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ON DEFORMATIONS OF HOLOMORPHIC MAPS

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§0. Introduction

The modern deformation theory has started with the splendid work of Kodaira-Spencer [1] followed by [2][3]. Moreover Kodaira has investigated families of submanifolds of a fixed compact complex manifold in [4]. In this paper the author propose to consider deformations of the structure "a compact complex manifold X plus a holomorphic map f into a fixed compact complex manifold Y". The fundamental restriction is that f is non-degenerate at some point or equivalently that the image f(X) has the same dimension as X. For this structure we can find the space of infinitesimal deformations $H^0(X, \mathcal{T})$ (for the definition of $\mathcal{T}$

see §1), and the obstructions for constructing a universal family (in the sense of Kodaira-Spencer) is in $H^1(X, \mathcal{T})$. The author has proved two fundamental theorems corresponding to the results of [2][3]. When f is an embedding this is nothing but the theory of displacements of Kodaira [4].

In addition, the same method as in the proof of the existence theorem can be applied to give a sufficient condition for the existence of a holomorphic map $\Phi : \mathcal{X} \longrightarrow \mathcal{Y}$ of families of compact complex manifolds extending $f: X \longrightarrow Y$. As an application of this result, we can prove that any sufficiently small deformation X_t of a monoidal transformation X of Y with non-singular center D is a monoidal transformation of a deformation Y_t of Y with non-singular center D_t.

Recently the author has succeeded in constructing the Kodaira-Spencer theory for families of holomorphic maps into a fixed family ($\mathcal{Y}$, q, S)of compact complex manifolds.

Throughout this paper, the ideas essentially belong to Professor Kodaira.

§1. Infinitesimal deformations

By a family of holomorphic maps into a fixed compact complex manifold Y, we mean a quadruplet $(\mathfrak{X}, \Phi, p, M)$ of complex manifolds $\mathfrak{X}$, M and holomorphic maps $\Phi : \mathfrak{X} \longrightarrow \mathfrak{Y} = Y \times M$, $p : \mathfrak{X} \longrightarrow M$ with following properties:

i) p is a surjective smooth proper holomorphic map,

ii) $q \circ \Phi = p$, where $q : \mathfrak{Y} \longrightarrow M$ is the projection onto the second factor.

We define the concept of completeness (as a family of holomorphic maps into Y) as in the theory of deformations of compact complex manifolds[1].

Let $(\mathfrak{X}, \Phi, p, M)$ be a family of holomorphic maps into Y, $o \in M$, $X = X_o = p^{-1}(o)$ and $f = \Phi_o : X \longrightarrow Y$. With only exception §3, we assume that f is non-degenerate. Then we have an exact sequence of sheaves on X:

$$0 \longrightarrow \Theta_X \xrightarrow{F} f^*\Theta_Y \xrightarrow{P} \mathcal{J} \longrightarrow 0$$

where Θ denotes the sheaf of germs of holomorphic vector fields, $\mathcal{J}$ is the cokernel of the canonical homomorphism F and P is the natural projection.

We investigate only "the germs of deformations". Restricting M on a neighborhood of o if necessary, we may assume that M is an open set in $\mathbb{C}^r$ with coordinates $t=(t_1,\dots,t_r)$ and that the prescribed point o is $(0,\dots,0)$. Taking a system of coordinates $(z^1,\dots,z^n,t_1,\dots,t_r)$ (resp. $(w^1,\dots,w^m)$) on $\mathfrak{X}$ (resp. on Y), we write explicitely $w=\Phi(z,\ t)$. Now we can define a linear map

$$\tau : T_o(M) \longrightarrow H^0(X, \mathcal{T})$$

(where $T_o(M)$ is the tangent space of M at o) by the formula

$$\tau\left(\frac{\partial}{\partial t}\right) = P\left(\sum \frac{\partial \Phi^\sigma}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial w^\sigma}\right) .$$

This is well defined and independent of the choice of local coordinates.

<u>Proposition</u> With notations as above, let ρ be the Kodaira-Spencer map for the deformation $(\mathfrak{X},\ p,\ M)$ of $X=X_o$, then the diagram

$$\begin{tikzcd} T_o(M) \arrow[r, "\tau"] \arrow[dr, "\rho"'] & H^0(X, \mathcal{T}) \arrow[d, "\delta"] \\ & H^1(X, \Theta_X) \end{tikzcd}$$

is commutative, where δ is the coboundary map of cohomology groups.

§2. Fundamental theorems

Following Kodaira-Spencer-Nirenberg we can prove:

<u>Theorem of completeness</u> Let $(\mathcal{X}, \Phi, p, M)$ be a family of non-degenerate holomorphic maps into Y, $o \in M$, $X = X_o$ and $f = \Phi_o : X \longrightarrow Y$. If

$$\tau : T_o(M) \longrightarrow H^0(X, \mathcal{T})$$

is surjective, then the family is complete at o.

