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Notes on the topology of analytic sets

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§0. Introduction.

In this note, we will collect some results on the topology of analytic sets. First, in §1 we will construct hypersurfaces

$$V^n \subset \mathbb{C}P^{n+1}$$

for all $n \geqq 4$, $2n$ is not of the form $2^a - 2$ for any a, having the following properties.

(i) V^n has exactly one Brieskorn type isolated singularity.

(ii) V^n is a compact topological manifold without boundary.

(iii) V^n admits no differentiable structure.

The construction is due to Kuiper [10], who has constructed this kind of hypersurfaces for the case $n = 4$. Indeed this was my starting point.

The proof of the property (iii) depends on the Brumfiel's work on $\pi_*(PL/O) = \Gamma_*$, the group of oriented differentiable structures on the spheres [3], and on Brumfiel, Madsen, Milgram's recent work [5].

In §2, we will consider the problem :

How to calculate various numerical invariants of compact complex analytic variety, such as various characteristic numbers (if they exist), the Euler characteristic and the signature?

This problem has been solved by Kato [9] for the projective hypersurfaces with isolated singularity and by Hirzebruch [7] for the signature of (some kind of) compact normal complex variety of complex dimension two.

Using Kato's work, we will calculate numerical invariants of the hypersurfaces constructed in § 1.

In § 3, we will remark some elementary properties of recently introduced characteristic homology classes for analytic varieties.

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§ 1. Some hypersurfaces in $\mathbb{C}P^{n+1}$.

We define a hypersurface $V^n(d, \lambda, a) \subset \mathbb{C}P^{n+1}$ $(n \geqq 4)$ by the following homogeneous polynomial of degree d

$$f^d_{\lambda,a}(z_0, z_1, \cdots, z_{n+1}) = z_0^{a_0} z_{n+1}^{d-a_0} + z_1^{a_1} z_{n+1}^{d-a_1} + \cdots + z_n^{a_n} z_{n+1}^{d-a_n} + \sum_{i=0}^{n} \lambda_i z_i^d$$

where a_i is an integer $\geqq 2$, $d \geqq a_i$ for all $i = 0, 1, \cdots, n$ and $\prod_{i=0}^{n} \lambda_i \neq 0$.

Then it is easy to verify that if we choose $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ carefully, then $V^n(d, \lambda, a)$ has exactly one Brieskorn type singularity defined by the following polynomial

$$g_a(z_0, \cdots, z_n) = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0$$

(cf. [10]).

We fix λ (which of course depends on n, d and a) so that the condition (i) is satisfied and write $V^n(d, a)$ for $V^n(d, \lambda, a)$.

Let $\Gamma(a)$ be the graph of a (in the sense of Milnor). Then by Brieskorn, $V^n(d, a)$ is a topological manifold if and only if $\Gamma(a)$ satisfies some condition, which we may call the "Brieskorn condition" [2], [8].

Henceforth we assume that $\Gamma(a)$ satisfies this condition. Thus $V^n(d, a)$ is a compact topological manifold without boundary. Moreover $V^n(d, a) - \{x_0\}$ has a structure of complex analytic manifold, where $x_0 = [0, 0, \cdots, 0, 1]$ is the singular point of $V^n(d, a)$.

To study the property (iii), we consider the following general problem.

Problem. Let M^{4n} $(n \geq 2)$ be a compact topological manifold without boundary such that $M^{4n} - \{\text{point}\}$ admits a differentiable structure. Then when M admits a global differentiable structure (which may not coincide the original one on $M - \{\text{point}\}$)?

A partial answer to this problem can be obtained from the following theorem of Brumfiel.

Theorem (Brumfiel [3]). The Kervaire-Milnor exact sequence

$$0 \longrightarrow \Gamma_{4n-1}(\partial\pi) \underset{f}{\rightleftarrows} \Gamma_{4n-1} \longrightarrow \operatorname{cok} J \longrightarrow 0$$

splits and a canonical splitting f can be obtained as follows. Let $\Sigma^{4n-1} \in \Gamma_{4n-1}$ be a homotopy sphere. Then by a result in the spin cobordism ring Ω_*^{spin}, Σ^{4n-1} bounds a spin manifold N^{4n},

$$\partial N^{4n} = \Sigma^{4n-1} .$$

Moreover, by a slightly extended version of the Hattori-Stong theorem, we may assume that all the decomposable Pontrjagin numbers of N^{4n} vanish. Then

$$f(\Sigma^{4n-1}) = \frac{1}{8}\operatorname{sign}(N^{4n}) \quad \operatorname{mod} \# \Gamma_{4n-1}(\partial\pi)$$

where $\operatorname{sign}(N^{4n})$ is the signature of N^{4n} and $\# \Gamma_{4n-1}(\partial\pi)$ is the order of the cyclic group $\Gamma_{4n-1}(\partial\pi)$.

