数理解析研究所講究録
第 158 巻 1972 年 70-77

On mod 3 characteristic classes for F_4-bundles.

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Let F_4 be simply connected exceptional Lie group of rank 4. The mod p characteristic classes for F_4-bundles are in one-to-one correspondence with the elements of the mod p cohomology ring $H^*(BF_4; Z_p)$ of the classifying space BF_4 of F_4.

For $p \neq 3$, the structure of $H^*(BF_4; Z_p)$ is determined from that of $H^*(F_4; Z_p)$ by use of Borel's theorems:

$$H^*(BF_4; Z_2) = Z_2[x_4, x_6, x_7, x_{16}, x_{24}],$$

$$H^*(BF_4; Z_p) = Z_p[x_4, x_{12}, x_{16}, x_{24}] \quad \text{for } p \neq 2,3,$$

where $x_i \in H^i$, $x_6 = Sq^2 x_4$, $x_7 = Sq^3 x_4$, $x_{24} = Sq^8 x_{16}$ for $p = 2$, and $\mathcal{P}^1 x_i = x_j$ if $j = i + 2p - 2$ and $p \geq 5$.

For $p = 3$, however, it seems very difficult to determine $H^*(BF_4; Z_3)$ directly from the result

(1) $H^*(F_4; Z_3) = Z_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15})$,

where $x_7 = \mathcal{P}^1 x_3$, $x_8 = \Delta x_7$ and $x_{15} = \mathcal{P}^1 x_{11}$.

In fact, S. Araki [1] showed that the elements x_{11} and x_{15}

are not transgressive, hence we cannot apply Borel's method to the case $p = 3$.

The purpose of the present speech is to determine the structure of $H^*(BF_4; Z_3)$ by use of the fibering

(2) $\Pi \longrightarrow BSpin(9) \xrightarrow{\rho} BF_4$,

where $\Pi = F_4/Spin(9)$ is Cayley plane. Since the natural map $BSpin(9) \longrightarrow BSO(9)$ has mod 3 trivial fibre BZ_2, the map induces an isomorphism of $H^*(\ ; Z_3)$. Thus we identify $H^*(BSpin(9); Z_3)$ with $H^*(BSO(9) : Z_3)$, which is the polynomial algebra of Pontrjagin classe $p_i \in H^{4i}$;

$$H^*(BSpin(9); Z_3) = Z_3 \left[p_1, p_2, p_3, p_4 \right].$$

First we prepare the following

Lemma 1. *The image of*

$$\rho^* : H^*(BF_4; Z_3) \longrightarrow H^*(BSpin(9); Z_3)$$

is contained in the subalgebra A generated by

$$p_1, \ \bar{p}_2 = p_2 - p_1^2, \quad \bar{p}_5 = p_4 p_1 + p_3 \bar{p}_2,$$

$$\bar{p}_9 = (p_3^2 - p_4 p_1^2)(p_3 + \bar{p}_2 p_1) \ \text{and} \ \bar{p}_{12} = p_4^3 + p_4^2 \bar{p}_2^2 + p_4 \bar{p}_2^4,$$

thus A is isomorphic to $Z_3[\bar{p}_1, \bar{p}_2, \bar{p}_5, \bar{p}_9, \bar{p}_{12}]/(r_{15})$

for $r_{15} = \bar{p}_5^3 + \bar{p}_5^2\bar{p}_2^2 p_1 - \bar{p}_{12}p_1^3 - \bar{p}_9\bar{p}_2^3$.