<u>Existence theorem</u> Let $f : X \longrightarrow Y$ be a non-degenerate holomorphic map. If $H^1(X, \mathcal{T}) = 0$, then there exists a family $(\mathcal{X}, \Phi, p, M)$ of holomorphic maps into Y and a point $o \in M$ such that

i) $\Phi_o : X_o \longrightarrow Y$ is equivalent to $f : X \longrightarrow Y$,

ii) $\tau : T_o(M) \longrightarrow H^0(X, \mathcal{T})$ is bijective.

§3. Extension of a holomorphic map

As a counterpart to the existence theorem we can prove:

<u>Extension theorem</u> Let $f : X \longrightarrow Y$ be a holomorphic map (not

necessarily non-degenerate). Suppose that

i) $f^*: H^1(Y, \Theta_Y) \longrightarrow H^1(X, f^*\Theta_Y)$ is surjective,

ii) $f^*: H^2(Y, \Theta_Y) \longrightarrow H^2(X, f^*\Theta_Y)$ is injective.

Then for any family $p: \mathfrak{X} \longrightarrow M$ of deformations of X with $X_o = X$, there exist an open neighborhood N of o in M, a complex analytic family $q: \mathfrak{Y} \longrightarrow N$ with $Y_o = Y$ and a holomorphic map $\Phi: \mathfrak{X}_{|N} \longrightarrow \mathfrak{Y}$ which satisfies $p = q \circ \Phi$ and coincides with f on fibres over $o \in M$.

From this follow two theorems:

<u>Stability of fibre structures</u> If $f: X \longrightarrow Y$ is a holomorphic map such that

$$f_* \mathcal{O}_X = \mathcal{O}_Y \quad \text{and} \quad R'f_* \mathcal{O}_X = 0$$

then the fibre structure is stable (cf.[Kodaira 5]).

<u>Equiblowing-down</u> Let $f: X \longrightarrow Y$ be a monoidal transformation with non-singular center D, $p: \mathfrak{X} \longrightarrow M$ be a family of deformations of $X = X_o$ with $o \in M$. Then there exist a open neighborhood N of o in M, a complex analytic family $q: \mathfrak{Y} \longrightarrow N$ with $Y = Y_o$, a sub-manifold $\mathfrak{D} \subset \mathfrak{Y}$ and a holomorphic map $\Phi: \mathfrak{X} \longrightarrow \mathfrak{Y}$ satisfying:

i) $q\circ\Phi=p$,

ii) X_t is the monoidal transformation with non-singular center $D_t=\mathfrak{D}\cap q^{-1}(t)$.

§4. Generalization

Let $(\mathcal{Y}, q, S)$ be a fixed family of compact complex manifolds. By a family of holomorphic maps into $(\mathcal{Y}, q, S)$, we mean a quintuplet $(\mathcal{X}, \Phi, p, M, s)$ of complex manifolds $\mathcal{X}$, M and holomorphic maps $\Phi:\mathcal{X}\longrightarrow\mathcal{Y}$, $s:M\longrightarrow S$ with following properties:

i) p is a surjective smooth proper holomorphic map,

ii) $s\circ p=q\circ\Phi$.

We define the concept of completeness (as a family of holomorphic maps into $(\mathcal{Y}, q, S)$) as usual.

Let $o\in M$, $o'=s(o)$, $X=X_o$, $Y=Y_{o'}$, and let $f=\Phi_o:X\longrightarrow Y$ be the holomorphic map induced by Φ. We assume that f is non-degenerate. In order to define the characteristic map we need something C^∞. For any locally free sheaf E we denote by $\mathcal{A}^{0,q}(E)$ the sheaf of germs of C^∞-differentiable $(0,q)$-forms with coefficients in E, and let $A^{0,q}(E)=H^0(\mathcal{A}^{0,q}(E))$. Moreover let

$$\mathcal{a}^{0,q}(\mathcal{J}) = \mathcal{a}^{0,q}(f^*\Theta_Y)/\mathcal{a}^{0,q}(\Theta_X)$$

$$A^{0,q}(\mathcal{J}) = H^0(X,\ \mathcal{a}^{0,q}(\mathcal{J})).$$

Then $(A^{0,*}(\mathcal{J}),\ \bar{\partial})$ forms a complex and we have "Dolbeault isomorphisms"