From this theorem, we conclude the following proposition (see also [4]).

Proposition 1-1. Let M^{4n} $(n \geqq 2)$ be a compact spin topological manifold with a differentiable structure on $M - \{\text{point}\}$. Let Σ_M be the homotopy sphere in the neighborhood of the "point". Then

$$f(\Sigma_M) = -2^{2n-2}(2^{2n-1} - 1)(\hat{\mathcal{A}}(M) - \hat{\mathcal{A}}(\bar{M})) \quad \operatorname{mod} \# \Gamma_{4n-1}(\partial\pi),$$

where $\bar{M}$ is a closed spin manifold whose decomposable Pontrjagin numbers are equal to those of M and $\hat{\mathcal{A}}(M)$ is the Borel-Hirzebruch's $\hat{\mathcal{A}}$-genus of M.

As corollaries, we obtain

Corollary 1-2. Let M be as above. If $f(\Sigma_M)$ is not diffeomorphic to the standard sphere S^{4n-1}, then M admits no differentiable structure.

Corollary 1-3. Let M^{4n} $(n = 2$ or $3)$ be a closed spin topological manifold which admits a differentiable structure on $M - \{\text{point}\}$. Then M admits a differentiable structure if and only if

$$\hat{\mathcal{A}}(M^{4n}) \in a_n M$$

where $a_n = 1$ if $n \equiv 0 \pmod 2$ and $a_n = 2$ if $n \equiv 1 \pmod 2$.

Now we go back to the original problem.

We constructed hypersurfaces

$$V^n(d, a) \subset \mathbb{C}P^{n+1}$$

such that $V^n(d, a)$ satisfies the conditions (i) and (ii). Thus $V^n(d, a)$ is a closed topological manifold. Let Σ_V be the homotopy sphere in the neighborhood of the singular point x_0. Then we have

Theorem 1-4.

(i) If $n \equiv d \equiv 0 \pmod 2$ with $n \geqq 4$, then $V^n(d, a)$ admits a differentiable structure if and only if Σ_V is diffeomorphic to the natural sphere.

(ii) If $n \equiv 1 \pmod 2$ and $2n$ is not of the form $2^a - 2$ for any a, then $V^n(d, a)$ admits a differentiable structure if and only if Σ_V is diffeomorphic to the natural sphere.

Proof. It is easy to verify that $V^n(d, a)$ is a spin manifold if and only if $n \equiv d \pmod 2$. Then (i) follows from Corollary 1-2. (ii) follows from the recent result of Brumfiel, Madsen and Milgram [5], that the Kervaire manifold K^{4n+2} ($4n+2$ is not of the form 2^a-2 for any a) is not topologically bordant to smooth manifold. Q. E. D.

§ 2. Numerical invariants of analytic varieties.

In this section, we will calculate various numerical invariants of $V^n(d, a)$ constructed in § 1. The calculation depends on the Kato's topological resolution theory [9].

Let $\tilde{V}^n(d, \mathbf{a}) \subset \mathbb{C}P^{n+1}$ be the Kato's topological resolution of $V^n(d, \mathbf{a})$. Roughly speaking, it can be obtained from $V^n(d, \mathbf{a})$ by deleting a closed small neighborhood of the singular point x_0 and "pasting" (in $\mathbb{C}P^{n+1}$) the non-singular affine variety $V_{\mathbf{a}}$ defined by

$$g_{\mathbf{a}} = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 1.$$

$\tilde{V}^n(d, \mathbf{a})$ is an almost complex submanifold of $\mathbb{C}P^{n+1}$ and the Poincaré dual of the fundamental class $[\tilde{V}^n]$ is $d\alpha$, where $\alpha \in H^2(\mathbb{C}P^{n+1}; \mathbb{Z})$ is the standard generator.

Therefore various numerical invariants of $\tilde{V}^n(d, \mathbf{a})$ such as characteristic number, the Euler characteristic and the signature are calculable and are equal to those of the non-singular hypersurface of degree d, which we will write $V^n(d)$.

