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SOME TOPICS IN THE THEORY OF OPERATOR ALGEBRAS

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§1 Asymptotic ratio set

Powers [14] has shown that a family of factors R_x, $0 \leqq x \leqq 1$ are mutually non* isomorphic. Araki and Woods [1] have introduced an asymptotic ratio set $r_\infty(R)$ for a W*algebra R as the set of all $x \in [0,\infty)$ such that $R \sim R \otimes R_x$ (*isomorphism) where $R_{(x^{-1})} = R_x$ for $x \neq 0$. $r_\infty(R) \cap (0,\infty)$ is a group. For R on a separable space, $r_\infty(R)$ is closed and is one of the following sets: S_ϕ = empty, $S_0 = \{0\}$, $S_1 = \{1\}$, $S_{01} = \{01\}$, $S_x = \{x^n;\ n \in Z\} \cup \{0\}$ for $0 < x \neq 1$ and $S_\infty = [0,\infty)$. [2,5,14]

Among the countable infinite tensor products of type I factors on separable spaces (abreviated as ITPEI's), there exists a unique R_∞ with $r_\infty(R_\infty) = S_\infty$, R_x satisfies $r_\infty(R_x) = S_x$ and $r_\infty(R_0 \otimes R_1) = S_{01}$. If an ITPFI R satisfies $r_\infty(R) = S_x$, $0 \leqq x \leqq \infty$, then $R \sim R_x$. If it is type II_∞ and satisfies $r_\infty(R) = S_{01}$, then $R \sim R_0 \otimes R_1$. Let (a,b) denote ∞ if a/b is irrational and the largest c if a and b are integer multiplies of c. For $x,y \in (0,\infty)$, $R_x \otimes R_y \sim R_z$ with $\log z = (\log x, \log y)$. $R_\infty \otimes R_x \sim R_\infty$ for any ITPFI R .

According to Powers, R has the property L_λ if for any $\varepsilon > 0$ and any normal state ω of R, there exists $N \in R$ such that $N^2 = 0$, $N^*N + NN^* = 1$ and $|(1-\lambda)\omega(QN) - \lambda\omega(NQ)| \leq \varepsilon\|Q\|$ for all $Q \in R$. We say that R has the property L'_λ if the statement holds for any finite collection of normal states $\omega_1 \cdots \omega_n$ of R. [5] R on a separable space has the property L'_λ if and only if $\lambda/(1-\lambda) \in r_\infty(R)$ $(0 \leq \lambda \leq 1/2)$. Any finite continuous von Neumann algebra has the property $L_{1/2}$. Property L_0, Property L'_0, $1 \in r_\infty(R)$ and R being properly infinite are equivalent. The property L_λ for R implies that R is purely infinite if $0 < \lambda < 1/2$, R is continuous if $\lambda = 1/2$.

The Property $L'_{1/2}$ for a finite von Neumann algebra on a separable space can be rephrased as the existence of a weakly central sequence of type I_2 factors and hence its equivalence with $R \sim R \otimes R_1$ is somewhat stronger that a result of McDuff.

If R does not have the property L , $r_\infty(R) = S_\phi$ or S_0 according as R is finite or infinite. (cf. [17]).

For ITPFI, there are no R with $r_\infty(R) = S_\phi$ and no purely infinite R with $r_\infty(R) = S_0$. All ITPFIexcept R_0 (type I_∞) has 1 in $r_\infty(R)$ and hence the property L . Question: do all hyperfinite factors share these properties. (There are non-hyperfinite counter-examples.)

There exist uncountably many $ITPFI_2$ with $r_\infty(R) = S_{01}$. Araki and Woods have introduced another invariant $\rho(R)$ which is the set of $x \in [0,\infty)$ such that $R \otimes R_x \sim R_x$. This invariant separates some ITPFI in the class S_{01}. Woods has shown that $\rho(R)$ has Lebesgue measure 0 for any type III ITPFI_2 (countable infinite tensor product of type I_2 factors).