$$H^p_{\bar{\partial}}(A^{0,*}(\mathcal{J})) \cong H^p(X,\ \mathcal{J}).$$

Now we may assume that M (resp.S) is an open set in $\mathbb{C}^r$ (resp.in $\mathbb{C}^{r'}$) with a system of coordinates $(t^1,\dots,t^r)$ (resp. $(s^1,\dots,s^{r'})$) and o (resp. o') is $(0,\dots,0)$. We regard $\mathfrak{X}$ (resp. $\mathfrak{Y}$) as a differentiable manifold $X\times M$ (resp. $Y\times S$) and suppose that the complex structure $\mathfrak{X}$ (resp. $\mathfrak{Y}$) is given by a vector (0,1)-form $\varphi(t)$ (resp. $\psi(s)$). First we define a linear map τ' as the composition

$$\tau' : T_o(S) \xrightarrow{\rho'} A^{0,1}(\Theta_Y) \xrightarrow{f^*} A^{0,1}(f^*\Theta_Y) \xrightarrow{P} A^{0,1}(\mathcal{J})$$

where ρ' is the Kodaira-Spencer map for the family $(\mathfrak{Y},\ q,\ S)$. Taking a system of coordinates $(z^1,\dots,z^n)$ (resp. $(w^1,\dots,w^m)$) on X (resp. on Y), we write Φ explicitely

$$w = \Phi(z,\ t), \quad s = s(t)$$

as a differentiable map from $X\times M$ to $Y\times S$. Then

$$\tau_t = \sum \frac{\partial \Phi^\sigma}{\partial t}\Big|_{t=0} \frac{\partial}{\partial w^\sigma}$$

defines an element in $A^{0,0}(f^*\Theta_Y)$ and satisfies the equality

$$(*) \qquad \bar{\partial}\tau_t - F(\rho(\frac{\partial}{\partial t})) + f^*(\frac{\partial s^\omega}{\partial t}\Big|_{t=0} \rho'(\frac{\partial}{\partial s^\omega})) = 0,$$

where ρ is the Kodaira-Spencer map for the family ($\mathfrak{X}$, p, M).

Let $\qquad D_{X/\mathfrak{Y}} = \bar{\partial}^{-1}(\tau'(T_o(S)) \subset A^{0,0}(\mathcal{T})$

$$\widetilde{D}_{X/\mathfrak{Y}} = \{(\tau, \theta) \in D_{X/\mathfrak{Y}} \times \mathbb{C}^{r'} \mid \bar{\partial}\tau = Pf^*(\theta^\omega \rho'(\frac{\partial}{\partial s^\omega}))\}.$$

Then by the equality (*), we can define a linear map

$$\tilde{\tau} : T_o(M) \longrightarrow \widetilde{D}_{X/\mathfrak{Y}}$$

by $\qquad \tilde{\tau}(\frac{\partial}{\partial t}) = (P\tau_t, \frac{\partial s^\omega}{\partial t}).$

With these preparations, we can state the fundamental theorems:

<u>Theorem of completeness</u> Let ($\mathfrak{X}$, Φ, p, M, s) be a family of holomorphic maps into a family ($\mathfrak{Y}$, q, S). With notations as above, assume that f is non-degenerate. If the map

$$\tilde{\tau} : T_o(M) \longrightarrow \widetilde{D}_{X/\mathfrak{Y}}$$

is surjective, then the family ($\mathfrak{X}$, Φ, p, M, s) is complete at o.

<u>Existence theorem</u> Let $f: X \longrightarrow Y$ be a non-degenerate

holomorphic map and $(\mathfrak{Y}, q, S)$ be a family of deformations $Y = Y_{o'}$ with $o' \in S$. Assume that the composition

$$\tau' : T_{o'}(S) \xrightarrow{\rho'} H^1(Y, \Theta_Y) \xrightarrow{f^*} H^1(X, f^*\Theta_Y) \xrightarrow{P} H^1(X, \mathcal{J})$$

is surjective, then there exists a family $(\mathfrak{X}, \Phi, p, M, s)$ of holomorphic maps into $(\mathfrak{Y}, q, S)$ and a point $o \in M$ with $s(o) = o'$ such that

i) $\Phi_o : X_o \longrightarrow Y_{o'}$ coincides with $f : X \longrightarrow Y$,

ii) $\tilde{\tau} : T_o(M) \longrightarrow \tilde{D}_{X/\mathfrak{Y}}$ is bijective.

§5. Remarks

1) We can prove two fundamental theorems when f is not necessarily non-degenerate.

2) We can prove a extension theorem (or it should be called a theorem of "costability") in the relative case.

3) As an application, we can give an example of algebraic manifolds with ample canonical bundle, for which the deformation problem is obstructed.

4) Let X be an algebraic manifold such that

i) the canonical bundle is ample, and

ii) the albanese map is an embedding.

Then the deformation problem for X is unobstructed.

For the formulation of theorems mentioned above, see a forthcoming paper [6]. Details will be published elsewhere.

References

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