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Local deformation of polynomials with isolated singularities

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§1. Introduction and statement of results

Let $f(z)$ be a polynomial in $\mathbb{C}^n$ which has an isolated critical point at the origin and $f(0)=0$. Then there is the Milnor fibering for a small number $\varepsilon > 0$: $f/|f| : S_\varepsilon^{2n-1} - K_{f,\varepsilon} \longrightarrow S^1$ where $S_\varepsilon^{2n-1} = \{z \in \mathbb{C}^n \mid \|z\| = \varepsilon\}$ and $K_{f,\varepsilon} = S_\varepsilon^{2n-1} \cap f^{-1}(0)$ ([3]). In this paper, we shall prove that the Milnor fibering is invariant under a certain "deformation" (See Definition 1 and Theorem 1). By applying the above result, we shall prove that the Milnor fibering of a weighted homogeneous polynomial is determined by its weight (see Theorem 2) and the Milnor fibering of a polynomial which has a "weighted homogeneous principal part" has a periodic monodromy (see Definition 2 and Theorem 3).

First we define some notations. A subset $\mathcal{W}\mathcal{C}(n)$ of $\mathbb{C}[z_1, \ldots, z_n]$ is defined by $\mathcal{W}\mathcal{C}(n) = \{ f \in \mathbb{C}[z_1, \ldots, z_n] \mid f$ has an isolated critical point at the origin$\}$. $W(n)$ is a subset of $\mathcal{W}\mathcal{C}(n)$ defined by $W(n) = \{ f \in \mathcal{W}\mathcal{C}(n) \mid f :$ weighted homogeneous$\}$.

Definition 1. Let f and g be polynomials in $\mathcal{W}\mathcal{C}(n)$. A family $\{f_t\}_{t \in I}$ of polynomials in $\mathcal{W}\mathcal{C}(n)$ is called <u>a deformation from f to g</u> if and only if (i) $f_0 = f$ and $f_1 = g$. (ii) (Piecewise Analyticity) There exists a finite sequence $0 = t_0 < t_1 < \cdots < t_N = 1$ such that for each $k = 0, 1, \cdots, N-1$, $f_t(\mathbb{z})$ is complex analytic in $\mathbb{z}$ and t for $(\mathbb{z}, t) \in \mathbb{C}^n \times [t_k, t_{k+1}]$ i.e. there exists an analytic function f_k in $\mathbb{C}^n \times \mathbb{C}$ such that $f_k \mid \mathbb{C}^n \times [t_k, t_{k+1}] = f_t \mid \mathbb{C}^n \times [t_k, t_{k+1}]$. (iii) (Equi-isolatedness) There exists a positive number ε such that $\operatorname{grad} f_t(\mathbb{z}) = (\frac{\partial f_t}{\partial z_1}(\mathbb{z}), \ldots, \frac{\partial f_t}{\partial z_n}(\mathbb{z}))$ is a non-zero vector for each $t \in I$ and $\mathbb{z}$ satisfying $0 < \|\mathbb{z}\| \leq \varepsilon$ and S_ε^{2n-1} is transverse to $f_t^{-1}(0)$ for each $t \in I$.

It is clear that the above deformation defines an equivalence relation in $\mathcal{W}\mathcal{C}(n)$ and we denote the corresponding equivalence class of f by $[f]$ and the quotient set by $\widetilde{\mathcal{W}}\mathcal{C}(n)$.

Definition 2. Let f be a polynomial in $\mathcal{W}\mathcal{C}(n)$. Assume that f can be expressed as $f(z) = f_0(z) + g(z)$ where f_0 is a polynomial in $W(n)$ with I-weight $(q_1, q_2, \cdots, q_n; d)$ (i.e. $f_0(z_1^{q_1}, \ldots, z_n^{q_n})$ is a homogeneous polynomial of degree d) and $g(z)$ is a polynomial such that each monomial $z_1^{\nu_1} z_2^{\nu_2} \cdots z_n^{\nu_n}$ of $g(z)$ satisfies $\sum_{k=1}^{n} q_k \nu_k > d$. Then we call f_0 <u>a weighted homogeneous principal part</u> of f.

Now, our results are as follows.

