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# The signature theorem for differentiable manifolds and some elementary number theory

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(Notes by S. Morita)

## §1. A formula for the signature of a compact normal complex surface.

Suppose we have a compact oriented manifold $X^{4k}$ with boundary $\partial X^{4k} = Y^{4k-1}$, which may be empty. Consider the following cohomology exact sequence (in these notes, the coefficients will be the rational numbers $\mathbb{Q}$, unless otherwise stated).

$$\cdots \longrightarrow H^{2k}(X, Y) \xrightarrow{\psi} H^{2k}(X) \longrightarrow H^{2k}(Y) \longrightarrow \cdots .$$

We define a quadratic form $B$ on $H^{2k}(X, Y)$ by

$$B(\alpha, \beta) = (\alpha \cup \beta)[g]$$

where $\alpha, \beta \in H^{2k}(X, Y)$ and $g \in H_{4k}(X, Y)$ is the orientation class.

Then the signature of $X$, sign $X$, is defined by

$$\text{sign } X = \text{sign } B = p^+ - p^-$$

where $p^+$ ($p^-$) is the dimension of a maximal subspace of $H^{2k}(X, Y)$ on which $B$ is positive (negative) definite.

Now it is easy to see that $\alpha \in \operatorname{Ker}\psi$ if and only if $B(\alpha, \beta) = 0$ for all $\beta \in H^{2k}(X, Y)$. Therefore the quadratic form $B$ is defined essentially on $\operatorname{Im}\psi \subset H^{2k}(X)$, and then it is non-degenerate.

The following lemma is a simple consequence of the Poincaré duality theorem.

Lemma 1. The quadratic form $B$ on $H^{2k}(X, Y)$ is non-degenerate if and only if $\psi$ is an isomorphism.

The definition of the signature above shows that, to define the signature for some space $X$ such that $H_{2k}(X)$ has finite dimension, we have only to give a "fundamental class" $[X] \in H_{4k}(X)$.

For if $[X]$ exists, then we can define the signature of $X$ exactly the same way as for compact oriented manifolds. Precisely, we define the signature of $X$ to be that of the quadratic form $B$ on $H^{2k}(X)$ defined by

$$B(\alpha, \beta) = (\alpha \cup \beta)[X], \qquad \alpha, \beta \in H^{2k}(X).$$

Now by Borel and Haefliger [1], any compact complex analytic variety $M$ of complex dimension $n$ has a fundamental class $[M] \in H_{2n}(M; \mathbb{Z})$. Therefore, if $n$ is even, we can define the signature of $M$.

Now we consider the case $n = 2$. Thus let $M$ be a compact normal complex surface and let $\Sigma M$ be the set of singular points of $M$. Since $M$ is normal and compact, $\Sigma M$ is a finite subset of $M$.

Now $M - \Sigma M$ is a complex manifold. Though it is not compact, it has a compact differentiable manifold $N$ with boundary as a deformation retract. Thus we have the signature of $M - \Sigma M$. On the other hand, we have the signature of $M$ itself. But since

$$H^2(M) \xrightarrow{\sim} H^2(M - \Sigma M)$$

and the quadratic forms on $H^2(M)$ and $H^2(M - \Sigma M)$ are the same under this identification, it follows that

$$\text{sign}\, M = \text{sign}\,(M - \Sigma M).$$

To calculate sign M, we blow up the singularities of M. Since the singular points of M are isolated, we can make the blowing up process locally. Thus let $U_P$ be a "nice" neighborhood of a singular point P so that $U_P - \{P\}$ is a manifold with compact boundary $\partial U_P$ and $U_P$ is homeomorphic to the cone over $\partial U_P$. Let

$$\pi : U_P' \longrightarrow U_P$$

be a resolution of the singularity P. Then we can write

$$U_P' = \pi^{-1}(U_P - \{P\}) \cup (S_1 \cup \cdots \cup S_r)$$

where $S_i$ is a compact irreducible curve in $U_P'$. Among the possible resolutions, there is a unique minimal one and any resolution can be obtained from it by successive blowing ups.

Now a classical theorem says that the intersection matrix $(S_i \circ S_j)$ is negative definite. Clearly $\partial U_P' = \partial U_P$, because the boundary was not changed.

