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DYNAMICS OF THE TWO-DIMENSIONAL STELLAR SYSTEM (I)

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Abstract

Dynamics of the two-dimensional stellar system with the frequency function of the generalized Schwartzschild type has been investigated by means of the non-axisymmetric, non-steady and collisionless Boltzmann equation. In the present paper, the dynamical features of the stellar system in which the quantities with the trigonometric functions of the longitude were assumed to be of the first order smallness, were examined under the requirement that the identity condition of the potential must hold for all values of the coordinates as well as of the time.

In general, two kinds of waves occur, the one advancing as a rigid rotation, while the other receding as if it compensated for the differential rotation, so that the resultant wave acts so as to develope the spiral arms. There exists, however, a special case in which the latter wave is not permissible. The one-armed and/or the two-armed patterns are possible, but the former appears on the looser condition than the latter.

Introduction

Hitherto, many investigations have been presented on the problem of the spiral structure found in the flattened galaxies. Most of theories have regarded the spiral phenomena as due to the effect from some perturbation deviating slightly from the overall steady state. And an approach that imposes the so-called spiral potential as the perturbation seems likely successful, as shown in the recent lecture given by Contopoulos (1970).

With respect to the origin of such perturbation, however, any reasonable explanation seems not to have been given up to the present. To this problem some kinds of approach may be conceibable, but we wanted to collect information on the multiformity of the dynamical features of the stellar system depending on the various different dynamical conditions in the hope that it might bring a general insight to the problem. So we planned detailed investigation into the two-dimensional stellar system being non-axisymmetric, non-steady, collisionless and being consisted of the frequency distribution function of the general Schwarzschild type after the way of Chandrasekhar (1942).

The present paper reports the result obtained under the requirement that the identity condition for the potential should hold always and everywhere in the self gravitating two-dimensional system.

Here, we add a brief mention on the contents of this paper. In § 1 the formal but strict relations among the coefficients of the velocity ellipse, the motion of the local standard of rest, the potential and the so-called weight factor to density χ are derived. The formulae therein are the basis for our investigation hereafter. Informations from the zeroth order approximation and the first order one are shown in § 2 and in § 3 respectively. In § 3, however, the so-called weight factor χ which is a function entered in the frequency function and depends on the coordinates and the time, has especially been assumed to be constant or negligible. Both § 4 and § 5 are devoted to the more general first approximation dropped the above-mentioned restriction to χ. And finally, main results and related discussions are given in § 6.

The most important conclusion of the present investigation is that under the above-mentioned requirement for the identity condition of the potential, the spiral patterns can only appear as leading. Whereas, as will be shown in the following paper, another requirement makes the spiral pattern either leading or trailing depending on some parameter values.

1. Formal Solution of Liouville Equation.

Dynamics of the disk-like stellar system is considered in two-dimension throughout in this paper by adopting the radial coordinates (r, θ) with their origin at the center of gravity of the system. We assume the frequency distribution F(P) as follows

$$F(P) = F\{\chi - h(\Pi - \Pi_0)^2 - k(\Theta - \Theta_0)^2 - 2l(\Pi - \Pi_0)(\Theta - \Theta_0)\}, \tag{1}$$

where Π, Θ: velocity components radial and tangential,

Π_0, Θ_0: velocity components radial and tangential of the local centroid,

h, k, l: coefficients of the velocity ellipse,

and all of the parameters χ; h, ...; Π_0, Θ_0 are functions of r, θ and time t.

If F(P)=const.exp(-P), then Π_0, Θ_0: 1/ 2h, 1/ 2k, 1/ 4l correspond to the motion of the local centroid, variances and covariance of the residual velocity at (r, θ) respectively. But we do not impose any special form of F(P) for the present, accordingly Π_0, Θ_0 are to be some representatives of motion of the local centroid, and h, k, l are to be some measures of the variances and the covariance of the residual motion.

Now, the two-dimensional Liouville equation in the polar coordinates is written as

$$\frac{\partial F}{\partial t} + \Pi \frac{\partial F}{\partial r} + \Theta \frac{\partial F}{r\partial \theta} + \left(\frac{\partial \mathfrak{V}}{\partial r} + \frac{\Theta^2}{r}\right)\frac{\partial F}{\partial \Pi} + \left(\frac{\partial \mathfrak{V}}{r\partial \theta} - \frac{\Pi\Theta}{r}\right)\frac{\partial F}{\partial \Theta} = 0, \tag{2}$$

in which $\mathfrak{V} = \mathfrak{V}(r, \theta; t)$ denotes the potential function. Putting (1) into (2), we have

$$\left.\begin{aligned}
&\frac{\partial h}{\partial r} = 0,\\
&\frac{\partial k}{r\partial \theta} + \frac{2l}{r} = 0,\\
&\frac{\partial h}{r\partial \theta} + 2\frac{\partial l}{\partial r} - \frac{2l}{r} = 0,\\
&\frac{\partial k}{\partial r} + 2\frac{\partial l}{r\partial \theta} + \frac{2h}{r} - \frac{2k}{r} = 0.
\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned}&\frac{\partial h}{\partial t}-2\frac{\partial\Delta_1}{\partial r}=0,\\&\frac{\partial k}{\partial t}-2\frac{\partial\Delta_2}{r\partial\theta}-\frac{2\Delta_1}{r}=0,\\&\frac{\partial l}{\partial t}-\frac{\partial\Delta_2}{\partial r}-\frac{\partial\Delta_1}{r\partial\theta}+\frac{\Delta_2}{r}=0.\end{aligned}\right\}\quad\begin{aligned}&\Delta_1\equiv h\Pi_0+l\Theta_0,\\&\Delta_2\equiv l\Pi_0+k\Theta_0.\end{aligned}\tag{4}$$

$$\left.\begin{aligned}&\frac{\partial\Delta_1}{\partial t}-\frac{\partial}{2\partial r}(X-\chi)=h\frac{\partial\Omega}{\partial r}+l\frac{\partial\Omega}{r\partial\theta},\\&\frac{\partial\Delta_2}{\partial t}-\frac{\partial}{2r\partial\theta}(X-\chi)=l\frac{\partial\Omega}{\partial r}+k\frac{\partial\Omega}{r\partial\theta},\\&\frac{\partial}{2\partial t}(X-\chi)=\Delta_1\frac{\partial\Omega}{\partial r}+\Delta_2\frac{\partial\Omega}{r\partial\theta}.\end{aligned}\right\}\quad\begin{aligned}X&\equiv\Pi_0\Delta_1+\Theta_0\Delta_2\\&=h\Pi_0^2+k\Theta_0^2+2l\Pi_0\Theta_0.\end{aligned}\tag{5}$$

The solutions of the partial differential equations in (3) and (4) are given after some calculations.

$$\left.\begin{aligned}&h=H(t)-2h_{2\theta},\\&k=H(t)+2h_{2\theta}-2rb_\theta+r^2K,\\&l=-h_{2\theta}'+rb_\theta',\end{aligned}\right\}\quad K:\text{const.},\quad x'\equiv\frac{\partial}{\partial\theta}x.\tag{6}$$

$$b_\theta\equiv b(t)\sin\{\theta+\gamma_1(t)\}=b_1(t)\sin\theta+b_2(t)\cos\theta,$$
$$h_{2\theta}\equiv h(t)\sin\{2\theta+\gamma_2(t)\}=h_1(t)\sin2\theta+h_2(t)\cos2\theta.$$

$$\left.\begin{aligned}&2\Delta_1=r\{\dot H(t)-2\dot h_{2\theta}\},\\&2\Delta_2=r\{D(t)-\dot h_{2\theta}'+2r\dot b_\theta'\},\end{aligned}\right\}\quad\dot x\equiv\frac{\partial}{\partial t}x.\tag{7}$$

$$\left.\begin{aligned}&\Pi_0=\frac{1}{Z}(k\Delta_1-l\Delta_2),\\&\Theta_0=\frac{1}{Z}(h\Delta_2-l\Delta_1),\end{aligned}\right\}\quad Z\equiv hk-l^2.\tag{8}$$

in which $\lim_{r\to0}\Pi_0=\lim_{r\to0}\Theta_0=0$ has been conditioned from the physical view point.

Ellimination of χ from (5) results with references to (3) and (4)

$$\frac{\partial\chi}{\partial t}+\Pi_0\frac{\partial\chi}{\partial r}+\Theta_0\frac{\partial\chi}{r\partial\theta}=0.\tag{9}$$

This represents the continuity of χ along the motion of (Π_0,Θ_0) signifying that all the particles labelled with χ move with the same velocity as the local centroid. Therefore, χ is regarded as something like a weight to be applied to $F(P-\chi)$.

As regards the gravitational force, we have from (5)

$$\left.\begin{aligned}\frac{\partial\mathcal{S}}{\partial r}&=\frac{\partial\mathcal{S}_1}{\partial r}+\frac{\partial\mathcal{S}_2}{\partial r},\\ \frac{\partial\mathcal{S}}{r\partial\theta}&=\frac{\partial\mathcal{S}_1}{r\partial\theta}+\frac{\partial\mathcal{S}_2}{r\partial\theta},\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial\mathcal{S}_1}{\partial r}&=\frac{k}{Z}\left(\dot{\Delta}_1-\frac{\partial\chi}{2\partial r}\right)-\frac{\ell}{Z}\left(\dot{\Delta}_2-\frac{\partial\chi}{2r\partial\theta}\right),\\ \frac{\partial\mathcal{S}_1}{r\partial\theta}&=\frac{h}{Z}\left(\dot{\Delta}_2-\frac{\partial\chi}{2r\partial\theta}\right)-\frac{\ell}{Z}\left(\dot{\Delta}_1-\frac{\partial\chi}{2\partial r}\right),\end{aligned}\right\}\quad\left.\begin{aligned}\frac{\partial\mathcal{S}_2}{\partial r}&=\frac{k}{Z}\frac{\partial\chi}{2\partial r}-\frac{\ell}{Z}\frac{\partial\chi}{2r\partial\theta},\\ \frac{\partial\mathcal{S}_2}{r\partial\theta}&=\frac{h}{Z}\frac{\partial\chi}{2r\partial\theta}-\frac{\ell}{Z}\frac{\partial\chi}{2\partial r},\end{aligned}\right\}\tag{10}$$

where $\mathcal{S}_1$ denotes the potential function taken no account of χ while $\mathcal{S}_2$ the one contributed from χ alone.

The local density $\rho(r,\theta,t)$ is, therefore, readily derived from Poisson's relation, namely

$$\rho=\rho_1+\rho_2$$

$$-4\pi G\rho_i=\frac{\partial^2\mathcal{S}_i}{\partial r^2}+\frac{1}{r}\frac{\partial\mathcal{S}_i}{\partial r}+\frac{1}{r^2}\frac{\partial^2\mathcal{S}_i}{\partial\theta^2},\quad(i=1,2)\tag{11}$$

where G is the gravitational constant.

In a self-gravitating system, however, some frequency distribution function with an argument P in (1), must be found really from the expression of ρ given in (11). But this is possible in general as mentioned below. A frequency distribution function is to be defined not only by its concrete functional form but also by its velocity range. As for the trancation, the one that leads the total mass of a stellar system has been discussed by Kurth (1957) but usually either $0\leq V=\sqrt{\Pi^2+\Theta^2}<\sqrt{2\mathcal{S}}$ or $0\leq V<\infty$ has hitherto been adopted. Either of these customary trancations, however, could not be accepted a priori, especially when a non-steady stellar system is under consideration. This circumstance may be illustrative in the numerical works of the many body problem. In this approach the initial velocity distribution is assigned rather arbitrarily, say, letting it be of a rectangular type with a velocity range of $0<A\leq V\leq B<\sqrt{2\mathcal{S}}$, and then an evolution of the system is looked for. So, if we change initially the type of the velocity distribution and/or the velocity range, a different way of the evolution is expected.

According to the authers' view, the density is to be expressed in general by

$$\int_{\Pi_1}^{\Pi_2}\int_{\Theta_1}^{\Theta_2} F(P)\,d\Pi\,d\Theta = \rho(r,\theta,t) \tag{12}$$

where Π_1, Π_2; Θ_1, Θ_2 are functions of r, θ and t. These four functions may be corresponded to the initial or the boundary conditions for the equations of motion which are equivalent to the subsidary equations of (2). In our case, ρ is given automatically with aids of Poissons's equation from the potential function which is also derivable from (2) by letting Π_0 and Θ_0 be inclusive in (1). In principle, there are varieties in selecting a possible set of F(P); Π_1, Π_2; Θ_1, Θ_2, but it is desirable that F(P) resembles the observed velocity frequency distribution and further, even though for sake of convenience, the means, the variances and covariance with respects to Π and Θ become nearer to Π_0, Θ_0; $1/2h$, $1/2k$ and $1/2\ell$ as possible. If some such suitable sets are found, it is considered that there exist respective dynamical states for the same density distribution. A detailed treatment on this matter, however, is not our present purpose, so it is reserved elsewhere.

By the way, we add another mention concerning our argument P in the frequency distribution function. Making use of the formulae (2)-(8) we can write P as follows

$$P = -H\left(2I - \frac{D}{H}J + \frac{K}{H}J^2\right) + (X - X - 2H\delta b)$$
$$+\left\{2h_{20}\Pi^2 - 2(h_{2\theta} - rb_\theta)\frac{J^2}{r^2} - 2(h'_{2\theta} + rb'_\theta)\frac{\Pi}{r}J + (\dot{H} - 2\dot{h}_{2\theta})\Pi - (\dot{h}'_{2\theta} - 2r\dot{b}'_\theta)J\right\}, \tag{13}$$

where $I \equiv \frac{1}{2}(\Pi^2 + \Theta^2) - \delta b$ is the gravitational energy and $J \equiv r\Theta$ is the angular momentum per unit mass. In a special case where the two-dimensional system is in a steady state and both of $h_{2\theta}$ and rb_θ are negligibly small, P in (13) is reduced to $-(aI+bJ+cJ^2)$, in which a, b, c are constants, as Jeans' theorem states. While, as for our general case of (13) too, P remains to be the first integral of a single particle's motion, since the formulae (6)-(10) have been the conditions which make P be the first integral.

2. Zeroth Order Approximation

In this section we consider a case where b_θ and $h_{2\theta}$ vanish, but $\chi^{(0)}$ or the "χ" in the zeroth order is not necessary constant. Then the formulae in (6)-(10) turn out into simple forms as follows.

$$h_1 = H(t), \quad h = H(t) + Kr^2, \quad \ell = 0, \tag{6'}$$

$$\Delta_1 = \frac{\dot{H}}{2} r, \quad \Delta_2 = \frac{D(t)}{2} r, \tag{7'}$$

$$\Pi_0 = \frac{\dot{H}}{2H} r, \quad \Theta_0 = \frac{Dr}{2(H+Kr^2)}\ ; \ Z \equiv H(H+Kr^2), \quad X \equiv \frac{\dot{H}^2}{4H} r^2 + \frac{D^2 r^2}{4(H+Kr^2)} \tag{8'}$$

$$\frac{\partial \chi^{(0)}}{\partial t} + \frac{\dot{H}}{2H} r \frac{\partial \chi^{(0)}}{\partial r} + \frac{D}{2(H+Kr^2)} \frac{\partial \chi^{(0)}}{\partial \theta} = 0. \tag{9'}$$

$$\left.\begin{aligned} &\frac{\partial \delta_1}{\partial r} = \left(\frac{\ddot{H}}{2} - \frac{\dot{H}^2}{4H}\right)\frac{r}{H} - \frac{D^2 r}{4(H+Kr^2)^2}, \qquad \frac{\partial \delta_1}{\partial \theta} = \frac{\dot{D} r^2}{2(H+Kr^2)}, \\ &\frac{\partial \delta_2}{\partial r} = \frac{1}{2H} \frac{\partial \chi^{(0)}}{\partial r}, \qquad \frac{\partial \delta_2}{\partial \theta} = \frac{1}{2(H+Kr^2)} \frac{\partial \chi^{(0)}}{\partial \theta}. \end{aligned}\right\} \tag{10'}$$

It follows from (10') that

$$\left.\begin{aligned} &\frac{\partial^2 \delta_1}{\partial r \partial \theta} - \frac{\partial^2 \delta_1}{\partial \theta \partial r} = \frac{\dot{D} H r}{(H+Kr^2)^2} \equiv \frac{\dot{D} r}{H}(1-Y)^2, \qquad Y \equiv \frac{Kr^2}{H+Kr^2}, \ r^2 \equiv \frac{HY}{K(1-Y)}, \\ &\frac{\partial^2 \delta_2}{\partial \theta \partial r} - \frac{\partial^2 \delta_2}{\partial r \partial \theta} = \frac{Kr}{(H+Kr^2)^2} \frac{\partial \chi^{(0)}}{\partial \theta} + \frac{Kr^2}{2H(H+Kr^2)} \frac{\partial^2 \chi^{(0)}}{\partial \theta \partial r} = \frac{K}{H^2}(1-Y)^2 r \frac{\partial \chi^{(0)}}{\partial \theta} + \frac{Y}{2H} \frac{\partial^2 \chi^{(0)}}{\partial \theta \partial r}. \end{aligned}\right\} \tag{14}$$

Since an identity $\frac{\partial^2 \delta}{\partial r \partial \theta} = \frac{\partial^2 \delta}{\partial \theta \partial r}$ or $\frac{\partial^2 \delta_1}{\partial r \partial \theta} - \frac{\partial^2 \delta_1}{\partial \theta \partial r} = \frac{\partial^2 \delta_2}{\partial \theta \partial r} - \frac{\partial^2 \delta_2}{\partial r \partial \theta}$ must hold at any place and at any time, we have with reference to (14)

$$\frac{K}{H^2}(1-Y)^2 r \frac{\partial \chi^{(0)}}{\partial \theta} + \frac{Y}{2H} \frac{\partial^2 \chi^{(0)}}{\partial \theta \partial r} = \frac{\dot{D} r}{H}(1-Y)^2. \tag{15}$$