<u>Theorem 1. Let $\{f_t\}_{t\in I}$ be a deformation from f to g $(f, g \in \mathcal{W}\mathcal{C}(n))$. Then there exists a positive number ε such that</u> (i) $K_{f,\varepsilon} = f^{-1}(0) \cap S_\varepsilon^{2n-1}$ <u>and</u> $K_{g,\varepsilon} = g^{-1}(0) \cap S_\varepsilon^{2n-1}$ <u>are isotopic i.e. there exists an isotopy</u> $h_t : S_\varepsilon^{2n-1} \to S_\varepsilon^{2n-1}$ <u>such that</u> $h_0 = id$. <u>and</u> $h_1(K_{f,\varepsilon}) = K_{g,\varepsilon}$. (ii) <u>There is a fibre-preserving diffeomorphism</u> $\psi : S_\varepsilon^{2n-1} - K_{f,\varepsilon} \to S_\varepsilon^{2n-1} - K_{g,\varepsilon}$ such <u>That is, the following diagram commutes.</u>

$$\begin{tikzcd} S_\varepsilon^{2n-1} - K_{f,\varepsilon} \arrow[rr, "\psi"] \arrow[dr, "f/|f|"'] & & S_\varepsilon^{2n-1} - K_{g,\varepsilon} \arrow[dl, "g/|g|"] \\ & S^1 & \end{tikzcd}$$

Theorem 2. If two polynomials f and g in $W(n)$ have the same weight, then $[f]=[g]$.

Theorem 3. Assume that a polynomial f in $\mathfrak{M}(n)$ has a weighted homogeneous principal part f_0. Then the monodromy of f is periodic.

The following corollaries are immediate consequences of Theorem 2.

Corollary 1. Let H be a non-singular hypersurface in $\mathbb{C}P^n$. Then H is isotopic to the standard hypersurface X^d where d is the degree of H and $X^d=\{z\in \mathbb{C}P^n \mid z_0^d+z_1^d+\cdots+z_n^d=0\}$.

Corollary 2. Let f be a polynomial in $W(n)$, having a weight $w=(w_1,\ldots,w_n)$. Assume that each w_k is an integer ≥ 1. Then f is equivalent to $g(z)=z_1^{w_1}+\cdots+z_n^{w_n}$.

§2 Proof of Theorem 1

Let $f_t(z)$ be a family of polynomials satisfying (i), (ii) and the first part of (iii) of the definition of deformations. The follows lemma is due to

Ramanujan.

Lemma 1. Let $\{f_t(z)\}_{t\in I}$ be as above. Then the monodromies of f_0 and f_1 coincide. If further $n\neq 3$, $K_{0,\varepsilon}=f_0^{-1}(0)\cap S^{2n-1}_\varepsilon$ and $K_{1,\varepsilon}=f_1^{-1}(0)\cap S^{2n-1}_\varepsilon$ are diffeomorphic.

proof: Let ε_0 be the stable radius of f_0. Then there is a $\eta>0$ such that $f_t^{-1}(0)$ and $S^{2n-1}_{\varepsilon_0}$ are transverse for each $t\in[0,\eta]$. Let W be $\{(z,t)\in S^{2n-1}_{\varepsilon_0}\times[0,\eta] \mid |f_t(z)|=\delta\}$ where δ is a small number such that $f_t^{-1}(\delta\cdot e^{i\theta})$ and S^{2n-1}_ε are transverse for each $t\in[0,\eta]$ and $\theta\in[0,2\pi]$. We consider the fibering $\varphi: W\longrightarrow S^1_\delta\doteq S^1$ defined by $\varphi(z,t)=f_t(z)$ and the projection $\pi: W\longrightarrow[0,\eta]$. It is easy to see that π is non-degenerate even if π is restricted to each fibre $F_\theta=\varphi^{-1}(\theta)$. Therefore, by using a connection vector field for π which is tangent to each fibres, we have a fibre-preserving diffeomorphism:

$$\begin{tikzcd} D^{2n}_{\varepsilon_0}\cap f_0^{-1}(S^1_\delta) \arrow[rr, "\exists\psi"] \arrow[dr, "f_0"'] & \arrow[d, phantom, "\circlearrowleft"] & f_\eta^{-1}(S^1_\delta)\cap D^{2n}_{\varepsilon_0} \arrow[dl, "f_\eta"] \\ & S^1 & \end{tikzcd}$$

Let ε_η be the stable radius of f_η. ($\varepsilon_\eta \le \varepsilon_0$). Then we have the following commutative diagram of fibration.