We have

$$H_2(U_P' ; \mathbb{Z}) \cong \mathbb{Z} S_1 \oplus \cdots \oplus \mathbb{Z} S_r$$

where $S_i$ is considered as a cycle. By the Poincaré-Lefschetz duality,

$$H_2(U_P' ; \mathbb{Z}) \cong H^2(U_P' , \partial U_P' )$$

and under this isomorphism, the quadratic form on $H_2(U_P')$ defined by the interesection numbers and that on $H^2(U_P', \partial U_P')$ defined by the cup-product (evaluated on the fundamental class) correspond each other.

Since the quadratic form of $U_p'$ is negative definite, by Lemma 1 and the above observation, the homomorphism

$$\varphi: H^2(U_p', \partial U_p') \longrightarrow H^2(U_p')$$

is an isomorphism and

$$\text{sign } U_p' = - r.$$

Now we recall the signature theorem for compact differentiable manifolds without boundary. It says that the signature can be expressed as a linear combination of various Pontrjagin numbers with rational coefficients.

Let $X$ be a compact complex manifold of complex dimension two. Then the signature theorem simply says that

$$\text{sign } X = \frac{1}{3}P_1[X]$$
$$= \frac{1}{3}(c_1^2[X] - e(X))$$

where $c_1^2[X]$ is the Chern number and $e(X)$ is the Euler number of $X$. Now we ask whether the signature theorem holds for compact almost complex manifold with non empty boundary. Thus let $X$ be a compact almost complex manifold of complex dimension two (which may have a boundary in the differentiable sense).

The first problem is to define the Chern number $c_1^2$ for $X$. In general, this is impossible. But if we assume

(*) $$c_1(X) \in \text{Im}(\varphi: H^2(X, Y) \longrightarrow H^2(X)),$$

then we can define the Chern number $c_1^2[X]$ as follows. Take an element $x \in H^2(X, Y)$ such that $\varphi(x) = c_1(X)$. Then we have

$$c_1^2[X] = x^2[g] \qquad [g] \in H_4(X, Y), \text{ the orientation class.}$$

This does not depend on the choice of $x$, for the quadratic form is defined essentially on Im $\wp$ .

Now a simple example; the 4-disk $D^4$ (with trivial almost complex structure) shows that the signature theorem is not true for compact manifolds with non-empty boundary. For

$$c_1^2[D^4] = 0$$
$$e(D^4) = 1$$

but

$$\operatorname{sign}(D^4) = 0.$$

Now assuming (*), we introduce an invariant;

$$\frac{1}{3}(c_1^2[X] - 2e(X)) - \operatorname{sign} X.$$

(Recall that the assumption (*) is satisfied for our case $U_P{}'$).

Now we go back to our original problem, i.e. to calculate the signature of compact normal complex surface.

We define for each singular point $P \in \Sigma M$, the invariant $\wp(P) \in Q$ by

$$\wp(P) = \frac{1}{3}(c_1^2[U_P{}'] - 2e(U_P{}')) - \operatorname{sign} U_P{}'$$

where $U_P{}'$ is a resolution of the singularity P. $\wp(P)$ depends only on the singularity. This can be checked as follows. As mentioned earlier, any resolution can be obtained from the minimal one by successive blowing ups. But differentio-topologically, blowing up one point is equivalent to the connected sum with $-\mathbb{C}P^2$. Now we check how the numbers $c_1^2$, e and the signature change by blowing up. It can be shown that $c_1^2$ goes down by one, e goes up by one and the signature goes down by one, therefore the value $\frac{c_1^2 - 2e}{3}$ - sign does not change.

We prove

Proposition 2. Let $M$ be a compact normal complex surface.

$$\text{sign } M = \frac{1}{3}(c_1^2[M - \Sigma M] - 2e(M - \Sigma M)) + \sum_{P \in \Sigma M} \varphi(P).$$

Proof. Let $M'$ be a resolution of $M$. We can write

$$M' = (M - \bigcup_{P \in \Sigma M} U_P) \cup (\bigcup_{P \in \Sigma M} U_P').$$

Now by the Novikov additivity of the signature, we have

(1) $\quad \text{sign } M' = \text{sign } (M - \Sigma M) + \sum_{P \in \Sigma M} \text{sign } U_P'$ ,

We also have the additivity of the Euler number and the Chern number which follows from a Mayer-Vietoris argument. Thus we have

(2) $\quad e(M') = e(M - \Sigma M) + \sum_{P \in \Sigma M} e(U_P')$

(3) $\quad c_1^2[M'] = c_1^2(M - \Sigma M) + \sum_{P \in \Sigma M} c_1^2[U_P']$ .