But this χ should also satisfy (9') simultaneously. Such a common solution of $\chi^{(0)}$ is given, as shown in §A1 of the Appendix, by

$$\chi^{(0)} = a(t)\theta + \frac{1+Ku}{Ku}\left\{F_0(u) + c\left(\theta - \frac{N(t)}{1+Ku}\right) - a(t)\theta + g_2\theta^2 + \frac{D^2(t)}{4K(1+Ku)^2}\right\}, \quad a(t) \equiv \frac{\dot{D}H}{K}, \ u \equiv \frac{r^2}{H}, \ N(t) \equiv \int_0^t \frac{D}{2H} dt. \tag{16}$$

Hence, we have on reflection of (10')

$$\left.\begin{aligned} &\frac{\partial \delta_2}{\partial r} = -\frac{r}{H^2 Ku^2}\left\{g_2\theta^2 - (a-c)\theta - cN + \frac{D^2(1+2Ku)}{4K(1+Ku)^2} + F_0(u) - u(1+Ku)\frac{\partial F_0}{\partial u}\right\}, \\ &\frac{\partial \delta_2}{\partial \theta} = \frac{1}{2HKu}\left(2g_2\theta - \frac{a}{1+Ku} + c\right). \end{aligned}\right\} \tag{17}$$

The density $\rho(r, \theta, t)$ is therefore obtained with the aids of (11), (10') and (17),

$$4\pi G \rho_1 = -\frac{d}{dt}\left(\frac{\dot{H}}{H}\right) - \frac{\dot{H}^2}{2H^2} + \frac{D^2(1-Ku)}{2H^2(1+Ku)^3}, \tag{18}$$

$$4\pi G \rho_2 = \frac{2}{H^2 Ku^2}\left\{-F_0(u) + u(1-Ku)\frac{\partial F_0}{\partial u} - u^2(1+Ku)\frac{\partial^2 F_0}{\partial u^2} - \frac{g_2}{2} + cN + \theta(a-c) - g_2\theta^2 - \frac{D^2(1+3Ku+4K^2u^2)}{4K(1+Ku)^3}\right\}$$

$$4\pi G\rho = -\frac{d}{dt}\left(\frac{\dot H}{H}\right) - \frac{\dot H^2}{2H^2} + \frac{2}{H^2 K u^2}\Big[-F_0(u) + u(1-Ku)\frac{\partial F_0}{\partial u} - u^2(1+Ku)\frac{\partial^2 F_0}{\partial u^2} \\ -\left(\frac{D^2}{4K} + \frac{g_2}{2}\right) + cN + \theta(a-c) - g_2\theta^2\Big] \tag{19}$$

This density distribution, however, even if a singularity at $u=0$ or $r=o$ is avoided by limitting as $r \geq \varepsilon > 0$
cannot be smooth along any circle around the origin because of the terms secular about θ. This may be forcibly saved by imposing some particular initial condition or by interpreting as a steep jump in the density distribution. But, let us concern here with a continuous distribution ρ by imposing the condition of $\rho(r, \theta, t) = \rho(r, \theta + 2\pi, t)$. Then we should have in (19)

$$a(t) = c = a_0 + g_{10} \quad \therefore\ a = a_0 = \frac{\dot D H}{K} : \text{const.}, \quad g_{10} = 0, \quad g_2 = 0, \tag{20}$$

whcih reduce the formulae (16)-(19) to

$$\chi^{(0)} = \frac{1+Ku}{Ku}F_0(u) + a_0\left(\theta - \frac{N}{1+Ku}\right) + \left(\frac{D^2}{4K} - a_0 N\right)\frac{1}{Ku(1+Ku)}, \quad \frac{D^2}{4K} - a_0 N \equiv \frac{D_0^2}{4K} : \text{const.}, \tag{16'}$$

$$\left.\begin{aligned} \frac{\partial \delta_3}{\partial r} &= \frac{r}{H^2 K u^2}\left\{-F_0(u) + u(1+Ku)F_0'(u) - \frac{D_0^2}{4K}\right\} + \frac{D^2 r}{4H^2(1+Ku)^2}, \\ \frac{\partial \delta_3}{\partial \theta} &= \frac{a_0}{2H(H+HKu)}, \end{aligned}\right\} \quad \frac{\partial F_0}{\partial u} \equiv F_0'(u) \tag{17'}$$

$$\left.\begin{aligned} 4\pi G\rho_1 &= -\frac{d}{dt}\left(\frac{\dot H}{H}\right) - \frac{\dot H^2}{2H^2} + \frac{D^2(1-Ku)}{2H^2(1+Ku)^3}, \\ 4\pi G\rho_2 &= \frac{2}{H^2 K u^2}\left\{-F_0(u) + u(1-Ku)F_0'(u) - u^2(1+Ku)F_0''(u) - \frac{D_0^2}{4K}\right\} + \frac{D_0^2(1-Ku)}{2H^2(1+Ku)^3} \end{aligned}\right\} \tag{18'}$$

$$\begin{aligned} 4\pi G\rho &= -\frac{d}{dt}\left(\frac{\dot H}{H}\right) - \frac{\dot H^2}{2H^2} + \frac{2}{H^2 K u^2}\left\{-F_0(u) + u(1-Ku)F_0'(u) - u^2(1+Ku)F_0''(u) - \frac{D_0^2}{4K}\right\} \\ &= \frac{2}{H^2 K}\frac{d}{du}\left[\frac{1}{u}\left\{F_0(u) + \frac{D_0^2}{4K}\right\} - (1+Ku)F_0'(u)\right] - \left\{\frac{d}{dt}\left(\frac{\dot H}{H}\right) + \frac{\dot H^2}{2H^2}\right\}. \end{aligned} \tag{19'}$$

As seen above this continuous stellar system is axisymmetric even though a tangential force is still acting unless $a_0=0$ or D=const. But there can be a variety in its density distribution because a functional form of $F_0(u)$ is chosen freely so far as

$$\frac{d}{du}\left[\frac{1}{u}\left\{F_0(u) + \frac{D_0^2}{4K}\right\} - (1+Ku)F_0'(u)\right] \geq \frac{H^2 K}{2}\left\{\frac{d}{dt}\left(\frac{\dot H}{H}\right) + \frac{\dot H^2}{2H^2}\right\}$$

is satisfied over a range of $0<r\leq R$*. Each of these systems is characterized with a finite dimension and with the kinematical

*A foot-note for this is given in the next page.

constants as follow

$$\left.\begin{aligned} A &\equiv \frac{1}{2}\left(\frac{\Theta_0}{r}-\frac{\partial \Theta_0}{\partial r}\right)=\frac{\dot{N}Ku}{(1+Ku)^2}, & B &\equiv -\frac{1}{2}\left(\frac{\Theta_0}{r}+\frac{\partial \Theta_0}{\partial r}\right)=-\frac{\dot{N}}{(1+Ku)^2} \\ \varkappa &\equiv \frac{1}{2}\left(\frac{\Pi_0}{r}+\frac{\partial \Pi_0}{\partial r}\right)=\frac{\dot{H}}{2H}, & \omega &\equiv A-B=\frac{\dot{N}}{1+Ku}, \quad u \equiv \frac{r^2}{H} \end{aligned}\right\} \tag{21}$$

which, except for $\varkappa$, are formally the same as in a steady state model usually given in text-books.

By the way, it may be interesting to check whether Lin's potential (1964, 1969) is consistent or not with our frequency function in the zeroth order. Let $\Delta\mathcal{B}_2$ be Lin's spiral-producing potential $S(r)\exp[i(\omega_n t-n\theta+f(r))]$ and ΔX be a corresponding increment of X , then we have from (10') and (9')

$$\frac{\partial \Delta\mathcal{B}}{\partial r}=\frac{1}{2H}\frac{\partial \Delta X}{\partial r}=\left(\frac{1}{S}\frac{dS}{dr}+i\frac{df}{dr}\right)\Delta\mathcal{B}_2, \quad \frac{\partial \Delta\mathcal{B}_2}{\partial \theta}=\frac{1}{2(H+Kr^2)}\frac{\partial \Delta X}{\partial \theta}=-in\Delta\mathcal{B}_2,$$

$$\left.\begin{aligned} &\therefore \frac{\partial \Delta X}{\partial r}=2H\left(\frac{1}{S}\frac{dS}{dr}+i\frac{df}{dr}\right)\Delta\mathcal{B}_2, \quad \frac{\partial \Delta X}{\partial \theta}=-2in(H+Kr^2)\Delta\mathcal{B}_2, \\ &\frac{\partial \Delta X}{\partial t}=-\frac{\dot{H}r}{2H}\frac{\partial \Delta X}{\partial r}-\frac{D}{2(H+Kr^2)}\overset{\left(\times\frac{\partial \Delta X}{\partial \theta}\right)}{=}\left\{-\dot{H}r\left(\frac{1}{S}\frac{dS}{dr}+i\frac{df}{dr}\right)+inD\right\}\Delta\mathcal{B}_2 \end{aligned}\right\} \tag{26}$$

Among these partial derivatives, however, there exist three identities such as

$$\left.\begin{aligned} &\frac{\partial^2 \Delta X}{\partial \theta \partial r}=\frac{\partial^2 \Delta X}{\partial r \partial \theta} \quad \text{or} \quad \left(\frac{1}{S}\frac{dS}{dr}+i\frac{df}{dr}\right)=-\frac{2}{r}, \\ &\frac{\partial^2 \Delta X}{\partial t \partial \theta}=\frac{\partial^2 \Delta X}{\partial \theta \partial t} \quad \text{or} \quad \frac{\omega_n}{n}=\frac{D}{2(H+Kr^2)}, \\ &\frac{\partial^2 \Delta X}{\partial t \partial r}=\frac{\partial^2 \Delta X}{\partial r \partial t} \quad \text{or} \quad \frac{\omega_n}{n}=\frac{D}{2H}. \end{aligned}\right. \tag{27}$$

Aside from the first identy showing a singularity at the origin, the last pair of identities contradict each other unless the velocity distribution is circular. Lin's potential, therefore, could not operate as a disturbance in the self gravitating two-dimensional system with the velocity distribution of Schwarzschild's type.

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* $F(u)+\frac{D_0^2}{4K}=C_1\frac{u}{1+Ku}$ (C_1: const.) gives $\frac{1}{u}\{F(u)+\frac{D_0^2}{4K}\}-(1+Ku)F'(u)=0$. Hence, for example, if we take $F(u)+\frac{D_0^2}{4K}$ as $\left[\frac{C_1}{1+Ku}+\frac{C_2}{(1+Ku)^2}\right]$, we can adjust both parameters C_1 and C_2 so as to satisfy the necessary condition for the finite dimension mentioned above.

3. The First Order Approximation Without χ.

If χ is a constant, it does not appear explicitely in all the formulae except for P in (1). So, our first order approximation is set up with this simple case where χ is a constant or is neglected. Taking the linear terms with respects to b_θ , $h_{2\theta}$ and their derivatives into account, we obtain the following formulae corresponding to (6)-(10).

$$\left.\begin{array}{l} h=H\left(1-\frac{2h_{2\theta}}{H}\right),\quad k=\frac{H}{1-Y}\left\{1+2(1-Y)\frac{-rb_\theta+h_{2\theta}}{H}\right\},\quad \ell=rb'_\theta-h'_{2\theta} \\ \quad H\equiv H(t),\quad K:\text{const.},\quad Y\equiv\frac{Kr^2}{H+Kr^2}\equiv\frac{Ku}{1+Ku} \end{array}\right\} \qquad (6'')$$

$$\Delta_1=\frac{\dot H r}{2}\left(1-\frac{2\dot h_{2\theta}}{\dot H}\right),\quad \Delta_2=\frac{Dr}{2}\left(1+\frac{2r\dot b'_\theta-\dot h'_{2\theta}}{D}\right). \qquad (7'')$$

$$\left.\begin{array}{l} \Pi_0=\frac{Y}{2H}\left\{\dot H-D(1-Y)\frac{rb'_\theta}{H}-2\dot h_{2\theta}+2\dot H\frac{h_{2\theta}}{H}+D(1-Y)\frac{h'_{2\theta}}{H}\right\}, \\ \Theta_0=\frac{Y}{2H}(1-Y)\left\{D+2r\dot b'_\theta-\dot H\frac{rb'_\theta}{H}+2D(1-Y)\frac{rb_\theta}{H}-\dot h'_{2\theta}+\frac{\dot H}{H}h'_{2\theta}-2D(1-Y)h_{2\theta}\right\}. \\ \frac{1}{Z}=\frac{1}{H^2}(1-Y)\left\{1+2(1-Y)\frac{rb_\theta}{H}+2Y\frac{h_{2\theta}}{H}\right\}, \\ X\equiv\frac{Y}{4K(1-Y)}\Big[\dot H^2+D^2(1-Y)+4D(1-Y)r\left\{\dot b'_\theta-\frac{\dot H}{2H}b'_\theta-\frac{D}{2H}(1-Y)b_\theta\right\} \\ \qquad -4\dot H\dot h_{2\theta}+2\left\{\dot H^2-D^2(1-Y)^2\right\}\frac{h_{2\theta}}{H}-2D(1-Y)\dot h'_{2\theta}+2D\dot H(1-Y)\frac{h'_{2\theta}}{H}\Big]. \end{array}\right\} \qquad (8'')$$

$$\left.\begin{array}{l} \frac{\partial\delta b_1}{\partial Y}=\frac{Y}{4H^2}\Big[\left\{2H\ddot H-\dot H^2-D^2(1-Y)^2\right\}-2D(1-Y)(3-2Y)r\dot b'_\theta-D^2(1-Y)^2(3-4Y)\frac{rb_\theta}{H} \\ \qquad +\left\{D\dot H(3-2Y)-2H\dot D\right\}(1-Y)\frac{rb'_\theta}{H}-4H\ddot h_{2\theta}+4\dot H\dot h_{2\theta}+2D(1-Y)^2\dot h'_{2\theta} \\ \qquad +4\left\{H\ddot H-\dot H^2-D^2Y(1-Y)^2\right\}\frac{h_{2\theta}}{H}+2\left\{H\dot D-D\dot H(1-Y)\right\}(1-Y)\frac{h'_{2\theta}}{H}\Big], \\ \frac{\partial\delta b_1}{r\partial\theta}=\frac{Y}{4H^2}\Big[2H\dot D(1-Y)+4H(1-Y)r\ddot b'_\theta+2D(1-Y)^2r\dot b_\theta+(4H\dot D-D\dot H)(1-Y)^2\frac{rb_\theta}{H} \\ \qquad +(\dot H^2-2H\ddot H)(1-Y)\frac{rb'_\theta}{H}-2H(1-Y)\ddot h'_{2\theta}-4D(1-Y)^2\dot h_{2\theta}+2\dot H(1-Y)\dot h'_{2\theta} \\ \qquad +4(D\dot H-H\dot D)(1-Y)^2\frac{h_{2\theta}}{H}+2(H\ddot H-\dot H^2)(1-Y)\frac{h'_{2\theta}}{H}\Big]. \end{array}\right\} \qquad (10'')$$

Cross-differentiations of (10") give the identity or

$$\begin{aligned} \frac{\partial^2\delta b_1}{\partial Y\, r\partial\theta}-\frac{\partial^2\delta b_1}{\partial\theta\partial Y}=&\ \frac{3Y}{K}\left[\left\{\ddot b'_\theta-\left(\frac{\ddot H}{2H}-\frac{\dot H^2+D^2}{4H^2}\right)b'_\theta\right\}+\frac{5\dot D}{6H}b_\theta\right] \\ &-\frac{2Y^2}{K}\left[\left\{\ddot b'_\theta-\left(\frac{\ddot H}{2H}-\frac{\dot H^2+D^2}{4H^2}\right)b'_\theta\right\}+\frac{5D}{4H}\left(\dot b_\theta-\frac{\dot H}{2H}b_\theta+\frac{D}{2H}b'_\theta\right)+\frac{7\dot D}{2H}b_\theta\right] \\ &+\frac{2Y^3}{K}\left(\frac{D}{H}\right)\left[\left(\dot b_\theta-\frac{\dot H}{2H}b_\theta+\frac{D}{2H}b'_\theta\right)+\frac{2\dot D}{D}b_\theta\right] \\ &+\frac{Y}{H}\dot D \\ &+\frac{2Yr}{H}\left[-\dot D+\left\{\ddot h'_{2\theta}-\frac{\dot H}{H}\dot h'_{2\theta}-\left(\frac{\ddot H}{H}-\frac{\dot H^2}{H^2}\right)h'_{2\theta}\right\}+\frac{2D}{H}\left(\dot h_{2\theta}-\frac{\dot H}{H}h_{2\theta}+\frac{D}{4H}h'_{2\theta}\right)+\frac{3\dot D}{H}h_{2\theta}\right] \\ &-\frac{Y^2r}{H}\left[-\dot D+\left\{\ddot h'_{2\theta}-\frac{\dot H}{H}\dot h'_{2\theta}-\left(\frac{\ddot H}{H}-\frac{\dot H^2}{H^2}\right)h'_{2\theta}\right\}+\frac{8D}{H}\left(\dot h_{2\theta}-\frac{\dot H}{H}h_{2\theta}+\frac{D}{4H}h'_{2\theta}\right)+\frac{10\dot D}{H}h_{2\theta}\right] \\ &+\frac{4Y^3r}{H}\left(\frac{D}{H}\right)\left[\left(\dot h_{2\theta}-\frac{\dot H}{H}h_{2\theta}+\frac{D}{4H}h'_{2\theta}\right)+\frac{\dot D}{D}h_{2\theta}\right]. \end{aligned} \qquad (28)$$

As regard the first or the higher order approximation, there can be alternative standpoints for making the potential identity satisfy, namely, the one imposes the condition for all values of r, θ and t, while the other requiers some functional relationship among these three variables satisfying the identity. Throughout this paper we adopt the former standpoint by reserving the approach from the latter alternative in the following paper.