$$\begin{tikzcd} f_\eta^{-1}(S^1_\delta)\cap D^{2n}_{\varepsilon_0} \arrow[dr, "f_\eta"'] & & f_\eta^{-1}(S^1_\delta)\cap D^{2n}_{\varepsilon_\eta} \arrow[ll, hook'] \arrow[dl, "f_\eta"] \\ & S^1_\delta & \end{tikzcd}$$

Let $\widetilde{F} = f_\eta^{-1}(\delta)\cap D^{2n}_{\varepsilon_0}$ and $F = f_\eta^{-1}(\delta)\cap D^{2n}_{\varepsilon_\eta}$. Because the local degree of f_t is constant and by [3], we know that $\widetilde{F}$ and F is homotopic to a bouquet $S^{n-1}\vee\cdots\vee S^{n-1}$, therefore to prove that $F \hookrightarrow \widetilde{F}$ is homotopy equivalent, it suffices to prove that $H_{n-1}(F) \longrightarrow H_{n-1}(\widetilde{F})$ is a bijection. By using a suitable Morse function, $\widetilde{F}$ is obtained by attaching handles of index $\le n-1$ to F. Therefore $H_n(\widetilde{F},F)=0$ and $H_{n-1}(\widetilde{F},F)$ is torsion free. Then it is clear that $H_{n-1}(F) \longrightarrow H_{n-1}(\widetilde{F})$ is a bijection from the homology exact sequence of the couple $(\widetilde{F},F)$. The latter part of the Lemma is the consequence of h-cobordism theorem. Q.E.D.

Proof of the Theorem 1. Let $\{f_t\}_{t\in I}$ be a deformation from f to g. and let $\widetilde{K} = \{(z,t) \in S_\varepsilon^{2n-1} \times I;\ f_t(z) = 0\}$. Let $\pi : S_\varepsilon^{2n-1} \times I \longrightarrow I$ be the projection, then by the assumption of the deformation, $\pi : (S_\varepsilon^{2n-1} \times I, \widetilde{K}) \longrightarrow I$ is non-degenerate. Therefore by constructing a connection vector field for π, we have a desired isotopy from $K_{f,\varepsilon} = f^{-1}(0) \cap S_\varepsilon^{2n-1}$ and $K_{g,\varepsilon} = g^{-1}(0) \cap S_\varepsilon^{2n-1}$ by the 1-parameter group of the connection. This completes the proof of the first part. Let δ be a small positive number such that $f_t^{-1}(\eta)$ and S_ε^{2n-1} are transverse for each $|\eta| = \delta$. Let $E = \{(z,t) \in D_\varepsilon^{2n} \times I \,|\, |f_t(z)| = \delta\}$ and $\varphi : E \to S_\delta^1 = S^1$ be defined by $\varphi(z,t) = f_t(z)$. Then φ is a fiber bundle. Let $\widehat{\pi}$ be the projection $W \to I$. Then $\widehat{\pi}$ is non-degenerate on each fibre of φ. Thus we have a fibre preserving diffeomorphism ψ

$$\begin{array}{ccc} f_0^{-1}(S_\delta^1) \cap D_\varepsilon^{2n} & \xrightarrow{\quad\psi\quad} & f_1^{-1}(S_\delta^1) \cap D_\varepsilon^{2n} \\ & {\scriptstyle f=f_0} \searrow \quad \swarrow {\scriptstyle f_1 = g} & \\ & S_\delta^1 & \end{array}$$

By Theorem 5.11 of [3], there is a fibre-preserving diffeomorphism φ_i $(i=0,1)$

$$\begin{tikzcd}
f_i^{-1}(S^1_\delta)\cap D^{2n}_\epsilon \arrow[rr, "\varphi_i"] \arrow[dr, "f_i"'] & \arrow[d, phantom, "\circlearrowleft"] & S^{2n-1}_\varepsilon - f_i^{-1}(\mathrm{Int}\, D^2_\delta) \arrow[dl, "f_i/|f_i|"] \\
 & S^1_\delta = S^1 &
\end{tikzcd}$$

Combining these diffeomorphisms, we obtain a fibre preserving diffeomorphism φ:

$$\begin{tikzcd}
S^{2n-1}_\varepsilon - K_{f,\varepsilon} \arrow[rr] \arrow[dr, "f/|f|"'] & \arrow[d, phantom, "\circlearrowleft"] & S^{2n-1}_\varepsilon - K_{g,\varepsilon} \arrow[dl, "g/|g|"] \\
 & S^1 &
\end{tikzcd}$$

Q.E.D.