From (1), (2) and (3), we obtain the formula. Q. E. D.

## § 2. Quotient and cyclic singularities and some connections with elementary number theory.

Let $G_p = \{\zeta \mid \zeta^p = 1\}$, the group of p-th roots of unity and let $q$ be a number relatively prime to $p$ $(0 < q < p)$.

$G_p$ acts on $\mathbb{C}^2$ by

$$\zeta\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \zeta^q & z_1 \\ \zeta & z_2 \end{pmatrix}, \quad z_1, z_2 \in \mathbb{C}.$$

Take the quotient space $\mathbb{C}^2/G_p$, then it is a complex space with one singularity, which we call the quotient singularity of type $(p, q)$. Let $\varphi(p; q)$ be the $\varphi$ of this singularity. Then we have

Theorem 3. $\quad \varphi(p; q) = \dfrac{\text{def}(p;q) - \frac{2}{3}}{p}$ where

$$\operatorname{def}(p;\, q) = -\sum_{j=1}^{p-1} \cot \frac{\pi j}{p} \cot \frac{\pi q j}{p} .$$

This theorem was motivated by the discussion of the equivariant signature theorem for 4-manifolds in [4]. The number $\operatorname{def}(p;\, q)$ is related to the classical Dedekind sum $(q, p)$, see [4]

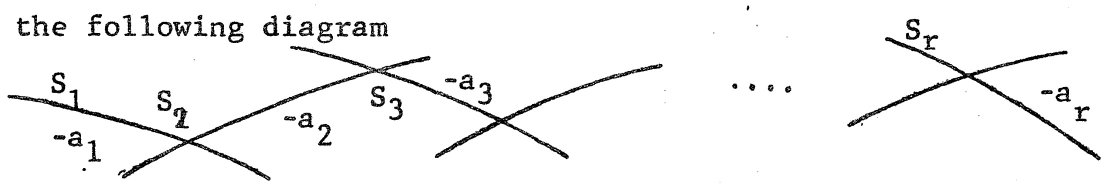
$$\operatorname{def}(p;\, q) = -\frac{2}{3}(q, p).$$

This is the first relation with number theory.

Now to calculate $\varphi(p;\, q)$, we blow up the quotient singularity. Let

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_r}}}$$

be the development of $\frac{p}{q}$ into the continued fraction with $a_i \geq 2$. Then the resolution of the quotient singularity can be represented by the following diagram




where $S_i$ is a rational curve and each $S_i$ intersects with the following $S_{i+1}$ transversely and the self-intersection number is

$$S_i \circ S_i = -a_i .$$

(See [3].)

Now since the intersection matrix is negative definite, we have

$$\varphi(p;\, q) = \frac{1}{3}(c_1^2 - 2(r+1)) + r .$$

We can write

$$\tilde{c}_1 = \sum \lambda_i S_i, \quad \lambda_i \in \mathbb{Q} ,$$

here $\tilde{c}_1$ is the Poincaré dual of $\varphi^{-1}(c_1)$. Then

$$(\sum_{i=1}^{r} \lambda_i S_i) \circ S_j = 2 - a_j .$$

This is the classical "adjunction formula". It can be checked as follows. Let $M$ be the manifold obtained by blowing up the given singularity and let $i : S_j \longrightarrow M$ be the inclusion. Then

$$\begin{aligned} \tilde{c}_1 \circ S_j &= \langle c_1(M), S_j \rangle \\ &= \langle i^* c_1(M), S_j \rangle \\ &= \langle c_1(\tau(M) \mid S_j), S_j \rangle \\ &= \langle c_1(S_j) + c_1(\nu), S_j \rangle \\ &= e(S_j) + S_j \circ S_j \\ &= 2 - a_j \end{aligned}$$

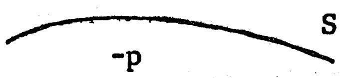
where $\nu$ is the normal bundle of $S_j$ in $M$.

Now since $\det(S_i \circ S_j) \neq 0$, we can determine $\lambda_i$ and hence $\tilde{c}_1$. Thus we can calculate $\varphi(p; q)$ to get the formula in Theorem 3. This was carried out by Don Zagier.