Then, it should be in (28) that $\dot{D}$ and a sum of the terms in every pair of the brackets are zero, and yet for each sum of the terms the coefficients of $\sin\theta$, $\cos\theta$, $\sin 2\theta$ and $\cos 2\theta$ vanish. With these requirements we have, by writing $b_\theta = b_1(t)\sin\theta + b_2(t)\cos\theta$ and $h_{2\theta} = h_1(t)\sin 2\theta + h_2(t)\cos 2\theta$, as in (6)

$$D(t) = D_0:\ \text{const.,} \tag{29}$$

$$(\mathrm{I})\begin{cases} \dot{b}_1 - \frac{\dot{H}}{2H}b_1 - \frac{D_0}{2H}b_2 = 0, \\ \dot{b}_2 - \frac{\dot{H}}{2H}b_2 + \frac{D_0}{2H}b_1 = 0, \end{cases} \quad (\mathrm{II})\begin{cases} \ddot{b}_2 - \left(\frac{\ddot{H}}{2H} - \frac{\dot{H}^2}{4H^2}\right)b_2 - \frac{D_0}{2H}\dot{b}_1 + \frac{D_0\dot{H}}{4H^2}b_1 = 0, \\ \ddot{b}_1 - \left(\frac{\ddot{H}}{2H} - \frac{\dot{H}^2}{4H^2}\right)b_1 + \frac{D_0}{2H}\dot{b}_2 - \frac{D_0\dot{H}}{4H^2}b_2 = 0. \end{cases}$$

$$(\mathrm{I})\begin{cases} \dot{h}_1 - \frac{\dot{H}}{H}h_1 - \frac{D_0}{2H}h_2 = 0, \\ \dot{h}_2 - \frac{\dot{H}}{H}h_2 + \frac{2D_0}{H}h_1 = 0, \end{cases} \quad (\mathrm{II})\begin{cases} \ddot{h}_1 - \frac{\dot{H}}{H}\dot{h}_1 - \left(\frac{\ddot{H}}{H} - \frac{\dot{H}^2}{H^2}\right)h_1 = 0, \\ \ddot{h}_2 - \frac{\dot{H}}{H}\dot{h}_2 - \left(\frac{\ddot{H}}{H} - \frac{\dot{H}^2}{H^2}\right)h_2 = 0. \end{cases} \tag{30}$$

As for b_θ, (I) and (II) in (29) are not independent, so that it is sufficient to consider only (I). Because of $b_1 = b(t)\cos\gamma_1(t)$, $b_2 = b(t)\sin\gamma_1$, both equations in (I) are written as

$$\left(\dot{b} - \frac{\dot{H}}{2H}b\right)\sin\gamma_1 + \left(\dot{\gamma}_1 + \frac{D}{2H}\right)b\cos\gamma_1 = 0,$$

$$\left(\dot{b} - \frac{\dot{H}}{2H}b\right)\cos\gamma_1 - \left(\dot{\gamma}_1 + \frac{D}{2H}\right)b\sin\gamma_1 = 0.$$

Since $\gamma_1(t)$ should be arbitrary,* we have

$$b(t) = b_0\sqrt{\frac{H(t)}{H_0}}, \quad \dot{\gamma}_1 = -\frac{D_0}{2H(t)}, \quad \gamma_1(t) = -\int\frac{D_0}{2H(t)}dt \equiv -N(t), \tag{31}$$

$$\therefore\ b_\theta = b_0\sqrt{\frac{H(t)}{H_0}}\sin\{\theta - N(t)\}.$$

Therefore, b_θ represents a sinewave having the wave-length of $2\pi r$ for any r and advancing with an angular velocity of $\dot{N}(t) = D_0/2H(t)$ in the positive θ-direction because θ must be increasing with time for keeping b_θ in the same phase.

* Without this restriction, the solution of (I) is to be $b_i = \sqrt{H}/\sqrt{\dot{N}}\,w_i$ $(i=1,2)$ where w_i is a solution of $\ddot{w} + \left(\dot{N}^2 - \frac{3\ddot{N}}{4\dot{N}} + \frac{\dddot{N}}{2\dot{N}}\right)w = 0$. Especially when $\dot{N}$: const., b_θ takes the same expression as (31).

While as for $h_{2\theta}$ the formulae corresponded to (31), or

$$h_{2\theta}=\frac{h_0}{H_0}H(t)\sin 2\left\{\theta-\frac{N(t)}{2}\right\},\qquad \dot{\gamma}_2=\frac{D_0}{2H(t)}=-\dot{N}, \tag{32}$$

are obtained from (I) in (30). Nevertheless, putting (32) into (II), we have

$$\frac{D_0^2}{4H^2}h(t)=0,$$

that is, $D_o=0$ or $h(t)=0$.

This unsatisfactory result for $h_{2\theta}$ is due to having taken no account of the second order terms about b_θ which contain the same bi-sectorial harmonics as the first order terms about $h_{2\theta}$. According to our present standpoint, even when the calculation is extended up to the second or higher order approximation, the terms with $\sin\theta$ or $\cos\theta$ should be independent of those with $\sin 2\theta$ or $\cos 2\theta$, so that no more information about b_θ cannot be obtained than given in (31).

Taking these situations in mind, we carried out the recalculation in which the products and the squars of b_θ , b_θ' and/or their time-derivatives are included in addition to the linear terms about $h_{2\theta}$ by making use of the known result for b_θ shown in (29) and (31).

Now the fundamental formulae, except for (6") and (7") which remain unchanged, should be replaced by the followings.

$$\left.\begin{aligned}
\Pi_0&=\frac{Y}{2H}\Big[\dot{H}-\frac{D_0}{H}(1-Y)rb_\theta-2\Big\{\dot{h}_{2\theta}-\frac{\dot{H}}{H}h_{2\theta}-\frac{D_0}{2H}(1-Y)h_{2\theta}'\Big\}-\frac{D_0}{KH}Y(3-2Y)b_\theta b_\theta'\Big],\\
\Theta_0&=\frac{Y(1-Y)}{2H}\Big[D_0+\frac{D_0}{H}(3-2Y)rb_\theta-\Big\{\dot{h}_{2\theta}'-\frac{\dot{H}}{H}h_{2\theta}'+\frac{2D_0}{H}(1-Y)h_{2\theta}\Big\}+\frac{2D_0}{KH}Y(3-2Y)b_\theta^2+\frac{D_0}{KH}Yb_\theta'^2\Big],\\
\frac{1}{Z}&=\frac{(1-Y)}{H^2}\Big\{1+2(1-Y)\frac{rb_\theta}{H}+2Y\frac{h_{2\theta}}{H}+4Y(1-Y)\frac{b_\theta^2}{KH}+Y\frac{b_\theta'^2}{KH}\Big\},\\
X&=\frac{Y}{4K(1-Y)}\Big[\dot{H}^2+D_0^2(1-Y)+\frac{2D_0^2}{H}(1-Y)(2-Y)rb_\theta-2D_0(1-Y)\Big\{\dot{h}_{2\theta}'-\frac{\dot{H}}{H}h_{2\theta}'+\frac{D_0}{H}(1-Y)h_{2\theta}\Big\}\\
&\qquad-4\dot{H}\Big(\dot{h}_{2\theta}-\frac{\dot{H}}{2H}h_{2\theta}\Big)+\frac{D_0^2}{KH}Y(3-2Y)^2b_\theta^2+\frac{D_0^2}{KH}Y(1-Y)b_\theta'^2\Big].
\end{aligned}\right\} \tag{8"}$$

$$\left.\begin{aligned}
\frac{\partial\delta b_1}{\partial Y}=-\frac{Y}{4H^2}\Big[&\dot{H}^2-2\ddot{H}H+D_0^2(1-Y)^2+\frac{D_0^2}{H}(1-Y)(6-9Y+4Y^2)rb_\theta\\
&+4H\Big\{\ddot{h}_{2\theta}-\frac{\dot{H}}{H}\dot{h}_{2\theta}-\Big(\frac{\ddot{H}}{H}-\frac{\dot{H}^2}{H^2}\Big)h_{2\theta}\Big\}-2D_0(1-Y)^2\Big(\dot{h}_{2\theta}'-\frac{\dot{H}}{H}h_{2\theta}'-\frac{2D_0}{H}Yh_{2\theta}\Big)\\
&+\frac{D_0^2}{KH}Y(3-2Y)(6-11Y+6Y^2)b_\theta^2-\frac{D_0^2}{KH}Y^2(3-2Y)b_\theta'^2\Big],
\end{aligned}\right\} \tag{10"}$$

$$\left.\begin{aligned}\frac{\partial \mathfrak{D}_1}{\partial r\partial\theta} = -\frac{r}{4H^2}\Big[&\frac{D_0^2}{H}(1-Y)(2-Y)rb_\theta' + 2H(1-Y)\{\ddot{h}_{2\theta}' - \frac{\dot{H}}{H}\dot{h}_{2\theta} - (\frac{\ddot{H}}{H} - \frac{\dot{H}^2}{H^2})h_{2\theta}\} \\ &+ 4D_0\dot{H}(1-Y)^2(\dot{h}_{2\theta} - \frac{\dot{H}}{H}h_{2\theta}) + \frac{2D_0^2}{KH}Y(1-Y)(3-2Y)b_\theta b_\theta'\Big],\end{aligned}\right)$$

where the time-derivatives of b_θ or b_θ' have been written in terms of b_θ and b_θ' by means of (29) (I) or

$$\dot{b}_\theta = \frac{\dot{H}}{2H}b_\theta - \frac{D_0}{2H}b_\theta', \qquad \dot{b}_\theta' = \frac{\dot{H}}{2H}b_\theta' + \frac{D_0}{2H}b_\theta. \tag{29$^{\prime\prime}$}$$

The identity condition in this case becomes

$$\begin{aligned}\frac{\partial^2 \mathfrak{D}_1}{\partial r\partial\theta} - \frac{\partial^2 \mathfrak{D}_1}{\partial\theta\partial r} = \frac{Y^2}{HK(1-Y)}\Big[&\dot{H}^2(2-Y)\{\ddot{h}_{2\theta}' - \frac{\dot{H}}{H}\dot{h}_{2\theta}' - (\frac{\ddot{H}}{H} - \frac{\dot{H}^2}{H^2})h_{2\theta}'\} \\ &+ 4D\dot{H}(1-Y)^2(\dot{h}_{2\theta} - \frac{\dot{H}}{H}h_{2\theta} + \frac{D_0}{4H}h_{2\theta}') + \frac{D_0^2}{KH}(3-2Y+Y^2)b_\theta b_\theta'\Big].\end{aligned} \tag{33}$$

Since the coefficients of Y^2, Y, Y^0 should be null, it results that

$$\begin{aligned}&(\dot{h}_{2\theta} - \frac{\dot{H}}{H}h_{2\theta} + \frac{D_0}{4H}h_{2\theta}') + \frac{D_0}{4KH}b_\theta b_\theta' = 0, \\ &(\dot{h}_{2\theta} - \frac{\dot{H}}{H}h_{2\theta} + \frac{D_0}{4H}h_{2\theta}') + \frac{H}{8D}\{\ddot{h}_{2\theta}' - \frac{\dot{H}}{H}\dot{h}_{2\theta}' - (\frac{\ddot{H}}{H} - \frac{\dot{H}^2}{H^2})h_{2\theta}'\} + \frac{3D_0}{8KH}b_\theta b_\theta' = 0, \\ &(\dot{h}_{2\theta} - \frac{\dot{H}}{H}h_{2\theta} + \frac{D_0}{4H}h_{2\theta}') + \frac{H}{2D}\{\ddot{h}_{2\theta}' - \frac{\dot{H}}{H}\dot{h}_{2\theta}' - (\frac{\ddot{H}}{H} - \frac{\dot{H}^2}{H^2})h_{2\theta}'\} + \frac{3D_0}{4KH}b_\theta b_\theta' = 0.\end{aligned} \tag{34}$$

Solving (34) in term of $b_\theta b_\theta'$ which is given by $\frac{b_0^2}{2H_0}H(t)\sin 2\{\theta - N(t)\}$ from (31), we obtain

$$\begin{aligned}&\dot{h}_{2\theta} - \frac{\dot{H}}{H}h_{2\theta} + \frac{D_0}{4H}h_{2\theta}' = -\frac{D_0 b_0^2}{8KH_0}\sin 2\{\theta - N(t)\}, \\ &\ddot{h}_{2\theta}' - \frac{\dot{H}}{H}\dot{h}_{2\theta}' - (\frac{\ddot{H}}{H} - \frac{\dot{H}^2}{H^2})h_{2\theta} = -\frac{D_0^2 b_0^2}{2KH_0H}\sin 2\{\theta - N(t)\}.\end{aligned} \tag{35}$$

Both equation in (35) are consistent with each other and are satisfied by

$$h_{2\theta} = -\frac{b_0^2}{4KH_0}H(t)\cos 2\{\theta - N(t)\}, \qquad h_{2\theta}' = \frac{b_0^2}{2KH_0}H(t)\sin 2\{\theta - N(t)\}, \tag{36}$$

Thus, h_{2_θ} is also a sine-wave with the wave-length of πr for any r and, with the same angular velocity as of b_θ

Now we can define the dynamical system in which both the waves b_θ and $h_{2\theta}$ shown in (31) and (36) respectively can occur, by the fundamental formulae substituted with (31) and (36) .

$$\left.\begin{aligned}&h = H\{1 + \frac{b_0^2}{2KH_0}\cos 2(\theta - N)\}, && H = H(t), \\ &k = H\{\frac{1}{1-Y} - \frac{2b_0 r}{\sqrt{H_0 H}}\sin(\theta - N) - \frac{b_0^2}{2KH_0}\cos 2(\theta - N)\}, && N(t) = \frac{D_0}{2}\int\frac{dt}{H(t)}, \\ &l = rb_0\sqrt{\frac{H}{H_0}}\cos(\theta - N) - \frac{b_0^2 H}{2KH_0}\sin 2(\theta - N). && Y \equiv \frac{Kr^2}{1+Kr^2} \equiv \frac{Ku}{1+Ku}.\end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned}&\Pi_0 = \frac{\dot{H}r}{2H} - \frac{D_0 r}{2H}(1-Y)\{\sqrt{\frac{Y}{1-Y}}\frac{b_0}{\sqrt{KH}}\cos(\theta - N) - (1-Y)\frac{b_0^2}{KH_0}\sin 2(\theta - N)\}, \\ &\Theta_0 = \frac{D_0 r}{2H}(1-Y)\Big[1 + \frac{Y}{2}\{3 + 4(1-Y)\}\frac{b_0^2}{KH} + \sqrt{\frac{Y}{1-Y}}\{1 + 2(1-Y)\}\frac{b_\theta}{\sqrt{KH_0}}\sin(\theta - N)\Big]\end{aligned}\right\} \tag{38}$$

$$-(1-Y)\{1-2(1-Y)\}\frac{b_0^2}{KH_0}\cos 2(\theta-N)\Big].$$

$$\left.\begin{aligned}\frac{\partial \delta_1}{\partial Y} &= -\frac{Y}{4H^2}(\dot{H}^2-2\ddot{H}H)-\frac{D^2Y}{4H^2}\Big[(1-Y)^2+3Y(1-Y)^2\{1+2(1-Y)\}\frac{b_0^2}{KH_0}\\ &\quad +(1-Y)\{1+(1-Y)+4(1-Y)^2\}\frac{b_0 Y}{\sqrt{H_0 H}}\sin(\theta-N)+2(1-Y)^3(1-3Y)\frac{b_0^2}{KH}\cos(\theta-N)\Big],\\ \frac{\partial \delta_1}{Y\partial\theta} &= -\frac{D_0^2Y}{4H^2}(1-Y)\Big[(2-Y)\frac{b_0 Y}{\sqrt{H_0 H}}\cos(\theta-N)-2(1-Y)^2\frac{b_0^2}{KH_0}\sin 2(\theta-N)\Big].\end{aligned}\right\} \quad (39)$$

$$\begin{aligned}4\pi G\rho_1 &= \frac{1}{2H^2}(\dot{H}^2-2\ddot{H}H)+\frac{D_0^4}{2H^4}(1-Y)^2\Big[(1-2Y)-3Y\{1+(1-Y)-8(1-Y)^2\}\frac{b_0^2}{KH}\\ &\quad -4\sqrt{Y(1-Y)}\{1-3(1-Y)\}\frac{b_0}{\sqrt{KH}}\sin(\theta-N)\\ &\quad +6Y(1-Y)\{1-4(1-Y)\}\frac{b_0^2}{KH}\cos 2(\theta-N)\Big].\end{aligned} \quad (40)$$

This model of the stellar system has the properties as follows:

(i) Even when the stellar system is steady, the vertex deviation, the angle of which is given by $\tan^{-1}\left(\frac{Yb_0\cos(\theta+\delta_1)-2h_0\cos(2\theta+\delta_2)}{-H_0Y/2(1-Y)+Yb_0\sin(\theta+\delta_1)-2h_0\sin(2\theta+\delta_2)}\right)$, ($H_0, b_0, h_0$: consts.), is possible owing to the assumption of non-axisymmetry. (ii) If the system is unsteady, the b_θ-wave alone or a pair of waves b_θ and $h_{2\theta}$ can appear so long as the vertex deviation exists ($\ell\neq 0$) for almost all t. (iii) In all of the fundamental physical quantities given in (37)-(40), the periodic fluctuations depending on the longitude are traced over the system. (iv) The patterns of such fluctuations rotate independently of the differential rotation with the angular velocity of $\dot{N}=D_0/2H$ by synchronizing with both waves of b_θ and $h_{2\theta}$ which also advance in longitude with the same angular velocity. (v) The theoretical fluctuating patterns, the modes of which are due to the initial conditions as well as the secular changes caused by the variation of $H(t)$, may by compared with the observations, especially the mean motion in (38) with the observed fluctuating rotation curves and the density distribution ρ in (40) with the observed pattern of structure.

