ,$$

because G_1, G_2 and G_3 are respectively majorants of g_1, g_2 and g_3. Therefore from Lemma 1 it follows

$$\sup|\bar{x}_m(\xi)| \leqq MK_m = S_m,\quad \sup|\bar{u}_m(\xi)|,\quad \sup|\bar{v}_m(\xi)| \leqq M\bar{K}_m = \bar{S}_m.$$

Thus, by mathematical induction, we complete the proof of (5.3).

§ 6. Properties of $\bar{x}_1(\xi)$. In Section 3, we derived the differential equation of $\bar{x}_1(\xi)$:

$$(6.1)\qquad \xi\frac{d\bar{x}_1}{d\xi} = (q_o+1)\bar{x}_1 + H_1(\xi),$$

whose solution satisfying the initial condition $\bar{x}_1(1) = 0$ is given by

$$(6.2)\qquad \bar{x}_1(\xi) = \xi^{q_o+1}\int_1^{\xi} t^{-q_o-2}\ H_1(t)\ dt.$$

Noticing that

$$\int_1^{\xi} t^{-q_o-2}H_1(t)dt = -\frac{1}{q_o+1}\Big[t^{-q_o-1}H_1(t)\Big]_1^{\xi} + \frac{1}{q_o+1}\int_1^{\xi} t^{-q_o-1}\ \frac{dH_1}{dt}dt$$

$$= \xi^{-q_o-1}\left\{ -\frac{H_1(0)}{q_o+1} + o(1)\right\}, \quad (\xi \to +0),$$

we get

(6.3) $$\bar{x}_1(\xi) = -\frac{H_1(0)}{q_o+1} + o(1), \quad (\xi \to +0).$$

§ 7. Case $C_2 p(0)\varphi_2(0) > 0$. Notice that $C_2 p(0)\varphi_2(0) = H_1(0)$. Substitutin (3.1),(3.5) in the function F defined by (2.8), we get $F = \xi + \sum_{m=1}^{\infty} \hat{x}_m(\xi)\zeta^m$, where in particular

$$\hat{x}_1(\xi) = \bar{x}_1(\xi) + H_1(\xi) = \frac{q_o H_1(0)}{q_o+1} + o(1), \quad (\xi \to +0).$$

Let $M_1(\geqq 1)$, ξ_1, δ_1 and δ be positive constants such that

(7.1) $$\begin{cases} |\hat{x}_1(\xi)| \leqq M_1 & \text{for } 0 \leqq \xi \leqq 1, \\ \hat{x}_1(\xi) \geqq \dfrac{q_o H_1(0)}{2(q_o+1)} & \text{for } 0 \leqq \xi \leqq \xi_1 \ (\leqq 1), \\ \left|\sum_{m=2}^{\infty} \hat{x}_m(\xi)\zeta^{m-2}\right| \leqq M_1 & \text{for } 0 \leqq \xi \leqq 1,\ 0 < \zeta \leqq \delta_1, \\ \delta = \operatorname{Min}\left\{\delta_1, \dfrac{\xi_1}{4M_1}, \dfrac{q_o H_1(0)}{4M_1(q_o+1)}\right\}. \end{cases}$$

From (7.1) it follows that if $\xi_1 \leqq \xi \leqq 1$, $0 < \zeta \leqq \delta$, we get

$$F \geqq \xi_1 - \left\{ |\hat{x}_1(\xi)|\zeta + \left|\sum_{m=2}^{\infty} \hat{x}_m(\xi)\zeta^{m-2}\right|\zeta^2 \right\}$$

$$\geqq \xi_1 - (M_1\delta + M_1\delta^2) \geqq \xi_1 - 2M\delta > 0,$$

and that if $0 \leqq \xi \leqq \xi_1$, $0 < \zeta \leqq \delta$, we get

$$F \geqq \hat{x}_1(\xi)\zeta + \sum_{m=2}^{\infty} \hat{x}_m(\xi)\zeta^m \geqq \frac{q_o H_1(0)}{2(q_o+1)}\zeta - M_1\zeta^2$$

$$\geqq \zeta^2\left\{ \frac{q_o H_1(0)}{2(q_o+1)}\zeta^{-1} - M_1 \right\}.$$

Since we have the inequality $\zeta = \varepsilon \xi^{-q_o} \leqq \delta$, we have

$$F \geqq \zeta^2 \left\{ \frac{q_o H_1(0)}{2\delta(q_o+1)} - M_1 \right\} = \frac{\zeta^2}{\delta} \left\{ \frac{q_o H_1(0)}{2(q_o+1)} - M_1 \delta \right\}$$

$$\geqq \frac{\zeta^2}{\delta} \left\{ \frac{q_o H_1(0)}{2(q_o+1)} - \frac{q_o H_1(0)}{4(q_o+1)} \right\} = \frac{\zeta^2 q_o H_1(0)}{4\delta(q_o+1)} > 0,$$

provided that ξ is in the interval

$$(\varepsilon/\delta)^{1/q_o} \leqq \xi \leqq \xi_1$$

for a sufficiently small ε , which prove the assertion (a) of Theorem 2. Since $\bar{x}_1 = - H_1(0)/(q_o+1) + o(1)$, $(\xi \rightarrow +0)$, the equation $x(\xi, \varepsilon \xi^{-q_o}) = 0$ has a solution $\xi = \hat{\xi}(\varepsilon)$:

$$\hat{\xi}(\varepsilon) = \left\{ \frac{H_1(0)}{q_o+1} \varepsilon + o(\varepsilon) \right\}^{1/(q_o+1)}, \quad (\varepsilon \rightarrow +0).$$

Therefore if $\varepsilon_o > 0$ is sufficiently small, we get

$$(\varepsilon/\delta)^{1/q_o} \leqq \hat{\xi}(\varepsilon) \leqq 1 \quad \text{for } 0 < \varepsilon \leqq \varepsilon_o.$$

Then it is easily seen that the function F is positive, namely $dx/d\xi > 0$ for (2.9) and (2.10). Hence $x(\xi, \varepsilon\xi^{-q_o})$ has one and only one zero there. This completes the proof of the assertion (i) of Theorem 2.

§ 8. Case $C_2 p(0) \varphi_2(0) < 0$. In the same manner as in § 7, we can complete the proof of the assertion (ii) of Theorem 2.

References

[1] M.J.Lighthill, A technique for rendering approximates solutions to physical problems uniformly valid, Phil. Mag., 40(1949), 1179-1201.

[2] H.L.Turrittin, A critique of the PLK-method,(presented in the

Serrin seminar, at University of Minnesota, January, 1958).

[3]W.A.Wasow, On the convergence of an approximation method of M.J. Lighthill, Jour. Rat. Mech. Anal., 4(1955), 751-767, and Correction.

[4]Y.Sibuya and K.Takahasi, On the differential equation $(x+\varepsilon u)du/dx + q(x)u-r(x)=0$, La Funk. Ekva., 9(1966),71-81.

[5]K.Takahasi, A note on PLK-method, M.R.C. Technical Summary Report 703(1966),1-17.

[6]K.Takahasi, A note on PLK-method, Proceedings U.S.A.-Japan seminar on diff. and func. equ.,University of Minnesota, (1967),507-517.