This lemma is proved as follows. Let BT be the classifying space of a maximal torus T of Spin(9). The Weyl group $\Phi' = \Phi(\mathrm{Spin}(9))$ of Spin(9) operates on $H^*(BT; Z_3) = Z_3[t_1, t_2, t_3, t_4]$ as parmutations and the changment of signs. As is well known, under the homomorphism ρ^{t*} induced by the natural map $\rho^t: BT \longrightarrow B\,\mathrm{Spin}(9)$, $H^*(B\,\mathrm{Spin}(9); Z_3)$ is mapped isomorphically onto the subalgebra $H^*(BT; Z_3)^{\Phi'}$ invariant under Φ' which is a polynomial algebra of the variables $\rho^{t*}(p_i)$(= the i-th elementary symmetric function on $t_1^2, t_2^2, t_3^2, t_4^2$). For the Weyle group $\Phi = \Phi(F_4)$ of F_4, the image of $(\rho\rho')^* = \rho'^*\rho^*$ is contained in the invariant subalgebra $H^*(BT; Z_3)^{\Phi} = \rho'^*A$. Φ is generated by Φ' and an element which operates on $Z_3[t_1, t_2, t_3, t_4]$ as the neflection with respect to the plane $t_1 + t_2 + t_3 + t_4 = 0$ (cf. [2]). Then the lemma is proved by purely algebraic computations.

Consider the mod 3 cohomology spectral sequence $(E_r^{*,*})$ associated with the fibering (2), where

$E_2^{*,*} = H^*(BF_4; Z_3) \otimes \{1, w, w^2\}$, $w \in H^8(\Pi; Z_3)$, and

$E_\infty^{*,*}$ is a graded algebra associated to $Z_3[p_1, p_2, p_3, p_4]$.

Thus the differential d_r is trivial unless $r = 9,17$.

By the aid of Lemma 1, we have the following

Theorem 1. $H^*(BF_4; Z_3) = Z_3[x_{36}, x_{48}] \otimes$

$$\left[Z_3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} + \Lambda(x_9) \otimes Z_3[x_{26}] \otimes \{1, x_{20}, x_{21}, x_{25}\}\right].$$

Schetch of the proof. For instance, we compute $H^*(BF_4; Z_3)$ step by step as follows.

(i) There exists x_4 such that $\rho^*(x_4) = p_1$. (easy)

(ii) There exists x_8 such that $\rho^*(x_8) = p_2$, since

$p_2 + p_1^2 = -\mathcal{P}^1(p_1) = \rho^*(-\mathcal{P}^1 x_4)$

(iii) The transgression image x_9 of w is non-zero, since $d_9(1 \otimes w) = 0$ implies rank $(E_\infty^{0,8} + E_\infty^{8,0}) = 3 >$ rank $(H^8(B\,Spin(9)))$ $= 2$.

(iv) The natural map

(3) $\quad Z_3[x_4, x_8] + \{x_9\} \to H^*(BF_4)$

is monic and it is epic for $\deg < 12$. The first half follows from (i), (ii), (iii) and $Z_3[p_1, p_2] \subset H^*(B\ Spin(9))$.

(v) The relation $x_4x_9 = 0$ holds and (3) is epic for deg < 16. Assume that $x_4x_9 = d_9(x_4 \otimes W) \neq 0$, then $E_\infty^{4,8} = 0$ and thus $\rho^* : H^{12}(BF_4) \to H^{12}(B\ Spin(9))$ is epic. But this contradicts to the result $p_3 \notin \mathrm{Im}\ \rho^*$ of Lemma 1. Thus we have the assestion of (v). Remark that $x_4 \otimes W \in E_\infty^{4,8}$ represents $p_3 (\mathrm{mod}\ \mathrm{Im}\ \rho^*)$. Similarly we have

(Vi) The relation $x_8x_9 = 0$ holds and (3) is epic for deg < 20. $x_8 \otimes W$ is a parmanent cycle and represents $p_4 (\mathrm{mod}\ \mathrm{Im}\ \rho^* + \{p_3p_1\})$.

(vii) There exists an element x_{20} such that $\rho^* x_{20} = \overline{p}_5 = p_4p_1 + p_3p_2 - p_3p_1^2$. The parmanent cycle $x_4x_8 \otimes W$ represents both of $p_2(\pm p_3 + \text{lower term})$ and $p_1\ (\pm p_4 + \text{lower term})$. This shows that $p_4p_1 \pm p_3p_2 + (\text{lower term})$ is in $E_\infty^{0,20}$, i.e., in the mage of ρ^*. Such an element should be $\overline{p}_5$ mod (p_1, p_2) by Lemma 1. Thus the existence of x_{20} is proved.