4. The First Order Approximation With A Variable χ.

Here we extend our calculation to a general case of the first order approximation by taking a variable χ in addition to the linear terms with respects to b_θ, $h_{2\theta}$ or each of their derivatives. As for χ, we write

$$\chi(r,\theta,t)=\chi^{(0)}(r,\theta,t)+\chi^{(1)}(r,\theta,t), \tag{41}$$

where $\chi^{(0)}$ is the zeroth order function given in (16') while $\chi^{(1)}$ is the first order function to be determined afterwards. The formulae (6"), (7"), (8") and (10") in the preceding section remain still valid as they are, but inclusion of χ requires evaluations of (9) and both of $\frac{\partial \delta b_2}{\partial r}$ and $\frac{\partial \delta b_2}{r\partial\theta}$ in (9). These are

$$\left.\begin{aligned}
&\left\{\frac{\partial\chi^{(0)}}{\partial t}+\Pi_0^{(0)}\frac{\partial\chi^{(0)}}{\partial r}+\Theta_0^{(0)}\frac{\partial\chi^{(0)}}{r\partial\theta}\right\}+\left\{\frac{\partial\chi^{(1)}}{\partial t}+\Pi_0^{(0)}\frac{\partial\chi^{(1)}}{\partial r}+\Theta_0^{(0)}\frac{\partial\chi^{(1)}}{r\partial\theta}+\Pi_0^{(1)}\frac{\partial\chi^{(0)}}{\partial r}+\Theta_0^{(1)}\frac{\partial\chi^{(0)}}{r\partial\theta}\right\}\\
&\quad=\frac{\partial\chi^{(1)}}{\partial t}+\frac{\dot H}{2H}r\frac{\partial\chi^{(1)}}{\partial r}+\dot N(1-r)\frac{\partial\chi^{(1)}}{\partial\theta}\\
&\qquad+\frac{r}{2H}\left\{-2\left(\dot h_{2\theta}-\frac{\dot H}{H}h_{2\theta}\right)+\frac{D}{H}(1-r)(h'_{2\theta}-rb'_\theta)\right\}\frac{\partial\chi^{(0)}}{\partial r}\\
&\qquad+\frac{(1-r)}{2H}\left\{-\left(\dot h'_{2\theta}-\frac{\dot H}{H}h'_{2\theta}\right)+2r\left(\dot b'_\theta-\frac{\dot H}{2H}b'_\theta\right)+\frac{2D}{H}(1-r)(-h_{2\theta}+rb_\theta)\right\}\frac{\partial\chi^{(0)}}{\partial\theta}=0,\\
&\chi_0^{(0)}=a_0\{\theta-N(1-r)\}+\frac{D_0^2}{4K}\frac{(1-r)^2}{r}+\frac{1}{r}F_0(u),
\end{aligned}\right\} \tag{9"}$$

$$\left.\begin{aligned}
\frac{\partial\delta b_2}{\partial r}&=\frac{\partial\delta b_2^{(0)}}{\partial r}+\left(\frac{k}{2Z}\right)^{(0)}\frac{\partial\chi^{(1)}}{\partial r}-\left(\frac{l}{2Z}\right)^{(0)}\frac{\partial\chi^{(1)}}{r\partial\theta}+\left(\frac{k}{2Z}\right)^{(1)}\frac{\partial\chi^{(0)}}{\partial r}-\left(\frac{l}{2Z}\right)^{(1)}\frac{\partial\chi^{(0)}}{r\partial\theta},\\
\frac{\partial\delta b_2}{r\partial\theta}&=\frac{\partial\delta b_2^{(0)}}{r\partial\theta}+\left(\frac{k}{2Z}\right)^{(0)}\frac{\partial\chi^{(1)}}{r\partial\theta}-\left(\frac{l}{2Z}\right)^{(0)}\frac{\partial\chi^{(1)}}{\partial r}+\left(\frac{k}{2Z}\right)^{(1)}\frac{\partial\chi^{(0)}}{r\partial\theta}-\left(\frac{l}{2Z}\right)^{(1)}\frac{\partial\chi^{(0)}}{\partial r}.
\end{aligned}\right\} \tag{10'''}$$

The mark at the shoulder (0) or (1) indicates that the marked quantity is of the zeroth order or of the first one respectively. In (9"), the terms in the first brackets vanish as already seen.

From cross-differentiations of (10) we have in view of (6")

$$\left.\begin{aligned}
\frac{\partial^2\delta b_2}{\partial\theta\partial r}-\frac{\partial^2\delta b_2}{\partial r\partial\theta}&=\left\{\frac{K}{H^2}(1-r)^2r\frac{\partial\chi^{(1)}}{\partial\theta}+\frac{r}{2H}\frac{\partial^2\chi^{(1)}}{\partial\theta\partial r}\right\}+\left\{\frac{K}{H^2}(1-r)^2r\frac{\partial\chi^{(0)}}{\partial\theta}+\frac{r}{2H}\frac{\partial^2\chi^{(0)}}{\partial\theta\partial r}\right\}\\
&\quad+\frac{1}{2H^2}\left\{2(1-r)^2rb'_\theta+(1-3r-2r^2)h'_{2\theta}\right\}\frac{\partial\chi^{(0)}}{\partial r}\\
&\quad+\frac{(1-r)}{2H^2r}\left\{-(1-10r+8r^2)rb_\theta-4(1+2r-2r^2)h_{2\theta}\right\}\frac{\partial\chi^{(0)}}{\partial\theta}\\
&\quad+\frac{1}{H^2}\left\{-(1-r)^3rb_\theta+(2-2r+r^2)h_{2\theta}\right\}\frac{\partial^2\chi^{(0)}}{\partial r\partial\theta}\\
&\quad+\frac{(1-r)r}{2H^2}(rb'_\theta-h'_{2\theta})\left(\frac{\partial^2\chi^{(0)}}{\partial r^2}-\frac{\partial^2\chi^{(0)}}{r^2\partial\theta^2}\right).
\end{aligned}\right\} \tag{42}$$

Subsitution of $\chi^{(0)}$, which corresponds to our continuous case in the zeroth order approximation characterized by (16')-(17'), into (42) follows with reference to (A1) in the Appendix

$$
\begin{aligned}
\frac{\partial^2 \delta_2}{\partial\theta\partial r}-\frac{\partial^2 \delta_2}{\partial r\partial\theta} &= \frac{K}{H^2}(1-Y)^2 Y\frac{\partial\chi^{(1)}}{\partial Y}+\frac{Y}{2H}\frac{\partial^2\chi^{(1)}}{\partial\theta\partial Y}+\frac{\dot{D}Y}{H}(1-Y)^2 \\
&-\frac{1}{2H^2}\{2(1-Y)^2 Y b_\theta'+(1+3Y-2Y)^2 h_{2\theta}'\}\Big[\frac{2K}{H}\Big(\frac{1-Y}{Y}\Big)^2 Y\Big\{-a_0 N Y^2+\frac{D_0^2}{4K}(1-Y^2)+\Big(F_0-Y\frac{\partial F_0}{\partial Y}\Big)\Big\}\Big] \\
&-\frac{(1-Y)}{2H^2 Y}\{(1-10Y+8Y^2)Y b_\theta+4(1+2Y-2Y^2)h_{2\theta}\}a_0 \\
&-\frac{(1-Y)}{2H^2}Y(-Y b_\theta'+h_{2\theta}')\Big[\frac{2K}{H}\Big(\frac{1-Y}{Y}\Big)^2\Big\{-3a_0 N+\frac{D^2}{4K}(3+Y^2-4Y^3)+3\Big(F_0-Y\frac{\partial F_0}{\partial Y}\Big)+2Y^2(1-Y)\frac{\partial^2 F_0}{\partial Y^2}\Big\}\Big].
\end{aligned}
\tag{42'}
$$

By equating (28) with (42') we get the identity condition as follows

$$
\begin{aligned}
\frac{K}{H^2}(1-Y)^2 Y\frac{\partial\chi^{(1)}}{\partial Y}+\frac{Y}{2H}\frac{\partial^2\chi^{(1)}}{\partial\theta\partial Y} &= \frac{1}{2H^2}\{2(1-Y)^2 Y b_\theta'+(1+3Y-2Y^2)h_{2\theta}'\}\Big[\frac{2K}{H}\Big(\frac{1-Y}{Y}\Big)^2 Y\Big\{-a_0 N Y^2+\frac{D_0^2}{4K}(1-Y^2)+\Big(F_0-Y\frac{\partial F_0}{\partial Y}\Big)\Big\}\Big] \\
&+\frac{K(1-Y)^2}{2H^3 Y}\{(1-10Y+8Y^2)Y b_\theta+4(1+2Y-2Y^2)h_{2\theta}\}a_0 \\
&+\frac{(1-Y)Y}{2H^2}(-Y b_\theta'+h_{2\theta}')\Big[\frac{2K}{H}\Big(\frac{1-Y}{Y}\Big)^2\Big\{-3a_0 N+\frac{D_0^2}{4K}(3+Y^2-4Y^3)+3\Big(F_0-Y\frac{\partial F_0}{\partial Y}\Big)+2Y^2(1-Y)\frac{\partial^2 F_0}{\partial Y^2}\Big\}\Big] \\
&+\frac{4\dot{D}Y}{H^2}Y(1-Y)^2\Big(\dot{h}_{2\theta}-\frac{\dot{H}}{H}h_{2\theta}\Big) \qquad +\frac{D}{2HK}Y^2(-5+4Y)\Big(\dot{b}_\theta-\frac{\dot{H}}{2H}b_\theta\Big) \\
&+\frac{D^2 Y}{H^3}Y(1-Y)^2 h_{2\theta}' \qquad +\frac{D^2}{4H^2K}Y(1-Y)(3-4Y)b_\theta' \\
&+\frac{2\dot{D}Y}{H^2}Y(1-Y)(3-2Y)h_{2\theta} \qquad +\frac{\dot{D}}{HK}Y\Big(\frac{5}{2}-7Y+4Y^2\Big)b_\theta \\
&+\frac{Y}{H}Y(2-Y)\frac{\partial}{\partial t}\Big(\dot{h}_{2\theta}'-\frac{\dot{H}}{H}h_{2\theta}'\Big) \qquad +\frac{1}{K}Y(3-2Y)\Big\{\ddot{b}_\theta'-\Big(\frac{\ddot{H}}{2H}-\frac{\dot{H}^2}{4H^2}\Big)b_\theta'\Big\}.
\end{aligned}
\tag{43}
$$

The first order function should satisfy simultaneously both partial differential equation (43) and (9") for any values of r, θ and t . With these requiements and on reflection of the different character between b_θ and $h_{2\theta}$ the solution for b_θ or / and that for $h_{2\theta}$ are obtained separately. Reserving troublesome calculations in A2 of the Appendix, we formulate only the result below.

b_θ : under a condition of $\dot{b}_\theta-\frac{\dot{H}}{2H}b_\theta+\frac{D}{2H}b_\theta'=0,$ (44)

$$
\left.
\begin{aligned}
\chi_b^{(1)} &= \Big(\frac{1-Y}{Y}\Big)^2\Big\{\frac{D^2}{2K}(1-Y^2)-2a_0 N+2\Big(F_0(Y)-Y\frac{\partial F_0}{\partial Y}\Big)\Big\}\frac{Y b_\theta}{H} \\
&\qquad -2a_0\Big(\frac{1-Y}{Y}\Big)\frac{Y b_\theta'}{H}+\frac{1}{Y}F^{(1)}(Y), \\
&a_0 \equiv \frac{\dot{D}H}{K}, \quad D_0:\text{const.};\quad D^2=D_0^2+4Ka_0 N(t),\quad \dot{N}=\frac{D}{2H}\ \text{or}\ N(t)=\int_0^t\frac{D(t)}{2H(t)}dt, \\
&\frac{Y}{H^2}\equiv u;\quad \theta-N(t)(1-Y)=\theta_0,\quad \theta_0:\text{const.},
\end{aligned}
\right\}
\tag{45}
$$

$F_0(Y)$, $F^{(1)}(Y)$: arbitrary functions of Y.

$h_{2\theta}$: under a condition of $\dot{h}_{2\theta}-\frac{\dot{H}}{H}h_{2\theta}+\frac{D_0}{2H}h_{2\theta}'=0,$ (46)

$$
\begin{aligned}
\chi_h^{(1)} &= \frac{1}{Y^2}\Big\{\frac{D_0^2}{2K}\Big(7-8Y+4Y^2-3Y^3+Y^4-\frac{1}{1-Y}\Big) \\
&\qquad -G_1 D_0(2-2Y+Y^2)-2G_3(2-3Y+Y^2)\Big\}\frac{h_{2\theta}}{H}+\frac{1}{Y}F^{(1)}(Y),
\end{aligned}
$$

$$\left.\begin{array}{l} a_0=0; \quad D=D_0; \qquad G_1,\ G_3: \text{consts.,} \\ \frac{r^2}{H}=u; \quad \theta-N(t)(1-Y)=\theta_0; \quad F^{(1)}(Y): \text{arbitrary func. of } Y. \\ \left[F_0=-\frac{D_0^2}{4K}\left(\frac{4-3Y}{1-Y}\right)+\frac{G_1 D_0}{2}+G_3(1-Y)\right]. \end{array}\right\} \tag{47}$$

Let us mention first about $\chi^{(1)}$ in (45) and (47) leaving b_θ and $h_{2\theta}$ in (45) and (47) respectively later on. Either of both $\chi_b^{(1)}$ and $\chi_h^{(1)}$ is available when the respective wave is exclusively acting. In order to make both cooperative, $\chi^{(1)}=\chi_b^{(1)}+\chi_h^{(1)}$ is to be adopted by replacing F_0 in (45) with F_0 in (47). Accordingly, when the parameter D characterizing the rotational motion Θ_0 keeps as D_0 or a constant and $F_0(Y)$ takes the particular form in (47), both waves b_θ and $h_{2\theta}$ can occur simultaneously. But, if D_0 turns out $D(t)$ or time-dependent, the $h_{2\theta}$-wave disappears at once, while the b_θ-wave remains as it is. It is noticed here that for getting the result up to the first order, $F_0(u)$ in the zeroth order formulae (16')-(19') should be made identical to the one in (45) or (47).

When $D=D_0$: const., the terms with a_0 drop out from the zeroth order terms in (16') - (19'), consequently the integral $r^2/H(t)=u$ remains but the other one $\theta - N(t)(1-Y) = v = \theta_0$ disappears from $\chi^{(o)}$. Nevertheless, even in such a case, if $\chi^{(1)}$ is not constant but variable, both integrals still exist for $\chi^{(1)}$, so also for $\chi = \chi^{(o)} + \chi^{(1)}$, because the subsidary equations of (9") come out

$$\frac{dt}{1}=\frac{dr}{\dot{H}r/2H}=\frac{d\theta}{D_0/2(H+Kr^2)}=-\frac{\partial\chi^{(1)}}{\varphi(r,\theta,t)}, \quad \varphi(r,\theta,t)\equiv \Pi_0^{(1)}\frac{\partial\chi^{(0)}}{\partial r}+\Theta_0^{(1)}\frac{\partial\chi^{(0)}}{r\partial\theta},$$

which readily provide both of the integrals.

Now we return to the waves in the present case. The b_θ-oscillation in (44) is the same as the one in the previous section given in (29), (29') and (31), expect for that D_0 is now replaced by $D(t)$. While for the $h_{2\theta}$- wave in (46), we obtain from (6) and (46)

$$h_{2\theta} = h(t)\sin(2\theta+\gamma_2)\,;\; h(t)=\frac{h_0}{H_0}H(t),\;\; \dot{\gamma}_2(t) = -\frac{D_0}{H(t)} = -2\dot{N}(t),\;\; h_0, H_0 : \text{consts.}, \qquad (48)$$

which is different from the one in the preceding section merely in the amplitude. Namely, the amplitude is here independent of b(t) so that either of both b_θ- and $h_{2\theta}$- waves can occurs even separately.

The waves in the case of the variable X, however, have an important property such that they carry not only the wave considered so far but also another wave caused by the differential rotation of the system at the same time. It is seen from the followings: As above-mentioned the presence of the variable X provides the integral $\theta = \theta_0 - N(t)(1-Y)$ where $\dot{N}(1-Y)$ corresponds to the angular velocity of the differential rotation at r, accordingly by subsituting this into b_θ and $h_{2\theta}$ we have

$$\left.\begin{aligned} b_\theta &= \frac{b_0}{\sqrt{H_0}}\sqrt{H(t)}\sin\{\theta_0 - N(1-Y)+N\} = \frac{b_0}{\sqrt{H_0}}\sqrt{H(t)}\sin(\theta_0+NY), \\ h_{2\theta} &= \frac{h_0}{H_0}H(t)\sin 2\{\theta_0 - N(1-Y)+N\} = \frac{h_0}{H_0}H(t)\sin 2(\theta_0+NY). \end{aligned}\right\} \qquad (49)$$

These indicate that there exists, in addition to the wave advancing with the uniform angular velocity $\dot{N}(t)$, the other wave retroceding with the same angular velocity as the galactic rotation, consequently the resultant wave at r moves in the leading direction with the angular velocity of $\dot{N}(t)Y$. For sake of convenience, we call the former wave as the " D-wave " and the latter as the " R-wave ".

