

Supplement to 'Minimal class generated by open compact  
and perfect mappings'

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This is a supplementary note on my lecture[4] at Kyoto Symposium on Topological Spaces, Dec.2, 1971. A space is called a weak p-space if it is regular( $T_1 + T_3$ ) and has a sequence  $\mathcal{U}_i$ ,  $i=1,2,\dots$ , of open coverings of  $X$  satisfying: If  $x \in U_i \in \mathcal{U}_i$ ,  $i=1,2,\dots$ , then i)  $\bigcap_{i=1}^{\infty} \overline{U}_i$  is compact and ii)  $\bigcap_{i=1}^{\infty} \overline{U}_i \subset U$  with  $U$  open implies  $\bigcap_{i=1}^{\infty} \overline{U}_i \subset \bigcap_{i=1}^n \overline{U}_i \subset U$  for some  $n$ . This sequence  $\{\mathcal{U}_i\}$  is called a defining one. This condition was introduced by Burke[2], Theorem 1.3, where he proved that a completely regular space is a p-space in the sense of Arhangel'skii[1], Definition 5, if and only if it is a weak p-space. However the space  $T$  in Engelking[3], Example 2.4.4, p.85, is not a p-space but a weak p-space, which proof is left to the reader. Since every Moore space is evidently a weak p-space, the class of weak p-spaces offers a class containing all p-spaces and all Moore spaces. The property to be a weak p-space is inherited under the operations taking  $G_\delta$  sets, closed sets, perfect preimages and countable products. The aim of this paper is to note that wherever 'p-spaces' are in [4] is replaced by 'weak p-spaces'. That can easily be verified with the aid of the following Lemmas 1 and 2 and

with trivial minor changes of the original proofs in [4]. For undefined terminologies refer to [4].

LEMMA 1. Let X be a weak p-space with a deriving sequence  
 $\{ \mathbb{U}_i = \{ U_{i\alpha} : \alpha \in A_i \} : i=1,2,\dots \}$ . Let  $B_i$  be a finite subset  
of  $A_i$  and  $\varphi_i^{i+1}: B_{i+1} \rightarrow B_i$  a transformation such that  $\varphi_i^{i+1}(\alpha)$   
 $= \beta$  implies  $\overline{U_{i+1,\alpha}} \subset U_{i\beta}$  and such that  $\langle \alpha_i \rangle \in \text{inv lim } \{ B_i;$

$\varphi_i^{i+1} \}$  implies  $\bigcap_{i=1}^{\infty} U_{i\alpha_i} \neq \emptyset$ . Set

$$U_i = \cup \{ U_{i\alpha} : \alpha \in B_i \}, \quad K = \cap U_i.$$

Then K is compact and  $\{ U_i \}$  forms a neighborhood base of K in X.

Proof. It suffices to consider the case:  $K \neq \emptyset$ . Let  $\mathbb{F}$  be a maximal filtre of subsets of K. Set

$$C_i = \{ \alpha \in B_i : U_{i\alpha} \cap K \in \mathbb{F} \}.$$

Then  $C_i \neq \emptyset$  and  $\{ C_i \}$  forms an inverse subsystem of  $\{ B_i \}$ .

Pick an element  $\langle \alpha_i \rangle$  from  $\text{inv lim } C_i$ . Set

$$L = \bigcap_{i=1}^{\infty} U_{i\alpha_i}.$$

Then L is a non-empty compact set with  $L \subset K$ . Let F be an arbitrary element of  $\mathbb{F}$ . Assume  $\overline{F} \cap L = \emptyset$ . Then  $\overline{F} \cap U_{j\alpha_j} = \emptyset$  for some j, a contradiction. Thus a point of L adheres  $\mathbb{F}$ , which proves that K is compact.

To prove  $\{ U_i \}$  forms a neighborhood base of K in X assume the contrary. Let U be an open set of X with  $K \subset U$  and with  $U_i - U \neq \emptyset$  for any i. Set

$$D_i = \{ \alpha \in B_i : U_{i\alpha} - U \neq \emptyset \}.$$

Then  $D_i \neq \emptyset$  and  $\{ D_i \}$  forms an inverse subsystem of  $\{ B_i \}$ .

Pick an element  $\langle \beta_i \rangle$  from  $\text{inv lim } D_i$ . Set

$$M = \bigcap_{i=1}^{\infty} U_{i\beta_i}.$$

Since  $M \subset K$ , then  $U_{k\beta_k} \subset U$  for some  $k$ , a contradiction. The proof is finished.

LEMMA 2. A weak p-space X is of countable type.

Proof. Let  $Q$  be a non-empty compact set of  $X$  and  $\{\mathbb{V}_i\}$  a defining sequence of open coverings of  $X$ . Let  $\mathbb{U}_i = \{U_{i\alpha} : \alpha \in A_i\}$  be an open covering of  $X$  such that i)  $\mathbb{U}_i$  refines  $\mathbb{V}_i$ , ii)  $\overline{\mathbb{U}_{i+1}}$  refines  $\mathbb{U}_i$ , and iii) all but a finite number of elements of  $\mathbb{U}_i$  do not meet  $Q$ . Then  $\{\mathbb{U}_i\}$  is also a defining one. Let  $\varphi_i^{i+1}: A_{i+1} \rightarrow A_i$  be a transformation such that

$$\varphi_i^{i+1}(\alpha) = \beta \text{ implies } \overline{U_{i+1,\alpha}} \subset U_{i\beta}.$$

$$B_i = \{ \alpha \in A_i : U_{i\alpha} \cap Q \neq \emptyset \}.$$

Then  $B_i \neq \emptyset$  and  $\{B_i\}$  forms an inverse subsystem of  $\{A_i; \varphi_i^{i+1}\}$ . Since the condition of Lemma 1 is satisfied,  $K$  defined in Lemma 1 is a compact set of countable character. Since  $Q \subset K$ ,  $X$  is of countable type and the proof is finished.

#### References

- [1] A. V. Arhangel'skii, On a class of spaces containing all metric and all locally compact spaces, Dokl. Akad. Nauk SSSR 151(1963)751-754; Soviet Math Dokl. 4(1963)1051-1055.
- [2] Dennis K. Burke, On p-spaces and  $w\Delta$ -spaces, Pacific J. Math. 35(1970)285-296.
- [3] R. Engelking, 'Outline of general topology', North-Holland

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- [4] K. Nagami, Minimal class generated by open compact and perfect mappings, forthcoming.