§3. Proof of the theorem 2.

Let $f(z)$ be a polynomial in $\mathbb{C}^n$. $f(z)$ is called a <u>weighted homogeneous polynomial</u> of type $w=(w_1,\dots,w_n)$

where each w_k is a positive rational number, if and only if each monomial $z_1^{\nu_1} z_2^{\nu_2} \cdots z_n^{\nu_n}$ of f satisfies $\sum_{k=1}^{n} \frac{\nu_k}{w_k} = 1$. We call w the <u>weight</u> of f. Expressing $w_k = \frac{u_k}{v_k}$, we define positive integers $q_1, \dots, q_n; d$ by $d = [u_1, \dots, u_n]$ (the least common multiple) and $q_k = d/w_k$. We call $(q_1, \dots, q_n; d)$ <u>I-weight</u> of $f(z)$. Then from the definition, $g(z) \equiv f(z_1^{q_1}, \dots, z_n^{q_n})$ is a homogeneous polynomial of degree d. More generally we consider $(Q^+)^n = Q^+ \times \cdots \times Q^+$ where Q^+ is the set of positive rational numbers and we call $w \in (Q^+)^n$ an <u>abstract weight</u>. For a given $w = (w_1, \dots, w_n) \in (Q^+)^n$, let $S(w) = \{ \nu = (\nu_1, \dots, \nu_n) \in N^n \mid \sum_{k=1}^{n} \frac{\nu_k}{w_k} = 1 \}$. ($N$: natural numbers). Clearly $S(w)$ is a finite set.

Definition 3. Let $\rho(w)$ be the number of elements of $S(w)$. For each point $t = (t_\nu)_{\nu \in S(w)} \in \mathbb{C}^{\rho(w)}$, we define a polynomial W_t by $W_t(z) = \sum_{\nu \in S(w)} t_\nu z_1^{\nu_1} z_2^{\nu_2} \cdots z_n^{\nu_n}$. A subset $V(w)$ of $\mathbb{C}^{\rho(w)}$ is defined by

$$V(w) = \{ t = (t_\nu) \in \mathbb{C}^{\rho(w)} \mid W_t(z) \notin \mathcal{M}(n) \}.$$

We call $V(w)$ <u>the characteristic set of w</u>.

<u>Lemma 6.</u> <u>For each weight $w = (w_1, w_2, \dots, w_n) \in (Q^+)^n$, $V(w)$ is an algebraic set defined by homogeneous polynomials.</u>

Proof: We consider the branched covering $\varphi: \mathbb{C}^n \to \mathbb{C}^n$ defined by $\varphi(x_1, \ldots, x_n) = (x_1^{q_1}, \ldots, x_n^{q_n})$ where $(q_1, \ldots, q_n; d)$ is the I-weight of w. Then $w_t(\varphi(x))$ and $\frac{\partial W_t}{\partial z_k}(\varphi(x))$ are homogeneous polynomials in x and their degrees are d and $d - q_k$ respectively. Let $\tilde{V}(w) = \{([x_1 : \cdots : x_n], t) \in \mathbb{C}P^{n-1} \times \mathbb{C}^{\rho(w)} \mid \frac{\partial W_t}{\partial z_k}(\varphi(x)) = 0,\ k = 1, 2, \ldots, n\}$ and $\pi : \mathbb{C}P^{n-1} \times \mathbb{C}^{\rho(w)} \to \mathbb{C}^{\rho(w)}$ be the projection. Then clearly $V(w) = \pi(\tilde{V}(w))$, therefore by the Remmert's proper mapping theorem ([2], p. 162), $V(w)$ is an analytic set. But $V(w)$ is clearly a cone at the origin, and therefore $V(w)$ is an algebraic set defined by homogeneous polynomials by the Chow's theorem (see for example Satz 14 of [6]). Q.E.D.

Remark: Lemma 6 can also be proved by a purely algebraic method. (see [8], §§ 80, 81).

Proof of Theorem 2: Let f and g be polynomials in $W(n)$, having the same weight $w = (w_1, \ldots, w_n)$. We consider the associated characteristic set $V(w)$. Let $t(f)$ and $t(g)$ be points in $\mathbb{C}^{\rho(w)}$ corresponding to f and g i.e. $W_{t(f)} = f$ and $W_{t(g)} = g$.