As regards the formulae of $\frac{\partial \delta_b}{\partial Y}$ and $\frac{\partial \delta_b}{Y\partial \theta}$ we can rewrite (10") and (10"') on reflections of (44)-(47)

$$\frac{\partial \delta_b^{(0)}}{\partial Y} = \frac{\partial \delta_{1b}^{(0)}}{\partial Y} + \frac{\partial \delta_{2b}^{(0)}}{\partial Y} = -\frac{Y}{4H^2}\Big[\dot{H}^2 - 2\ddot{H}H + \frac{4}{\kappa u^2}\Big\{F_0(u) - u(1+\kappa u)F_0'(u) + \frac{D_0^2}{4\kappa}\Big\}\Big] \qquad (51)$$

where $\frac{\partial \delta_{1b}}{\partial Y} = -\frac{Y}{4H^2}\Big\{\dot{H}^2 - 2\ddot{H}H + \frac{D^2}{(1+\kappa u)^2}\Big\}, \qquad \kappa u \equiv \frac{KY^2}{H} = \frac{Y}{1-Y},$

$$\frac{\partial \delta_b^{(0)}}{Y\partial \theta} = \frac{\partial \delta_{1b}^{(0)}}{Y\partial \theta} + \frac{\partial \delta_{2b}^{(0)}}{Y\partial \theta} = \frac{\dot{D}Y}{2HKu}, \qquad (52)$$

where $\dfrac{\partial \delta b_{1b}^{(0)}}{r\partial\theta} = \dfrac{\dot{D} r}{2H(1+\kappa u)}.$

$$\frac{\partial \delta b_b^{(1)}}{\partial r} = \frac{\partial \delta b_{1b}^{(1)}}{\partial r} + \frac{\partial \delta b_{2b}^{(1)}}{\partial r} = -\frac{b_\theta}{4H^2}\Big[\frac{D^2}{4\kappa} + \frac{4}{\kappa u^2}\{3F_0(u) - u(3-\kappa u)F_0'(u) + 2u^2(1+\kappa u)F_0''(u) + \frac{3D_0^2}{4\kappa}\}\Big]$$
$$-\frac{b_\theta'}{2H^2}a_0\Big(1+\frac{1}{\kappa u}\Big) - \frac{r}{H^2\kappa u^2}\Big[F^{(1)}(u) - u(1+\kappa u)\frac{\partial F^{(1)}}{\partial u}\Big], \qquad (53)$$

where $\dfrac{\partial \delta b_{1b}^{(1)}}{\partial r} = -\dfrac{b_\theta}{4H^2}\dfrac{D^2}{\kappa}\Big\{1 + \dfrac{3}{(1+\kappa u)^2} - \dfrac{4}{(1+\kappa u)^3}\Big\} - \dfrac{b_\theta'}{2H^2}a_0\Big(1 - \dfrac{1}{1+\kappa u}\Big),$

$$\frac{\partial \delta b_b^{(1)}}{r\partial\theta} = \frac{\partial \delta b_{1b}^{(1)}}{r\partial\theta} + \frac{\partial \delta b_{2b}^{(1)}}{r\partial\theta} = \frac{b_\theta}{2H^2}a_0\Big(1+\frac{1}{\kappa u}\Big) + \frac{b_\theta'}{4H^2}\Big[-\frac{D^2}{\kappa} + \frac{4}{\kappa^2 u^2}\{F_0(u) - u(1+\kappa u)F_0'(u) + \frac{D_0^2}{4\kappa}\}\Big], \qquad (54)$$

where $\dfrac{\partial \delta b_{1b}^{(1)}}{r\partial\theta} = \dfrac{b_\theta}{2H^2}a_0\Big\{1 + \dfrac{1}{1+\kappa u} - \dfrac{2}{(1+\kappa u)^2}\Big\} + \dfrac{b_\theta'}{4H^2}\dfrac{D^2}{\kappa}\Big\{-1 + \dfrac{1}{(1+\kappa u)^2}\Big\},$

$$\frac{\partial \delta b_h^{(0)}}{\partial r} = \frac{\partial \delta b_{1h}^{(0)}}{\partial r} + \frac{\partial \delta b_{2h}^{(0)}}{\partial r} = -\frac{r}{4H^2}\Big\{\dot{H}^2 + D_0^2 - 2\ddot{H}H\Big) + \frac{1}{\kappa u^2}\big(2G_1D_0 + 4G_3 - \frac{3D_0^2}{\kappa}\big)\Big\}, \qquad (55)$$

where $\dfrac{\partial \delta b_{1h}^{(0)}}{\partial r} = -\dfrac{r}{4H^2}\Big\{\dot{H}^2 - 2\ddot{H}H + \dfrac{D_0^2}{(1+\kappa u)^2}\Big\},$

$$\frac{\partial \delta b_h^{(0)}}{r\partial\theta} = \frac{\partial \delta b_{1h}^{(0)}}{r\partial\theta} + \frac{\partial \delta b_{2h}^{(0)}}{r\partial\theta} = 0, \qquad (56)$$

Where $\dfrac{\partial \delta b_{1h}^{(0)}}{r\partial\theta} = 0.$

$$\frac{\partial \delta b_h^{(1)}}{\partial r} = \frac{\partial \delta b_{1h}^{(1)}}{\partial r} + \frac{\partial \delta b_{2h}^{(1)}}{\partial r} = \frac{r h_{2\theta}}{H^3}\Big\{\frac{G_1 D_0}{\kappa u^2} + \frac{2}{\kappa^2 u^3}\big(4G_3 + 2G_1 D_0 - \frac{3D_0^2}{\kappa}\big)\Big\} + \frac{r}{H^2}\Big\{-\frac{1}{\kappa u^2}F^{(1)}(u) + \Big(1+\frac{1}{\kappa u}\Big)\frac{\partial F^{(1)}(u)}{\partial u}\Big\}, \qquad (57)$$

Where $\dfrac{\partial \delta b_{1h}^{(1)}}{\partial r} = \dfrac{r h_{2\theta}}{H^3}D_0^2\Big\{1 + \dfrac{1}{(1+\kappa u)^3}\Big\}$

$$\frac{\partial \delta b_h^{(1)}}{r\partial\theta} = \frac{\partial \delta b_{1h}^{(1)}}{r\partial\theta} + \frac{\partial \delta b_{2h}^{(1)}}{r\partial\theta} = -\frac{r h_{2\theta}'}{2H^2}\Big\{\frac{G_1 D_0}{\kappa u^2} + \frac{1}{\kappa^2 u^3}\big(4G_3 + 2G_1 D_0 - \frac{3D_0^2}{\kappa}\big)\Big\}, \qquad (58)$$

where $\dfrac{\partial \delta b_{1h}^{(1)}}{r\partial\theta} = \dfrac{r h_{2\theta}'}{2H^2}D_0^2\Big\{\dfrac{1}{1+\kappa u} + \dfrac{1}{(1+\kappa u)^2}\Big\}.$

These give the corresponding densities by means of Poisson equation as follows.

$$\left.\begin{aligned} 4\pi G\rho_b^{(0)} &= 4\pi G(\rho_{1b}^{(0)} + \rho_{2b}^{(0)}) = \frac{1}{2H^2}\Big[\dot{H}^2 - 2\ddot{H}H - \frac{4}{\kappa u^2}\{F_0(u) - u(1-\kappa u)F_0' + u^2(1+\kappa u)F_0'' + \frac{D_0^2}{4\kappa}\}\Big], \\ 4\pi G\rho_{1b}^{(0)} &= \frac{1}{2H^2}\Big[\dot{H}^2 - 2\ddot{H}H - D^2\Big\{\frac{1}{(1+\kappa u)^2} - \frac{2}{(1+\kappa u)^3}\Big\}\Big], \end{aligned}\right\} \qquad (59)$$

$$\left.\begin{aligned} 4\pi G\rho_b^{(1)} &= 4\pi G(\rho_{1b}^{(1)} + \rho_{2b}^{(1)}) = -\frac{4 r b_\theta}{H^3\kappa u^3}\{2F_0(u) - 2uF_0' + u^2(1-2\kappa u)F_0'' - u^3(1+\kappa u)F_0''' + \frac{D_0^2}{2\kappa}\} \\ &\quad - \frac{2}{H^2\kappa u^3}\Big\{F^{(1)}(u) - u(1+3\kappa u)\frac{\partial F^{(1)}}{\partial u} + u^2(1+\kappa u)\frac{\partial^2 F^{(1)}(u)}{\partial u^2}\Big\}, \end{aligned}\right\} \qquad (60)$$

where $$4\pi G\varrho_{1b} = \frac{2\gamma b_0}{H^3} D^2 \left\{ -\frac{1}{(1+\kappa u)^3} + \frac{3}{(1+\kappa u)^4} \right\}.$$

$$4\pi G\varrho_{h}^{(0)} = 4\pi G(\varrho_{1h}^{(0)} + \varrho_{2h}^{(0)}) = \frac{1}{2H^2}\left\{(\dot{H}^2 + D_0^2 - 2\ddot{H}H) + \frac{1}{\kappa u^2}\left(\frac{3D_0^2}{\kappa} - 2G_1 D_0 - 4G_3\right)\right\},$$

where $$4\pi G\varrho_{1h}^{(0)} = \frac{1}{2H^2}\left\{ \dot{H}^2 - 2\ddot{H}H - D_0^2 \left\{ \frac{1}{(1+\kappa u)^2} - \frac{2}{(1+\kappa u)^3} \right\}\right\}. \tag{61}$$

$$4\pi G\varrho_{h}^{(1)} = 4\pi G(\varrho_{1n}^{(1)} + \varrho_{2h}^{(1)}) = \frac{6 h_{2\theta}}{H^3 \kappa^2 u^3}\left(4G_3 + 2G_1 D_0 - \frac{3D_0^2}{\kappa}\right)$$
$$+ \frac{1}{H^2 \kappa u^2}\left\{3F^{(1)}(u) + u(1+\kappa u)\frac{\partial F^{(1)}(u)}{\partial u} + 2\kappa u^2 (1+\kappa u)\frac{\partial^2 F^{(1)}(u)}{\partial u^2}\right\}, \tag{62}$$

where $$4\pi G\varrho_{1h}^{(1)} = \frac{2 h_{2\theta}}{H^3} D_0 \left\{ -1 + \frac{1}{1+\kappa u} + \frac{1}{(1+\kappa u)^2} + \frac{2}{(1+\kappa u)^3} - \frac{3}{(1+\kappa u)^4} \right\}.$$

Followings are said from the above formulae. (i) The zeroth order quantities $\Omega_{2h}^{(0)}$ and $\rho_{2h}^{(0)}$ for $h_{2\theta}$ correspond to those for b_θ in a special case when $D=D_o$ and $F_0(u) = -\frac{D_0^2}{4\kappa}\left(1+\frac{4}{\kappa u}\right) + \frac{G_1 D_0}{2} + \frac{G_3}{1+\kappa u}$ ($F_o(Y)$ in (47)). (ii) Under the condition of (44) or (46) inclusion of the variable χ into F(P) in (1) leads to replacements of the essential terms which characterize Ω and ρ, by the terms with $1/u^n$ or $1/r^{2n}$ (n:an integer). But, this effect can be made less by modifying the conditinn (44) or (46) as in the next section. (iii) In the case of b_θ such replacements can be, at least partially, compensated by adjusting the arbitrary function F_o, while as for $h_{2\theta}$, $F_o(u)$ is specified in the above-mentioned way so that the $h_{2\theta}$-wave cannot happen unless the stellar system has a particular potential field defined by (55)-(58).

5. The First Order Approximation With A Variable χ. (Continued)

In the preceding section we found an example permitting the b_θ - or/and the $h_{2\theta}$-wave in the case of variable χ. Here, our concern is to look for the other examples and to examine the corresponding dynamical conditions.

The case of b_θ is considered first by neglecting all the terms with $h_{2\theta}$ and with use of the abbreviations,

$$
\begin{aligned}
&b_\theta = b_1\sin\theta + b_2\cos\theta, \qquad b_\theta' = b_1\cos\theta - b_2\sin\theta,\\
&\mathbb{A} \equiv \dot{b}_\theta - \frac{\dot{H}}{2H}b_\theta = (\dot{b}_1 - \frac{\dot{H}}{2H}b_1)\sin\theta + (\dot{b}_2 - \frac{\dot{H}}{2H}b_2)\cos\theta \equiv \mathcal{A}_1\sin\theta + \mathcal{A}_2\cos\theta,\\
&\mathbb{A}' \equiv \dot{b}_\theta' - \frac{\dot{H}}{2H}b_\theta' = -(\dot{b}_2 - \frac{\dot{H}}{2H}b_2)\sin\theta + (\dot{b}_1 - \frac{\dot{H}}{2H}b_1)\cos\theta \equiv -\mathcal{A}_2\sin\theta + \mathcal{A}_1\cos\theta,\\
&\mathbb{B} \equiv \frac{D}{H}b_\theta = \frac{D}{H}b_1\sin\theta + \frac{D}{H}b_2\cos\theta \equiv \mathcal{B}_1\sin\theta + \mathcal{B}_2\cos\theta,\\
&\mathbb{B}' \equiv \frac{D}{H}b_\theta' = -\frac{D}{H}b_2\sin\theta + \frac{D}{H}b_1\sin\theta \equiv -\mathcal{B}_2\sin\theta + \mathcal{B}_1\cos\theta.
\end{aligned}
\tag{63}
$$

Then, the identity condition (A15) is transformed to

$$
\begin{aligned}
&\sin\theta\Big[-2(\mathcal{A}_1 - \tfrac{\mathcal{B}_2}{2})Y^2(1-Y)^2\frac{\partial}{\partial Y}(\frac{F_0}{Y}) + \frac{D_0^2}{2K}(\mathcal{A}_1 - \tfrac{\mathcal{B}_2}{2})(1-Y)^2 + a_0(\mathcal{A}_2 + \tfrac{\mathcal{B}_1}{2})Y(1-Y)\\
&\qquad + Y^2\Big\{-\frac{DH}{K}(\dot{\mathcal{A}}_2 + \frac{\dot{H}}{2H}\mathcal{A}_2) + \frac{3H\dot{H}}{K}(\dot{\mathcal{A}}_1 + \frac{\dot{H}}{2H}\mathcal{A}_1) + \frac{2H^2}{K}\frac{\partial}{\partial t}(\dot{\mathcal{A}}_1 + \frac{\dot{H}}{2H}\mathcal{A}_1) - 2a_0\mathcal{A}_2\Big\}\Big]\\
&+\cos\theta\Big[-2(\mathcal{A}_2 + \tfrac{\mathcal{B}_1}{2})Y^2(1-Y)^2\frac{\partial}{\partial Y}(\frac{F_0}{Y}) + \frac{D_0^2}{2K}(\mathcal{A}_2 + \tfrac{\mathcal{B}_1}{2})(1-Y)^2 - a_0(\mathcal{A}_1 - \tfrac{\mathcal{B}_2}{2})Y(1-Y)\\
&\qquad + Y^2\Big\{\frac{DH}{K}(\dot{\mathcal{A}}_1 + \frac{\dot{H}}{2H}\mathcal{A}_1) + \frac{3H\dot{H}}{K}(\dot{\mathcal{A}}_2 + \frac{\dot{H}}{2H}\mathcal{A}_2) + \frac{2H^2}{K}\frac{\partial}{\partial t}(\dot{\mathcal{A}}_2 + \frac{\dot{H}}{2H}\mathcal{A}_2) + 2a_0\mathcal{A}_1\Big\}\Big] = 0.
\end{aligned}
\tag{64}
$$

Both coefficients of $\sin\theta$ and $\cos\theta$ should vanish, so after respective integration about Y each of the coefficients turns out

$$
\begin{aligned}
&-2(\mathcal{A}_1 - \tfrac{\mathcal{B}_2}{2})\Big\{F_0(Y) + \frac{D_0^2}{4K}\Big\} + a_0(\mathcal{A}_2 + \tfrac{\mathcal{B}_1}{2})Y\log\frac{Y}{1-Y} + f_1(t)Y\\
&\quad + (\frac{Y}{1-Y})\Big[-\frac{DH}{K}(\dot{\mathcal{A}}_2 + \frac{\dot{H}}{2H}\mathcal{A}_2) + \frac{3H\dot{H}}{K}(\dot{\mathcal{A}}_1 + \frac{\dot{H}}{2H}\mathcal{A}_1) + \frac{2H^2}{K}\frac{\partial}{\partial t}(\dot{\mathcal{A}}_1 + \frac{\dot{H}}{2H}) - 2a_0\mathcal{A}_2\Big] = 0,\\
&-2(\mathcal{A}_2 + \tfrac{\mathcal{B}_1}{2})\Big\{F_0(Y) + \frac{D_0^2}{4K}\Big\} - a_0(\mathcal{A}_1 - \tfrac{\mathcal{B}_2}{2})Y\log\frac{Y}{1-Y} + f_2(t)Y\\
&\quad + (\frac{Y}{1-Y})\Big[\frac{DH}{K}(\dot{\mathcal{A}}_1 + \frac{\dot{H}}{2H}\mathcal{A}_2) + \frac{3H\dot{H}}{K}(\dot{\mathcal{A}}_2 + \frac{\dot{H}}{2H}\mathcal{A}_2) + \frac{2H^2}{K}\frac{\partial}{\partial t}(\dot{\mathcal{A}}_2 + \frac{\dot{H}}{2H}) + 2a_0\mathcal{A}_1\Big] = 0,
\end{aligned}
\tag{65}
$$

where f_1 and f_2 are to be arbitrary functions of t which is regarded here as a parameter.

$F_0(Y)$ does not contain t, accordingly we should have a relation between the respective coefficients of $Y\log Y/(1-Y)$ as follows

$$\frac{a_0(\mathfrak{A}_2+\mathfrak{B}_1/2)}{\mathfrak{A}_1-\mathfrak{B}_2/2}=-\frac{a_0(\mathfrak{A}_1-\mathfrak{B}_2/2)}{\mathfrak{A}_2+\mathfrak{B}_1/2}\quad\text{or}\quad\left\{\left(\mathfrak{A}_2+\frac{\mathfrak{B}_1}{2}\right)^2+\left(\mathfrak{A}_1-\frac{\mathfrak{B}_2}{2}\right)^2\right\}a_0=0.$$

Therefore, if $a_0\neq 0$ or $D(t)$ is not a constant, it is required that

$$\mathfrak{A}_1-\frac{\mathfrak{B}_2}{2}=0,\ \mathfrak{A}_2+\frac{\mathfrak{B}_1}{2}=0\ ;\ f_1(t)=f_2(t)=0;\ F_0:\ \text{undetermined}, \tag{66}$$

which correspond just to the case of $\mathfrak{E}=0$ treated in the preceding section.

A remark may be necessary that the terms in the brackets in (65) are proved to be cancelled out by making use of the first pair of equations in (66).