Instead of doing the complete calculation, we only check the following simple example.

Example. The quotient singularity of type $(p, 1)$.

The resolution configulation is




The Chern class is

$$\tilde{c}_1 = \frac{p-2}{p} S .$$

Therefore

$$c_1^2 = -\frac{(p-2)^2}{p} .$$

Hence

$$\varphi(p, 1) = \frac{c_1^2 - 4}{3} + 1$$

$$= - \frac{p^2-3p+4}{3p} .$$

A simple formula for def(p, 1), see [4], shows that $\varphi(p, 1)$ is equal to $\frac{\mathrm{def}(p,1) - \frac{2}{3}}{p}$ as stated in the theorem.

Now let M be a compact normal complex surface which has only quotient singularities. Then we can improve the statement in Proposition 2 for such M as follows.

First we introduce the Euler number $\tilde{e}(M)$ in the sense of Satake [10]. For this we consider a triangulation of M for which all singular points are vertices. Then

$$\tilde{e}(M) = \tilde{s}_0 - s_1 + s_2$$

where $s_j$ $(j \geqq 1)$ equals the number of j-dimensional simplices of the triangulation. In $\tilde{s}_0$ we count each 0-simplex with multiplicity 1 if it is a non-singular point and with multiplicity $\frac{1}{p}$ if it is a quotient singularity of order p. Thus

$$e(M) = \tilde{e}(M) + \sum_p \frac{p-1}{p} a_p$$

where $a_p$ is the number of quotient singularities of order p. Clearly,

$$e(M - \Sigma M) = e(M) - \sum_p a_p .$$

Therefore,

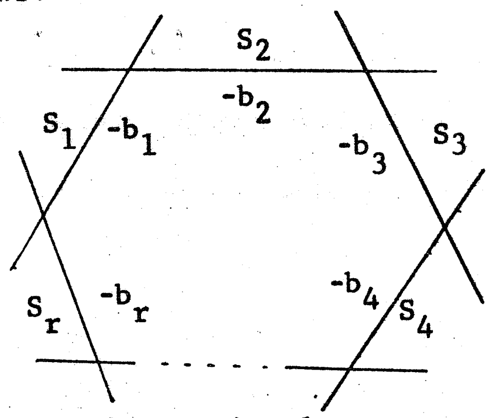
$$e(M - \Sigma M) = \tilde{e}(M) - \sum_p \frac{1}{p} a_p .$$

We define $c_1^2[M]$ to be equal to $c_1^2[M-\Sigma M]$. Then we can write the formula in Proposition 2 as follows

$$\text{sign } M = \frac{1}{3}(c_1^2[M] - 2\tilde{e}(M)) + \sum_{P\in\Sigma M} \tilde{\varphi}(P)$$

where $\tilde{\varphi}(P) = \frac{1}{p}\text{def}(p; q)$.

Next suppose we have a singularity whose resolution configuration looks as follows.




$r \geq 2$

where $S_j$ is a non-singular rational curve, one curve intersects with the following one transversely with $S_j \circ S_{j+1} = 1$ and $S_j \circ S_j = -b_j \leqq -2$. Since the intersection matrix must be negative definite, there must be at least one $j$ such that $S_j \circ S_j \leqq -3$.

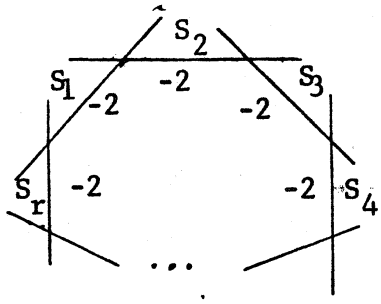
We will write $((b_1, b_2, \cdots, b_r))$ for this configuration.

We propose the following question.

<u>Question</u>. Given $((b_1, b_2, \cdots, b_r))$ with $b_j$ : natural number $\geqq 2$ and there is at least one $j$ such that $b_j \geqq 3$.

Does there exist cyclic singularity with the required resolution configuration?