Then by the assumption, $t(f) \notin V(w)$ and $t(g) \notin V(w)$. Thus $V(w)$ is a proper algebraic set of $\mathbb{C}^{P(w)}$ and there is a piecewise linear path $P: I \longrightarrow \mathbb{C}^{P(w)} - V(w)$ from $t(f)$ to $t(g)$. We define $f_t = W_{P(t)}$ for each $t \in I$. Then it is clear that $\{f_t\}$ satisfies i), ii) and the first part of iii). Assume $\lambda \bar{z} = \operatorname{grad} f_t(z)$ for some $z \neq 0$ and $\lambda \in \mathbb{C}^*$ where $f_t(z) = 0$. Because $f_t(z)$ satisfies the equation:

$$f_t(z) = \sum_{k=1}^{n} \frac{z_k}{w_k} \cdot \frac{\partial f_t}{\partial z_k}(z)$$

This means $\sum \frac{\lambda}{w_k} |z_k|^2 = 0$. This is a contradiction. Q.E.D.

§4. Proof of Theorem 3.

Let f be a polynomial in $\mathcal{M}(n)$ and assume that f has a weighted homogeneous principal part f_0. Let $(q_0, \ldots, q_n; d)$ be the I-weight of f_0 and consider the mapping $\varphi: \mathbb{C}^n \to \mathbb{C}^n$ $(\varphi(x_1, \ldots, x_n) = (x_1^{q_1}, \ldots, x_n^{q_n}))$. Then $f_0(\varphi(x))$ and $\frac{\partial f_0}{\partial z_k}(\varphi(x))$ are homogeneous polynomials of degree d and $d - q_k$ respectively. We prove that $f_t(z) = f_0(z) + t\,g(z)$, where $g(z) = f(z) - f_0(z)$, satisfies the condition of the Lemma 1. Let $M = \underset{x \in S^{2n-1}}{\text{minimum}} \|\operatorname{grad} f_0(\varphi(x))\|$ where $\operatorname{grad} f_0(\varphi(x)) = \left(\overline{\frac{\partial f_0}{\partial z_1}(\varphi(x))}, \ldots, \overline{\frac{\partial f_0}{\partial z_n}(\varphi(x))}\right)$. Then M is a positive number. Let $U_k = \{x \in S^{2n-1} \mid |\frac{\partial f_0}{\partial z_k}(\varphi(x))| > M/2\sqrt{n}\}$, and $CU_k' = \{rx \mid 0 < r \leq 1,\ x \in U_k\}$. Then $\bigcup_k U_k = S^{2n-1}$. For each fixed k, let $x \in CU_k'$ and let $\eta = \|x\|$. Then by the above observation $|\frac{\partial f_0}{\partial z_k}(\varphi(x))| > M/2\sqrt{n} \cdot \eta^{d-q_k}$, while $|\frac{\partial g}{\partial z_k}(\varphi(x))| / |\eta|^{d-q_k+1}$ is bounded. Therefore we can find $\varepsilon_k > 0$ so that $|\frac{\partial f_0}{\partial z_k}(\varphi(x))| > 2 \cdot |\frac{\partial g}{\partial z_k}(\varphi(x))|$ for each $x \in CU_k' \cap D^{2n}_{\varepsilon_k}$. Let $\varepsilon' = \underset{1 \leq k \leq n}{\text{minimum}}\ \varepsilon_k$ and choose ε such that $D^{2n}_{\varepsilon} = \{z \in \mathbb{C}^n \mid \|z\| \leq \varepsilon\}$ is contained in $\varphi(D^{2n}_{\varepsilon'}) = \{z \in \mathbb{C}^n \mid |z_1|^{2/q_1} + \cdots + |z_n|^{\frac{2}{q_n}} \leq \varepsilon'^2\}$. It is obvious that ε satisfies the condition of the Lemma 1. Q.E.D.

Example: Let $f(z) = z_1^p + z_2^q + z_1^r z_2^s$. Then $f \in \mathcal{WC}(2)$. Case 1: $r/p + s/q > 1$ and let $f_0(z) = z_1^p + z_2^q$. Then f_0 is a weighted homogeneous principal part of f. Case 2. $r/p + s/q = 1$.

Then $f \in W(2)$ and $f_0 \in W(2)$ and $[f] = [f_0]$.

Case 3. $r/p + s/q < 1$ and $r, s \geqq 2$. Then f has ~~non~~ no weighted homogeneous principal part.

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