On the other hand, if $a_0=0$ or $D(t)=D_0$:const., two alternative cases are considered. A case where $\mathfrak{A}_1-\frac{\mathfrak{B}_2}{2}=0$ and $\mathfrak{A}_2+\frac{\mathfrak{B}_1}{2}=0$ hold furthermore corresponds to a special case of the above-mentioned one, consequently $F_0(Y)$ is left undetermined. In another case where $\mathfrak{A}_1-\frac{\mathfrak{B}_2}{2}\neq 0$, $\mathfrak{A}_2+\frac{\mathfrak{B}_1}{2}\neq 0$

we have from (65)

$$\begin{aligned}F_0(Y)&=-\frac{D_0^2}{4K}+\frac{2f_1(t)}{2\mathfrak{A}_1-\mathfrak{B}_2}Y+\left(\frac{Y}{1-Y}\right)\left\{-\frac{DH}{K}\frac{\dot{\mathfrak{A}}_2+\frac{\dot H}{2H}\mathfrak{A}_2}{2\mathfrak{A}_1-\mathfrak{B}_2}+\frac{3\dot H H}{K}\frac{\dot{\mathfrak{A}}_1+\frac{\dot H}{2H}\mathfrak{A}_1}{2\mathfrak{A}_1-\mathfrak{B}_2}+\frac{2H^2}{K}\frac{\frac{\partial}{\partial t}(\dot{\mathfrak{A}}_1+\frac{\dot H}{2H}\mathfrak{A}_1)}{2\mathfrak{A}_1-\mathfrak{B}_2}\right\}\\&=-\frac{D_0^2}{4K}+\frac{2f_2(t)}{2\mathfrak{A}_2+\mathfrak{B}_1}Y+\left(\frac{Y}{1-Y}\right)\left\{\frac{DH}{K}\frac{\dot{\mathfrak{A}}_1+\frac{\dot H}{2H}\mathfrak{A}_1}{2\mathfrak{A}_2+\mathfrak{B}_1}+\frac{3\dot H H}{K}\frac{\dot{\mathfrak{A}}_2+\frac{\dot H}{2H}\mathfrak{A}_2}{2\mathfrak{A}_2+\mathfrak{B}_1}+\frac{2H^2}{K}\frac{\frac{\partial}{\partial t}(\dot{\mathfrak{A}}_2+\frac{\dot H}{2H}\mathfrak{A}_2)}{2\mathfrak{A}_2+\mathfrak{B}_1}\right\},\end{aligned}$$

which results

$$\frac{f_1(t)}{2\mathfrak{A}_1-\mathfrak{B}_2}=\frac{f_2(t)}{2\mathfrak{A}_2+\mathfrak{B}_1}=k_1,\qquad k_1,\ k_2:\ \text{consts.}, \tag{67}$$

$$\left.\begin{aligned}&-D_0\left(\dot{\mathfrak{A}}_2+\frac{\dot H}{2H}\mathfrak{A}_2\right)+3\dot H\left(\dot{\mathfrak{A}}_1+\frac{\dot H}{2H}\mathfrak{A}_1\right)+2H\frac{\partial}{\partial t}\left(\dot{\mathfrak{A}}_1+\frac{\dot H}{2H}\mathfrak{A}_1\right)=k_2\frac{K}{H}(2\mathfrak{A}_1-\mathfrak{B}_2),\\&D_0\left(\dot{\mathfrak{A}}_1+\frac{\dot H}{2H}\mathfrak{A}_1\right)+3\dot H\left(\dot{\mathfrak{A}}_2+\frac{\dot H}{2H}\mathfrak{A}_2\right)+2H\frac{\partial}{\partial t}\left(\dot{\mathfrak{A}}_2+\frac{\dot H}{2H}\mathfrak{A}_2\right)=k_2\frac{K}{H}(2\mathfrak{A}_2+\mathfrak{B}_1).\end{aligned}\right\} \tag{68}$$

Putting

$$\mathfrak{A}_1=m(t)\mathfrak{B}_2,\qquad \mathfrak{A}_2=-n(t)\mathfrak{B}_1\qquad \left(m\neq\frac{1}{2},\ n\neq\frac{1}{2}\right), \tag{69}$$

we obtain from (68) and (67)

$$\left.\begin{aligned}&-\{4\dot m n+(2m-1)\dot n\}D_0\mathfrak{B}_1+\left\{2\dot m\dot H+2\ddot m H-(2m-1)\left(mn\frac{D_0^2}{H}+k_2\frac{K}{H}\right)\right\}\mathfrak{B}_2=0,\\&-\left\{2\dot n\dot H+2\ddot n H+(2n-1)\left(mn\frac{D_0^2}{H}+k_2\frac{K}{H}\right)\right\}\mathfrak{B}_1-\{4m\dot n+\dot m(2n-1)\}D_0\mathfrak{B}_2=0,\\&f_1(t)=k_1(2m-1)\mathfrak{B}_2,\quad f_2(t)=-k_1(2n-1)\mathfrak{B}_1.\end{aligned}\right\} \tag{70}$$

Therefore, there is a variety of the dynamical state permitting the b_{θ}-wave. Especially when both $m(\neq\frac{1}{2})$ and $n(\neq\frac{1}{2})$ are constant, it follows from (69) that

$$b_i(t)=\frac{b_{io}}{\sqrt{H_0}}\sqrt{H(t)}\quad(i=1,2),\quad H_0,\ b_{io},\ m=n:\ \text{consts.},\quad \dot{\gamma}_1=-m\frac{D_0}{H}. \tag{71}$$

Therefore, on reflection of (70) we have

$$k_2=-m^2\frac{D_0^2}{K}(<0),\qquad f_1(t)=k_1(2m-1)\frac{D_0}{H}b_2(t),\qquad f_2(t)=-k_1(2m-1)\frac{D_0}{H}b_1(t), \tag{72}$$

for which $F_0(Y)$ becomes

$$F_0(Y)=-\frac{D_0^2}{4K}+2k_1Y+k_2\left(\frac{Y}{1-Y}\right)=\frac{D_0^2}{K}\left\{\left(m^2-\frac{1}{4}\right)-\frac{m^2}{1-Y}\right\}+2k_1Y. \tag{73}$$

With respect to the $h_{2\theta}$- wave too, we use the abbreviations as

$$\begin{aligned}
&h_{2\theta}=h_1\sin2\theta+h_2\cos2\theta,\qquad h'_{2\theta}=2h_1\cos2\theta-2h_2\sin2\theta,\\
&\mathcal{C}\equiv\dot{h}_{2\theta}-\frac{\dot{H}}{H}h_{2\theta}=\left(\dot{h}_1-\frac{\dot{H}}{H}h_1\right)\sin2\theta+\left(\dot{h}_2-\frac{\dot{H}}{H}h_2\right)\cos2\theta\equiv\mathcal{C}_1\sin2\theta+\mathcal{C}_2\cos2\theta,\\
&\mathcal{C}'\equiv\dot{h}'_{2\theta}-\frac{\dot{H}}{H}h'_{2\theta}=-2\left(\dot{h}_2-\frac{\dot{H}}{H}h_2\right)\sin2\theta+2\left(\dot{h}_1-\frac{\dot{H}}{H}h_1\right)\cos2\theta\equiv-2\mathcal{C}_2\sin2\theta+2\mathcal{C}_1\cos2\theta,\\
&\mathcal{D}\equiv\frac{D}{H}h_{2\theta}=\frac{D}{H}h_1\sin2\theta+\frac{D}{H}h_2\cos2\theta\equiv\vartheta_1\sin2\theta+\vartheta_2\cos2\theta,\\
&\mathcal{D}'\equiv\frac{D}{H}h'_{2\theta}=-\frac{2D}{H}h_2\sin2\theta+\frac{2D}{H}h_1\cos2\theta\equiv-2\vartheta_2\sin2\theta+2\vartheta_1\cos2\theta,
\end{aligned} \tag{74}$$

and assume

$$\mathcal{C}_1=p(t)\vartheta_2\qquad \mathcal{C}_2=-q(t)\vartheta_1, \tag{75}$$

Then by letting the terms with b_{θ} be neglected we have from the identity condition (A15)

$$\begin{aligned}
\sin2\theta\Big[&2\frac{\partial F_0}{\partial Y}\{p(1+3Y-2Y^2)-(1+2Y-Y^2)\}\vartheta_2-2F_0\{p(3-2Y)-(3-Y)\}\vartheta_2\\
&+\frac{D_0^2}{2K}\frac{1}{Y}\{p(2-Y+2Y^2)-(2-Y-Y^2)\}\vartheta_2+a_0\left\{q\left(2Y+\log\frac{Y}{1-Y}\right)-(1-Y)\log\frac{Y}{1-Y}\right\}\vartheta_1\\
&+\frac{H^2}{K}\left(\frac{1}{1-Y}-Y\right)\left\{(p\dot{q}+2\dot{p}q)\frac{D}{H}\vartheta_1+3pq\frac{a_0K}{H^2}\vartheta_1+p^2q\frac{D^2}{H^2}\vartheta_2-\dot{p}\left(\frac{2a_0K}{DH}+\frac{\dot{H}}{H}\right)\vartheta_2-\ddot{p}\vartheta_2\right\}\\
&-\frac{DH}{K}(1-Y)\left\{pq\frac{D}{H}\vartheta_2+\left(\dot{q}+q\frac{a_0K}{DH}\right)\vartheta_1-H\left(\dot{G}_1-\frac{D}{H}G_2+\frac{D}{H}G_2Y\right)\right\}\Big],\\
-\cos2\theta\Big[&2\frac{\partial F_0}{\partial Y}\{q(1+3Y-2Y^2)-(1+2Y-Y^2)\}\vartheta_1-2F_0\{q(3-2Y)-(3-Y)\}\vartheta_1\\
&+\frac{D_0^2}{2K}\frac{1}{Y}\{q(2-Y+2Y^2)-(2-Y-Y^2)\}\vartheta_1-a_0\left\{p\left(2Y+\log\frac{Y}{1-Y}\right)-(1-Y)\log\frac{Y}{1-Y}\right\}\vartheta_2\\
&+\frac{H^2}{K}\left(\frac{1}{1-Y}-Y\right)\left\{-(\dot{p}q+2p\dot{q})\frac{D}{H}\vartheta_2-3pq\frac{a_0K}{H^2}\vartheta_2+pq^2\frac{D^2}{H^2}\vartheta_1-\dot{q}\left(\frac{2a_0K}{DH}+\frac{\dot{H}}{H}\right)\vartheta_1-\ddot{q}\vartheta_1\right\}\\
&-\frac{DH}{K}(1-Y)\left\{pq\frac{D}{H}\vartheta_1-\left(\dot{p}+p\frac{a_0K}{DH}\right)\vartheta_2\right\}+H\left(\dot{G}_2+\frac{D}{H}G_1-\frac{DG_1}{H}Y\right)\Big]=0,
\end{aligned}$$

where $G^{(1)}$ is replaced by $G_1(t)\sin 2\theta + G_2(t)\cos 2\theta$ as it should be (cf. A2).

Since both the coefficients of $\sin 2\theta$ and $\cos 2\theta$ must vanish, we have $(\vartheta_1 \neq 0, \vartheta_2 \neq 0)$

$$\begin{aligned}
&\frac{\partial F_0}{\partial Y}\{(p-1)+(3p-2)Y-(2p-1)Y^2\}-F_0\{3(p-1)-(2p-1)Y\} \\
&= -\frac{D_0^2}{4K}\frac{1}{Y}\{2(p-1)-(p-1)Y+(2p+1)Y^2\}+\frac{a_0\vartheta_1}{2\vartheta_2}\{-\frac{3pq}{1-Y}+q+3q(p-1)Y-[(q-1)+Y]\log\frac{Y}{1-Y}\} \\
&\quad +\frac{D^2}{2K}\{-\frac{p^2q}{1-Y}+\dot{p}q+pq(p-1)Y\}+(\frac{H^2}{2K}\ddot{p}+\frac{\dot{H}H}{2K}\dot{p}+\frac{a_0H}{D}\dot{p})(\frac{1}{1-Y}-Y)+\frac{\dot{G}_1H-G_2D}{2\vartheta_2}+Y\frac{G_2D}{2\vartheta_2} \\
&\quad +\frac{DH}{2K}\frac{\vartheta_1}{\vartheta_2}\{\dot{q}+(p\dot{q}+2\dot{p}q-\dot{q})Y-\frac{p\dot{q}-2\dot{p}q}{1-Y}\},
\end{aligned} \tag{76}$$

$$\begin{aligned}
&\frac{\partial F_0}{\partial Y}\{(q-1)+(3q-2)Y-(2q-1)Y^2\}-F_0\{3(q-1)-(2q-1)Y\} \\
&= -\frac{D_0^2}{4K}\frac{1}{Y}\{2(q-1)-(q-1)Y+(2q+1)Y^2\}-\frac{a_0\vartheta_2}{2\vartheta_1}\{-\frac{3pq}{1-Y}+p+3p(q-1)Y-[(p-1)+Y]\log\frac{Y}{1-Y}\} \\
&\quad +\frac{D^2}{2K}\{-\frac{pq^2}{1-Y}+p\dot{q}+pq(q-1)Y\}+(\frac{H^2}{2K}\ddot{q}+\frac{H\dot{H}}{2K}\dot{q}+\frac{a_0H}{D}\dot{q})(\frac{1}{1-Y}-Y)+\frac{-\dot{G}_2H-G_1D}{2\vartheta_1}+Y\frac{G_1D}{2\vartheta_1} \\
&\quad -\frac{HD}{2K}\frac{\vartheta_2}{\vartheta_1}\{\dot{p}+(\dot{p}q+2p\dot{q}-\dot{p})Y-\frac{\dot{p}q+2p\dot{q}}{1-Y}\}.
\end{aligned}$$

Comparison of both equations in (76) indicates that first p=q, second $a_0=0$ otherwise the coefficients of Ylog Y/(1-Y) in both equations cannot vanish, third $\dot{p}=\dot{q}=0$ from the coefficients of $\frac{HD\vartheta_2}{2K\vartheta_1}$ and $\frac{HD\vartheta_1}{2K\vartheta_2}$, and fourth

$$\frac{\dot{G}_1H-G_2D}{\vartheta_2}=\frac{-\dot{G}_2H-G_1D}{\vartheta_1}, \quad \frac{G_1}{\vartheta_1}=\frac{G_2}{\vartheta_2},$$

which provide

$$\begin{aligned}
G_1(t)&=\frac{k\vartheta_1}{\sqrt{\vartheta_1^2+\vartheta_2^2}}=k\cos\gamma_2(t) \\
G_2(t)&=\frac{k\vartheta_2}{\sqrt{\vartheta_1^2+\vartheta_2^2}}=k\sin\gamma_2(t)
\end{aligned}
\quad \text{where} \quad
\begin{aligned}
h_1&=h(t)\cos\gamma_2 \\
h_2&=h(t)\sin\gamma_2
\end{aligned}
\quad \text{k:arbitrary constant.} \tag{77}$$

On the other hand, the $h_{2\theta}$- wave assumed in (75) is characterized by

$$h(t)=\frac{h_0}{H_0}H(t), \quad \dot{\gamma}_2(t)=-p\frac{D_0}{H}, \tag{78}$$

to which the function F(Y) defined below is corresponded

$$\begin{aligned}
&\frac{\partial F_0}{\partial Y}+\frac{-3(p-1)+(2p-1)Y}{(p-1)+(3p-2)Y-(2p-1)Y^2}F_0=\frac{1}{(p-1)+(3p-2)Y-(2p-1)Y^2}\Big[\frac{D_0^2}{4K}\{-\frac{2p^3}{1-Y}-\frac{2(p-1)}{Y} \\
&\quad +(p+1)(2p-1)+(2p^3-2p^2-2p-1)Y\}+\frac{kH_0}{2h_0}\{(p-1)+Y\}\Big]
\end{aligned} \tag{79}$$

This first order differential equation can be solved in a straightforward way though its general expression is omitted here. Especially for $p=1$ we have

$$F_0=-\frac{D_0^2(4-3Y)}{4K(1-Y)}+G_3(1-Y)+\frac{kH_0}{2h_0} \tag{80}$$

which is nothing but the formula in (47).

In the above treatment, we examined the cases in which a combination type of the different sub-indices such as $\mathcal{A}_1 = m(t)\mathcal{B}_2$, $\mathcal{A}_2 = -n(t)\mathcal{B}_1$ and $\mathcal{C}_1 = p(t)\mathcal{D}_2$, $\mathcal{C}_2 = -q(t)\mathcal{D}_1$ were assumed. For these cases, if m=n:const. or p=q:const., (71) or (78) indicate that the angular velocity of the respective wave depends on the multiple factor m or p, while the corresponding amplitude suffers no effect.

The amplitude, however, can also be changed by adopting another combination type with the same subdindex such as

$$\left.\begin{aligned} &\mathcal{A}_1 = m(t)\mathcal{B}_1, \quad \mathcal{A}_2 = m(t)\mathcal{B}_2, \\ &\mathcal{C}_1 = p(t)\mathcal{D}_1, \quad \mathcal{C}_2 = p(t)\mathcal{D}_2. \end{aligned}\right\} \tag{81}$$

Because, these give at once

$$b(t) = b_0\left(\frac{H(t)}{H_0}\right)^{\frac{1}{2}} e^{2\int m(t)\dot{N}(t)dt}, \quad h(t) = h_0\left(\frac{H(t)}{H_0}\right) e^{2\int p(t)\dot{N}(t)dt} \tag{82}$$

and especially when $m(t) = m_0\frac{\dot{H}}{D}$, $p(t) = p_0\frac{\dot{H}}{D_0}$ (m_0, p_0:constants) respectively, (82) are reduced to the simple forms of

$$b(t) = b_0\left(\frac{H(t)}{H_0}\right)^{m_0+\frac{1}{2}}, \qquad h(t) = h_0\left(\frac{H(t)}{H_0}\right)^{p_0+1}. \tag{82'}$$

Generally speaking, the identity condition (A15) is linear with repects to $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$ and $\mathcal{D}_i$ (i=1,2), so that the most general case is expected to be of the type mixed from the above two ones, namely

$$\left.\begin{aligned} &\mathcal{A}_1 = m_1(t)\mathcal{B}_1 + n_1(t)\mathcal{B}_2, \quad \mathcal{A}_2 = m_2(t)\mathcal{B}_2 + n_2(t)\mathcal{B}_1 \\ &\mathcal{C}_1 = p_1(t)\mathcal{D}_1 + q_1(t)\mathcal{D}_2, \quad \mathcal{C}_2 = p_2(t)\mathcal{D}_2 + q_2(t)\mathcal{D}_1 \end{aligned}\right\} \tag{83}$$

These may provide further more possible examples of both b_θ- and $h_{2\theta}$- oscillations, since both or either of the amplitude and the angular velocity, are adjustable at the same time. But, we refrain here from examining (83) so as to illustrate exhaustively the possible examples with their characteristic formulae, and we are contented with having got an important information that there can be a variety of both b_θ- and $h_{2\theta}$- oscillations depending on the different conditions.