Since $V_{\mathbf{a}}$ is parallelizable, we have the following result essentially due to Kato.

Proposition 2-1.

(i) All the Stiefel-Whitney classes except W_{2n} of $V^n(d, \mathbf{a})$ pulled back to $\mathbb{C}P^{n+1}$ by the Gysin homomorphism are equal to those of $V^n(d)$.

In particular all the decomposable Stiefel-Whitney numbers of $V^n(d, \mathbf{a})$ are equal to those of $V^n(d)$.

(ii) The Euler characteristic is given by

$$\chi(V^n(d, \mathbf{a})) = \frac{1}{d}\left\{ (1-d)^{n+2} + (n+2)d - 1\right\} + (-1)^{n+1} \prod_{i=0}^{n} (a_i - 1).$$

Now assume n is even, say $n = 2k$. Then

(iii) All the Pontrjagin classes except P_k of $V^n(d, a)$ pulled back to $\mathbb{C}P^{n+1}$ by the Gysin homomorphism are equal to those of $V^n(d)$. In particular, all the decomposable Pontrjagin numbers of $V^n(d, a)$ are equal to those of $V^n(d)$.

(iv) The signature of $V^n(d, a)$ is given by

$$\operatorname{sign} V^n(d, a) = \operatorname{sign} V^n(d) - \operatorname{sign} V_a .$$

(v) The Pontrjagin class $P_k(V^n(d, a))$ is determined by (iii) and the requirement that sign $V^n(d, a)$ is equal to the L-genus of $V^n(d, a)$.

§ 3. Some remarks on the characteristic homology classes for analytic variety.

In this section, we will study the recently introduced characteristic homology classes for analytic varieties. But we can say something only for the varieties whose singularities are isolated.

Now we recall the definition of the Stiefel-Sullivan homology classes for compact real analytic variety [11].

Definition 3-1. A triangulated compact pair (K, L) is said to be a relative Euler (resp. mod 2 Euler) space if the Euler characteristic (resp. mod 2 Euler characteristic) of the link of any vertex of $K - L$ is equal to zero. In case $L = \emptyset$, we say that K is an Euler (resp. mod 2 Euler) space.

This definition was motivated by the following theorem.

Theorem (Sullivan [11]). Let V be a compact complex (real) analytic variety and fix a Lojasiewicz triangulation. Then V is an Euler (mod 2 Euler) space.

Now the Stiefel-Sullivan homology classes for relative mod 2 Euler space (K, L) are defined as follows. Let $S_d K$ be the first barycentric subdivision of K. Let $c_i \in C_i(S_d K, S_d L; \mathbb{Z}/2)$ be the chain defined by

$$c_i = \sum_{\sigma^i \in S_d K - S_d L} \sigma^i .$$

Then it can be shown that c_i is actually a cycle. We define the i-th Stiefel-Sullivan class $s_i(K, L) \in H_i(K, L; \mathbb{Z}/2)$ by

$$s_i(K, L) = [c_i].$$

Now let M^n be a closed smooth manifold and fix a C^1-triangulation. Then clearly M is a mod 2 Euler space. Thus we have the Stiefel-Sullivan class $s_i(M) \in H_i(M; \mathbb{Z}/2)$. But this is nothing but the Poincaré dual of the Stiefel Whitney class $w_{n-i}(M) \in H^{n-i}(M; \mathbb{Z}/2)$. This fact was first observed by Whitney and recently proved by Cheeger.

Now assume M^n is a compact complex manifold of complex dimension n. Then there is the "Chern homology class"

$$\tilde{c}_i \in H^{2i}(M; \mathbb{Z})$$

which is the Poincaré dual of the ordinary Chern class $c_{n-i} \in H^{2n-2i}(M; \mathbb{Z})$. Observing the following facts

(i) $s_{2i+1}(M) = 0$ for all i,

(ii) $s_{2i}(M) = \tilde{c}_i(M)$ mod 2,

we consider the following question :

Question 3-2. Does this situation hold also for compact complex variety? i.e. if V^n is a compact complex variety of complex dimension n, then

(i) $s_{2i+1}(V) = 0$ for all i?

(ii) Can we define the "Chern homology class" $\tilde{c}_i \in H_{2i}(V; \mathbb{Z})$ so that

$$\tilde{c}_i \bmod 2 = s_{2i}(M) \ ?$$

The notion of "Chern homology class", I take from Sullivan's note [11], in which he says Deligne has constructed a candidate for the Chern homology classes.

Now we remark the following observation.