(viii) $x_4^2 \otimes w^2$ represents x_{12}^2 since $x_4 \otimes W$ does x_{12}.

By the same reason as (iii) we have

(ix) $x_{21} = d_{17}(x_4 \otimes W^2) \neq 0$, $x_{25} = d_{17}(x_8 \otimes w^2) \neq 0$

and $x_{26} = d_{17}(x_9 \otimes W^2) \neq 0$.

Continuing the computation a little more, we have the existence of the elements x_{36}, x_{48} such that

(x) $\rho^*(x_{36}) = \overline{p}_9$ and $\rho^*(x_{48}) = \overline{p}_{12}$.

Finally we prove the theorem by the aid of comparison theorem.

Corollary Ker ρ^* is an ideal generated by x_9, x_{21}, x_{25}, x_{26}, and ρ^* maps the subalgebra generated by x_4, x_8, x_{20}, x_{36}, x_{48} isomorphically onto Im $\rho^* = A$.

Let $\widetilde{BF}_4$ be the 4-connective fibre space over BF_4. we have a fibering

(4) $\widetilde{BF}_4 \rightarrow BF_4 \xrightarrow{p} K(Z, 4)$.

The loop-space of BF_4 is a 3-connective fibre space over F_4, say $\widetilde{F}_4$. From [3]

$$H^*(\widetilde{F}_4; Z_3) = \Lambda(x_{11}, x_{15}, x_{19}, x_{23}) \otimes Z_3[x_{18}]$$

where $x_{15} = \mathcal{P}^1 x_{11}$, $x_{19} = \Delta x_{18}$ and $x_{23} = \mathcal{P}^1 x_{19}$. By spectral sequence arguments we have

$$H^*(\widetilde{BF}_4;\ Z_3) = \Lambda(y_{19}) \otimes Z_3[y_{12}, y_{16}, y_{20}, y_{24}]$$

for deg $<$ 55. On the others hand

$$H^*(K(Z,4);\ Z_3) = Z_3[x_4, x_8, x_{20}, x_{26}, \ldots] \otimes \Lambda(x_9, x_{21}, x_{25}, \ldots),$$

where $x_8 = \mathcal{P}^1 x_4$, $x_{20} = \mathcal{P}^3 x_8$, $x_9 = \Delta x_8$, $x_{21} = \Delta x_{20}$, $x_{25} = \mathcal{P}^4 x_9$, $x_{26} = \Delta x_{25}$. Consider the cohomology spectral sequence associated with the fibering, then by dimensional reasons we have

<u>Lemma 2.</u> $\Delta \mathcal{P}^4 \Delta \mathcal{P}^1 x_4 = \pm\, p^*(x_{26}) \neq 0.$

This leads us to the following

<u>Theorem 2. In Theorem 1, we can choose</u>

$$x_8 = \mathcal{P}^1 x_4, \qquad x_9 = \Delta \mathcal{P}^1 x_4,$$

$$x_{20} = \mathcal{P}^3 \mathcal{P}^1 x_4, \qquad x_{21} = \Delta \mathcal{P}^3 \mathcal{P}^1 x_4,$$

$$x_{25} = \mathcal{P}^4 \Delta \mathcal{P}^1 x_4, \qquad x_{26} = \Delta \mathcal{P}^4 \Delta \mathcal{P}^1 x_4$$

<u>and</u> $x_{48} = \mathcal{P}^3 x_{36}$

<u>Corollary, For the above choice of generators, the following relations hold:</u>

$$x_4 x_9 = x_8 x_9 = x_4 x_{21} = x_8 x_{25} = x_{20} x_{21} = x_{20} x_{25} = 0,$$

$$x_4 x_{25} = -x_8 x_{21} = x_9 x_{20}, \qquad x_4 x_{26} = -x_9 x_{21},$$

$$x_8 x_{26} = -x_9 x_{25}, \qquad x_{21} x_{25} = -x_{20} x_{26}.$$

These relations are obtained from the relation $x_4 x_9 = 0$ by applying cohomology operations.

References

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