Now, let us review the case of variable χ throughout both sections precedent and present by forcussing our attention to a question which kind of models are permessible as of the two-dimensional stellar system. Concerning it the followings are mentioned: (i) The waves b_θ and $h_{2\theta}$ consist of two-component-waves, namely the " D-wave " advancing with a uniform angular velocity over the system and the " R-wave " receding as if it compensated for the differential rotation. (ii) The waves of b_θ or/and $h_{2\theta}$, therefore, advance so as to develope the arm-structure. Whereas, the models in §3 in which χ was kept constant, the latter component-wave was lacking so that the waves were forced to move as a rigid rotation. (iii) If D is time-dependent, only the b_θ-wave is possible, though there is as much allowance for the model as for the arbitrary functions of $F^{(0)}(Y)$ and $F^{(1)}(Y)$. In § 3, however, we found no model with a time dependent D. (vi) For a constant D on the other hand, there exists a variety of model which permits either or both waves of b_θ and $h_{2\theta}$ on the contrary to the models in § 3 where the $h_{2\theta}$-wave alone could not occur. Generally speaking, freedom in the model is more for b_θ than for $h_{2\theta}$ as suspected from conparisons between (45) and (47) and also between (70) and (79).

(v) The gravitational field and the density distribution for any of various models can be evaluated if we compute $\frac{\partial \delta b_3}{\partial r}$, $\frac{\partial \delta b_3}{r \partial \theta}$, φ_2 for an assigned $F^{(0)}(Y)$ which is determined consistently, say, from (67) or/and (69), by making use of the formulae such as h, k, l in (6"), Σ in (8"), $\frac{\partial \delta b_3}{\partial r}$, $\frac{\partial \delta b_2}{r \partial \theta}$ in (10") and $\chi^{(1)}$ in (A13), and then if we combine these $\frac{\partial \delta b_3}{\partial r}$, $\frac{\partial \delta b_2}{r \partial \theta}$ with $\frac{\partial \delta b_1}{\partial r}$, $\frac{\partial \delta b_1}{r \partial \theta}$ shown in (51)-(62).

6 Summary

Here we summarize the main results in the present investigation. As for our fundamental assumption refer to § Introduction.

1) In the self-gravitating stellar system with the given density distribution, the velocity distribution cannot be determined uniquely. But, if some velocity distribution is chosen adequately at some epoch, its evolutional change can be persuited. Accordingly for the same density distribution, different evolutions are possible.

2) The so-called weight factor to the density, χ , which is introduced into the argument of the frequency function f(P) as a function of coordinate and time, is defined by the continuity equation following the motion of the local standard of rest or (Π_0, Θ_0). (§1)

3) In the present treatment, letting be the normal galaxies in mind, we regarded the parameters relating on the vertex deviation as the small quantities of the first order. In the zeroth order approximation dropped the first order terms, it was shown that there could be a variety of permissible models which were finite in the dimension as well as in the total mass. (2)

4) With the help of the density factor χ , it was proved that Lin's spiral producing potential could not operate as a disturbance in the self-gravitating two-dimensional system unless the velocity distribution was circular.

5) If the weight factor χ is constant or negligible, the identity condition which requires consistency between $\frac{\partial \mathfrak{S}}{\partial r}$ and $\frac{\partial \mathfrak{S}}{r\partial \theta}$ can not hold except when the parameter D(t) characterizing the mean rotation ($\Theta_0 = \frac{Dr}{2(H+Kr^2)}$ + the first order terms) is kept constant. In this special case of $D=D_0$:const., the calculation up to the first order showed that there existed one non-steady

finite model in which the oscillatory variations in the velocity dispersion, b_θ and/or $h_{2\theta}$ were possible with the respective periods of 2π and π in longitude, unless the vertex deviation did not vanish. Presence of these oscillations causes fluctuations depending on longitude with respects to the dynamical quantities, such as the motion of the local standard of rest, the velocity dispersion, the density distribution etc.. (§3)

6) Especially as for density distribution, the above mentioned fluctuations appear as a patterned structure which consists of one and two arms due to b_θ and $h_{2\theta}$ respectively or only one arm from b_θ. Each of these arms is produced by the wave, which we call the D-wave , advancing with the angular velocity of ($\frac{D_0}{2H}$ + the first order terms) so that the patterned structure remains nearly unchanged irrespective of the differential rotation. It is conceivable, therefore, the main feature of patterned structure is ascribable to the initial state. (§3)

7) In the above-mentioned model, the one armed pattern of b_θ can appear alone while the two-armed one of $h_{2\theta}$ occurs only when the one-armed one exists. In order to make the density fluctuation in the two-armed pattern comparable with that in the one-armed pattern, the amplitude of $h_{2\theta}$ oscillation should amount to a few tenths of H . This forces the position of the maximum density to displace away from the center of the system. Therefore, this model may not be a representative of the normal spirals with two-armed spirals. (§3)

8) If χ is variable, such an integral as $\theta=\theta_0 + \frac{D}{2(H+Kr^2)}$ exsists so far as up to the first order approximation concerns. Therefore, in addition to the D-wave, another kind of wave or the R-wave receding reversely to the differential rotation is produced by either of oscillations b_θ and $h_{2\theta}$. Consequently the resultant wave moves with the difference angular velocity

between both waves. (§4)

9) For the variable χ, a simple case where both b_θ and $h_{2\theta}$ oscillations take the same form as in the model of the constant χ was examined in details at first. In this simple case, if $D=D_0$:const., both of the oscillations occur in the models which are the same in the zeroth order but differ in the first order, while if $D=D(t)$, only the b -oscillation can appear in the models differing in both zeroth and first orders. (§4)

10) In a general case with a variable χ, however, either of both oscillations b_θ and $h_{2\theta}$ can occur with a variety in the angular velocity (of the density wave) as well as in the amplitude, though the $h_{2\theta}$-oscillation vanishes unless D is kept constant. But none of these possible angular velocities contain the radial coordinate in the zeroth order, so that the effect of the wave due to the differential rotation leads the resultant wave move in the preceding direction. Therefore, the patterned structures of all of the various models with variable χ will develope into the leading spiral arms with different time-scales corresponding to the respective dynamical circumstances. (§4, §5)

11) All of the present results have been derived under the requirement that the identity condition of the potential must hold everywhere and every time. The oscillations b_θ and $h_{2\theta}$, therefore, appear as sinuous with the periods of 2π and π in longitude but not some of segments from a sine-curve, accordingly the spiral arms are not bandlike. However, if the above-mentioned requirement is loosen, the different aspect is seen, for example, either of the the leading arm or the trailing one occurs rather evenly as bandlike arms. The investigation on such line will be reserved in the forthcoming paper.

12) It is added a note relating on the so-called anti-spiral theorem due to Lynden-Bell and Ostricker (1967). These authors proved that the patterns

of a spiral form did not generally exist as normal modes of oscillations for a differentially rotating nondissipative gaseous system. Yabushita (1969) and Miyamoto (1969) ascertained that this theorem held as well in their results from hydrodynamical treatment. On the other hand Shu (1970) demonstrated that the theorem did not necessarily apply to a stellar system in which there were resonant stars. Our above result, however, admits the leading spiral patterns even when the waves are stable. This discrepancy seems to be in a fact that the perturbed frequency function (or the perturbed density distribution function) as well as the perturbed potential in our case can hardly be assumed to have a common factor $\exp[i(m\theta+\nu t)]$ as in the approach from normal modes, since our those functions would contain, even if all of the time-dependent quantities were expressible by const.$\times\exp[i\nu t]$, the terms with factors of $\exp[i(m\theta+\nu t)]$, $\exp[i\nu t]$, $\exp[im\theta]$ and 1. Our treatment, therefore, is more general in the above sense than that due to the normal modes, though our frequency function F(P) is restricted to have a form of (1) and is required to be consistent with the potential by adjusting the trancation.

Appendix

A1. Derivation of the expression for $\chi^{(0)}$.

Let us find a common solution $\chi^{(0)}$ for the simultaneous partial differential equations of (9) and (15), or

$$\left.\begin{aligned} &\frac{\partial\chi^{(0)}}{\partial t}+\gamma\frac{\dot{H}}{2H}\frac{\partial\chi^{(0)}}{\partial\gamma}+\dot{N}(1-Y)\frac{\partial\chi^{(0)}}{\partial\theta}=0, \\ &\frac{Y}{2H}\frac{\partial^2\chi}{\partial\theta\partial\gamma}+\frac{K}{H^2}(1-Y)^2\gamma\frac{\partial\chi^{(0)}}{\partial\theta}-\frac{\dot{D}\gamma}{H}(1-Y)^2=0. \end{aligned}\right\} N(t)\equiv\int_0^t\frac{D}{2H}dt,\ Y\equiv\frac{K\gamma^2}{H+K\gamma^2} \tag{A1}$$

The subsidary equation of the first equation in (A1) becomes

$$\frac{dt}{1}=\frac{d\gamma}{\gamma\dot{H}/2H}=\frac{d\theta}{\dot{N}(1-Y)}. \tag{A2}$$

From the first pair and another pair consisting of the first and third terms, we get

$$\frac{\gamma^2}{H(t)}=\frac{\gamma_0^2}{H_0}\equiv u,\qquad \theta-\frac{N(t)}{1+Ku}=\theta_0\equiv v \tag{A3}$$

respectively. Therefore, $\chi^{(0)}$ is given by $\chi^{(0)}(u,v)$ which includes every integral of the first equation in (A1).

On the other hand, since the second equation in (A1) is transformed into

$$\frac{1}{2H}\frac{\partial}{\partial\gamma}\left(Y\frac{\partial\chi^{(0)}}{\partial\theta}\right)=\frac{\dot{D}H}{2HK}\frac{\partial Y}{\partial\gamma}\quad\text{or}\quad\frac{\partial^2}{\partial\theta\partial\gamma}\left\{Y\left(\chi^{(0)}-\frac{\dot{D}H}{K}\theta\right)+F(\gamma,t)+G(\theta,t)\right\}=0,$$

in which $F(r,t)$ and $G(\theta,t)$ are arbitrary functions including t as a parameter. This follows at once for $Y\neq0$ or $r\neq0$

$$\chi^{(0)}=a\theta_0+\frac{1}{Y}\{F(\gamma,t)+G(\theta,t)\},\qquad \frac{\dot{D}H}{K}\equiv a(t). \tag{A4}$$

Put (A4) into the second equation of (A1), we have

$$\left\{\frac{1}{Y}\left(\frac{\partial F}{\partial t}+\gamma\frac{\dot{H}}{2H}\frac{\partial F}{\partial\gamma}\right)+a\dot{N}(1-Y)\right\}+\left\{\frac{1}{Y}\left(\frac{\partial G}{\partial t}+\dot{N}\frac{\partial G}{\partial\theta}\right)+\left(\dot{a}\theta-\dot{N}\frac{\partial G}{\partial\theta}\right)\right\}=0. \tag{A5}$$

This is decomposable into two parts, or

$$\begin{aligned} &\frac{\partial F}{\partial t}+\gamma\frac{\dot{H}}{2H}\frac{\partial F}{\partial\gamma}+a\dot{N}Y(1-Y)=\varphi_1(t)+Y\varphi_2(t), \\ &\frac{\partial G}{\partial t}+\dot{N}\frac{\partial G}{\partial\theta}+Y\left(\dot{a}\theta-\dot{N}\frac{\partial G}{\partial\theta}\right)=-\{\varphi_1(t)+Y\varphi_2(t)\}, \end{aligned} \tag{A6}$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are unknown functions.

Now, let us express $G(\theta,t)$ by a power series of θ, or

$$G(\theta,t)=\sum_{n=0}^{\infty}g_n(t)\theta^n, \tag{A7}$$

and substitute it into the second equation of (A6), we have

$$\Big[\{\dot{g}_0+\dot{N}g_1+\varphi_1(t)\}+(\dot{g}_1+2\dot{N}g_2)\theta+\sum_{n=2}^{\infty}\{\dot{g}_n+\dot{N}(n+1)g_{n+1}\}\theta^n\Big]$$
$$+\Upsilon\Big[\{-\dot{N}g_1+\varphi_2(t)\}+(\dot{a}-2\dot{N}g_2)\theta-\sum_{n=2}^{\infty}\dot{N}(n+1)g_{n+1}\theta^n\Big]=0.$$

The above equation should hold identically for any values of r and θ, hence it follows that

$$g_3=g_4=\cdots=0,\quad g_2=\frac{\dot{a}}{2\dot{N}}=\text{const.},\quad \dot{g}_1=-\dot{a},\quad \dot{g}_0=-\{\varphi_1(t)+\varphi_2(t)\},$$

consequently we have

$$\left.\begin{aligned}&G(\theta,t)=g_0(t)+g_1(t)\theta+g_2\theta^2=g_0(t)+\{g_{10}+a_0-a(t)\}\theta+g_2\theta^2;\\&g_2=\dot{a}/2\dot{N},\ g_{10}\ \text{and}\ a_0:\ \text{consts.}.\end{aligned}\right\}\qquad \text{(A8)}$$

Substitution of (A8) into the first equation of (A6) results

$$\frac{\partial F}{\partial t}+\gamma\frac{\dot{H}}{2H}\frac{\partial F}{\partial \gamma}+a\dot{N}\Upsilon(1-\Upsilon)+\dot{g}_0+\dot{N}(1-\Upsilon)(c-a)=\frac{\partial F}{\partial t}+\gamma\frac{\dot{H}}{2H}\frac{\partial F}{\partial \gamma}-a\dot{N}(1-\Upsilon)^2+c\dot{N}(1-\Upsilon)+\dot{g}_0=0$$
$$g_{10}+a_0\equiv c:\ \text{const.},$$

which readily provides

$$F(\gamma,t)=F_0(u)-g_0(t)+\frac{D^2(t)}{4k(1+ku)^2}-\frac{cN(t)}{1+ku}. \qquad \text{(A9)}$$

With reference to (A4), (A8) and (A9), we obtain the final expression of $\chi^{(0)}$ as follows.

$$\chi^{(0)}=a\theta+\frac{1+ku}{ku}\Big\{F_0(u)+\frac{D^2(t)}{4k(1+ku)^2}-\frac{cN(t)}{1+ku}+(c-a)\theta+g_2\theta^2\Big\}. \qquad \text{(A10)}$$

By the way, let us ascertain that $\chi^{(0)}$ obtained above is actually expressed in a form of $\chi^{(0)}(u,v)$ proved at the beginning. The formula (A10) is rewriten as

$$\chi^{(0)}+\frac{1+ku}{ku}+\frac{cv}{ku}=-\frac{a\theta}{ku}+\frac{D^2}{4k^2u(1+ku)}+\frac{1+ku}{ku}g_2\theta^2;$$

the right-hand side of which is transformed into in succession

$$-\frac{a}{ku}\Big(v+\frac{N}{1+ku}\Big)+\frac{D^2}{4k^2u(1+ku)}+\frac{1+ku}{ku}g_2\Big(v+\frac{N}{1+ku}\Big)^2$$
$$=-\frac{a}{ku}v-\frac{aN}{ku(1+ku)}+\frac{D^2}{4k^2u(1+ku)}+\frac{1+ku}{ku}g_2v^2+\frac{2g_2N}{ku}v+\frac{g_2N^2}{ku(1+ku)}$$
$$=\frac{v}{ku}(g_{10}-c)+g_2\frac{1+ku}{ku}v^2+\frac{1}{ku(1+ku)}\Big(\frac{D^2}{4k}-aN+g_2N^2\Big),\quad (2g_2N-a=g_{10}-c).$$

But it is proved that $\Big(\frac{D^2}{4k}-aN+g_2N^2\Big)$ =const. or alternatively

$$\frac{D\dot{D}}{2k}-a\dot{N}-\dot{a}N+2g_2N\dot{N}=\dot{N}(a+g_{10}-c)-\dot{a}N=0,\ \text{that is},\ \frac{\dot{N}}{N}=\frac{\dot{a}}{(g_{10}-c)+a}=\frac{\dot{a}}{a-a_0},$$

since the last relation follows at once from $2g_2\dot{N}=\dot{a}$. Therefore, we have finally.

$$\chi^{(0)}=\frac{1+\kappa u}{\kappa u}F_0(u)-\frac{v}{\kappa u}(g_{10}+2a_0)+g_2\frac{1+\kappa u}{\kappa u}v^2+\left(\frac{D^2}{4\kappa}-aN+g_2N^2\right)\frac{1}{\kappa u(1+\kappa u)},$$
$$\frac{D^2}{4\kappa}-aN+g_2N^2=\text{const.},\tag{A11}$$

where all the parameters except u and v are constants, just as expected.

It may be worthy of notice that an expansion of G by a Fourier series, which satisfies $G(\theta,t)=G(\theta+2\pi,t)$, of course, provides no admissible term.

A2. Derivation of the expression for $\chi^{(1)}$.