Proposition 3-3. Let V^n be a compact complex variety of complex dimension n. Then

(i) $s_{2i+1}(V) = 0$ for all i such that $2i+1 > 2 \dim_{\mathbb{C}} \Sigma V$.

(ii) If $i > \dim_{\mathbb{C}} \Sigma V$, then we can uniquely define the Chern homology class

$$\tilde{c}_i \in H_{2i}(V; \mathbb{Z}).$$

In particular, $s_{2n-1}(V) = 0$ for any V and if ΣV is isolated, then the question 3-2 is solved. (We put $\tilde{c}_0(V) = \chi(V)$, if V is connected.)

Proof. We first observe Lemma 3-4. Let M^n be a compact differentiable manifold with boundary $\partial M \neq \emptyset$. Fix a C^1-triangulation on M. Then

$$s_i(M, \partial M) \in H_i(M, \partial M; \mathbb{Z}/2)$$

is the Poincaré dual of the Stiefel Whitney class

$$w_{n-i}(M) \in H^{n-i}(M; \mathbb{Z}/2).$$

Proof. This can be reduced to the absolute case by considering the double of M.

Proof of Proposition 3-3. Let N be a regular neighborhood of ΣV and put $W = V - \mathrm{Int}\, N$. We may assume that W is an almost complex manifold. Consider the following exact sequence. (Coefficient is $\mathbb{Z}/2$ or $\mathbb{Z}$.)

$$\cdots \longrightarrow H_i(N) \xrightarrow{i_*} H_i(V) \xrightarrow{j_*} H_i(V, N) \xrightarrow{\partial} H_{i-1}(N) \longrightarrow \cdots .$$

$$H_i(V, N) \cong H_i(W, \partial W)$$

Now clearly

$$j_*(s_i(V)) = s_i(W, \partial W).$$

But since W is an almost complex manifold, by Lemma 3-4, we have

$$s_i(W, \partial W) = \begin{cases} 0 & i \equiv 1 \pmod 2 \\ \tilde{c}_k(W) \bmod 2 & i = 2k, \end{cases}$$

where $\tilde{c}_k(W)$ is the Poincaré dual of the Chern class $c_{n-k}(W) \in H^{2n-2k}(W; \mathbb{Z})$.

Now if $i > 2 \dim_{\mathbb{C}} \Sigma V$, then j_* is a monomorphism, for

$$H_i(N) \cong H_i(\Sigma V) = 0.$$

Hence we have

$$s_{2i+1}(V) = 0 \quad \text{for} \quad 2i+1 > 2 \dim_{\mathbb{C}} \Sigma V.$$

Next we define $\tilde{c}_i(V) \in H_{2i}(V; \mathbb{Z})$ for $i > \dim_{\mathbb{C}} \Sigma V$. Clearly if we could define $\tilde{c}_i(V)$, then we should have

$$j_*(\tilde{c}_i(V)) = \tilde{c}_i(W).$$

Since j_* is a monomorphism, we have only to show that

$$\partial c_i(W) = 0 .$$

We prove this for all i. Let $\pi: \tilde{V} \longrightarrow V$ be a resolution of V. Consider the following diagram [6].

$$\begin{array}{ccccccccc} & & & & H_i(W, \partial W) & & & & \\ & & & & \Vert\wr & & & & \\ \cdots \longrightarrow & H_i(V) & \xrightarrow{j_*} & & H_i(V, N) & \xrightarrow{\partial} & H_{i-1}(N) & \longrightarrow \cdots \\ & \uparrow \pi_* & & & \uparrow\wr\, \pi_* & & \uparrow \pi_* & \\ \cdots \longrightarrow & H_i(V) & \xrightarrow[\tilde{j}_*]{} & & H_i(\tilde{V}, \tilde{N}) & \xrightarrow{\partial} & H_{i-1}(\tilde{N}) & \longrightarrow \cdots \end{array}$$

where $\tilde{N} = \pi^{-1}(N)$.

Now to show $\partial \tilde{c}_i(W) = 0$, it suffices to show that $\partial \pi_*^{-1}(\tilde{c}_i(W)) = 0$. But clearly,

$$\pi_*^{-1}(\tilde{c}_i(W)) = \tilde{j}_* \tilde{c}_i(\tilde{V}),$$

where $\tilde{c}_i(\tilde{V})$ is the i-th Chern homology class of $\tilde{V}$. Hence $\partial \tilde{c}_i(W) = 0$. Q. E. D.

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