Krieger [10,11,12] has constructed for every $x \in (0,1)$, an ITPFI such that $r_\infty(R) = S_{01}$ and $R \otimes R \sim R_x$, hyperfinite factors $A_{x,p}$, $1/2 \leq p < 1$ such that $r_\infty(A_{x,p}) = S_{01}$, $A_{x,p} \otimes A_{x,p} \sim R_x$ and $A_{x,p} \not\sim A_{x,q}$ for $p \neq q$, and a hyperfinite factor such that $r_\infty(R) = S_{01}$ and $R \otimes R \sim R_\infty$.

Williams [18] has shown that $A \otimes R_x$ does not have the property L_λ except for $\lambda/(1-\lambda) \in S_x$ if A has a restricted semifinite part. Using the free group with two generators, he has non hyperfinite $A \otimes R_x$ with $r_\infty(A \otimes R_x) = S_x$, $0 \leq x \leq 1$. (See also [8], [16].) He has also whown that a countable ITP of finite factors is * isomorphic to $F \otimes I$ where F is finite, I is an ITPFIand if $I \neq R_0$, F is an ITP of given finite factors with respect to cyclic trace vectors.

Nielsen [13] has shown that any W*-algebra R on a separable space has a unique decomposition $R = R_{(\phi)} \oplus R_{(01)} \oplus R_{(\infty)} \oplus \int_0^1 d\mu(x) R_{(x)}$, coarser than the central decomposition, where μ is a Borel measure on $[0,1]$ and $R_{(\alpha)}$ is of pure type S_α in the sense that almost all factors in its central decomposition have the asymptotic ratio set S_α.

§2 Representations of the CCR (canonical commutation relations)

For an isomorphism ϕ of a group G into topological group H, there exists the weakest group topology on G which makes ϕ continuous. It can serve as an invariant in the classification of representations. For a unitary representation U of a real vector space V_ϕ, such that $U(\lambda f)$ is continuous in $\lambda \in R$ for $f \in V_\phi$, the weakest vector topology τ_ϕ, making U continuous, is given by a collection of distances $d_\Phi(f) = \sup_{0<\lambda<1} \|\{U(\lambda f)-1\}\Phi\|$. It is the weakest group topology on V_ϕ making the representation of $f \in V_\phi$ by the infinitesimal generator $\phi(f)$ of $U(\lambda f) = \exp i\lambda\phi(f)$ continuous relative to the topology of resolvent convergence. If Ω is separating for $\{U(f)\}''$, then τ_ϕ is metrizable by d_Ω. [7,9,19].

If a unitary operator $V(g)$ for some $g \in V_\phi^*$ (algebraic dual) satisfies $V(g)U(f)V(g)^*U(f)^* = \exp ig(f)$ and $|(\Phi, V(g)\Phi)| > 1/2$, then $d_\Phi(f)^{-1}g(f)$ is uniformly bounded for $d_\Phi(f) < 1/2$. Hence $g(f)$ is τ_ϕ continuous in f. If a subspace $V_\pi \subset V_\phi^*$ has a unitary representation $V(g)$, $g \in V_\pi$ having this commutation property, the pair U,V is called a representation of CCR over V_ϕ, V_π. The above boundedness implies the non-existence of a representation of CCR over V_ϕ, V_ϕ^*. Hence V_ϕ^* does not have any V_ϕ^*-quasi-invariant measures-a special case of a known result.

A representation U of V_ϕ can be extended uniquely to the topological completion $\overline{(V_\phi, \tau_\phi)}$. If there exists a separating vector in the common domain of $\phi(f)$, $f \in \overline{V}_\phi$, then τ_ϕ is a Hilbert space topology. If a pair V_ϕ, $V_\pi (\subset V_\phi^*)$ can be imbedded in a real Hilbert space algebraically

$(g(f) = (g,f))$, then it has a representation of CCR. The converses of both statements hold for a countable infinite tensor product of Schrodinger representations of one dimensional CCR.

