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A Refinement to a Theorem of Davenport and Lewis.

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The theorem of H. Davenport and D. J. Lewis cited is in the paper "Exponential Sums in Many Variables" in American Journal of Mathematics, vol. 84, 1962, pp. 649/665. Let F_p be the Galois field with p elements, where p is a large prime. Let $F(X)$ be a cubic polynomial in n variables $X = (X_1, \cdots, X_n)$ with coefficients in F_p. Here we suppose $F(X)$ to be non-degenerate, i.e. that $F(X)$ cannot be transformed into a polynomial with fewer variables by any non-singular linear transformations. Let $F(X)$ be expressed as $F(X) = C(X) + Q(X) + L(X)$ + constant term, where C, Q and L are cubic, quadratic and linear part respectively. We define $h = h(C)$ to be the least number for which $C(X)$ is representable identically as

$$L_1 \cdot Q_1 + \cdots + L_h \cdot Q_h,$$

where $L_1, \cdots$ and $Q_1, \cdots$ are linear and quadratic forms respectively with coefficients in F_p. Obviously $h(C)$ is an invariant of C, and $0 \le h(C) \le C$, and $h(C) = 0$ if and only if C vanishes identically, and $h(C) = n$ if and only if C does not represent 0 non-trivially in F_p. Then they state, as Theorem 1 of their paper, the

[*Theorem*] (Davenport-Lewis) For a non-degenerate cubic polynomial $F(X)$ with coefficients in F_p, we have

$$\left| \sum_{X \in F_p^n} e\left(\frac{1}{p} F(X)\right) \right| \ll p^{n - \frac{1}{4} h(C)}.$$

Here and in the followings n is supposed to be fixed and the implied constants depend only on n. As usual, $e(x) = e^{2\pi\sqrt{-1}x}$. For the proof they use a polarization of $C(X)$.

We show that, for cubic forms, the exponent can be diminished by $\frac{1}{4}$ if $h(C) > 0$, by using Gauss sums, i.e., we have

[*Theorem* 1] Let $C(X)$ be a non-degenerate cubic form with coefficients in F_p, then we have

$$\left| \sum_{X \in F_p^n} e\left(\frac{1}{p} C(X)\right) \right| \ll p^{n - \frac{1}{4}(h(C)+1)},$$

if $h(C) \geqq 1$.

In the followings we suppose $C(X)$ to be non-degenerate in F_p, and $p > 3$.

(*Lemma* 1) Let $Q(X)$ be a quadratic form with coefficients in F_p, then we have

$$\sum_{X \in F_p^n} e\left(\frac{1}{p} Q(X)\right) = \varepsilon_p^{n-\lambda} \cdot \varepsilon \cdot p^{\lambda} \times p^{\frac{n-\lambda}{2}}$$

where $n - \lambda$ is the rank Q, $\varepsilon = \pm 1$ in general, $\varepsilon = \left(\frac{\Delta}{p}\right)$ if $\lambda = 0$ with $\Delta = \det Q$, and $\varepsilon_p = 1$ or i according as $p \equiv 1$ or 3 mod 4.

Proof : Well-known.

Let $C(X)$ be expressed as $\sum_{i,j,k}^{n} c_{ijk} X_i X_j X_k$, where the coefficients

c_{ijk} are symmetrical in i,j,k. Define $n \times n$ matrix $\mathfrak{h}(X)$ with $\sum_{k}^{n} c_{ijk}X_k$ as its (i, j)-th entry. The determinant $H(X)$ of $\mathfrak{h}(X)$ is the Hessian of $C(X)$.

(*Lemma* 2) The number of points $y \in F_p^n$ for which the matrix $\mathfrak{h}(y)$ has the rank $n - \lambda$ in F_p is of an order $0(p^{2n-h(C)-\lambda})$ if $n > \lambda \geq n-h(C)+1$, 1 if $\lambda = n$, and $0(p^{n-1})$ if $n - h(C) \geqq \lambda \geqq 1$.

Proof : The first is a restatement of Lemma 3 in the paper of Davenport and Lewis. The case $\lambda = n$ is suggested on the page 662 between the 11^{th} and 8^{th} lines from below. For the last statement we use the fact that, if $H(y) = 0$ as a polynomial and $p \geqq n+1$, then $C(X)$ is degenerate in F_p.

(*Lemma* 3) We have

$$\sum_{x \in F_p^n} \left(\frac{H(6x)}{p}\right) \cdot e\left(\frac{2}{p} C(x)\right) \ll p^{n-\frac{1}{2}}$$

if $n \geqq 1$ and $C(x)$ has non-trivial coefficients. Here $\left(\frac{*}{p}\right)$ is the Legendre symbol.

Proof : Easy.

Now we proceed to the proof of the Theorem.

We have

$$\left| \sum_{x \in F_p^n} \left(\frac{1}{p} C(x)\right) \right|^2 = \sum_{\bar{x}, \bar{\bar{x}}} \sum e\left(\frac{1}{p}(C(\bar{x}) - C(\bar{\bar{x}}))\right)$$

$$= \sum_{x,y} \sum e\left(\frac{1}{p} \sum_{ij} (6 \cdot \sum_k c_{ijk} y_k) x_i x_j + \frac{2}{p} C(y)\right)$$

by putting $\bar{x} = x + y$, $\bar{\bar{x}} = x - y$. If $n > h(C) \geq 1$, the above sum is equal to

$$\varepsilon_p^n \cdot \sum_{y \in F_p^n} \left(\frac{H(6y)}{p}\right) p^{\frac{n}{2}} \cdot e\left(\frac{2}{p} C(y)\right) + \sum_{\lambda=1}^{n-h(C)} 0\left(p^{\frac{n+\lambda}{2}}\right) \times 0\left(p^{n-1}\right)$$

$$+ \sum_{\lambda=n-h(C)+1}^{n-1} 0\left(p^{\frac{n+\lambda}{2}}\right) \times 0\left(p^{2n-h(C)-\lambda}\right) + p^n$$

$$= 0\left(p^{\frac{n}{2}}\right) \times 0\left(p^{n-\frac{1}{2}}\right) + 0\left(p^{\frac{1}{2}n+\frac{1}{2}(n-h(C))}\right) \times 0\left(p^{n-1}\right)$$

$$+ 0\left(p^{\frac{n}{2}+(2n-h(C))-\frac{1}{2}(n-h(C)+1)}\right) + p^n$$

$$= 0\left(p^{2n-\frac{1}{2}(h(C)+1)}\right).$$

If $h(C) = n$, the sum is equal to

$$\varepsilon_p^n \cdot \sum_{y \in F_p^n} \left(\frac{H(6y)}{p}\right) p^{\frac{n}{2}} \cdot e\left(\frac{2}{p} C(y)\right) + \sum_{\lambda=1}^{n-1} 0\left(p^{\frac{n+\lambda}{2}}\right) \times 0\left(p^{n-\lambda}\right) + p^n$$

$$= 0\left(p^{2n-\frac{1}{2}(n+1)}\right).$$

And we have the stated result.