The answer is yes. Here we show the existence as follows. By Kodaira ([6], especially diagram $I_b$ on p.565 and Table I, type $I_b$ on p.604) there is a configuration




for any $r$, and all the self-intersection numbers are equal to $-2$. On the curve $S_j$ we blow up $(b_j - 2)$ points, which must be different from the points where two curves intersect. Then the wanted configuration is obtained. For, since the intersection matrix $(S_i \circ S_j)$ is negative definite, by a theorem of Grauert [2], we can blow down the curves $\{S_j\}$ to obtain a singularity. But it is not known whether the singularity obtained above is equivalent to the "canonical" one constructed in the Tokyo-IMU lecture (see [5]).

Its structure as cyclic singularity may also depend on the choice of the points to be blown down.

Now we calculate the $\varphi$ for a cyclic singularity of type $((b_1, \cdots, b_r))$ (as mentioned above, the singularity does not depend only on $((b_1, \cdots, b_r))$, but $\varphi$ depends only on $((b_1, \cdots, b_r))$ ).

Since all the curves are rational, we have

$$\bar{c}_1 \circ S_j = S_j \circ S_j + 2.$$

Hence

$$\tilde{c}_1 = \sum_{j=1}^{r} S_j \ .$$

Therefore the Chern number $c_1^2$ of our singularity is

$$c_1^2 = \sum_{j=1}^{r} - b_j + 2r \ .$$

On the other hand, since the zero-th and the first Betti numbers are equal to 1, the Euler number is

$$e = r.$$

Therefore

$$\varphi((b_1, \cdots, b_r)) = \frac{-\sum_{j=1}^{r} b_j}{3} + r .$$

This gives many relations to number theory and is important for the study of the Hilbert modular group. As one example, we give a special case of a theorem mentioned in [5].

Theorem 4. Let $p$ be a prime number such that $p \equiv 3 \pmod 4$, $p > 3$ and $h(\mathbb{Q}(\sqrt{p})) = 1$ and let $((b_1, \cdots, b_r))$ be the primitive period of the continued fraction development of $\sqrt{p}$,

$$\sqrt{p} = a_1 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ddots \cfrac{1}{b_r - \cfrac{1}{b_1 - \ddots}}}}$$

with $a_1, b_j \geq 2$, $b_r = 2a_1$. Then

$$((b_1, \cdots, b_r)) = -h(\mathbb{Q}(\sqrt{-p}))$$

where $h(k)$ is the class number of the field $k$.

This is the second connection with number theory.

Finally we remark that our invariant $\varphi$ for a cyclic singularity is related to some other algebraic or number-theoretic invariants.

Precisely, for a cyclic singularity of type $((b_1, \cdots, b_r))$, the boundary of a "nice" neighborhood of $S_1 \cup \cdots \cup S_r$ is a torus bundle over the circle. If we represent the torus by $\mathbb{R}^2/\mathbb{Z}^2$, this bundle can be given by an element $A$ of $SL(2, \mathbb{Z})$, which represents the identification of the two boundary components of $[0, 1] \times T^2$. (Of course, A can be replaced by a conjugate element in $SL(2, \mathbb{Z})$).

As matrix A for the torus bundle belonging to the period $((b_1, \cdots, b_r))$, we can take

$$A = \begin{pmatrix} 0 & -1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & b_r \end{pmatrix}$$

$$= \begin{pmatrix} -q' & -q \\ p' & p \end{pmatrix}$$

where

$$p/q = b_1 - \cfrac{1}{b_2 - \cdots - \cfrac{1}{b_r}}$$

$$p'/q' = b_1 - \cfrac{1}{b_2 - \cdots - \cfrac{1}{b_{r-1}}} .$$

For $SL(2, \mathbb{Z})$, Rademacher [9] and C. Meyer [7] have studied a function

$$\varphi : SL(2, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

defined by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a + d - 2(d, c)}{c} - 3 \operatorname{sgn} c \, (a + d) \qquad \text{(for } c \neq 0\text{)}$$

where $(d, c)$ is again the Dedekind sum [4]. The value of $\varphi$ depends only on the conjugacy class of the element of $SL(2, \mathbb{Z})$.

The following formula is related to work of W. Meyer [8] on torus bundles.

For the matrix $A$ above

$$\varphi(A) = \sum_{j=1}^{r} b_j - 3r.$$

Therefore,

$$\varphi(A) = -3\varphi((b_1, \cdots, b_r)).$$

This relates our invariant $\varphi$ for cyclic singularities with the function $\varphi$ of Rademacher and C. Meyer. The relation to L-functions of real quadratic number fields was explained in [5].

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