One usually requires that $U(\lambda f)$ and $V(\lambda g)$ are continuous in λ . If B is a σ-field generated by cylinder sets over V_ϕ in V_ϕ^* , μ is a V_π-quasi-invariant measure on (V_ϕ^*, B) , $H_\mu = L_2(V_\phi^*, B, \mu)$, $U_\mu(f)$ is multiplication $\exp i\xi(f)$, $\xi \in V_\phi^*$ and $[V_\mu(g)\psi](\xi) = [d\mu(\xi+g)/d\mu(\xi)]^{1/2}\psi(\xi+g)$, then $U_\mu(\lambda f)$ and $V_\mu(\lambda g)$ are continuous in λ . [4] If V_ϕ and V_π are finite linear span of countable dual bases, then all multipliers (first order cocycles) can be explicitly given and hence concrete structure of all representations are known.

One usually requires in addition that the bilinear form $(g,f) = g(f)$ on $V_\pi \times V_\phi$ be nondegenerate (V_π and V_ϕ separate each other). It can be uniquely extended to $V_\pi \times V_\phi$ (the closure relative to $\tau_\pi \times \tau_\phi$) but may fail to be non-degenerate. We call a representation of CCR closable or non-closable according as (g,f) is nondegenerate or not on $V_\pi \times V_\phi$.

§3 Quasiequivalence criterion for quasifree states.

Let K be a complex linear space, Γ an involution of K , and γ a nondegenerate hermitian form on K satisfying $\gamma(\Gamma h, \Gamma h') = \sigma\gamma(h',h)$, σ = +or-(CAR or CCR). For $\sigma = +, \gamma(h,h) > 0$ is assumed for $h \neq 0$. $\mathfrak{A}(K,\Gamma,\gamma)$ denotes a free $*$ algebra over the symbols $B(f)$, $f \in K$ adjoined by an identity 1 and divided by the two-sided $*$ ideal generated by $B(cf+dg) - CB(f) - dB(g)$, $B(f)^* - B(\Gamma f)$, $B(f)^*B(g) + B(g)B(f)^* - \gamma(f,g)1$.

Any state ϕ defines a hermitian form $S(f,g)=\phi(B(f)^*B(g))$, satisfying $S(f,g) + \sigma S(\Gamma g,\Gamma f) = \gamma(f,g)$ and $S(f,f) \geqq 0$. Conversely, there exists a unique quasifree state ϕ_S giving rise to any such S. In the associated representation π_S, $\pi_S(B(f))$ for $f \in \mathrm{Re}K = \{h;\Gamma h = h\}$ is essentially selfadjoint (bounded for $\sigma = +$) and defines an induced vector topology τ_S on ReK and hence on K. It is given by a positive definite form $(f,g)_S \equiv S(f,g) + S(\Gamma g,\Gamma f)$. Let $\overline{K}_S = \overline{(K,\tau_S)}$, $S(f,g) = (f,Sg)_S$, $f,g \in \overline{K}_S$. Then $1 \geqq S \geqq 0$. For $\sigma = -$, the representation is closable if and only if $1/2$ is not a discrete eigenvalue of S, which we shall assume.

If S is a projection, ϕ_S is called a Fock state. Any ϕ_S is a restriction of a Fock state $\phi_{P(S)}$ of $\mathfrak{A}(\hat{K},\hat{\Gamma},\hat{\gamma})$, $\hat{K} = K \oplus K$, $\hat{\Gamma} = \Gamma \oplus - \sigma\Gamma$, $\hat{\gamma} = \gamma \oplus \sigma\gamma$. ϕ_S and $\phi_{S'}$, are quasiequivalent [3,6] if and only if (1) $\tau_S \sim \tau_{S'}$, which implies $\tau_{P(S)} \sim \tau_{P(S')}$ on $\hat{K}$, and (2) $P(S) - P(S')$ is in the Hilbert Schmidt class relative to any Hilbert space norm equivalent to $\tau_{P(S)}$. For $\sigma = +$, (1) always holds and (2) is equivalent to $S^{1/2} - (S')^{1/2}$ being in the HS class. If ϕ_S and $\phi_{S'}$ are gauge invariant (relative to $K = L \oplus \Gamma L$), then $S = S_1 \quad (1-S_1)$, $S' = S_1' \oplus (1-S_1')$ and the result agrees with [15].

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