A similar treatment to the case of $\chi^{(0)}$ is possible $\underbrace{}_{\text{for}}$ getting a common solution $\chi^{(1)}$ for both the partial differential equations (9") and (43), but inclusion of many additional terms to each equation makes calculations troublesome.

We rearrange (43) for sake of convenience as follows

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial r}\left(\frac{Y}{2H}\chi^{(1)}\right)\right\}=&\,a_0N\left\{\frac{1}{H^2}\left(\frac{1-Y}{Y}\right)(1+Y-2Y^2)b_0'-\frac{\kappa}{H^3}\left(\frac{1-Y}{Y}\right)^2(4-2Y^2)rh_{20}'\right\}\\
&+a_0\left\{\frac{1}{2H^2}(1-6Y+4Y^2)b_0+\frac{2\kappa}{H^3}\left(\frac{1-Y}{Y}\right)(1+Y-Y^2)rh_{20}\right\}\\
&+\frac{D^2}{\kappa}\left\{\frac{1}{4H^2}\left(\frac{1-Y}{Y}\right)(-1-Y+2Y^2+5Y^3-6Y^4)b_0'+\frac{\kappa}{2H^3}\left(\frac{1-Y}{Y}\right)^2(2-Y^2-2Y^3+3Y^4)rh_{20}'\right\}\\
&+\frac{DH}{\kappa}\left\{\frac{1}{2H^2}Y^2(-5+4Y)\left(\dot b_0-\frac{\dot H}{2H}b_0\right)+\frac{4\kappa}{H^3}Y(1-Y)^2r\left(\dot h_{20}-\frac{\dot H}{H}h_{20}\right)\right\}\\
&+\frac{1}{H}\left[\frac{H}{\kappa}Y(3-2Y)\left\{\ddot b_0'-\left(\frac{\ddot H}{2H}-\frac{\dot H^2}{4H^2}\right)b_0'\right\}+Y(2-Y)r\frac{\partial}{\partial t}\left(\dot h_{20}'-\frac{\dot H}{H}h_{20}'\right)\right]\\
&+\left\{F_0(u)-Y\frac{\partial F_0}{\partial Y}\right\}\left\{\frac{1}{H^2}\left(\frac{1-Y}{Y}\right)(-1-Y+2Y^2)b_0'+\frac{\kappa}{H^3}\left(\frac{1-Y}{Y}\right)^2(4-2Y)rh_{20}'\right\}\\
&+\frac{\partial^2F_0}{\partial Y^2}\left\{-\frac{2}{H^2}Y(1-Y)^3b_0'+\frac{2\kappa}{H^3}(1-Y)^4rh_{20}'\right\}.
\end{aligned}\tag{A12}$$

Its repeated integrations with respects to θ and r gives

$$\begin{aligned}
\chi^{(1)}=&\,\frac{a_0N}{H}\left\{\left(-\frac{2}{Y^2}+\frac{4}{Y}-2\right)rb_0+\left(\frac{4}{Y^2}+2\right)h_{20}\right\}\\
&+\frac{a_0}{H}\left\{\left(-\frac{1}{Y}+2\right)rb_0'+\left(-\frac{1}{2Y}\log u-\frac{1}{2}\right)h_{20}'\right\}\\
&+\frac{D^2}{\kappa H}\left\{\left(\frac{1}{2Y^2}-\frac{1}{Y}+\frac{1}{2}+\frac{Y}{2}-\frac{Y^2}{2}\right)rb_0+\left(-\frac{1}{Y^2}-\frac{1}{2}-\frac{Y}{2}+\frac{Y^2}{2}\right)h_{20}\right\}\\
&+\frac{D}{\kappa}\left\{Yr\left(\dot b_0'-\frac{\dot H}{2H}b_0'\right)-\frac{Y}{2}\left(\dot h_{20}'-\frac{\dot H}{H}h_{20}'\right)\right\}\\
&+\frac{H}{\kappa}\left[2r\left\{\ddot b_0-\left(\frac{\ddot H}{2H}-\frac{\dot H^2}{4H^2}\right)b_0\right\}+\left(\frac{1}{1-Y}+\frac{1}{Y}-1\right)\frac{\partial}{\partial t}\left(\dot h_{20}-\frac{\dot H}{H}h_{20}\right)\right]\\
&+\frac{1}{H}\left[\left\{F_0(u)\left(\frac{2}{Y^2}-\frac{4}{Y}+2\right)+\frac{\partial F_0}{\partial Y}\left(-\frac{2}{Y}+4-2Y\right)\right\}rb_0\right.\\
&\qquad\left.+\left\{F_0(u)\left(-\frac{4}{Y^2}+\frac{4}{Y}+2\right)+\frac{\partial F_0}{\partial Y}\left(\frac{2}{Y}-4+2Y\right)\right\}h_{20}\right]\\
&+\frac{1}{Y}\left\{F^{(1)}(r,t)+G^{(1)}(\theta,t)\right\},
\end{aligned}\tag{A13}$$

where $F^{(1)}$ and $G^{(1)}$ are arbitrary functions in the first order.

We substitute (A13) into ($\mathfrak{D}''$) and arrange the terms in the order of the power of Y with the use of the abbreviations such as

$$\dot{b}_\theta-\frac{\dot{H}}{2H}b_\theta\equiv\mathbb{A},\quad \frac{D}{H}b_\theta\equiv\mathbb{B}\,;\quad \dot{h}_{\omega\theta}-\frac{\dot{H}}{H}h_{\omega\theta}\equiv\mathbb{C},\quad \frac{D}{H}h_{\omega\theta}\equiv\mathbb{D}. \tag{A14}$$

Then it results that

$$\begin{aligned}
&\frac{r}{H}\Big[2\Big\{\frac{D_0}{4K}+\Big(F_0(Y)-Y\frac{\partial F_0}{\partial Y}\Big)\Big\}\Big(\mathbb{A}+\frac{\mathbb{B}'}{2}\Big)+a_0\Big(3\mathbb{A}'-\frac{\mathbb{B}}{2}\Big)\\
&\qquad+\frac{DH}{K}\Big(\dot{\mathbb{A}}'+\frac{\dot{H}}{2H}\mathbb{A}'\Big)+\frac{2H^2}{K}\frac{\partial}{\partial t}\Big(\dot{\mathbb{A}}+\frac{\dot{H}}{2H}\mathbb{A}\Big)+\frac{3H\dot{H}}{K}\Big(\dot{\mathbb{A}}+\frac{\dot{H}}{2H}\mathbb{A}\Big)\Big]\\
&+\frac{r}{HY}\Big[-4\Big\{\frac{D_0^2}{4K}+\Big(F_0(Y)-Y\frac{\partial F_0}{\partial Y}\Big)\Big\}\Big(\mathbb{A}+\frac{\mathbb{B}'}{2}\Big)-a_0\Big(\mathbb{A}'-\frac{\mathbb{B}}{2}\Big)\Big]\\
&+\frac{r}{HY^2}\Big[2\Big\{\frac{D_0^2}{4K}+\Big(F_0(Y)-Y\frac{\partial F_0}{\partial Y}\Big)\Big\}\Big(\mathbb{A}+\frac{\mathbb{B}'}{2}\Big)\Big]\\
&+\frac{1}{HY}\log u\Big[-a_0\Big(\frac{\mathbb{C}'}{2}-\mathbb{D}\Big)\Big]+\frac{1}{H}\log u\big[-a_0\mathbb{D}\big]+\frac{1}{H(1-Y)}\Big[\frac{H\dot{H}}{K}\dot{\mathbb{C}}+\frac{H^2}{K}\ddot{\mathbb{C}}\Big]\\
&+\frac{1}{H}\Big[-\frac{D_0^2}{K}\Big(\mathbb{C}-\frac{\mathbb{D}'}{4}\Big)-a_0\mathbb{C}'-\frac{DH}{2K}\dot{\mathbb{C}}'-\frac{H}{K}(\dot{H}\dot{\mathbb{C}}+H\ddot{\mathbb{C}})\\
&\qquad-4\Big(F_0(Y)-Y\frac{\partial F_0}{\partial Y}\Big)\Big(\mathbb{C}+\frac{\mathbb{D}'}{4}\Big)\qquad+H\Big(-\frac{D}{2H}\frac{\partial G^{(1)}}{\partial\theta}\Big)\Big]\\
&+\frac{1}{HY}\Big[\frac{D_0^2}{2K}\Big(\mathbb{C}+\frac{\mathbb{D}'}{2}\Big)\qquad+\frac{DH}{2K}\dot{\mathbb{C}}'+\frac{H}{K}(\dot{H}\dot{\mathbb{C}}+H\ddot{\mathbb{C}})\\
&\qquad+6\Big(F_0(Y)-Y\frac{\partial F_0}{\partial Y}\Big)\Big(\mathbb{C}+\frac{\mathbb{D}'}{2}\Big)\qquad+H\Big(\frac{\partial G^{(1)}}{\partial t}+\frac{D}{2H}\frac{\partial G^{(1)}}{\partial\theta}\Big)\Big]\\
&+\frac{1}{HY^2}\Big[-\frac{D_0^2}{K}\Big(\mathbb{C}+\frac{\mathbb{D}'}{2}\Big)\\
&\qquad+2\Big(F_0(Y)-Y\frac{\partial F_0}{\partial Y}\Big)\Big(\mathbb{C}+\frac{\mathbb{D}'}{2}\Big)\Big]\\
&+\frac{1}{H}\Big[\frac{\partial F_0}{\partial Y}\mathbb{D}'\Big]-\frac{2}{HY^2}\Big[F_0\Big(\mathbb{C}+\frac{\mathbb{D}'}{2}\Big)\Big]\qquad+\frac{1}{r}\Big[\frac{\partial F^{(1)}}{\partial t}+\frac{\dot{H}}{2H}r\frac{\partial F^{(1)}}{\partial r}\Big]=0.
\end{aligned} \tag{A15}$$

In order to make the above equation hold everywhere and always, it is necessary that any set of terms in every pairs of the brackets vanishes for an arbitrary value of θ. Let us get such a necessary condition for b_θ at first. If we put

$$\mathbb{A}+\frac{\mathbb{B}'}{2}=\dot{b}_\theta-\frac{\dot{H}}{2H}b_\theta+\frac{D}{2H}b_\theta'\equiv\mathbb{E},\quad \mathbb{A}'-\frac{\mathbb{B}}{2}=\dot{b}_\theta'-\frac{\dot{H}}{2H}b_\theta'-\frac{D}{2H}b_\theta\equiv\mathbb{E}',$$

then the coefficients of r/H are written as

$$\frac{2H^2}{K}\ddot{\mathbb{E}}+\frac{4H\dot{H}}{K}\dot{\mathbb{E}}+\Big\{\frac{1}{K}\Big(H\ddot{H}+\frac{\dot{H}^2+D_0^2}{2}\Big)-2a_0N+2\Big(F_0(Y)-Y\frac{\partial F_0}{\partial Y}\Big)\Big\}\mathbb{E}+a_0\mathbb{E}'=0,$$

while all the coefficients of both r/HY and r/HY^2 are expressible by some multiple of E or E'. Therefore, when

$$\mathbb{E} = \dot{b}_\theta - \frac{\dot{H}}{2H} b_\theta + \frac{D}{2H} b'_\theta = 0 \qquad \text{(A16)}$$

holds without any restriction except for $F_0 = F_0(u)$, $F^{(1)} = F^{(1)}(u)$ and $G^{(1)} = G_{b\theta}$:const.*, the whole terms concerning b_θ vanish, or in other words, an oscillatory motion of b_θ defined by (A16) occurs in reality.

Secondly as regards $h_{2\theta}$, the coefficients of $1/HY^2$ gives**

$$\mathbb{F} \equiv \mathbb{C} + \frac{\mathbb{D}'}{2} \equiv \dot{h}_{2\theta} - \frac{\dot{H}}{H} h_{2\theta} + \frac{D}{2H} l_{2\theta} = 0. \qquad \text{(A17)}$$

Then the equation (A15) becomes as follows by letting the terms for b_θ be aside

$$-a_0 \mathbb{D}\left(\log u + \frac{3}{1-Y} + \frac{2}{Y}\right) + \mathbb{D}'\left\{\frac{D^2}{2k(1-Y)} + \frac{3D_0^2}{4k} + F_0 + (1-Y)\frac{\partial F_0}{\partial Y}\right\}$$
$$+ \frac{H}{Y}\left\{\frac{\partial G^{(1)}}{\partial t} + \frac{D}{2H}(1-Y)\frac{\partial G^{(1)}}{\partial \theta}\right\} + \frac{H}{Y}\left(\frac{\partial F^{(1)}}{\partial t} + \frac{\dot{H}}{2H} Y \frac{\partial F^{(1)}}{\partial Y}\right) = 0, \qquad \text{(A18)}$$

where $G^{(1)}(\theta, t)$ is represented by

$$G^{(1)} = G_1(t)\sin 2\theta + G_2(t)\cdot\cos 2\theta,$$

since $\frac{\partial G^{(1)}}{\partial \theta}$ and $\frac{\partial G^{(1)}}{\partial t}$ have a periodicity of π as readily seen by taking differences for both cases of $\theta=0$ and $\theta=2\pi$ with respect to the coefficient of $1/H$ or $1/HY$ respectively. Hence (A18) is rewritten as

$$\mathbb{D}\left\{-a_0\left(\log u + \frac{3}{1-Y}\right) + \frac{1}{Y}\left(\dot{G}_1 H + G_1 \frac{a_0 K}{D} - 2a_0\right) + 2G_2 D\right\} + \frac{H}{Y}\left(\frac{\partial F^{(1)}}{\partial t} + \frac{\dot{H}Y}{2H}\frac{\partial F^{(1)}}{\partial Y}\right)$$
$$+ \mathbb{D}'\left\{\left(\frac{3D_0^2}{4K} - \frac{G_1 D}{2}\right) + \frac{1}{Y}\left(\dot{G}_2 H + G_2 \frac{a_0 K}{D}\right) + \frac{D^2}{2K(1-Y)} + (1-Y)^2 \frac{\partial}{\partial Y}\left(\frac{F_0(Y)}{1-Y}\right)\right\} = 0. \qquad \text{(A18')}$$

We have, therefore, from the coefficients of $\mathbb{D}$

$$a_0 = 0 \; (D = D_0 : \text{const.}), \quad G_2 = 0, \quad G_1 : \text{const.}, \qquad \text{(A19)}$$

and from the coefficients of $\mathbb{D}'$

$$\frac{\partial}{\partial Y}\left(\frac{F_0}{1-Y}\right) + \frac{1}{(1-Y)^2}\left(\frac{3D_0^2}{4K} - \frac{G_1 D_0}{2}\right) + \frac{D_0^2}{2K(1-Y)^3} = 0$$

---

* In order to satisfy $\frac{\partial^2 \mathfrak{H}_{b2}}{\partial Y \partial \theta} = \frac{\partial^2 \mathfrak{H}_{b2}}{\partial \theta \partial Y}$, $G^{(1)}$ should be a linear function of $v = \theta - \frac{N}{1+Ku}$, nevertheless, if $G^{(1)}$ contains v, a discontinuity as mentioned in §2 occurs at $\theta = 2\pi$. We can put $G^{(1)} \doteq 0$ since $G_{b\theta}$ is included in $F^{(1)}(u)$.

** No care is necessary here about that F_0 in the second term of the last line of (A15) is an arbitrary function of Y .

integration of which results

$$F_0(Y) = -\frac{D_0^2}{4K}\frac{4-3Y}{1-Y} + \frac{G_1 D_0}{2} + G_3(1-Y), \qquad G_1, G_3: \textit{arbitrary consts.,} \tag{A20}$$

while lastly from the remaing terms without θ we have

$$F^{(1)}(r,t) = F^{(1)}(u), \quad u \equiv \frac{r^2}{H(t)}. \tag{A21}$$

Thus, the $h_{2\theta}$ -oscillation occurs when $\mathbb{F}=0$ is satisfied under the restrictions such as $D=D_0$: const., $G^{(1)}=G_1\mathbb{D}$(G_1:const.) and F_0 being in a form of (A20).

As a final form of $\chi^{(1)}$, we obtain with references to (A13), (A16), (A17), (A19), (A20), and (A21) as follows

$$\begin{aligned} \chi_b^{(1)} &= \left(\frac{1-Y}{Y}\right)^2 \left\{-2a_0 N + \frac{D^2}{2K}(1-Y^2) + 2\left(F_0(Y) - Y\frac{\partial F_0}{\partial Y}\right)\right\}\frac{Y b_{2\theta}}{H} - a_0\left(\frac{1-Y}{Y}\right)\frac{Y b_{2\theta}'}{H} + \frac{1}{Y}F^{(1)}(u), \\ \chi_h^{(1)} &= \frac{1}{Y^2}\left\{\frac{D_0^2}{2K}\left(7 - 8Y + 4Y^2 - 3Y^3 + Y^4 - \frac{1}{1-Y}\right)\right. \\ &\qquad \left. - G_1 D_0(2-2Y+Y^2) - 2G_3(2-3Y+Y^2)\right\}\frac{h_{2\theta}}{H} + \frac{1}{Y}F^{(1)}(u). \end{aligned} \tag{A22}$$

where H, D, N are the zeroth order functions of t while K, G_1, G_2 are the first order constants.

References

Chandrasekhar, S., 1940, Principles of Stellar Dynamics, The University of Chicago Press.

Contopoulos, G., 1969, Gravitational Theories of Spiral Structure, University of Chicago.

Kurth, R., 1957, Introduction to the Mechanics of Stellar Systems, Pergamon Press.

Lin, C.C. and Shu, F.H., 1964, Ap.J., 140, 646.

Lin, C.C., Yuan, C., and Shu, F.H., 1969, Ap.J., 155, 721.

Lynden-Bell, D. and Ostricker, J.D., 1967, M.N.R.A.S., 136, 273.

Miyamoto, M., 1969, Publ.Ast.Soc.Jap., 21, 319.

Shu, H., 1970, Ap.J., 160, 89;99.

Yabushita, S., 1969, M.N.R.A.S., 141, 231.